

**NODAL SOLUTIONS
TO SUPERLINEAR BIHARMONIC EQUATIONS
VIA DECOMPOSITION IN DUAL CONES**

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ABSTRACT. We present an abstract approach to locate multiple solutions of some superlinear variational problems in a Hilbert space H . The approach has many points in common with existing methods, but we add a new tool by using a decomposition technique related to dual cones in H which goes back to Moreau. As an application we deduce new existence results for sign changing solutions for some superlinear biharmonic boundary value problems.

1. Introduction

In this paper we are concerned with locating critical points of a functional defined on a real Hilbert space in the presence of invariant cones. The abstract approach we present provides a framework for proving new existence results for nodal (i.e. sign changing) solutions of superlinear biharmonic equations. We recall that, for a C^1 -functional Φ defined on a Hilbert space H , it is well known that the invariance properties of the vector field $A := \text{Id} - \nabla\Phi: H \rightarrow H$ are closely related to the number and location of critical points of Φ . In many applications the Hilbert space H has a partial order induced by a closed cone \mathcal{K} , and A leaves the cone \mathcal{K} or more general order intervals invariant. Depending on

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further assumptions, different methods like the classical sub- and supersolution technique, topological fixed point theory or variational methods have been used in order to locate critical points of Φ (which are precisely the fixed points of A). For the first two methods we refer the reader to the survey papers by Amann [1] and Dancer [13]. The variational methods are particularly important in the cases where no a priori bounds on the set of critical points of Φ are available and thus a global fixed point index for A cannot be defined. The combined use of variational tools and invariance information goes back to the pioneering papers [23], [18], [8], [9]. More recently, a number of different techniques were developed to locate critical points inside and outside of invariant sets for A , see e.g. [2]–[4], [14]–[17], [11], [10], [24], [26], [7].

In the present paper we generalize some results obtained recently in [3]. We assume that the functional Φ is of class $C^{1,1}$, and the vector field A leaves a nonempty closed cone $\mathcal{K} \subset H$ and its reflection $-\mathcal{K} = \{-u : u \in \mathcal{K}\}$ invariant. Actually we need slightly stronger invariance conditions, see Section 2 below. Under additional superlinearity conditions on A , we then obtain existence of three nontrivial critical points u_1, u_2, u_3 of Φ , where $u_1 \in \mathcal{K}$, $u_2 \in -\mathcal{K}$ and $u_3 \in H \setminus (\mathcal{K} \cup -\mathcal{K})$. Assuming in addition that Φ is even, we obtain infinitely many critical points located in $H \setminus (\mathcal{K} \cup -\mathcal{K})$. Similar results in special cases or under different assumptions can be found in [3], [24], [26], [2]. In the proof of our results we generalize the approach of [3], and we simplify some of the arguments. To pass from the Sobolev space $H^1(\mathbb{R}^N)$ considered in [3] to an arbitrary Hilbert space H , we implement ideas related to the decomposition method in dual cones. This method goes back to Moreau [28], and it was rediscovered and elaborated by Gazzola and Grunau [19] in the context of variational problems for polyharmonic operators (see also Miersemann [27]). Its main advantage is the fact that it provides an abstract substitute for the decomposition $u = u^+ + u^-$ of functions $u \in H^1(\Omega)$ (where $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$). In particular, this is useful for variational problems in subspaces of $H^2(\Omega)$ where the decomposition $u = u^+ + u^-$ is not available.

In the second part of the paper we consider an application of this type. More precisely, we study the nonlinear biharmonic equation

$$(1.1) \quad \Delta^2 u = f(x, u), \quad x \in \Omega,$$

subject either to Navier boundary conditions

$$(1.2) \quad u = \Delta u = 0 \quad \text{on } \partial\Omega$$

or to Dirichlet boundary conditions

$$(1.3) \quad u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

Here f is a superlinear, subcritical nonlinearity which is nondecreasing in u and such that $f(0) = 0$. We obtain new existence and multiplicity results for sign changing solutions of (1.1), (1.2) and (1.1), (1.3). However, while for the boundary conditions (1.2) we get results on every smooth bounded domain, this is not the case for (1.3). In fact, our results strongly depend on the assumption that the Green function of the biharmonic operator Δ^2 on Ω corresponding to the boundary conditions (1.2), (1.3), respectively, is positive. While for (1.2) this is true on an arbitrary domain, it fails to be true for (1.3) even on simple bounded domains (see [22] for a survey on this type of problems). Nevertheless, by a classical result of Boggio [6], it is true in the case where $\Omega = B$ is a ball in \mathbb{R}^N . Other examples for domains where the Green function corresponding to (1.3) is positive are planar domains close to a disk [21], [29] and, for a certain range of parameters, the limaçon [12]. On these domains we obtain existence of sign changing solutions of (1.1), (1.3). In case of the ball $\Omega = B$ in \mathbb{R}^N , we do *not* need to assume that f is a radial function in x .

Finally, we also prove that, if $f \in C^1(\mathbb{R})$ and $f'(u) > f(u)/u > 0$ for all $u \neq 0$, then every sign changing solution of (1.1), (1.2) has Morse index greater than or equal to two with respect to the corresponding energy functional Φ . The same is true for problem (1.1), (1.3) for domains with a positive Green function. This complements a well known result for the second order case, see [5]. Again the proof relies on the decomposition in dual cones. We remark that, in some situations, the decomposition can also be used to bound the energy of sign changing solutions from below, see [20, Lemma 4].

The paper is organized as follows. In Section 2 we develop the abstract variational framework, and we prove two results on the existence and location of multiple critical points, see Theorems 2.6 and 2.7. In Section 3 we are concerned with the biharmonic equation (1.1), and we prove our main results on the existence, multiplicity and the signs of solutions, see Theorems 3.5 and 3.9. We close this section with the above-mentioned estimate for the Morse index of sign changing solutions. In the appendix we give a short proof of a topological fact used in the proof of Theorem 2.6.

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2. The abstract framework

Let H be a real Hilbert space with scalar product (\cdot, \cdot) and norm $\|\cdot\|$. For a subset D of H we write $\text{int}(D)$ resp. \overline{D} resp. ∂D for the interior, the closure and the boundary of D , respectively. Moreover, for $\varepsilon > 0$ we denote by $B_\varepsilon(D)$ the open ε -neighbourhood of D , i.e. $B_\varepsilon(D) := \{u \in H : \text{dist}(u, D) < \varepsilon\}$. If

$D = \{u\}$ consists of a single point, we write $B_\varepsilon(u)$ for the ε -neighbourhood of u . We consider the fixed point problem

$$(2.1) \quad A(u) = u, \quad u \in H,$$

where $A: H \rightarrow H$ is a nonlinear operator satisfying the following assumptions.

(A₁) $A: H \rightarrow H$ is compact and Lipschitz continuous, and $A(0) = 0$.

(A₂) $A = \nabla\Psi$ for some functional $\Psi \in C^{1,1}(H, \mathbb{R})$, and there are constants $C_0, q^* > 0, p > 1, \eta > 2$ and $q_* \in (0, 1)$ such that

$$(2.2) \quad (A(u), u) \geq \eta\Psi(u) - C_0 \quad \text{for } u \in H,$$

$$(2.3) \quad |(A(u), v)| \leq (q_*\|u\| + q^*\|u\|^p)\|v\| \quad \text{for } u, v \in H.$$

We assume that (A₁) and (A₂) are in force from now on. Then solutions of (2.1) are precisely the critical points of the $C^{1,1}$ -functional

$$\Phi: H \rightarrow \mathbb{R}, \quad \Phi(u) = \frac{1}{2}\|u\|^2 - \Psi(u).$$

Moreover, we have:

LEMMA 2.1. Φ satisfies the Palais–Smale condition.

