POSITIVE PERIODIC SOLUTIONS
OF SUPERLINEAR SYSTEMS OF INTEGRAL EQUATIONS
DEPENDING ON PARAMETERS

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Abstract. A class of superlinear system of integral equations depending on multi parameters is considered. It is shown that there are three mutually exclusive and exhaustive subsets $\Theta_1$, $\Gamma$ and $\Theta_2$ of the parameter space such that there exist at least two positive periodic solutions associated with elements in $\Theta_1$, at least one positive periodic solution associated with $\Gamma$ and none associated with $\Theta_2$.

1. Introduction

Coupled differential systems arise in a number of biological, ecological, economical and other models which describe interactions. In [3], a coupled differential system of the form

\[
\begin{align*}
x'(t) &= -a(t)x(t) + \lambda k(t)f(x(t - \tau_1(t)), y(t - \sigma_1(t))), \\
y'(t) &= -b(t)y(t) + \nu h(t)g(x(t - \tau_2(t)), y(t - \sigma_2(t))),
\end{align*}
\]

is studied and the existence of positive periodic solutions corresponding to different values of the parameters $\lambda$ and $\nu$ are derived by transforming the above

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system into an equivalent coupled system of integral equations

\[(1.1) \quad x(t) = \lambda \int_{t}^{t+\omega} K(t, s)k(s)f(x(s-\tau_1(s)), y(s-\sigma_1(s))) \, ds,\]

\[(1.2) \quad y(t) = \nu \int_{t}^{t+\omega} H(t, s)h(s)g(x(s-\tau_2(s)), y(s-\sigma_2(s))) \, ds.\]

This prompts us to study more general coupled systems of integral equations. For this purpose, we follow some of the ideas developed by the authors in [2] in setting up our problem: First, $R^N$ is the $N$-dimensional Euclidean space endowed with componentwise ordering $\leq$. For any $u, v \in R^N$, the interval $[u, v]$ is the set $\{x \in R^N | u \leq x \leq v\}$. Let $T = (t_1, \ldots, t_N) \in R^N$ with positive components and let $e^{(i)} = (1, 0, \ldots, 0), \ldots, e^{(N)} = (0, \ldots, 0, 1)$ be the standard orthonormal vectors in $R^N$. Let $G$ be a closed subset of $R^N$ which has the following “periodic” structure: for each $x \in G$,

$$x + t e^{(i)} \in G,$$

and for each pair $y, z \in G$,

$$\mu([y, y + T] \cap G) = \mu([z, z + T] \cap G) > 0,$$

where $\mu$ is the Lebesgue measure, and we set

$$G(x) = [x, x + T] \cap G.$$

Examples of nontrivial $G$ can be found in [2].

The system of integral equations of the form

\[(1.3) \quad \phi_j(x) = \lambda_j \int_{G(x)} K_j(x, s)f_j(s, \phi_1(s-\tau_1(s)), \ldots, \phi_\omega(s-\tau_\omega(s))) \, ds,\]

for $x \in G$, $j = 1, \ldots, \omega$, will be considered. Here, the functions $K_j, f_j, \tau_{jk}$, where $j, k \in \{1, \ldots, \omega\}$, satisfy the following ‘periodic’ conditions:

- for $j \in \{1, \ldots, \omega\}$, $K_j \in C(G \times G, R^+)$, $K_j(x + t e^{(i)}, y + t e^{(i)}) = K_j(x, y)$ for any $(x, y) \in G \times G$ and $i \in \{1, \ldots, N\}$,
- for $j \in \{1, \ldots, \omega\}$, $f_j \in C(G \times R^+, R)$, $f_j(x + t e^{(i)}, u_1, \ldots, u_\omega) = f_j(x, u_1, \ldots, u_\omega)$ for $i \in \{1, \ldots, N\}$ and any $x \in G$,
- for $j, k \in \{1, \ldots, \omega\}$, $\tau_{jk}: G \to G$ is continuous, $\tau_{jk}(x + t e^{(i)}) = \tau_{jk}(x)$ for any $x \in G$ and $i \in \{1, \ldots, N\}$,

