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SOME GENERAL CONCEPTS OF SUB- AND SUPERSOLUTIONS FOR NONLINEAR ELLIPTIC PROBLEMS

Vy Khoi Le — Klaus Schmitt

Abstract. We propose general and unified concepts of sub- supersolutions for boundary value problems that encompass several types of boundary conditions for nonlinear elliptic equations and variational inequalities. Various, by now classical, sub- and supersolution existence and comparison results are covered by the general theory presented here.

1. Introduction — Problem settings

We are interested here in sub-supersolution results for boundary value problems with second order principal operators and general boundary conditions. The problems may or may not contain obstacles or constraints. Based on the weak (variational) formulation of the problem, we deduce that the boundary conditions (or at least parts of them) may usually be encoded into the set of test (admissible) functions.

The goal of this paper is to show that in several cases (covering those that have been studied in the literature), by formulating the problem as a variational inequality, even if it is a smooth equation, we may give simple, unified, and general definitions of sub- and supersolutions. These concepts of sub- and supersolutions extend the classical definitions for equations subject to Dirichlet,

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Neumann, Robin, or No-Flux (periodic boundary conditions for the one space dimensional problem) boundary conditions (see e.g. [16]) and are motived by the recent definitions of sub-supersolutions for variational inequalities in [11], [13], [14]. Also, we can demonstrate the existence of solutions and extremal solutions between sub- and supersolutions and other properties of the solution sets when sub- and supersolutions exist.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary and $W^{1,p}(\Omega)$ be the usual first-order Sobolev space with the norm

$$\|u\| = \|u\|_{W^{1,p}(\Omega)} = (\|u\|^p_{L^p(\Omega)} + \||\nabla u|\|^p_{L^p(\Omega)})^{1/p}, \quad u \in W^{1,p}(\Omega). \tag{1.1}$$

Assume that K is a closed, convex subset of $W^{1,p}(\Omega)$. We consider the following variational inequality on K:

$$\begin{cases} \displaystyle\int_\Omega A(x,\nabla u)\cdot(\nabla v - \nabla u)\,dx + \int_\Omega f(x,u)(v-u)\,dx \\ \qquad\displaystyle + \int_{\partial\Omega} g(x,u)(v-u)\,dS \geq 0, \quad \text{for all } v \in K, \\ u \in K. \end{cases} \tag{1.2}$$

We remark that in order to simplify the notation we use u and v instead of $u|_{\partial\Omega}$ and $v|_{\partial\Omega}$ for the trace of u and v on $\partial\Omega$ in the surface integral in (1.2). This simplification will also be used in the sequel in other instances when it is clear from the context. In the variational inequality (1.2), A is an elliptic operator, f is the lower order term, and g is a boundary term.

Problems such as (1.2), in the case of (smooth) equations, i.e. K is a subspace of $W^{1,p}(\Omega)$, have been studied by sub-supersolution methods in, e.g. [7], [5], [16] and some of the references therein, subject to different boundary conditions (usually homogeneous ones). In previous papers, sub- and supersolutions are defined using inequality conditions on the boundary. Therefore, different boundary conditions require different definitions of sub- and supersolutions. As a consequence, separate arguments and calculations are needed to study the existence and properties of solutions between sub- and supersolutions. In what follows, we show that common, unified definitions of sub- and supersolutions may be given for various types of boundary conditions (including unilateral constraints). Thus a common, comprehensive general existence theorem is possible for many different types of boundary value problems. The sub-supersolution approach for variational inequalities with homogeneous Dirichlet boundary conditions was studied in a systematic way in [13]. Our results here are motivated by and also generalize those in the papers [11], [13], [14].

We begin with the assumptions on the principal operator. Assume that

$$A\colon \Omega \times \mathbb{R}^N \to \mathbb{R}$$

is a Carathéodory function satisfying the growth condition

$$|A(x,\xi)| \le a_1(x) + b_1|\xi|^{p-1}, \quad \text{for a.e. } x \in \Omega, \text{ all } \xi \in \mathbb{R}^N, \tag{1.3}$$

with $p \in [1,\infty)$ (fixed), $a_1 \in L^{p'}(\Omega)$, $1/p + 1/p' = 1$, and $b_1 > 0$. Moreover, A is monotone, i.e.

$$(A(x,\xi) - A(x,\xi')) \cdot (\xi - \xi') \ge 0, \quad \text{for a.e. } x \in \Omega, \text{ all } \xi, \xi' \in \mathbb{R}^N, \tag{1.4}$$

and A is coercive in the following sense: there exist $a_2 \in L^1(\Omega)$ and $b_2 > 0$ such that

$$A(x,\xi) \cdot \xi \ge b_2|\xi|^p - a_2(x), \quad \text{for a.e. } x \in \Omega, \text{ all } \xi \in \mathbb{R}^N. \tag{1.5}$$

We also suppose that $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $g\colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions subject to certain growth conditions to be specified later.

In what is to follow we shall use the standard notation $u \wedge v = \min\{u, v\}$, $u \vee v = \max\{u, v\}$, $U * V = \{u * v : u \in U,\ v \in V\}$, and $u * V = \{u\} * V$, where $u, v \in W^{1,p}(\Omega)$, $U, V \subset W^{1,p}(\Omega)$ and $* \in \{\wedge, \vee\}$.

We propose the following general definitions of sub- and supersolutions of inequality (1.2).