PROOF. Let (u_n) be a Palais–Smale sequence for Φ , i.e. $C := \sup_{n \in \mathbb{N}} |\Phi(u_n)| < \infty$ and $\nabla\Phi(u_n) \rightarrow 0$ as $n \rightarrow \infty$. Then

$$\begin{aligned} \eta C + o(1)\|u_n\| &\geq \eta\Phi(u_n) - (\nabla\Phi(u_n), u_n) \\ &= \frac{\eta-2}{2}\|u_n\|^2 + (A(u_n), u_n) - \eta\Psi(u_n) \geq \frac{\eta-2}{2}\|u_n\|^2 - C_0 \end{aligned}$$

by (2.2), so that (u_n) is bounded. Since A is compact, we may pass to a subsequence satisfying $A(u_n) \rightarrow u_0 \in H$. But since $u_n - A(u_n) = \nabla\Phi(u_n) \rightarrow 0$ as $n \rightarrow \infty$, we conclude that $u_n \rightarrow u_0$. \square

Our aim is to locate critical points of Φ by applying dynamical systems arguments to the negative gradient flow of Φ . Since A is locally Lipschitz continuous, this flow $\varphi: \mathcal{G} \rightarrow H$ is well defined by

$$\begin{cases} \frac{\partial}{\partial t}\varphi(t, u) = -\nabla\Phi(\varphi(t, u)) = A(\varphi(t, u)) - \varphi(t, u), \\ \varphi(0, u) = u, \end{cases}$$

where $\mathcal{G} = \{(t, u) : u \in H, 0 \leq t < T(u)\}$ and $T(u) \in (0, \infty]$ is the maximal existence time for the trajectory $t \mapsto \varphi(t, u)$. In the following, we will write φ^t

instead of $\varphi(t, \cdot)$. Note that for $0 \leq s < t < T(u)$ we have the estimate

$$\begin{aligned} \|\varphi^t(u) - \varphi^s(u)\| &\leq \int_s^t \|\nabla\Phi(\varphi^\tau(u))\| d\tau \\ &\leq \sqrt{t-s} \left(\int_s^t \|\nabla\Phi(\varphi^\tau(u))\|^2 d\tau \right)^{1/2} \\ &= \sqrt{(t-s)[\Phi(\varphi^s(u)) - \Phi(\varphi^t(u))]} \end{aligned}$$

From this we conclude that, if for some $u \in H$ the energy Φ is bounded from below on the trajectory $\{\varphi^t(u) : t \in [0, T(u))\}$, then $T(u) = \infty$. In this case the ω -limit set

$$\omega(u) = \bigcap_{0 \leq t < \infty} \overline{\bigcup_{t \leq s < \infty} \varphi^s(u)}$$

of u is a nonempty compact subset of H consisting of critical points of Φ , as follows in a standard way from Lemma 2.1. A subset D of H is called *positively invariant* (under φ) if

$$\varphi^t(u) \in D \quad \text{for every } u \in D \text{ and every } t \in [0, T(u)).$$

For a positively invariant set D we define the *domain of absorption* by

$$\mathcal{A}(D) := \{u \in H : \varphi^t(u) \in D \text{ for some } t \in [0, T(u))\}.$$

We note that, if $D \subset H$ is open, then $\mathcal{A}(D)$ is also an open subset of H . We also define

$$\mathcal{A}_0 := \{u \in H : T(u) = \infty, \varphi^t(u) \rightarrow 0 \text{ as } t \rightarrow \infty\}.$$

LEMMA 2.2. $\mathcal{A}_0 \subset H$ is an open neighbourhood of zero.

PROOF. Put $\alpha_0 = ((1 - q_*)/2q^*)^{1/(p-1)}$. Then, for all $u \in B_{\alpha_0}(0)$ we have by (2.3)

$$\begin{aligned} \Psi(u) &= \int_0^1 \frac{\partial}{\partial t} \Psi(tu) dt = \int_0^1 (A(tu), u) dt \leq \int_0^1 (q_* \|tu\| + q^* \|tu\|^p) \|u\| dt \\ &\leq \int_0^1 t \|u\|^2 (q_* + q^* \|u\|^{p-1}) dt = \frac{\|u\|^2}{2} (q_* + q^* \|u\|^{p-1}) \\ &\leq \frac{\|u\|^2}{2} (q_* + q^* \alpha_0^{p-1}) = \frac{q_* + 1}{4} \|u\|^2, \end{aligned}$$

so that

$$\Phi(u) = \frac{1}{2} \|u\|^2 - \Psi(u) \geq \left(\frac{1}{2} - \frac{q_* + 1}{4} \right) \|u\|^2 \geq 0.$$

Moreover,

$$(2.4) \quad \Phi(u) \geq \left(\frac{1}{2} - \frac{q_* + 1}{4} \right) \alpha_0^2 =: \beta_0 \quad \text{for } u \in \partial B_{\alpha_0}(0).$$

Since $\Phi(0) = 0$ and Φ is continuous in 0, there is $r \in (0, \alpha_0)$ such that

$$\Phi(u) < \beta_0 \quad \text{for } u \in B_r(0) \subset H.$$

Now if $u \in B_r(0)$, then $\Phi(\varphi^t(u)) < \beta_0$ for every $t \in [0, T(u))$, so that $\varphi^t(u) \in B_{\alpha_0}(0)$ for all $t \in [0, T(u))$ by (2.4). Hence $\Phi(\varphi^t(u)) \geq 0$ for all $t \in [0, T(u))$, which implies that $T(u) = \infty$. Moreover, $\omega(u) \subset B_{\alpha_0}(0)$ is a nonempty compact set consisting of critical points of Φ . However, if $v \in B_{\alpha_0}(0)$ is a critical point of Φ , then

$$\|v\|^2 = (A(v), v) \leq \|v\|^2(q_* + q^*\|v\|^{p-1}) \leq \|v\|^2(q_* + q^*\alpha_0^{p-1}) = \frac{q_* + 1}{2}\|v\|^2.$$

Since $q_* < 1$, we conclude $v = 0$. Thus $\omega(u) = \{0\}$ for every $u \in B_r(0)$, which implies that $B_r(0) \subset \mathcal{A}_0$. From this we deduce that \mathcal{A}_0 is an open neighbourhood of 0, as claimed. \square

In the following, we look for nontrivial critical points of Φ on the closed set $\partial\mathcal{A}_0$. Obviously this set is positively invariant for the flow φ , and $\inf_{u \in \mathcal{A}_0} \Phi(u) \geq 0$. In fact one can prove strict positivity here, but we do not need this. Consequently, we have $T(u) = \infty$ for every $u \in \mathcal{A}_0$, and $\omega(u)$ is a nontrivial compact subset of \mathcal{A}_0 consisting of (nontrivial) critical points of Φ .