the boundedness conditions:

$$\inf_{x, y \in G(t), t \in G} K_j(x, y) \geq m_j > 0, \quad M_j = \sup_{x, y \in G(t), t \in G} K_j(x, y) < \infty,$$

for $j \in \{1, \ldots, \omega\}$, and the “superlinear” conditions:

\[(H1) \quad \text{for } j \in \{1, \ldots, \omega\}, f_j(x, 0, \ldots, 0) > 0 \text{ for any } x \in G, f_j(x, u_1, \ldots, u_\omega) \text{ is nondecreasing on } (u_1, \ldots, u_\omega) \in [0, \infty) \times \ldots \times [0, \infty) \text{ (i.e. } f_j(x, u_1, \ldots, u_\omega) \leq f_j(x, v_1, \ldots, v_\omega) \text{ for } 0 \leq u_j \leq v_j, j \in \{1, \ldots, \omega\}, \text{ and } x \in G),\]
(H2) for \( j \in \{1, \ldots, n\} \), \( \lim_{u_t + \ldots + u_\omega \to \infty} f_j(x, u_1, \ldots, u_\omega)/(u_1 + \ldots + u_\omega) = \infty \) uniformly with respect to all \( x \in G \).

The numbers \( \lambda_1, \ldots, \lambda_\omega \) will be assumed to be nonnegative and treated as parameters. Note that when \( \lambda_1 = \ldots = \lambda_\omega = 0 \), our system reduces to a system of decoupled equations. For this reason, the case \( \lambda_1 = \ldots = \lambda_\omega = 0 \) will be avoided in our subsequent discussions. Therefore our system (1.3) may be regarded as a multi-state interactive model depending on the parameter vector \( \lambda = (\lambda_1, \ldots, \lambda_\omega) \) in the set

\[
\Xi = \{ (\lambda_1, \ldots, \lambda_\omega) : \lambda_j \geq 0, \; j = 1, \ldots, \omega \} \setminus \{(0, \ldots, 0)\}.
\]

For any \( (a_1, \ldots, a_m) \) and \( (b_1, \ldots, b_m) \) in \( \mathbb{R}^m \), we will write \( (a_1, \ldots, a_m) \geq (b_1, \ldots, b_m) \) if \( a_j \geq b_j \) for \( j \in \{1, \ldots, m\} \). If \( (a_1, \ldots, a_m) \geq (b_1, \ldots, b_m) \) and if \( a_k > b_k \) or some \( k \in \{1, \ldots, m\} \), we will write \( (a_1, \ldots, a_m) > (b_1, \ldots, b_m) \).

A vector function \( (\phi_1, \ldots, \phi_\omega) : G \to \mathbb{R}^\omega \) is said to be positive if \( (\phi_1(x), \ldots, \phi_\omega(x)) \geq (0, \ldots, 0) \) for all \( x \in G \) and \( (\phi_1(x_0), \ldots, \phi_\omega(x_0)) > (0, \ldots, 0) \) for some \( x_0 \in G \).

It is said to be \( T \)-periodic if \( \phi_1, \ldots, \phi_\omega \) are \( T \)-periodic, that is, \( \phi_j(x + te^{(i)}) = \phi_j(x) \) for \( x \in G, \; j \in \{1, \ldots, \omega\} \) and \( i \in \{1, \ldots, N\} \).

By a solution of (1.3) associated with the parameter vector \( (\alpha_1, \ldots, \alpha_\omega) \in \Xi \), we mean a continuous vector function \( \phi : G \to \mathbb{R}^\omega \) which satisfies (1.3) for \( \lambda_j = \alpha_j \) for \( j \in \{1, \ldots, \omega\} \). As in [3], we will prove there exists a continuous surface \( \Gamma \) splitting \( \Xi \) into disjoint subsets \( \Theta_1, \Gamma \) and \( \Theta_2 \) such that the system (1.3) has at least two, at least one, or no positive \( T \)-periodic solutions according whether \( \lambda \) is in \( \Theta_1, \Gamma \) or \( \Theta_2 \), respectively. We remark, however, that, the result in [3] is only good for the coupled system (1.1)–(1.2) which is much less general than our results below.