Definition 1.1. A function $\underline{u} \in W^{1,p}(\Omega)$ is called a subsolution of (1.2) if the following conditions are satisfied:

$$f(\,\cdot\,, \underline{u}) \in L^q(\Omega), g(\,\cdot\,, \underline{u}) \in L^{\widetilde{q}}(\partial\Omega), \tag{1.6}$$

where $q \in (1, p^*)$ and $\widetilde{q} \in (1, \widetilde{p}^*)$,

$$\underline{u} \vee K \subset K, \tag{1.7}$$

and, for all $v \in \underline{u} \wedge K$,

$$\int_\Omega A(x, \nabla\underline{u}) \cdot \nabla(v - \underline{u})\, dx + \int_\Omega f(x, \underline{u})(v - \underline{u})\, dx + \int_{\partial\Omega} g(x, \underline{u})(v - \underline{u})\, dS \ge 0. \tag{1.8}$$

Here, p^* is the Sobolev conjugate exponent of p

$$p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } N > p \text{ and } N > 1, \\ \infty & \text{if } N \le p \text{ or } N = 1. \end{cases}$$

and

$$\widetilde{p}^* = \begin{cases} \dfrac{(N-1)p}{N-p} & \text{if } N > p \text{ and } N > 1, \\ \infty & \text{if } N \le p \text{ or } N = 1. \end{cases}$$

We have a similar definition for supersolutions of (1.2).

Definition 1.2. A function $\overline{u} \in W^{1,p}(\Omega)$ is called a supersolution of (1.2) if the following conditions are satisfied:

$$f(\,\cdot\,,\overline{u}) \in L^{q}(\Omega), \qquad g(\,\cdot\,,\overline{u}) \in L^{\widetilde{q}}(\partial\Omega), \tag{1.9}$$

where $q \in (1,p^*)$ and $\widetilde{q} \in (1,\widetilde{p}^*)$,

$$\overline{u} \wedge K \subset K, \tag{1.10}$$

and, for all $v \in \overline{u} \vee K$,

$$\int_\Omega A(x,\nabla \overline{u}) \cdot \nabla(v-\overline{u})\,dx + \int_\Omega f(x,\overline{u})(v-\overline{u})\,dx + \int_{\partial\Omega} g(x,\overline{u})(v-\overline{u})\,dS \geq 0.$$

The following is our main existence theorem, it's proof is patterned after the arguments used in [10], [11], [13].

Theorem 1.3. *Assume there exists a pair of sub- and supersolution of* (1.2) *such that* $\underline{u} \leq \overline{u}$ *and that* f *and* g *satisfy the following growth conditions between* $\underline{u}$ *and* $\overline{u}$*:*

$$|f(x,u)| \leq a_3(x), \quad |g(\xi,v)| \leq \widetilde{a}_3(\xi),$$

for almost all $x \in \Omega$, $\xi \in \partial\Omega$, *all* $u \in [\underline{u}(x),\overline{u}(x)]$, $v \in [\underline{u}(\xi),\overline{u}(\xi)]$, *where* $a_3 \in L^{q'}(\Omega)$, $\widetilde{a}_3 \in L^{\widetilde{q}'}(\partial\Omega)$, $q \in (1,p^*)$, $\widetilde{q} \in (1,\widetilde{p}^*)$, *and* p^*, $\widetilde{p}^*$ *are defined as in Definition* 1.1. *Then, there exists a solution* u *of* (1.2) *such that* $\underline{u} \leq u \leq \overline{u}$.

Proof. Let $r = \max\{p,q,\widetilde{q}\}(< p^*)$ and put

$$b(x,u) = \begin{cases} [u-\overline{u}(x)]^{r-1} & \text{if } u > \overline{u}(x), \\ 0 & \text{if } \underline{u}(x) \leq u \leq \overline{u}(x), \\ -[\underline{u}(x)-u]^{r-1} & \text{if } u < \underline{u}(x), \end{cases} \tag{1.13}$$

for $x \in \Omega$, $u \in \mathbb{R}$, and

$$(Tu)(x) = \begin{cases} \overline{u}(x) & \text{if } u(x) > \overline{u}(x), \\ u(x) & \text{if } \underline{u}(x) \leq u \leq \overline{u}(x), \\ \underline{u}(x) & \text{if } u < \underline{u}(x), \end{cases}$$

for $x \in \overline{\Omega}$, $u \in W^{1,p}(\Omega)$. Note that (1.14) is understood almost everywhere with respect to the Lebesgue measure in Ω when $x \in \Omega$ and with respect to the surface measure on $\partial\Omega$ when $x \in \partial\Omega$. Straightforward calculations show that

$$|b(x,u)| \leq a_4(x) + b_4|u|^{r-1}, \tag{1.15}$$

for a.e. $x \in \Omega$, $u \in \mathbb{R}$, where $a_4 \in L^{r'}(\Omega)$. Because of the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$, the operator $B\colon W^{1,p}(\Omega) \to [W^{1,p}(\Omega)]^*$, given by

$$\langle Bu,v\rangle = \int_\Omega b(x,u)v\,dx \quad (u,v \in W^{1,p}(\Omega))$$

is well defined, bounded, and completely continuous. Moreover, there are $a_5, b_5 > 0$ such that

$$\langle Bu, u\rangle \geq b_5\|u\|^r_{L^r(\Omega)} - a_5, \quad \text{for all } u \in W^{1,p}(\Omega). \tag{1.16}$$

In fact, for $u \in W^{1,p}(\Omega)(\subset L^r(\Omega))$, we have

$$\begin{aligned}
\langle Bu, u\rangle &= \int_{\{x\in\Omega: u(x)>\overline{u}(x)\}} (u-\overline{u})^{r-1}u\,dx - \int_{\{x\in\Omega: u(x)<\underline{u}(x)\}} (\underline{u}-u)^{r-1}u\,dx \\
&\geq \int_{\{x\in\Omega: u(x)>\overline{u}(x)\}} (C_1|u|^r - C_2|\overline{u}|^{r-1}|u|)\,dx \\
&\quad + \int_{\{x\in\Omega: u(x)<\underline{u}(x)\}} (C_1|u|^r - C_2|\underline{u}|^{r-1}|u|)\,dx \\
&\geq C_1\int_\Omega |u|^r\,dx - C_1\int_\Omega (|\underline{u}| + |\overline{u}|)^r\,dx \\
&\quad - \frac{C_1}{2}\int_\Omega |u|^r\,dx - C_2\int_\Omega (|\underline{u}|^{r-1} + |\overline{u}|^{r-1})^{r/(r-1)}\,dx \\
&= b_5\int_\Omega |u|^r\,dx - a_5.
\end{aligned}$$