To obtain more information on the location of some critical points, we investigate further invariance properties of the flow φ . For this we consider a fixed closed cone in H . So $\mathcal{K} = \bar{\mathcal{K}}$ is convex, $\mathbb{R}^+ \cdot \mathcal{K} \subset \mathcal{K}$ and $\mathcal{K} \cap (-\mathcal{K}) = 0$. The dual cone \mathcal{K}^* of \mathcal{K} is then defined by

$$\mathcal{K}^* := \{u \in H : (u, v) \leq 0 \text{ for all } v \in \mathcal{K}\}.$$

Let $\mathcal{P}: H \rightarrow \mathcal{K}$ be the projection on \mathcal{K} which is uniquely determined by the property

$$\|u - \mathcal{P}u\| = \min_{v \in \mathcal{K}} \|u - v\|.$$

Moreover, we put $\mathcal{P}^* = Id - \mathcal{P}: H \rightarrow H$. Then, as observed in [28], [19], we have

$$(2.5) \quad \mathcal{P}^*u \in \mathcal{K}^* \quad \text{and} \quad (\mathcal{P}u, \mathcal{P}^*u) = 0 \quad \text{for all } u \in H.$$

We need the following crucial invariance assumption for the vector field A .

$$(A_3) \quad (A(u), v) \leq (A(\mathcal{P}^*u), v) \text{ for } u \in H \text{ and } v \in \mathcal{K}^*.$$

Note that (A₃) in particular implies that $A(\mathcal{K}) \subset \mathcal{K}$. Indeed, for $u \in \mathcal{K}$ we have by (2.5) and (A₃)

$$\begin{aligned} \|\mathcal{P}^*(A(u))\|^2 &= (A(u), \mathcal{P}^*(A(u))) \\ &\leq (A(\mathcal{P}^*u), \mathcal{P}^*(A(u))) = (A(0), \mathcal{P}^*(A(u))) = 0. \end{aligned}$$

For us the following invariance properties are of interest.

LEMMA 2.3.

- (a) For $\alpha > 0$ sufficiently small, the set $B_\alpha(\mathcal{K})$ is positively invariant for the flow φ . Moreover, every critical point of Φ in $\overline{B_\alpha(\mathcal{K})}$ belongs to \mathcal{K} .
- (b) \mathcal{K} is positively invariant for φ .

PROOF. (a) Let $u \in H$. Then, by (2.5), (A₂) and (A₃),

$$\begin{aligned} \|\mathcal{P}^*(A(u))\|^2 &= (A(u), \mathcal{P}^*(A(u))) \leq (A(\mathcal{P}^*u), \mathcal{P}^*(A(u))) \\ &\leq (q_* \|\mathcal{P}^*u\| + q^* \|\mathcal{P}^*u\|^p) \|\mathcal{P}^*(A(u))\|, \end{aligned}$$

so that

$$\|\mathcal{P}^*(A(u))\| \leq \|\mathcal{P}^*u\| (q_* + q^* \|\mathcal{P}^*u\|^{p-1}).$$

Hence, if $0 < \|\mathcal{P}^*u\| < \alpha_0 := ((1 - q_*)/2q^*)^{1/(p-1)}$, then $\|\mathcal{P}^*(A(u))\| < \|\mathcal{P}^*u\|$. Consequently, for $\alpha < \alpha_0$, we have $A(\partial B_\alpha(\mathcal{K})) \subset \text{int}(B_\alpha(\mathcal{K}))$, and every fixed point $u \in \overline{B_\alpha(\mathcal{K})}$ of A satisfies $\|\mathcal{P}^*(A(u))\| = \|\mathcal{P}^*(u)\| = 0$, hence it belongs to \mathcal{K} .

We now show that, for $\alpha < \alpha_0$, $B_\alpha(\mathcal{K})$ is positively invariant. Assume, by contradiction, that there is $u_0 \in B_\alpha(\mathcal{K})$ such that $\varphi^{t_0}(u_0) \in \partial B_\alpha(\mathcal{K})$ for some $t_0 \in (0, T(u_0))$, and that t_0 is the smallest positive time where the trajectory $t \mapsto \varphi^t(u_0)$ meets $\partial B_\alpha(\mathcal{K})$. Since $B_\alpha(\mathcal{K})$ is open and convex, by Mazur's separation theorem there exists a continuous linear functional $\ell: H \rightarrow \mathbb{R}$ and $\beta \in \mathbb{R}$ such that $\ell(\varphi^{t_0}(u_0)) = \beta$ and $\ell(u) > \beta$ for $u \in B_\alpha(\mathcal{K})$. It follows that

$$\left. \frac{\partial}{\partial t} \right|_{t=t_0} \ell(\varphi^t(u_0)) = \ell(-\nabla \Phi(\varphi^{t_0}(u_0))) = \ell(A(\varphi^{t_0}(u_0))) - \beta > 0.$$

Hence there exists $\varepsilon > 0$ such that $\ell(\varphi^t(u_0)) < \beta$ for $t \in (t_0 - \varepsilon, t_0)$. Thus, $\varphi^t(u_0) \notin B_\alpha(\mathcal{K})$ for $t \in (t_0 - \varepsilon, t_0)$. This contradicts our choice of t_0 and proves that $B_\alpha(\mathcal{K})$ is positively invariant under φ .

(b) This follows from (a), since $\mathcal{K} = \bigcap_{\alpha > 0} B_\alpha(\mathcal{K})$. \square

PROPOSITION 2.4. *Suppose that there is $u_0 \in \mathcal{K} \setminus \{0\}$ such that $\Phi(u_0) < 0$. Then Φ has a critical point in \mathcal{K} . Hence there is a nontrivial solution $u \in \mathcal{K}$ of (2.1).*

PROOF. Since $\Phi(u_0) < 0$, u_0 is not contained in $\overline{\mathcal{A}_0}$. Therefore, by Lemma 2.2, there is $s \in (0, 1)$ such that $su_0 \in \partial \mathcal{A}_0 \cap \mathcal{K}$. By Lemma 2.3(b) the ω -limit set $\omega(su_0)$ is contained in $\partial \mathcal{A}_0 \cap \mathcal{K}$, and it is nonempty. Since any element of $\omega(su_0)$ is a critical point of Φ , the assertion follows. \square

Next we are concerned with multiple critical points of Φ . For this we also consider the cone $-\mathcal{K}$ and its dual cone

$$(-\mathcal{K})^* = \{u \in H : (u, v) \geq 0 \text{ for all } v \in \mathcal{K}\}.$$

We let $\mathcal{Q}: H \rightarrow -\mathcal{K}$ be the projection on $-\mathcal{K}$ which is uniquely determined by the relation

$$\|u - \mathcal{Q}u\| = \min_{v \in -\mathcal{K}} \|u - v\|.$$

Moreover, we put $\mathcal{Q}^* = Id - \mathcal{Q}: H \rightarrow H$. Again we find that, for all $u \in H$,

$$\mathcal{Q}^*u \in (-\mathcal{K})^* \quad \text{and} \quad (\mathcal{Q}u, \mathcal{Q}^*u) = 0.$$

We make the following additional assumption:

$$(A_4) \quad (A(u), v) \leq (A(\mathcal{Q}^*u), v) \text{ for } u \in H \text{ and } v \in (-\mathcal{K})^*.$$

We remark that, if $A: H \rightarrow H$ is an odd vector field, then (A_4) follows from (A_3) .

Applying Lemma 2.3(a) to the cone $-\mathcal{K}$, we find that, for $\alpha > 0$ sufficiently small, $B_\alpha(-\mathcal{K})$ is also a positively invariant set for the flow φ . We now fix $\alpha > 0$ such that the statement of Lemma 2.3(a) holds for $B_\alpha(\mathcal{K})$ and $B_\alpha(-\mathcal{K})$, and we put

$$\mathcal{A}_+ := \mathcal{A}(B_\alpha(\mathcal{K})) \cap \partial\mathcal{A}_0, \quad \mathcal{A}_- := \mathcal{A}(B_\alpha(-\mathcal{K})) \cap \partial\mathcal{A}_0.$$

Then we have the following.