2. Some basic lemmas

Let \( X \) be the set of all real \( T \)-periodic continuous functions defined on \( G \) which is endowed with the usual linear structure as well as the norm

\[
\| \psi \| = \sup_{x \in G(t), \; t \in G} |\psi(x)|.
\]

Then \( X^\omega \) is also a Banach space with the norm

\[
\| (\phi_1, \ldots, \phi_\omega) \| = \| \phi_1 \| + \ldots + \| \phi_\omega \|.
\]

Furthermore, let \( \Phi \) and \( \Omega \) be defined respectively by

\[
\Phi = \{ (\phi_1, \ldots, \phi_\omega) \in X^\omega : \phi_j(x) \geq 0, \; x \in G, \; j = 1, \ldots, \omega \},
\]

\[
\Omega = \{ (\phi_1, \ldots, \phi_\omega) \in \Phi : (\phi_1(x) + \ldots + \phi_\omega(x) \geq \alpha^* (\| \phi_1 \| + \ldots + \| \phi_\omega \|)), \; x \in G \},
\]

where \( \alpha^* = \min_{j=1, \ldots, \omega} \{ m_j/M_j \} \). Then \( \Phi \) and \( \Omega \) are cones in \( X^\omega \).
Define, for each \( \phi = (\phi_1, \ldots, \phi_\omega) \in X^\omega \),

\[
T_\lambda(\phi)(x) = (A_{\lambda_1}(\phi)(x), \ldots, A_{\lambda_\omega}(\phi)(x)),
\]

where

\[
A_{\lambda_j}(\phi)(x) = \lambda_j \int_{G(x)} K_j(x, s)f_j(s, \phi_1(s - \tau_{j_1}(s)), \ldots, \phi_\omega(s - \tau_{j_\omega}(s))) \, ds,
\]

for \( j = 1, \ldots, \omega \). Then our system (1.3) can be written as

\[
\phi(x) = T_\lambda(\phi)(x).
\]

For the sake of convenience, we will set

\[
f_j(s, \phi(s)) := f_j(s, \phi_1(s - \tau_{j_1}(s)), \ldots, \phi_\omega(s - \tau_{j_\omega}(s)))
\]

in the following discussions.

Let \( \phi = (\phi_1, \ldots, \phi_\omega) \in \Phi \). For each \( j \in \{1, \ldots, \omega\} \),

\[
A_{\lambda_j}(\phi)(x) = \lambda_j \int_{G(x)} K_j(x, s)f_j(s, \phi(s)) \, ds \leq \lambda_j M_j \int_{G(x)} f_j(s, \phi(s)) \, ds
\]

so that

\[
\frac{1}{M_j} \|A_{\lambda_j}(\phi)\| \leq \lambda_j \int_{G(x)} f_j(s, \phi(s)) \, ds
\]

and

\[
A_{\lambda_j}(\phi)(x) \geq \lambda_j m_j \int_{G(x)} f_j(s, \phi(s)) \, ds \geq \alpha^* \|A_{\lambda_j}(\phi)\|.
\]

That is, for each \( \lambda \in \Xi \), \( T_\lambda \Phi \) is contained in \( \Omega \).

Furthermore, by standard arguments, we may also show that \( T_\lambda \) is completely continuous. To see this, we may assume for the sake of simplicity that \( G \) is a subset in \( \mathbb{R}^2 \). Recall that the interval \([u, v]\) is the set \( \{x \in \mathbb{R}^2 \mid u \leq x \leq v\} \). Let \( A = (x_1, y_1), B = (x_2, y_2) \) in \( G \). We consider the case where \((x_1, y_1) \leq (x_2, y_2)\), while the other cases can similarly be treated. We set \( C = (x_2, y_1) \), \( D = (x_1, y_2) \), \( E = (x_2, y_1 + t_2) \), \( F = (x_1 + t_1, y_2 + t_2) \), \( K = (x_1 + t_1, y_1 + t_2) \), \( H = (x_1 + t_1, y_2) \), \( I = (x_2 + t_1, y_1 + t_2) \), \( J = (x_2 + t_1, y_2 + t_2) \), and \( G_1 = [A, B] \), \( G_2 = [D, E] \), \( G_3 = [C, H] \), \( G_4 = [B, K] \), \( G_5 = [E, F] \), \( G_6 = [H, I] \), \( G_7 = [K, J] \).