Let us define

$$\langle F(u), v\rangle = \int_\Omega f(x,u)v\,dx,$$

and

$$\langle G(u), v\rangle = \int_{\partial\Omega} g(x,u)v\,dS, \quad u, v \in W^{1,p}(\Omega).$$

It follows from (1.12), (1.14), the continuity of the mapping $T\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)$, the compactness of the mappings

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega), \quad u \mapsto u,$$

and

$$W^{1,p}(\Omega) \hookrightarrow L^{\widetilde{q}}(\partial\Omega), \quad u \mapsto u|_{\partial\Omega},$$

that the mappings

$$u \mapsto F(Tu) = (F\circ T)(u) \quad \text{and} \quad u \mapsto G(Tu) = (G\circ T)(u)$$

are bounded and completely continuous from $W^{1,p}(\Omega)$ to its dual. Let us now define $\mathcal{A}$ from $W^{1,p}(\Omega)$ to $[W^{1,p}(\Omega)]^*$ by

$$\langle \mathcal{A}u, v\rangle = \int_\Omega A(x,\nabla u)\cdot\nabla v\,dx \quad (u, v \in W^{1,p}(\Omega)).$$

From our assumptions (1.3)–(1.5), it can be proved that $\mathcal{A}$ is bounded, continuous, monotone, and coercive in the following sense:

$$\langle \mathcal{A}u, v\rangle \geq b_2\||\nabla u|\|^p_{L^p(\Omega)} - \|a_2\|_{L^1(\Omega)}, \quad \text{for all } u \in W^{1,p}(\Omega). \tag{1.17}$$

Consider the following variational inequality on K:

$$\begin{cases} \langle \mathcal{A}u + Bu + F(Tu) + G(Tu), v - u\rangle \geq 0, & \text{for all } v \in K, \\ u \in K. \end{cases} \tag{1.18}$$

To show that (1.18) has solutions, we first observe that because $\mathcal{A}$ is monotone, bounded, and continuous, and $B, F \circ T$, and $G \circ T$ are completely continuous and bounded on $W^{1,p}(\Omega)$, the operator $\mathcal{A} + B + F \circ T + G \circ T$ is pseudomonotone on that function space. Next, let us check that $\mathcal{A} + B + F \circ T + G \circ T$ is coercive in the following sense:

$$\lim_{\|u\| \to \infty} \frac{\langle \mathcal{A}u + Bu + F(Tu) + G(Tu), u - u_0\rangle}{\|u\|} = \infty, \tag{1.19}$$

for any $u_0 \in K$ fixed. In fact, we have

$$\begin{aligned} &\langle \mathcal{A}u + Bu + F(Tu) + G(Tu), u - u_0\rangle \\ &\quad \geq \langle \mathcal{A}u, u\rangle + \langle Bu, u\rangle - |\langle \mathcal{A}u, u_0\rangle| - |\langle Bu, u_0\rangle| \\ &\qquad - |\langle F(Tu), u - u_0\rangle| - |\langle G(Tu), u - u_0\rangle| \\ &\quad \geq b_2 \||\nabla u|\|^p_{L^p(\Omega)} - \|a_2\|_{L^1(\Omega)} + b_5 \|u\|^r_{L^r(\Omega)} - a_5 \\ &\qquad - \int_\Omega |A(x, \nabla u)||\nabla u_0|\, dx - \int_\Omega |b(x,u)||u_0|\, dx \\ &\qquad - \int_\Omega |f(x, Tu)|(|u| + |u_0|)\, dx - \int_{\partial\Omega} |g(x, Tu)|(|u| + |u_0|)\, dS. \end{aligned} \tag{1.20}$$

With C being a generic positive constant, we have the following estimates:

$$\begin{aligned} \int_\Omega |A(x,\nabla u)||\nabla u_0|\, dx &\leq b_1 \int_\Omega |\nabla u|^{p-1} |\nabla u_0|\, dx + \int_\Omega a_1 |\nabla u_0|\, dx \\ &\leq b_1 \||\nabla u|\|^{p-1}_{L^p(\Omega)} \||\nabla u_0|\|_{L^p(\Omega)} + \|a_1\|_{L^{p'}(\Omega)} \||\nabla u_0|\|_{L^p(\Omega)} \\ &\leq \frac{b_2}{2} \||\nabla u|\|^p_{L^p(\Omega)} + C \||\nabla u_0|\|^p_{L^p(\Omega)} + \|a_1\|_{L^{p'}(\Omega)} \||\nabla u_0|\|_{L^p(\Omega)} \\ &\leq \frac{b_2}{2} \||\nabla u|\|^p_{L^p(\Omega)} + C. \end{aligned} \tag{1.21}$$

It follows from (1.15) that

$$\begin{aligned} \int_\Omega |b(x,u)||u_0|\, dx &\leq b_4 \|u\|^{r-1}_{L^r(\Omega)} \|u_0\|_{L^r(\Omega)} + \|a_4\|_{L^{r'}(\Omega)} \|u_0\|_{L^r(\Omega)} \\ &\leq \frac{b_5}{3} \|u\|^r_{L^r(\Omega)} + C. \end{aligned} \tag{1.22}$$

From (1.14) and (1.12), one obtains

$$\begin{aligned} \int_\Omega |f(x, Tu)|(|u| + |u_0|)\, dx &\leq \|a_3\|_{L^{q'}(\Omega)} (\|u\|_{L^q(\Omega)} + \|u_0\|_{L^q(\Omega)}) \\ &\leq C(\|a_3\|_{L^{q'}(\Omega)} \|u\|_{L^r(\Omega)} + 1) \leq \frac{b_5}{3} \|u\|^r_{L^r(\Omega)} + C. \end{aligned} \tag{1.23}$$

Also,

$$\int_{\partial\Omega} |g(x,Tu)|(|u|+|u_0|)\,dS \le \|\widetilde{a}_3\|_{L^{q'}(\partial\Omega)}(\|u\|_{L^{\widetilde{q}}(\partial\Omega)}+\|u_0\|_{L^{\widetilde{q}}(\partial\Omega)}) \le C(\|u\|_{W^{1,p}(\Omega)}+1) \tag{1.24}$$