LEMMA 2.5. *The sets \mathcal{A}_+ and \mathcal{A}_- are disjoint relatively open subsets of $\partial\mathcal{A}_0$.*

PROOF. Since $B_\alpha(\pm\mathcal{K})$ are open subsets of H , $\mathcal{A}(B_\alpha(\pm\mathcal{K}))$ are also open in H . Hence \mathcal{A}_\pm are relatively open in $\partial\mathcal{A}_0$. Now suppose by contradiction that $\mathcal{A}_+ \cap \mathcal{A}_- \neq \emptyset$, and let $u \in \mathcal{A}_+ \cap \mathcal{A}_-$. Since $u \in \partial\mathcal{A}_0$, we have $T(u) = \infty$ and $\omega(u) \neq \emptyset$. Since $u \in \mathcal{A}(B_\alpha(\mathcal{K})) \cap \mathcal{A}(B_\alpha(-\mathcal{K}))$, we have $\omega(u) \subset \overline{B_\alpha(\mathcal{K})} \cap \overline{B_\alpha(-\mathcal{K})}$ and therefore $\omega(u) \subset \mathcal{K} \cap (-\mathcal{K}) = \{0\}$ by Lemma 2.3(a), since $\omega(u)$ consists of critical points of Φ . Hence $\varphi^t(u) \rightarrow 0$ as $t \rightarrow \infty$, but this contradicts the fact that $u \in \partial\mathcal{A}_0$. The lemma is proved. \square

THEOREM 2.6. *Suppose that the assumptions (A_1) – (A_4) are satisfied. Suppose furthermore that there exists a continuous path $h: [0, 1] \rightarrow H$ with $h(0) \in \mathcal{K}$, $h(1) \in -\mathcal{K}$, and $\Phi(h(t)) < 0$ for all $t \in [0, 1]$. Then Φ has at least three critical points u_1, u_2, u_3 , where $u_1 \in \mathcal{K}$, $u_2 \in -\mathcal{K}$ and $u_3 \in H \setminus (\mathcal{K} \cup -\mathcal{K})$.*

PROOF. The existence of u_1 and u_2 follows from Proposition 2.4 applied to \mathcal{K} and $-\mathcal{K}$. To get u_3 , we note that $\overline{\mathcal{A}_0} \cap h([0, 1]) = \emptyset$ by assumption. We put $Q := [0, 1]^2$, and we let $\mathcal{B} \subset Q$ be defined by

$$\mathcal{B} = \{(s_1, s_2) \in Q : s_1 h(s_2) \in \mathcal{A}_0\}.$$

Then $\mathcal{B} \subset Q$ is a relatively open set such that $(\{0\} \times [0, 1]) \subset \mathcal{B}$ and $(\{1\} \times [0, 1]) \cap \overline{\mathcal{B}} = \emptyset$. Hence there is a connected component Γ of the relative boundary $\partial\mathcal{B}$ of \mathcal{B} in Q such that $\Gamma \cap ([0, 1] \times \{0\}) \neq \emptyset$ and $\Gamma \cap ([0, 1] \times \{1\}) \neq \emptyset$. This topological fact has been proved in [26, Lemma 3.1], and we give a different short proof in the appendix of this paper. Put $\Gamma_0 = \{s_1 h(s_2) : (s_1, s_2) \in \Gamma\}$. Then Γ_0 is a connected subset of $\partial\mathcal{A}_0$, and $\Gamma_0 \cap \pm\mathcal{K} \neq \emptyset$. By Corollary 2.5, the sets $\Gamma_0 \cap \mathcal{A}_\pm$

are disjoint relatively open nonempty subsets of Γ_0 , so that, by connectivity, there must be $u \in \Gamma_0 \setminus (\mathcal{A}_+ \cup \mathcal{A}_-)$. Now the trajectory $\{\varphi^t(u) : t > 0\}$ is contained in the closed set $\mathcal{A}_0 \setminus (\mathcal{A}_+ \cup \mathcal{A}_-)$, and so is the ω -limit set $\omega(u)$. In particular, $\omega(u) \cap (\mathcal{K} \cup -\mathcal{K}) = \emptyset$, so any point $u_3 \in \omega(u)$ has the asserted property. \square

THEOREM 2.7. *Suppose that the assumptions (A₁)–(A₃) are satisfied, and that $A(-u) = -A(u)$ and $\Psi(-u) = \Psi(u)$ holds for all $u \in H$. Suppose also that there is a subspace $V \subset H$ of dimension n such that*

$$\limsup_{u \in V, \|u\| \rightarrow \infty} \Phi(u) < 0.$$

Then Φ has at least $n - 1$ pairs $\pm u$ of nontrivial critical points located in $H \setminus (\mathcal{K} \cup -\mathcal{K})$.

PROOF. We first note that, by the oddness of A , assumption (A₄) is also satisfied. For a closed and symmetric subset D of H we denote by $\gamma(D)$ the usual Krasnoselski genus of D . For a definition and basic properties of γ , see [30, pp. 94]. Since, by assumption, Φ is an even functional, $\partial\mathcal{A}_0$ is a closed and symmetric subset of H , and $\mathcal{A}_- = -\mathcal{A}_+$. We wish to estimate the number of critical points of Φ in the symmetric set

$$\mathcal{E} := \partial\mathcal{A}_0 \setminus (\mathcal{A}_+ \cup \mathcal{A}_-).$$

Note that $\mathcal{E} \subset H$ is closed by Lemma 2.5, and it is positively invariant for the flow φ . For a real number c , we consider the closed and symmetric subsets

$$\mathcal{E}^c := \{u \in \mathcal{E} : \Phi(u) \leq c\}, \quad K_c := \{u \in \mathcal{E} : \Phi(u) = c, \nabla\Phi(u) = 0\}.$$

We note that K_c is compact by Lemma 2.1. We consider the nondecreasing sequence of values

$$c_k := \inf\{c \in \mathbb{R} : \gamma(\mathcal{E}^c) \geq k\}, \quad k \in \mathbb{N}.$$

If $c_k < \infty$, then c_k is a critical value of Φ . In fact, we will prove the following stronger statement.

(2.6)

If $c_k = c_{k+1} = \dots = c_{k+l} < \infty$ for some k, l , then $\gamma(K_c) \geq l + 1$ for $c := c_k$.