We suppose that \( \Delta \) is a bounded set of \( X^\omega \). Then there exists constant \( \tilde{T} > 0 \), such that \( \|\phi\| \leq \tilde{T} \) for any \( \phi \in \Delta \). In view of the theorem of Arzela–Ascoli, we
only need to show that $A_{\lambda_j}(\Delta)$ is equicontinuous for any $j \in \{1, \ldots, \omega\}$. Indeed,

$$A_{\lambda_j}(\phi)(B) - A_{\lambda_j}(\phi)(A) = \lambda_j \left\{ \int_{G_7} + \int_{G_6} + \int_{G_5} \right\} K_j(B, s)f_j(s, \phi(\ast)) \, ds$$

$$+ \lambda_j \int_{G_4} [K_j(B, s) - K_j(A, s)]f_j(s, \phi(\ast)) \, ds$$

$$- \lambda_j \left\{ \int_{G_3} + \int_{G_2} + \int_{G_1} \right\} K_j(A, s)f_j(s, \phi(\ast)) \, ds.$$

Furthermore, $f_j \in C(G(x) \times [-\hat{T}, \hat{T}] \times \ldots \times [-\hat{T}, \hat{T}] \times \{ R \})$ and $f_j(x + t, e_i, u_1, \ldots, u_\omega) = f_j(x, u_1, \ldots, u_\omega)$ for any $x \in G$, then there exists constant $\hat{H}$, such that

$$|f_j(s, \phi(\ast))| \leq \hat{H}, \quad \text{for } s \in \bigcup_{j=1}^{\omega} G_j,$$

thus

$$|\lambda_j \int_{G_7} K_j(B, s)f_j(s, \phi(\ast)) \, ds| \leq \lambda_j M_j \hat{H}|x_2 - x_1| |y_2 - y_1|,$$

$$|\lambda_j \int_{G_6} K_j(B, s)f_j(s, \phi(\ast)) \, ds| \leq \lambda_j M_j \hat{H}|x_2 - x_1|,$$

$$|\lambda_j \int_{G_5} K_j(B, s)f_j(s, \phi(\ast)) \, ds| \leq \lambda_j M_j \hat{H}|y_2 - y_1|,$$

$$|\lambda_j \int_{G_4} K_j(A, s)f_j(s, \phi(\ast)) \, ds| \leq \lambda_j M_j \hat{H}|x_1 - x_2|,$$

$$|\lambda_j \int_{G_3} K_j(A, s)f_j(s, \phi(\ast)) \, ds| \leq \lambda_j M_j \hat{H}|y_1 - y_2|,$$

and

$$|\lambda_j \int_{G_2} [K_j(B, s) - K_j(A, s)]f_j(s, \phi(\ast)) \, ds| \leq \lambda_j \hat{H} \int_{G(B)} |K_j(B, s) - K_j(A, s)| \, ds.$$