(because of the continuity of the trace operator from $W^{1,p}(\Omega)$ to $L^{\widetilde{q}}(\partial\Omega)$). Combining the estimates in (1.20)–(1.24) with (1.20), we get

$$\begin{aligned}\langle \mathcal{A}u+Bu+F(Tu)+G(Tu),u-u_0\rangle &\ge \frac{b_2}{2}\||\nabla u|\|^p_{L^p(\Omega)}+\frac{b_5}{3}\|u\|^r_{L^r(\Omega)}-C(\|u\|+1)\\ &\ge b_6\|u\|^p-a_6(\|u\|+1) \quad (\text{since } r\ge p),\end{aligned} \tag{1.25}$$

for all $u\in W^{1,p}(\Omega)$, where $a_6,b_6>0$. Since $p>1$, (1.25) immediately implies (1.19).

The existence of solutions of (1.18) now follows from classical existence results for elliptic variational inequalities (cf. e.g. [15]). Let u be any solution of (1.18). We shall prove that

$$\underline{u}\le u\le \overline{u}, \tag{1.26}$$

and thus u is also a solution of (1.2). To check the first inequality of (1.26), let us consider the function $v=\underline{u}\vee u=u+(\underline{u}-u)^+$. Since $u\in K$, we have $v\in K$ (cf. (1.7)). Using v in (1.18), one gets

$$\begin{aligned}0\le &\int_\Omega A(x,\nabla u)\cdot\nabla[(\underline{u}-u)^+]\,dx+\int_\Omega b(x,u)(\underline{u}-u)^+\,dx\\ &+\int_\Omega f(x,Tu)(\underline{u}-u)^+\,dx+\int_{\partial\Omega} g(x,Tu)(\underline{u}-u)^+\,dS.\end{aligned} \tag{1.27}$$

On the other hand, choosing $v=\underline{u}\wedge u=\underline{u}-(\underline{u}-u)^+$ in (1.8) yields

$$\begin{aligned}-\int_\Omega A(x,\nabla\underline{u})\cdot\nabla[(\underline{u}-u)^+]\,dx-\int_\Omega f(x,\underline{u})(\underline{u}-u)^+\,dx&\\ -\int_{\partial\Omega} g(x,\underline{u})(\underline{u}-u)^+\,dS\ge 0.&\end{aligned} \tag{1.28}$$

Adding (1.27) and (1.28), we obtain

$$\begin{aligned}0\le &\int_\Omega [A(x,\nabla u)-A(x,\nabla\underline{u})]\cdot\nabla[(\underline{u}-u)^+]\,dx\\ &+\int_\Omega [f(x,Tu)-f(x,\underline{u})](\underline{u}-u)^+\,dx+\int_\Omega b(x,u)(\underline{u}-u)^+\,dx\\ &+\int_{\partial\Omega}[g(x,Tu)-g(x,\underline{u})](\underline{u}-u)^+\,dS.\end{aligned} \tag{1.29}$$

Note that

$$\begin{aligned}(1.30)\qquad &\int_\Omega [A(x,\nabla u)-A(x,\nabla \underline{u})]\cdot\nabla[(\underline{u}-u)^+]\,dx\\ &\qquad=\int_{\{x\in\Omega:u(x)<\underline{u}(x)\}}[A(x,\nabla u)-A(x,\nabla \underline{u})]\cdot(\nabla\underline{u}-\nabla u)\,dx\le 0.\end{aligned}$$

For $x\in\Omega$ such that $u(x)<\underline{u}(x)$, we have

$$(Tu)(x)=\underline{u}(x), \tag{1.31}$$

and thus $f(x,Tu(x))=f(x,\underline{u}(x))$. Therefore,

$$\begin{aligned}(1.32)\qquad &\int_\Omega [f(x,Tu)-f(x,\underline{u})](\underline{u}-u)^+\,dx\\ &\qquad=\int_{\{x\in\Omega:u(x)<\underline{u}(x)\}}[f(x,Tu)-f(x,\underline{u})](\underline{u}-u)\,dx=0.\end{aligned}$$

Similarly, for $x\in\partial\Omega$ such that $u(x)<\underline{u}(x)$, (1.31) still holds and one has $g(x,Tu(x))=g(x,\underline{u}(x))$. Again, we have

$$\begin{aligned}(1.33)\qquad &\int_{\partial\Omega} [g(x,Tu)-g(x,\underline{u})](\underline{u}-u)^+\,dS\\ &\qquad=\int_{\{x\in\partial\Omega:u(x)<\underline{u}(x)\}}[g(x,Tu)-g(x,\underline{u})](\underline{u}-u)\,dS=0.\end{aligned}$$

Combining the estimates in (1.29)–(1.33), one gets

$$0\le\int_\Omega b(x,u)(\underline{u}-u)^+\,dx1.39=-\int_{\{x\in\Omega:u(x)<\underline{u}(x)\}}[\underline{u}(x)-u(x)]^r\,dx\le 0.$$

Hence, the set $\{x\in\Omega:u(x)<\underline{u}(x)\}$ has Lebesgue measure 0, that is, $u(x)\ge\underline{u}(x)$ a.e. on Ω and the first inequality in (1.26) is proved. The second inequality there is proved analogously.

From (1.26) and the definition of b and T, we see that $b(x,u)=0$ and $Tu=u$. Hence, the variational inequality (1.18) reduces to (1.2), implying that u is also a solution of (1.2). □

The above result may be generalized in the following theorem. The technique of proof again follows ideas already used in [9], [10].