In particular, K_c is an infinite set if $l \geq 1$. To show (2.6), we let $\varepsilon > 0$ be such that $\gamma(\bar{U}) = \gamma(K_c)$, where $U = B_\varepsilon(K_c)$. Since Φ satisfies the Palais–Smale condition, there is $\delta > 0$ such that for

$$M = \{u \in \mathcal{E} \setminus B_{\varepsilon/2}(K_c) : \Phi(u) \in [c - \delta, c + \delta]\}$$

we have

$$(2.7) \quad \tau := \inf_{u \in M} \|\nabla\Phi(u)\| > 0.$$

Now let $0 < \delta_1 < \min\{\delta, \tau\varepsilon/4\}$, and let $u \in \mathcal{E}^{c+\delta_1} \setminus [U \cup \mathcal{E}^{c-\delta_1}] \subset M$. Then we claim:

$$(2.8) \quad \varphi^t(u) \in \mathcal{E}^{c-\delta_1} \quad \text{for some } t \in (0, T(u)).$$

Indeed, (2.7) immediately implies that the trajectory $t \mapsto \varphi^t(u)$ cannot stay in M for all times $t \in [0, T(u)]$. On the other hand, if, for some $t > 0$, $\{\varphi^s(u) : 0 \leq s \leq t\} \subset M$ and $\varphi^t(u) \in \partial M$, then either $\Phi(\varphi^t(u)) = c - \delta < c - \delta_1$ or $\varphi^t(u) \in \partial B_{\varepsilon/2}(K_c)$. In the latter case, we conclude

$$\begin{aligned} \frac{\varepsilon}{2} &\leq \|u - \varphi^t(u)\| \leq \int_0^t \left\| \frac{\partial \varphi^s(u)}{\partial s} \right\| ds \\ &\leq \frac{1}{\tau} \int_0^t \|\nabla \Phi(\varphi^s(u))\|^2 ds \leq \frac{1}{\tau} [\Phi(u) - \Phi(\varphi^t(u))], \end{aligned}$$

and hence

$$\Phi(\varphi^t(u)) \leq \Phi(u) - \frac{\tau\varepsilon}{2} < c - \delta_1.$$

Thus (2.8) follows. Now, for $u \in \mathcal{E}^{c+\delta_1} \setminus U$, let $t^*(u)$ be the smallest $t \in [0, T(u)]$ such that $\varphi^t(u) \in \mathcal{E}^{c-\delta_1}$. Then the function $t^*: \mathcal{E}^{c+\delta_1} \setminus U \rightarrow [0, \infty)$ is even and lower semicontinuous, since $\mathcal{E}^{c-\delta_1}$ is closed. In fact, t^* is also upper semicontinuous. To see this, let $u \in \mathcal{E}^{c+\delta_1} \setminus U$. We may assume that $\Phi(u) \geq c - \delta_1$, since otherwise $t^* \equiv 0$ in a neighbourhood of u . Then $\Phi(\varphi^{t^*(u)}(u)) = c - \delta_1$, and the estimate from above shows that $\varphi^{t^*(u)}(u) \in M$, hence $\varphi^{t^*(u)}(u)$ is not a critical point of Φ by (2.7). Consequently, if $\rho > 0$ is given, then $\Phi(\varphi^{t^*(u)+\rho}(u)) < c - \delta_1$, and thus $\Phi(\varphi^{t^*(u)+\rho}(v)) < c - \delta_1$ for v sufficiently close to u . We conclude that $t^*(v) \leq t^*(u) + \rho$ for v sufficiently close to u , which shows upper semicontinuity. Having thus proved that t^* is a continuous function, we find that the map

$$\vartheta: \mathcal{E}^{c+\delta_1} \setminus U \rightarrow \mathcal{E}^{c-\delta_1}, \quad \vartheta(u) = \varphi^{t^*(u)}(u)$$

is odd and continuous. By definition of $c = c_k = c_{k+l}$ and properties of the genus we conclude that

$$k - 1 \geq \gamma(\mathcal{E}^{c-\delta_1}) \geq \gamma(\mathcal{E}^{c+\delta_1} \setminus U) \geq \gamma(\mathcal{E}^{c+\delta_1}) - \gamma(\bar{U}) \geq k + l - \gamma(K_c),$$

and (2.6) follows. Next we prove that

$$(2.9) \quad c_{n-1} < \infty.$$

Indeed, by assumption we can choose $R > 0$ such that

$$\Phi(u) < 0 \quad \text{for } u \in V, \|u\| \geq R.$$

This implies that $W := V \cap \mathcal{A}_0$ is an open bounded and symmetric neighbourhood of 0 in V . Since the dimension of V is n , the boundary $\partial_V W \subset \partial \mathcal{A}_0$ of W in V satisfies $\gamma(\partial_V W) = n$. Let $C = \partial_V W \setminus (\mathcal{A}_+ \cup \mathcal{A}_-)$, which is a compact symmetric set, and let $N \subset H$ be an open symmetric neighbourhood of C such

that $\gamma(\overline{N}) = \gamma(C)$. Then the set $\widehat{N} = \partial_V W \setminus N$ is closed and symmetric, and it is contained in $\mathcal{A}_+ \cup \mathcal{A}_-$. Since \mathcal{A}_+ and \mathcal{A}_- are disjoint relatively open subsets of $\partial\mathcal{A}_0$ by Lemma 2.5, we can define an odd and continuous map $h: \widehat{N} \rightarrow \mathbb{R} \setminus \{0\}$ by $h(u) = 1$ for $u \in \mathcal{A}_+$ and $h(u) = -1$ for $u \in \mathcal{A}_-$. We thus conclude that $\gamma(\widehat{N}) \leq 1$. Now from the subadditivity of the genus we infer

$$\gamma(C) = \gamma(\overline{N}) \geq \gamma(\partial_V W) - \gamma(\widehat{N}) \geq n - 1.$$

Since $C \subset \mathcal{E} \cap V$, we conclude that $c_{n-1} \leq \sup \Phi(V) < \infty$, as claimed.

Since the assertion is a direct consequence of (2.6) and (2.9), the proof is complete. \square

REMARK 2.8. Theorem 2.7 can be seen as an abstract version of the multiplicity result in [3]. In [3], a relative genus was introduced and used in the proof. We believe that the proof given here is somewhat more direct since it only uses the classical Krasnoselski genus.

3. Application to superlinear biharmonic boundary value problems

We are interested in nontrivial solutions of the biharmonic equation

$$(3.1) \quad \Delta^2 u = f(x, u), \quad x \in \Omega$$

on a smooth bounded domain $\Omega \subset \mathbb{R}^N$ subject either to Navier boundary conditions

$$(3.2) \quad u = \Delta u = 0 \quad \text{on } \partial\Omega$$

or to Dirichlet boundary conditions

$$(3.3) \quad u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

Here $\partial/\partial\nu$ denotes the exterior normal derivative at the boundary. For the nonlinearity f we require the following assumptions:

- (f₁) $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function, and $f(x, 0) = 0$ for a.e. $x \in \Omega$.
- (f₂) There are constants $q^* > 0$, $q_* \in (0, \lambda_1)$, and $0 < p < 8/(N-4)$ for $N > 4$, resp. $p > 0$ for $N \leq 4$, such that

$$|f(x, t) - f(x, s)| \leq [q_* + q^*(|t|^p + |s|^p)]|t - s| \quad \text{for a.e. } x \in \Omega, \quad t \in \mathbb{R}.$$

- (f₃) There is $R > 0$ and $\eta > 2$ such that

$$0 < \eta F(x, t) \leq f(x, t)t \quad \text{for a.e. } x \in \Omega, \quad |t| \geq R,$$

- (f₄) f is nondecreasing in $t \in \mathbb{R}$ for a.e. $x \in \Omega$.

Here $F(x, t) := \int_0^t f(x, s) ds$, and λ_1 denotes the first eigenvalue of Δ^2 on Ω relative to the boundary conditions (3.2), (3.3), respectively.

REMARK 3.1. A simple example for (f₁)–(f₄) is given by nonlinearities of the form $f(x, t) = \sum_{i=1}^k a_i(x)|t|^{p_i}t$ with nonnegative weight functions $a_i \in L^\infty(\Omega)$ which are positive on a set of positive measure in Ω . Here we require that $0 < p_i < 8/(N-4)$ for $N > 4$ and $p_i > 0$ for $N \leq 4$, $i = 1, \dots, k$.