In view of the uniformity of $K_j(x, y)$ in $G(B)$, for any $\varepsilon > 0$, there is $\delta$ which satisfies

$$0 < \delta < \min \left\{ \frac{\varepsilon}{\lambda M_j \hat{H} t_2}, \frac{\varepsilon}{\lambda M_j \hat{H} t_1}, \frac{\varepsilon}{\lambda M_j \hat{H}} \right\},$$

and for $0 < x_2 - x_1 < \delta, 0 < y_2 - y_1 < \delta$, we have

$$|K_j(B, s) - K_j(A, s)| < \frac{\varepsilon}{\lambda_j \hat{H} t_2 t_1}, \quad \text{for } s \in G(B).$$
Thus
\[
|A_{\lambda}(\phi)(B) - A_{\lambda}(\phi)(A)| \leq \left| \lambda \int_{G} K_{j}(B, s)f_{j}(s, \phi(*)(s))ight| ds
\]
\[
+ \lambda \int_{G_{a}} K_{j}(B, s)f_{j}(s, \phi(*)(s)) ds + \lambda \int_{G_{b}} K_{j}(B, s)f_{j}(s, \phi(*)(s)) ds
\]
\[
+ \lambda \int_{G_{a}} [K_{j}(B, s) - K_{j}(A, s)]f_{j}(s, \phi(*)(s)) ds
\]
\[
+ \lambda \int_{G_{a}} K_{j}(A, s)f_{j}(s, \phi(*)(s)) ds + \lambda \int_{G_{a}} K_{j}(A, s)f_{j}(s, \phi(*)(s)) ds
\]
\[
+ \lambda \int_{G_{a}} K_{j}(A, s)f_{j}(s, \phi(*)(s)) ds \leq \Delta
\]
for any \( \phi \in \Delta \). This means that \( A_{\lambda}(\Delta) \) is equicontinuous.

**Lemma 2.1.** For any compact subset \( D \) of \( \Xi \), there exists a constant \( b_{D} > 0 \) such that any positive \( T \)-periodic solution \( \phi = (\phi_{1}, \ldots, \phi_{\omega}) \) of (1.3) associated with \( \lambda = (\lambda_{1}, \ldots, \lambda_{\omega}) \in D \) will satisfy \( \|\phi\| < b_{D} \).

**Proof.** Suppose to the contrary that there is a sequence
\[
\{\phi^{(n)}\} = \{(\phi_{1}^{(n)}, \ldots, \phi_{\omega}^{(n)})\}_{n=1}^{\infty}
\]
of positive \( T \)-periodic solutions of (1.3) associated with \( \lambda^{(n)} = (\lambda_{1}^{(n)}, \ldots, \lambda_{\omega}^{(n)}) \) such that \( \lambda^{(n)} \in D \) for all \( n \) and \( \lim_{n \to \infty} \|\phi^{(n)}\| = \infty \).

Since \( \phi^{(n)} = T_{\lambda^{(n)}}(\phi^{(n)}) \in \Omega \), thus
\[
\phi_{1}^{(n)}(x) + \ldots + \phi_{\omega}^{(n)}(x) \geq \alpha^{*}\|\phi^{(n)}\|
\]
for \( n \geq 1 \). Since \( \lambda^{(n)} \in D \) for all \( n \), there is some \( k \) such that \( \lambda_{k}^{(n)} > 0 \) for all sufficiently large \( n \). Then in view of (H2), we may choose \( R_{k} > 0, \eta_{k} \) and \( n_{0} \geq 1 \) such that
\[
f_{k}(x, u_{1}, \ldots, u_{\omega}) \geq \eta(u_{1} + \ldots + u_{\omega}) \text{ for all nonnegative } u_{1}, \ldots, u_{\omega}
\]
and \( x \in G \) which satisfy \( u_{1} + \ldots + u_{\omega} \geq R_{k} \), \( \alpha^{*}(||\phi_{1}^{(n_{0})}|| + \ldots + ||\phi_{\omega}^{(n_{0})}|| \geq R_{k}, \text{ and}
\]
\[
\alpha^{*}\eta_{k}m_{k}\lambda_{k}^{(n_{0})}, \mu G(x) > 1.
\]

Thus, we have
\[
\|\phi_{k}^{(n_{0})}|| \geq \phi_{k}^{(n_{0})}(x)
\]
\[
= \lambda_{k}^{(n_{0})} \int_{G(x)} K_{k}(x, s)f_{k}(s, \phi_{1}^{(n_{0})}(s - \tau_{k_{1}}(s)), \ldots, \phi_{\omega}^{(n_{0})}(s - \tau_{k_{\omega}}(s))) ds
\]
\[
\geq \alpha^{*}\eta_{k}m_{k}\lambda_{k}^{(n_{0})}, \mu G(x)(||\phi_{1}^{(n_{0})}|| + \ldots + ||\phi_{\omega}^{(n_{0})}||) > \|\phi_{k}^{(n_{0})}||.
\]