Theorem 1.4. *Assume that* $\underline{u}_1,\dots,\underline{u}_k$ (*resp.* $\overline{u}_1,\dots,\overline{u}_m$) *are subsolutions* (*resp. supersolutions*) *of* (1.2) *such that*

$$\underline{u}_0=:\max\{\underline{u}_1,\dots,\underline{u}_k\}\le\min\{\overline{u}_1,\dots,\overline{u}_m\}:=\overline{u}_0, \tag{1.34}$$

and that f and g have the following growth conditions between the sub- and supersolutions:

$$|f(x,u)|\le a_3(x),\qquad |g(\xi,v)|\le\widetilde{a}_3(\xi), \tag{1.35}$$

for a.a. $x \in \Omega$, $\xi \in \partial\Omega$, *all* $u \in [\min\{\underline{u}_1(x), \dots, \underline{u}_k(x)\}, \max\{\overline{u}_1(x), \dots, \overline{u}_m(x)\}]$, *all* $v \in [\min\{\underline{u}_1(\xi), \dots, \underline{u}_k(\xi)\}, \max\{\overline{u}_1(\xi), \dots, \overline{u}_m(\xi)\}]$, *where where* a_3 *and* $\widetilde{a}_3$ *are as in Theorem* 1.3. *Then, there exists a solution* u *of* (1.2) *such that* $\underline{u}_0 \le u \le \overline{u}_0$.

Proof. The proof of this theorem follows the same lines as those in Theorem 1.3, with the following modifications. Let b be defined as in (1.13) with $\underline{u}_0$ and $\overline{u}_0$ instead of $\underline{u}$ and $\overline{u}$. For $i \in \{0, \dots, k\}$ and $j \in \{0, \dots, m\}$, $x \in \overline{\Omega}$, and $u \in W^{1,p}(\Omega)$, we put

$$(T_{ij}u)(x) = \begin{cases} \overline{u}_j(x) & \text{if } u(x) > \overline{u}_j(x), \\ u(x) & \text{if } \underline{u}_i(x) \le u(x) \le \overline{u}_j(x), \\ \underline{u}_i(x) & \text{if } u(x) < \underline{u}_i(x). \end{cases} \tag{1.36}$$

Also, we define Γ from $W^{1,p}(\Omega)$ to $[W^{1,p}(\Omega)]^*$ by

$$\begin{aligned} \langle \Gamma(u), v\rangle = {}& \int_\Omega f(x, T_{00}u) v\,dx + \int_{\partial\Omega} g(x, T_{00}u) v\,dS \\ &+ \sum_{j=1}^m \biggl(\int_\Omega |f(x, T_{0j}u) - f(x, T_{00}u)| v\,dx \\ &+ \int_{\partial\Omega} |g(x, T_{0j}u) - g(x, T_{00}u)| v\,dS \biggr) \\ &- \sum_{i=1}^k \biggl(\int_\Omega |f(x, T_{i0}u) - f(x, T_{00}u)| v\,dx \\ &+ \int_{\partial\Omega} |g(x, T_{i0}u) - g(x, T_{00}u)| v\,dS \biggr), \end{aligned} \tag{1.37}$$

for $u, v \in W^{1,p}(\Omega)$. It is clear from this definition that Γ is well defined and is completely continuous on $W^{1,p}(\Omega)$. Consider the following variational inequality:

$$\begin{cases} \langle \mathcal{A}u + Bu + \Gamma u, v - u\rangle \ge 0, & \text{for all } v \in K, \\ u \in K. \end{cases} \tag{1.38}$$

As in the proof of Theorem 1.3, we can check that $\mathcal{A} + B + \Gamma$ is pseudomonotone and coercive on $W^{1,p}(\Omega)$ and thus on K. Hence, the inequality (1.38) has solutions in K. Let $u \in K$ be any solution of (1.38). We show that

$$\underline{u}_s \le u \le \overline{u}_j, \tag{1.39}$$

for every $s \in \{1, \dots, k\}$, $j \in \{1, \dots, m\}$.

To prove the first inequality, we note that because $u \in K$ and $\underline{u}_s$ is a subsolution, we have $v = u + (\underline{u}_s - u)^+ = \underline{u}_s \vee u \in K$. Using v in (1.38), one obtains

$$
\begin{aligned}
(1.40)\qquad 0 \le \int_\Omega A(x,\nabla u)\cdot\nabla[(\underline{u}_s - u^+)]\,dx &+ \int_\Omega b(x,u)(\underline{u}_s - u^+)\,dx \\
&+ \int_\Omega f(x,T_{00}u)(\underline{u}_s - u)^+\,dx + \int_{\partial\Omega} g(x,T_{00}u)(\underline{u}_s - u)^+\,dS \\
&+ \sum_{j=1}^m \bigg(\int_\Omega |f(x,T_{0j}u) - f(x,T_{00}u)|(\underline{u}_s - u)^+\,dx \\
&+ \int_{\partial\Omega} |g(x,T_{0j}u) - g(x,T_{00}u)|(\underline{u}_s - u)^+\,dS\bigg) \\
&- \sum_{i=1}^k \bigg(\int_\Omega |f(x,T_{i0}u) - f(x,T_{00}u)|(\underline{u}_s - u)^+\,dx \\
&+ \int_{\partial\Omega} |g(x,T_{i0}u) - g(x,T_{00}u)|(\underline{u}_s - u)^+\,dS\bigg).
\end{aligned}
$$

On the other hand, noting that $\underline{u}_s$ is a subsolution and choosing $v = \underline{u}_s - (\underline{u}_s - u)^+ = \underline{u}_s \wedge u \in \underline{u}_s \wedge K$ in (1.8) with $\underline{u}_s$ instead of $\underline{u}$, we get

$$
\begin{aligned}
(1.41)\quad 0 \le -\int_\Omega A(x,\nabla\underline{u}_s)\cdot\nabla[(\underline{u}_s - u)^+]\,dx& \\
- \int_\Omega f(x,\underline{u}_s)(\underline{u}_s - u)^+\,dx &- \int_{\partial\Omega} g(x,\underline{u}_s)(\underline{u}_s - u)^+\,dS.
\end{aligned}
$$