3.1. Navier boundary conditions. In this section we deal with the boundary value problem (3.1), (3.2). Under assumptions (f₁)–(f₄), we can cast this problem in the abstract framework of Section 2. We consider the Hilbert space $H := H^2(\Omega) \cap H_0^1(\Omega)$, endowed with the scalar product

$$(3.4) \quad (u, v) = \int_{\Omega} \Delta u \Delta v, \quad u, v \in H$$

and the corresponding norm $\|\cdot\|$. Then we have compact Sobolev embeddings

$$H \hookrightarrow L^s(\mathbb{R}^N) \quad \text{for } 1 \leq s < 2_*.$$

Here 2_* denotes the critical Sobolev exponent for the biharmonic operator, i.e. $2_* = 2N/(N-4)$ for $N > 4$ and $2_* = \infty$ for $N = 1, \dots, 4$. Using (f₂), (f₃) and these Sobolev embeddings, it is easy to see that the functional

$$\Psi: H \rightarrow \mathbb{R}, \quad \Psi(u) = \int_{\Omega} F(x, u(x)) dx.$$

is of class $C^{1,1}$, and that $A = \nabla \Psi: H \rightarrow H$ is uniquely determined by

$$(A(u), v) = \int_{\Omega} f(x, u(x))v(x) dx \quad \text{for all } u, v \in H.$$

Fixed points of A are precisely the critical points of the $C^{1,1}$ -functional

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \Psi(u),$$

and these are precisely the weak solutions of (3.1), (3.2). Since it is standard to deduce the abstract conditions (A₁) and (A₂) from assumptions (f₁)–(f₃), we omit the details at this point. Instead we recall another consequence of (f₃). For some constant $c_0 > 0$ and some bounded positive function $c: \Omega \rightarrow \mathbb{R}$, we have

$$(3.5) \quad F(x, u) \geq c(x)|u|^\eta - c_0 \quad \text{for } x \in \Omega, \quad u \in \mathbb{R},$$

see e.g. [30, p. 111]. From (3.5) we deduce the following property of Φ .

LEMMA 3.2. *If $S \subset H \setminus \{0\}$ is a compact subset and $\tilde{S} := \{tu : u \in S, t \geq 0\}$, then*

$$\lim_{u \in \tilde{S}, \|u\| \rightarrow \infty} \Phi(u) = -\infty$$

PROOF. Let $(u_n)_n \subset \tilde{S}$ be a sequence with $\|u_n\| \rightarrow \infty$, and let $v_n \in S, t_n \geq 0$ be such that $u_n = t_n v_n$. By compactness of S , we may pass to a subsequence such that $v_n \rightarrow v \in S$. Since $\eta \leq p + 1 < 2_*$ by (f₂), (f₃) and (3.5), the compactness of the Sobolev embedding $H \hookrightarrow L^\eta(\Omega)$ and the boundedness of the function c imply that

$$\int_{\Omega} c(x)|v_n|^\eta dx \rightarrow \int_{\Omega} c(x)|v|^\eta dx > 0 \quad \text{as } n \rightarrow \infty.$$

Moreover, $t_n \rightarrow \infty$, since $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$. From (3.5) we thus infer

$$\begin{aligned} \Phi(u_n) &= \frac{\|u_n\|^2}{2} - \int_{\Omega} F(x, u_n) dx \leq \frac{\|u_n\|^2}{2} - \int_{\Omega} c(x)|u_n|^\eta dx + c_0|\Omega| \\ &= t_n^\eta \left(t_n^{2-\eta} \frac{\|v_n\|^2}{2} - \int_{\Omega} c(x)|v_n|^\eta dx + t_n^{-\eta} c_0|\Omega| \right) \\ &= t_n^\eta \left(o(1) - \int_{\Omega} c(x)|v|^\eta dx \right). \end{aligned}$$

Hence $\Phi(u_n) \rightarrow -\infty$ as $n \rightarrow \infty$, as claimed. \square

Next we consider the closed cone $\mathcal{K} := \{u \in H : u \geq 0 \text{ a.e. on } \Omega\}$. For this choice we also consider the dual cones \mathcal{K}^* , $(-\mathcal{K})^*$ and the projections \mathcal{P} , \mathcal{P}^* , \mathcal{Q} , \mathcal{Q}^* as defined in Section 2, and we want to verify the invariance conditions (A₃) and (A₄). We recall the crucial fact that, on *any* smooth bounded domain Ω , the Navier boundary conditions (3.2) allow a strong maximum principle to hold for the biharmonic operator Δ^2 on Ω . More precisely, for any nonnegative function $h \in L^\infty(\Omega)$, $h \not\equiv 0$, the unique solution $v \in H$ of the problem $\Delta^2 v = h$ subject to (3.2) is a $C^3(\bar{\Omega})$ -function which satisfies

$$(3.6) \quad v > 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial v}{\partial \nu} < 0 \quad \text{on } \partial\Omega.$$

This follows by applying twice the maximum principle for the harmonic operator $-\Delta$ and once the Hopf boundary lemma. The following lemma collects useful consequences of this fact.

LEMMA 3.3.

- (a) $\mathcal{K}^* \subset -\mathcal{K}$,
- (a') if $u \in \mathcal{K}^* \setminus \{0\}$, then $u < 0$ a.e. in Ω ,
- (b) $(-\mathcal{K})^* \subset \mathcal{K}$,
- (c) $\mathcal{P}(u) \geq u^+$, $\mathcal{P}^*(u) \leq u^-$,
- (d) $\mathcal{Q}(u) \leq u^-$, $\mathcal{Q}^*(u) \geq u^+$.

For the case of Dirichlet boundary conditions on the unit ball B , part (a) has already been noted in [19, Lemma 2].

PROOF OF LEMMA 3.3. Since $u = \mathcal{P}u + \mathcal{P}^*u = \mathcal{Q}u + \mathcal{Q}^*u$ with $\mathcal{P}u \geq 0$ and $\mathcal{Q}u \leq 0$, properties (a)–(d) are readily seen to be equivalent. Property (a) can

be proved precisely as [19, Lemma 2]. We now prove the stronger statement (a'). Let $u \in \mathcal{K}^* \setminus \{0\}$. Since the space $\mathcal{C} := \{w \in C^2(\overline{\Omega}) : w = 0 \text{ on } \partial\Omega\}$ is dense in $H = H^2(\Omega) \cap H_0^1(\Omega)$, there is $u_0 \in \mathcal{C}$ such that $(u, u_0) > 0$. Consider an arbitrary function $h \in L^\infty(\Omega)$, $h \geq 0$, $h \not\equiv 0$, and let $v \in H$ be the unique solution of the problem $\Delta^2 v = h$ subject to the boundary conditions (3.2). Then $v \in C^3(\overline{\Omega})$ and $v > 0$ in Ω , $\partial v / \partial \nu < 0$ on $\partial\Omega$ by (3.6). Hence there exists $\varepsilon = \varepsilon(h, u_0)$ such that $v + \varepsilon u_0 \in \mathcal{K}$. Consequently,

$$0 \geq (u, v + \varepsilon u_0) = \int_{\Omega} u h \, dx + \varepsilon (u, u_0) > \int_{\Omega} u h \, dx.$$

We conclude that $\int_{\Omega} u h < 0$ for all $h \in L^\infty(\Omega)$ with $h \geq 0$, $h \not\equiv 0$, and this implies that $u(x) < 0$ a.e. on Ω . \square