This is a contradiction. The proof is complete. \(\square\)
LEMMA 2.2. If (1.3) has a positive $T$-periodic solution associated with $\lambda^* = (\lambda_1^*, \ldots, \lambda_\omega^*) > (0, \ldots, 0)$, then for any $\lambda = (\lambda_1, \ldots, \lambda_\omega) \in \Xi$ that satisfies $\lambda \leq \lambda^*$, equation (1.3) also has a positive $T$-periodic solution associated with $\lambda$. The system (1.3) has a positive $T$-periodic solution associated with some $\lambda^* = (\lambda_1^*, \ldots, \lambda_\omega^*)$ satisfying $\lambda_j^* > 0$ for $j = 1, \ldots, \omega$.

PROOF. Let $\phi^* = (\phi_1^*, \ldots, \phi_\omega^*)$ be a positive $T$-periodic solution of (1.3) associated with $\lambda^*$. Since $\lambda_j^* \leq \lambda_j^*$, we have

$$\phi^*_j(x) = A\lambda_j^*(\phi^*)(x) \geq A\lambda_j(\phi^*)(x)$$

for $j \in \{1, \ldots, \omega\}$. Let $\phi^{(0)} = (\phi_1^*, \ldots, \phi_\omega^*)$ and

$$\phi^{(n+1)} = T_\lambda(\phi^{(n)}), \quad \text{for } n = 0, 1, \ldots$$

Clearly, we have

$$\phi^{(0)}(x) \geq \phi^{(1)}(x) \geq \ldots \geq \phi^{(n)}(x) \geq (0, \ldots, 0).$$

Let $\phi(x) = \lim_{n \to \infty} \phi^{(n)}(x)$. In view of the Lebesgue dominated convergence theorem, we see from (2.1) that $\phi$ is a nonnegative $T$-periodic function that satisfies

$$\phi(x) = T_\lambda(\phi)(x).$$

It will thus be a solution of (1.3) if we can show it is continuous. To see the proof, assume for the sake of simplicity that $G$ is a subset of $\mathbb{R}^2$. Then we define $A, \ldots, J, G_1, \ldots, G_7$ as in the proof of the complete continuity of $T_\lambda$. Then

$$\phi_j(B) - \phi_j(A) = \lambda_j \left\{ \int_{G_5} + \int_{G_6} + \int_{G_7} \right\} K_j(B, s) f_j(s, \phi(s)) \, ds$$

$$+ \lambda_j \int_{G_1} [K_j(B, s) - K_j(A, s)] f_j(s, \phi(s)) \, ds$$

$$- \lambda_j \left\{ \int_{G_1} + \int_{G_2} + \int_{G_3} \right\} K_j(A, s) f_j(s, \phi(s)) \, ds$$

for $j = 1, \ldots, \omega$. Since $\phi^{(0)}(x) \geq \phi^{(1)}(x) \geq \ldots \geq \phi^{(n)}(x) \geq (0, \ldots, 0)$, we see that $|\phi_j(x)| \leq |\phi^*_j(x)| \leq \|\phi^*\|$ for all $x \in G$. Furthermore, $f_j \in C(G(x) \times [-\|\phi^*\|, \|\phi^*\|] \times \ldots \times [-\|\phi^*\|, \|\phi^*\|], R)$ and $f_j(x + t_j e_i, u_1, \ldots, u_\omega) = f_j(x, u_1, \ldots, u_\omega)$ for any $x \in G$, thus there exists constant $\tilde{H}$, such that

$$|f_j(s, \phi(s))| \leq \tilde{H}, \quad s \in \bigcup_{j=1}^7 G_j, \quad j = 1, \ldots, \omega.$$  

By estimates similar to those in the proof of the complete continuity of $T_\lambda$, we may then arrive at

$$|\phi_j(B) - \phi_j(A)| \leq 7\varepsilon.$$
Now that we have shown $\phi$ is a solution of (1.3), we need to show it is positive. Indeed, since $\phi'$ is positive, $\phi(x) \geq 0$ for $x \in G$. Since each $f_j(x, 0, \ldots, 0) > 0$ for $x \in G$ by our assumption, $\phi$ cannot be the trivial solution. Thus, $\phi$ is positive.