Adding (1.40) and (1.41) yields the following inequality:

$$
\begin{aligned}
(1.42)\qquad 0 \le \int_\Omega &[A(x,\nabla u) - A(x,\nabla\underline{u}_s)]\cdot\nabla[(\underline{u}_s - u^+)]\,dx \\
&+ \int_\Omega [f(x,T_{00}u) - f(x,\underline{u}_s)](\underline{u}_s - u)^+\,dx \\
&+ \int_{\partial\Omega} [g(x,T_{00}u) - g(x,\underline{u}_s)](\underline{u}_s - u)^+\,dS \\
&+ \int_\Omega b(x,u)(\underline{u}_s - u)^+\,dx \\
&+ \sum_{j=1}^m \bigg(\int_\Omega |f(x,T_{0j}u) - f(x,T_{00}u)|(\underline{u}_s - u)^+\,dx \\
&+ \int_{\partial\Omega} |g(x,T_{0j}u) - g(x,T_{00}u)|(\underline{u}_s - u)^+\,dS\bigg) \\
&- \sum_{i=1}^k \bigg(\int_\Omega |f(x,T_{i0}u) - f(x,T_{00}u)|(\underline{u}_s - u)^+\,dx \\
&+ \int_{\partial\Omega} |g(x,T_{i0}u) - g(x,T_{00}u)|(\underline{u}_s - u)^+\,dS\bigg).
\end{aligned}
$$

For $x \in \overline{\Omega}$ such that $u(x) < \underline{u}_s(x)$, we have

$$u(x) < \underline{u}_s(x) \le \underline{u}_0(x) \le \overline{u}_0(x) \le \overline{u}_j(x),$$

for every $j \in \{1, \dots, m\}$. Therefore, $T_{00}u(x) = T_{0j}u(x) = \underline{u}_0(x)$ and $T_{s0}u(x) = \underline{u}_s(x)$. These imply that

$$\begin{aligned}(1.43)\quad &\int_\Omega [f(x, T_{00}u) - f(x, \underline{u}_s)](\underline{u}_s - u)^+ \, dx \\ &\quad - \sum_{i=1}^k \int_\Omega |f(x, T_{i0}u) - f(x, T_{00}u)|(\underline{u}_s - u)^+ \, dx \\ &\le \int_{\{x \in \Omega : u(x) < \underline{u}_s(x)\}} [f(x, T_{00}u) - f(x, \underline{u}_s)](\underline{u}_s - u)^+ \, dx \\ &\quad - \int_{\{x \in \Omega : u(x) < \underline{u}_s(x)\}} |f(x, T_{s0}u) - f(x, T_{00}u)|(\underline{u}_s - u)^+ \, dx \\ &= \int_{\{x \in \Omega : u(x) < \underline{u}_s(x)\}} [f(x, \underline{u}_0) - f(x, \underline{u}_s) - |f(x, \underline{u}_s) - f(x, \underline{u}_0)|](\underline{u}_s - u) \, dx \\ &\le 0.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\int_{\partial\Omega} [g(x, T_{00}u) - g(x, \underline{u}_s)](\underline{u}_s - u)^+ \, dS \\ &\quad - \sum_{i=1}^k \int_{\partial\Omega} |g(x, T_{i0}u) - g(x, T_{00}u)|(\underline{u}_s - u)^+ \, dS \\ &\le \int_{\{x \in \partial\Omega : u(x) < \underline{u}_s(x)\}} [g(x, T_{00}u) - g(x, \underline{u}_s)](\underline{u}_s - u)^+ \, dS \\ &\quad - \int_{\{x \in \partial\Omega : u(x) < \underline{u}_s(x)\}} |g(x, T_{s0}u) - g(x, T_{00}u)|(\underline{u}_s - u)^+ \, dS \le 0.\end{aligned}$$

Moreover, for $j \in \{1, \dots, m\}$,

$$\begin{aligned}&\int_\Omega |f(x, T_{0j}u) - f(x, T_{00}u)|(\underline{u}_s - u)^+ \, dx \\ &= \int_{\{x \in \Omega : u(x) < \underline{u}_s(x)\}} |f(x, \underline{u}_0(x)) - f(x, \underline{u}_0(x))|(\underline{u}_s - u)^+ \, dx = 0,\end{aligned}$$

and also,

$$(1.45)\qquad \int_{\partial\Omega} |g(x, T_{0j}u) - g(x, T_{00}u)|(\underline{u}_s - u)^+ \, dS = 0,$$

As above, it follows from the monotonicity of A in (1.4) that

$$(1.46)\qquad \int_\Omega [A(x, \nabla u) - A(x, \nabla \underline{u}_s)] \cdot \nabla[(\underline{u}_s - u)^+] \, dx \le 0.$$

From (1.41)–(1.46), one obtains

$$0 \le \int_\Omega b(x,u)(\underline{u}_s - u)^+ \, dx \le -\int_{\{x\in\Omega: u(x)<\underline{u}_s(x)\}} [\underline{u}_s(x) - u(x)]^r \, dx \le 0.$$

Therefore, $\int_{\{x\in\Omega:u(x)<\underline{u}_s(x)\}} [\underline{u}_s(x) - u(x)]^r \, dx = 0$ and thus $u(x) \ge \underline{u}_s(x)$ for a.e. $x \in \Omega$ and $s \in \{1, \dots, k\}$.