COROLLARY 3.4. *If (f₄) holds, then $(A(u), v) \leq (A(P^*u), v)$ and $(A(u), w) \leq (A(Q^*u), w)$ for $u \in H$, $v \in \mathcal{K}^*$ and $w \in (-\mathcal{K})^*$. Thus the assumptions (A₃) and (A₄) from Section 2 are satisfied.*

PROOF. Let $u \in H$, $v \in \mathcal{K}^*$, $w \in (-\mathcal{K})^*$. By Lemma 3.3(a), (b) we have $v \leq 0$ and $w \geq 0$. Moreover, by Lemma 3.3(c), (d) and (f₄) we have

$$f(x, [\mathcal{P}^*(u)](x)) \leq f(x, u^-(x)) \leq 0 \leq f(x, u^+(x)) \leq f(x, [\mathcal{Q}^*(u)](x))$$

for $x \in \Omega$, and therefore

$$\begin{aligned} (A(u), v) &= \int_{\Omega} f(x, u) v \, dx \leq \int_{\Omega} f(x, u^-) v \, dx \\ &\leq \int_{\Omega} f(x, \mathcal{P}^*(u)) v \, dx = (A(P^*(u)), v), \\ (A(u), w) &= \int_{\Omega} f(x, u) w \, dx \leq \int_{\Omega} f(x, u^+) w \, dx \\ &\leq \int_{\Omega} f(x, \mathcal{Q}^*(u)) w \, dx = (A(Q^*(u)), w), \end{aligned}$$

as claimed. \square

Now we can state our main result.

THEOREM 3.5. *Suppose that (f₁)–(f₄) are satisfied. Then problem (3.1), (3.2) has at least three nontrivial solutions u_1 , u_2 , u_3 , where u_1 is positive, u_2 is negative and u_3 changes sign. If in addition $f(x, -t) = -f(x, t)$ holds for all $x \in \Omega$, $t \in \mathbb{R}$, then problem (3.1), (3.2) has infinitely many sign changing solutions.*

PROOF. By Corollary 3.4 and the preceding discussion, we know that the abstract conditions (A₁)–(A₄) are satisfied. We apply Theorem 2.6. For this we let $u \in \mathcal{K}$, $v \in -\mathcal{K}$ be two linearly independent functions, and define $h_s: [0, 1] \rightarrow H \setminus \{0\}$ for $s > 0$ by $h_s(t) = s(tv + (1-t)u)$. Then $h_s(0) \in \mathcal{K}$, $h_s(1) \in -\mathcal{K}$ for all s . Moreover, if s is sufficiently large, then $\Phi(h_s(t)) < 0$ for all $t \in [0, 1]$

by Lemma 3.2. Hence all assumptions of Theorem 2.6 are satisfied, and thus we obtain the existence of nontrivial critical points $u_1 \in \mathcal{K}$, $u_2 \in -\mathcal{K}$ and $u_3 \in H \setminus (\mathcal{K} \cup -\mathcal{K})$ of Φ . Hence u_1, u_2, u_3 are nontrivial solutions of (3.1), (3.2). Applying the strong maximum principle stated in (3.6), we conclude that u_1 is positive and u_2 is negative, whereas u_3 changes sign.

Now we suppose that, in addition, $f(x, -t) = -f(x, t)$ for all $x \in \Omega$, $t \in \mathbb{R}$. Then $A(-u) = -A(u)$ and $\Psi(-u) = \Psi(u)$ for all $u \in H$. We fix $n \in \mathbb{N}$ and an arbitrary n -dimensional subspace $V \subset H$. Then

$$\lim_{u \in V, \|u\| \rightarrow \infty} \Phi(u) = -\infty$$

by Lemma 3.2. Hence all assumptions of Theorem 2.6 are satisfied, and this Theorem yields $n - 1$ pairs of critical points of Φ located in $H \setminus (\mathcal{K} \cup -\mathcal{K})$. These critical points are sign changing solutions of (3.1), (3.2). Since $n \in \mathbb{N}$ was arbitrary, the assertion follows. \square

We close this section with a result concerning the Morse index of sign changing solutions when Φ is a C^2 -functional and the nonlinearity satisfies a strict monotonicity assumption.

PROPOSITION 3.6. *Suppose that, in addition to (f₁)–(f₄), the nonlinearity $f \in C^1(\mathbb{R})$ satisfies $f'(t) > f(t)/t > 0$ for $t \in \mathbb{R} \setminus \{0\}$. Then every sign changing solution of (3.1), (3.2) has Morse index greater than or equal to two with respect to Φ .*

REMARK 3.7. The assumption $f'(t) > f(t)/t > 0$ for $t \in \mathbb{R} \setminus \{0\}$ is satisfied for the class of nonlinearities given in Remark 3.1.

PROOF OF PROPOSITION 3.6. Since $f \in C^1(\mathbb{R})$ satisfies (f₂), it is easy to see that $\Phi \in C^2(H, \mathbb{R})$ and

$$\Phi''(u)(v, w) = \int_{\Omega} (\Delta v \Delta w - f'(u)vw) dx \quad \text{for } v, w \in H.$$

Let u be a sign changing solution of (3.1), (3.2), and let $u_1 = \mathcal{P}u$, $u_2 = \mathcal{P}^*u$. First we note that

$$(3.7) \quad \Phi''(u)(u, u) = \int_{\Omega} ((\Delta u)^2 - f'(u)u^2) dx = \int_{\Omega} (f(u)u - f'(u)u^2) dx < 0$$

by assumption. Next, let $\Omega^+ = \{x \in \Omega : u(x) > 0\}$. By Lemma 3.3(a'), (c) we have $u_1 u_2 \leq 0$ a.e. on Ω and $u_1 u_2 < 0$ a.e. on Ω^+ . Therefore

$$(3.8) \quad \int_{\Omega} \frac{f(u)}{u} u_1 u_2 dx \leq \int_{\Omega^+} \frac{f(u)}{u} u_1 u_2 dx < 0$$

by assumption. Now for $\alpha, \beta \in \mathbb{R}$ we have

$$(3.9) \quad \begin{aligned} \Phi''(u)(\alpha u_1 + \beta u_2, \alpha u_1 + \beta u_2) \\ = \Phi''(u)(u, \alpha^2 u_1 + \beta^2 u_2) - (\alpha - \beta)^2 \Phi''(u)(u_1, u_2), \end{aligned}$$

where

$$(3.10) \quad \Phi''(u)(u_1, u_2) = \int_{\Omega} \Delta u_1 \Delta u_2 \, dx - \int_{\Omega} f'(u) u_1 u_2 \, dx = - \int_{\Omega} f'(u) u_1 u_2 \, dx$$

by the orthogonality of u_1, u_2 and

$$(3.11) \quad \begin{aligned} \Phi''(u)(u, \alpha^2 u_1 + \beta^2 u_2) \\ = \int_{\Omega} \Delta u \Delta(\alpha^2 u_1 + \beta^2 u_2) \, dx - \int_{\Omega} f'(u) u (\alpha^2 u_1 + \beta^2 u_2) \, dx \\ = \int_{\Omega} f(u) (\alpha^2 u_1 + \beta^2 u_2) \, dx - \int_{\Omega} f'(u) u (\alpha^2 u_1 + \beta^2 u_2) \, dx \\ = \int_{\Omega} \left[\frac{f(u)}{u} - f'(u) \right] [(\alpha u_1 + \beta u_2)^2 + (\alpha - \beta)^2 u_1 u_2] \, dx \\ \leq \int_{\Omega} \left[\frac{f(u)}{u} - f'(u) \right] (\alpha - \beta)^2 u_1 u_2 \, dx. \end{aligned}$$

Combining (3.8)–(3.11), we get

$$(3.12) \quad \Phi''(u)(\alpha u_1 + \beta u_2, \alpha u_1 + \beta u_2) \leq (\alpha - \beta)^2 \int_{\Omega} \frac{f(u)}{u} u_1 u_2 \, dx < 0 \quad \text{if } \alpha \neq \beta.$$

Now from (3.7) and (3.12) it follows that Φ'' is negative definite on the two-dimensional subspace spanned by u_1 and u_2 . Hence the Morse index of u with respect to Φ is at least two. \square

3.2. Dirichlet boundary conditions. In this section we briefly discuss the boundary value problem (3.1), (3.3). So now we consider the Hilbert space $H := H_0^2(\Omega)$, which we also endow with the scalar product given by (3.4). We can define Φ, Ψ and A in a completely analogous way as in the last section, relative to the new underlying space $H = H_0^2(\Omega)$. Then we get the precise analogue of Lemma 3.2 for compact subsets $S \subset H \setminus \{0\}$. Considering again the closed cone \mathcal{K} of nonnegative functions in H , we need a result similar to Lemma 3.3. For this we restrict our attention to domains Ω such that the Dirichlet boundary conditions (3.3) allow a maximum principle to hold for the biharmonic operator Δ^2 on Ω . In other words, we require that the Green function of the biharmonic operator Δ^2 corresponding to the boundary conditions (3.3) is *positive*. By a similar argument as in the last section, we obtain the following (cf. also [19, Lemma 2]).

LEMMA 3.8. *Suppose that $\Omega \subset \mathbb{R}^N$ is such that the Green function of Δ^2 corresponding to the boundary conditions (3.3) is positive. Then we have:*

- (a) $\mathcal{K}^* \subset -\mathcal{K}$,
- (a') if $u \in \mathcal{K}^* \setminus \{0\}$, then $u < 0$ a.e. in Ω ,
- (b) $(-\mathcal{K})^* \subset \mathcal{K}$,
- (c) $\mathcal{P}(u) \geq u^+$, $\mathcal{P}^*(u) \leq u^-$,
- (d) $\mathcal{Q}(u) \leq u^-$, $\mathcal{Q}^*(u) \geq u^+$.

PROOF. As in the proof of Lemma 3.3, it suffices to show (a'). Let $u \in \mathcal{K}^* \setminus \{0\}$. Since the space $C_0^\infty(\Omega)$ of test functions is dense in $H = H_0^2(\Omega)$, there is $u_0 \in C_0^\infty(\Omega)$ such that $(u, u_0) > 0$. Consider an arbitrary function $h \in L^\infty(\Omega)$, $h \geq 0$, $h \not\equiv 0$, and let $v \in H$ be the unique solution of the problem $\Delta^2 v = h$ subject to the boundary conditions (3.3). Then v is continuous and positive in Ω (since the Green function is positive). Hence there exists $\varepsilon = \varepsilon(h, u_0)$ such that $v + \varepsilon u_0 \in \mathcal{K}$. Consequently,

$$0 \geq (u, v + \varepsilon u_0) = \int_{\Omega} u h \, dx + \varepsilon (u, u_0) > \int_{\Omega} u h \, dx$$

We conclude that $\int_{\Omega} u h < 0$ for all $h \in L^\infty(\Omega)$ with $h \geq 0$, $h \not\equiv 0$, and this implies that $u(x) < 0$ a.e. on Ω . \square

As already remarked in the introduction, the additional assumption on the Green function in Lemma 3.8 does not hold for any domain Ω . However, it is true if Ω is a ball in \mathbb{R}^N . In fact, in this case, the Green function is known explicitly, and it is positive, see [6, p. 126]. In two dimensions, this is also true for perturbations of a disk, see [21], [29], and it is true for the limaçon in a certain parameter range, see [12]. Now we can proceed as in the last section to obtain the following result. We omit the details.

THEOREM 3.9. *Suppose that $\Omega \subset \mathbb{R}^N$ is a domain such that the Green function of the biharmonic operator Δ^2 corresponding to the boundary conditions (3.3) is positive, and suppose that the nonlinearity f satisfies (f₁)–(f₄). Then problem (3.1), (3.3) has at least three nontrivial solutions u_1, u_2, u_3 , where u_1 is positive, u_2 is negative and u_3 changes sign. If in addition $f(x, -t) = -f(x, t)$ holds for all $x \in \Omega$, $t \in \mathbb{R}$, then problem (3.1), (3.3) has infinitely many sign changing solutions.*

By precisely the same argument as in the proof of Proposition 3.6, we also get the following.

PROPOSITION 3.10. *Suppose that $\Omega \subset \mathbb{R}^N$ is a domain such that the Green function of the biharmonic operator Δ^2 corresponding to the boundary conditions (3.3) is positive. Suppose furthermore that, in addition to (f₁)–(f₄), the nonlinearity $f \in C^1(\mathbb{R})$ satisfies $f'(t) > f(t)/t > 0$ for $t \in \mathbb{R} \setminus \{0\}$. Then every sign*

changing solution of (3.1), (3.3) has Morse index greater than or equal to two with respect to Φ .

4. Appendix

Here we give a short proof (using the Leray Schauder continuation principle) of a topological lemma due to Liu (see [25] and [26, Lemma 3.1]).

LEMMA 4.1. *Let $Q := [0, 1]^2$ be the unit square, and let $\mathcal{B} \subset Q$ be a relatively open set such that $(\{0\} \times [0, 1]) \subset \mathcal{B}$ and $(\{1\} \times [0, 1]) \cap \overline{\mathcal{B}} = \emptyset$. Then there is a connected component Γ of the relative boundary $\partial\mathcal{B}$ of \mathcal{B} in Q such that $\Gamma \cap ([0, 1] \times \{0\}) \neq \emptyset$ and $\Gamma \cap ([0, 1] \times \{1\}) \neq \emptyset$.*

PROOF. Consider the continuous map $g: Q \rightarrow \mathbb{R}$ defined by

$$g: Q \rightarrow \mathbb{R}, \quad g(x) = \begin{cases} -\text{dist}(x, \partial\mathcal{B}) & \text{for } x \in \overline{\mathcal{B}}, \\ \text{dist}(x, \partial\mathcal{B}) & \text{for } x \in Q \setminus \overline{\mathcal{B}}, \end{cases}$$

and for each $\lambda \in [0, 1]$ let $g_\lambda: [0, 1] \rightarrow \mathbb{R}$ be the continuous function defined by $g_\lambda(s) = g(s, \lambda)$. By assumption we have $g_\lambda(0) < 0 < g_\lambda(1)$ for all $\lambda \in [0, 1]$, and this implies

$$\deg(g_\lambda, [0, 1], 0) = 1.$$

where \deg denotes the usual Leray-Schauder degree, see e.g. [31]. Now the Leray-Schauder continuation principle (see [31, Theorem 14.C]) implies that there exists a connected set $\Gamma \subset [0, 1] \times [0, 1]$ such that

$$g_\lambda(s) = 0 \quad \text{for all } (s, \lambda) \in \Gamma, \\ \Gamma \cap ([0, 1] \times \{0\}) \neq \emptyset, \quad \Gamma \cap ([0, 1] \times \{1\}) \neq \emptyset.$$

Since $g_\lambda(s) = g(s, \lambda) = 0$ if and only if $(s, \lambda) \in \partial\mathcal{B}$, the assertion follows. \square

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