Analogous arguments may be used to show the second inequality in (1.39), from which it follows that $\underline{u}_0 \le u \le \overline{u}_0$ a.e. on Ω. Finally, we have $b(x,u(x)) = 0$ a.e. on Ω and

$$T_{ij}u(x) = u(x) \quad \text{a.e. on } \Omega,$$

for all $i \in \{0,\dots,k\}$, $j \in \{0,\dots,m\}$. Thus,

$$\langle \Gamma(u), v\rangle = \int_\Omega f(x,u)v \, dx, \quad \text{for all } v \in K,$$

and (1.38) becomes (1.2). □

Remark 1.5. (a) The above theorem suggests more general definitions of sub- and supersolutions. Namely: An element $\alpha \in W^{1,p}(\Omega)$ is a subsolution if it is the supremum of a finite number of functions each of which is a subsolution satisfying Definition 1.1 and an element $\beta \in W^{1,p}(\Omega)$ is a supersolution if it is the infimum of a finite number of supersolutions each of which is a supersolution satisfying Definition 1.2. In this case the set of subsolutions is closed with respect to the operation $\vee$ and the set of supersolutions is closed with respect to the operation $\wedge$, and, of course, Theorem 1.4 is simply a restatement of Theorem 1.3. Thus, if we let $\mathcal{S}$ be the set of solutions of (1.2) between $\underline{u}_0$ and $\overline{u}_0$. Theorem 1.4 means that $\mathcal{S} \ne \emptyset$ and under the above assumptions, one can prove (cf. [10], [14]) that $\mathcal{S}$ is compact and directed. As a consequence, $\mathcal{S}$ has greatest (the supremum of all subsolutions) and smallest (the infimum of all supersolutions) elements with respect to the standard ordering, which are the extremal solutions of (1.2) between $\underline{u}_0$ and $\overline{u}_0$. Such results also have a long history and likely go back to [1], see also [5], [9], [16].

(b) Using ideas of [12] it is possible to show that $\underline{u}_0$ is actually a subsolution as defined in Definition 1.1 and $\overline{u}_0$ is a supersolution as defined in Definition 1.2, provided the problem considered is an equation, i.e. K is a subspace of $W^{1,p}(\Omega)$ containing the test functions. Whether this result also holds in the general case is an open question.

(c) If only a subsolution (or a supersolution) of (1.2) exists and f and g satisfy certain one-sided growth conditions then we can also show the existence of solutions of (1.2) above the subsolution (or below the supersolution). We can also show the existence of a minimal solution above that subsolution (or a maximal solution below that supersolution) (see e.g. [14]).

2. Some examples

2.1. Problems with Dirichlet boundary conditions. Consider the boundary value problem

$$-\mathrm{div}[A(x,\nabla u)] + f(x,u) = 0 \quad \text{in } \Omega, \tag{2.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{2.2}$$

the variational form of which is the inequality (1.2) with $g = 0$ and

$$K = W_0^{1,p}(\Omega), \tag{2.3}$$

which is equivalent to the variational equality:

$$\begin{cases} \displaystyle\int_\Omega A(x,\nabla u)\cdot\nabla v\,dx + \int_\Omega f(x,u)v\,dx = 0, \quad \text{for all } v \in W_0^{1,p}(\Omega), \\ u \in W_0^{1,p}(\Omega). \end{cases}$$

In this case, for assumption (1.7) (respectively, (1.10) to be fulfilled, we need that

$$\underline{u} \le 0 \quad \text{on } \partial\Omega \quad (\text{resp. } \overline{u} \ge 0 \text{ on } \partial\Omega).$$

Concerning condition (1.8), it can be checked that the set $\{v - \underline{u} : v \in \underline{u} \wedge W_0^{1,p}(\Omega)\}$ is dense in the negative cone of $W_0^{1,p}(\Omega)$:

$$W_-^{1,p}(\Omega) := \{w \in W_0^{1,p}(\Omega) : w \le 0 \text{ a.e. on } \Omega\}.$$

Therefore, condition (1.8), in this particular case, becomes the following condition

$$\int_\Omega A(x,\nabla\underline{u})\cdot\nabla v\,dx + \int_\Omega f(x,\underline{u})v\,dx \ge 0, \qquad \text{for all } v \in W_0^{1,p}(\Omega),\ v \le 0 \text{ on } \Omega. \tag{2.5}$$

In view of (2.4) and (2.5), we re-obtain the classical concept of sub- and supersolution for equations with homogeneous Dirichlet boundary condition (cf. e.g. [7], [5], [9], [10]).

For problems with nonhomogeneous Dirichlet conditions, we have equation (2.1) together with

$$u = h \quad \text{on } \partial\Omega, \tag{2.6}$$

instead of (2.2), where $h \in W^{1-\frac{1}{p},p}(\partial\Omega)$ is the trace of a function in $W^{1,p}(\Omega)$, still denoted by h, for simplicity. In this case, problem (2.1)–(2.6) is, in the variational form, the inequality (1.2) with $g = 0$ and

$$K = \{h\} \oplus W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : u = h,\ \text{on } \partial\Omega\}.$$

The condition $\underline{u} \vee K \subset K$ is satisfied if and only if $\underline{u}$ satisfies the boundary condition $\underline{u} \le h$ a.e. on $\partial\Omega$. The set

$$\{v - \underline{u} : v \in \underline{u} \wedge [\{h\} \oplus W_0^{1,p}(\Omega)]\} = \{w - (\underline{u} - h) : w \in (\underline{u} - h) \wedge W_0^{1,p}(\Omega)\}$$

is dense in the negative cone $W_-^{1,p}(\Omega)$ (because $\underline{u} - h \le 0$ on $\partial\Omega$). Condition (1.8) is again equivalent to (2.5).

2.2. Problems with Neumann and Robin boundary conditions. In the case where $K = W^{1,p}(\Omega)$, (1.2) reduces to the variational equality

$$\begin{cases} \displaystyle\int_\Omega A(x,\nabla u)\cdot\nabla v\,dx + \int_\Omega f(x,u)v\,dx + \int_{\partial\Omega} g(x,u)v\,dS = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } v \in W^{1,p}(\Omega), \\ u \in W^{1,p}(\Omega), \end{cases} \tag{2.7}$$

which is the weak form of the boundary value problem

$$\begin{cases} -\mathrm{div}[A(x,\nabla u)] + f(x,u) = 0 & \text{in } \Omega, \\ A(x,\nabla u)\cdot n = -g(x,u) & \text{on } \partial\Omega. \end{cases}$$

When $g = 0$ on $\partial\Omega$, we have a homogeneous Neumann boundary condition. Otherwise, one has a nonhomogeneous Neumann boundary condition which also may depend on u. It is clear that condition (1.7) always holds. Also, for any $\underline{u}$ in $W^{1,p}(\Omega)$, we have $\underline{u} \wedge W^{1,p}(\Omega) = \{v \in W^{1,p}(\Omega) : v \le \underline{u}$ a.e. on $\Omega\}$. Therefore, (1.8) is equivalent to the inequality

$$\int_\Omega A(x,\nabla \underline{u})\cdot\nabla w\,dx + \int_\Omega f(x,\underline{u})w\,dx + \int_{\partial\Omega} g(x,\underline{u})w\,dS \ge 0,$$

for all $w \in W^{1,p}(\Omega)$ such that $w \le 0$ a.e. on Ω, which is, in its turn, equivalent to

$$\int_\Omega A(x,\nabla \underline{u})\cdot\nabla w\,dx + \int_\Omega f(x,\underline{u})w\,dx + \int_{\partial\Omega} g(x,\underline{u})w\,dS \le 0, \tag{2.8}$$
$$\text{for all } w \in W^{1,p}(\Omega),\ w \ge 0 \text{ a.e. on } \Omega.$$

We have a similar condition for supersolutions of (2.7). These concepts of sub- and supersolutions here coincide with the classical ones for sub- and supersolutions in Neumann problems. Our definitions here also cover the cases where the Neumann conditions also depend on u. In fact, when $g(x,u) = a\,u$, we have a Robin boundary condition.

2.3. By choosing $K = \{u \in W^{1,p}(\Omega) : u = h$ on $\Gamma\}$, where Γ is a measurable subset of $\partial\Omega$, we have the equation (2.1) with a mixed boundary condition consisting of a Dirichlet condition on Γ and a Neumann/Robin condition on $\partial\Omega \setminus \Gamma$.

2.4. Let K be a convex subset of $W_0^{1,p}(\Omega)$. The inequality (1.2), in this case, formulates problems with unilateral constraints (such as obstacle problems) and homogeneous Dirichlet boundary conditions, which were discussed in [13]. Many results in that paper are particular cases of those discussed here. In fact, the general definitions of sub- supersolutions presented here are motivated in part by the concepts and arguments in [13].

2.5. Let us consider the choice

$$K = \{u \in W^{1,p}(\Omega) : u = \text{const. on } \partial\Omega\}.$$

For $u \in W^{1,p}(\Omega)$, we note that $u \vee K \subset K$ (resp. $u \wedge K \subset K$) if and only if $u \in K$. In fact, it is clear that if $u \in K$, then $u \vee K, u \wedge K \subset K$. Conversely, assume that $u \vee K \subset K$. For any constant function c, we have $u \vee c = \max\{u, c\} =$ constant on $\partial\Omega$. Therefore, either $u \leq c$ a.e. on $\partial\Omega$ or $u \geq c$ a.e. on $\partial\Omega$ (with respect to the Hausdorff measure). Since this is true for any $c \in \mathbb{R}$, we must have $u =$ constant on $\partial\Omega$, that is, $u \in K$. For $\underline{u} \in K$, we have

$$\underline{u} \wedge K = \{v \in K : v \leq \underline{u} \text{ a.e. on } \Omega\} = \{\underline{u} - w : w \in K, w \geq 0 \text{ a.e. on } \Omega\}.$$

Therefore, the inequality (1.8) is equivalent to

$$\int_\Omega A(x, \nabla \underline{u}) \cdot \nabla v \, dx + \int_\Omega f(x, \underline{u}) v \, dx + \int_{\partial\Omega} g(x, \underline{u}) v \, dS \leq 0,$$

for all $v \in K$ such that $v \geq 0$ on Ω. One has a similar equivalence for supersolutions. Note that in this particular case, the definitions for sub- and supersolutions here reduce to those in [11]. We note that in the case that $g \equiv 0$ the problem considered here is the boundary value problem

$$\begin{cases} -\mathrm{div} A(x, \nabla u) + f(x, u) = 0, \\ u|_{\partial\Omega} = \text{const.}, \quad \displaystyle\int_{\partial\Omega} A(x, \nabla u) \cdot n \, dS = 0, \end{cases}$$

where the constant boundary data are not specified. This problem in dimension $N = 1$ (the periodic boundary value problem) was first studied by sub- and supersolution methods by Knobloch [8]. (See also [2], [3], where free boundary problems of this type are studied.)

2.6. For another example, let us consider the boundary value problem consisting of (2.1) and the following unilateral boundary condition on the boundary:

$$\begin{cases} u \geq \psi, \\ A(x, \nabla u) \cdot n \geq 0, \\ (u - \psi)[A(x, \nabla u) \cdot n] = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.9}$$

(ψ is a measurable function on $\partial\Omega$) which occur in problems with semi-permeable media (cf. [6]). The problem can be formulated as the variational inequality (1.2) with $g = 0$ and

$$K = \{u \in W^{1,p}(\Omega) : u \geq \psi \text{ a.e. on } \partial\Omega\}.$$

It is worth noting that in this case, there is a non-symmetry concerning conditions (1.7) and (1.10) in the definitions of sub- and supersolutions. In fact, it is easy to see that for $\underline{u}, \overline{u} \in W^{1,p}(\Omega)$, $\underline{u}$ always satisfies (1.7), while (1.10) holds if and only if $\overline{u} \geq \psi$ a.e. on $\partial\Omega$, that is $\overline{u} \in K$.

Furthermore, problems for which the domain Ω is unbounded may be tackled in a similar vein by using classical approaches (see e.g. [9]).

Remark 2.1. The above definitions and approach could be extended in a straightforward manner to problems with lower terms depending also on the gradient of u, i.e. $f = f(x, u, \nabla u)$. We can also extend them to problems with locally Lipschitz constraints together with convex constraints (variational hemi-variational inequalities) such as those considered, for example, in [4] and the references therein.

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VY KHOI LE
Department of Mathematics and Statistics
University of Missouri–Rolla
Rolla, MO 65401, USA

E-mail address: vy@umr.edu

KLAUS SCHMITT
Department of Mathematics
University of Utah
155 South 1400 East
Salt Lake City, UT 84112, USA

E-mail address: schmitt@math.utah.edu