To show the existence of a positive periodic solution associated with some $\lambda^*$, let

$$\alpha_j(x) = \int_{G(x)} K_j(x, s) ds, \quad j = 1, \ldots, \omega,$$

and

$$M_{f_j} = \max_{x \in G(x), t \in G} f_j(x, \alpha_1(x - \tau_1), \ldots, \alpha_\omega(x - \tau_\omega)), \quad j = 1, \ldots, \omega.$$

Then clearly $M_{f_j} > 0$ for $j \in \{1, \ldots, \omega\}$.

Let $(\lambda_1^*, \ldots, \lambda_\omega^*) = (1/M_{f_1}, \ldots, 1/M_{f_\omega}).$ We have

$$\alpha_j(x) = \int_{G(x)} K_j(x, s) ds \geq \lambda_j^* \int_{G(x)} K_j(x, s) f_j(s, \alpha_1(s - \tau_1(s)), \ldots, \alpha_\omega(s - \tau_\omega(s))) ds,$$

for $j = 1, \ldots, \omega$. Now let $\phi^{(0)} = (\alpha_1(x), \ldots, \alpha_\omega(x))$ and $\phi^{(n+1)} = T_{\lambda^*} \phi^{(n)}(x)$ as in (2.1). Then the same argument shows that $\phi(x) = \lim_{n \to \infty} \phi^{(n)}(x)$ is a nonnegative $T$-periodic solution of (1.3) which satisfies $\phi(x) > (0, \ldots, 0)$. The proof is complete. \hfill \Box

Let $\Pi$ be the subset of $\Xi$ such that (1.3) has a positive $T$-periodic solution associated with $\lambda = (\lambda_1, \ldots, \lambda_\omega)$. Then by Lemma 2.2, $\Pi$ contains some $\lambda^* = (\lambda_1^*, \ldots, \lambda_\omega^*)$ such that (1.3) has a positive $T$-periodic solution associated it, and hence it contains the subset

$$\Pi_* = \{(\lambda_1, \ldots, \lambda_\omega) : (\lambda_1, \ldots, \lambda_\omega) > (0, \ldots, 0), \lambda_j \leq \lambda_j^*, j = 1, \ldots, \omega\}.

**Lemma 2.3.** The subset $\Pi_*$ of $\Xi$ is bounded.

**Proof.** Suppose to the contrary that there is a sequence

$$\phi^{(n)} = \{\phi_1^{(n)}, \ldots, \phi_\omega^{(n)}\}$$

of positive $T$-periodic solutions of (1.3) associated with $\lambda^{(n)} = \{(\lambda_1^{(n)}, \ldots, \lambda_\omega^{(n)})\}$ such that $\lim_{n \to \infty} \lambda_j^{(n)} = \infty$ for some $k \in \{1, \ldots, \omega\}$. Then either there exists a subsequence $\phi^{(n_j)} = \{(\phi_1^{(n_j)}, \ldots, \phi_\omega^{(n_j)})\}$ such that $\|\phi^{(n_j)}\| \to \infty$ as $j \to \infty$ or there is $M > 0$ such that $\|\phi^{(n)}\| \leq M$ for all $n$. Since $\phi^{(n)} \in \Omega$, thus

$$\phi_1^{(n)}(x) + \ldots + \phi_\omega^{(n)}(x) \geq \alpha^* \|\phi^{(n)}\|.$$

By (H2), we may choose $R_{f_k} > 0$ such that $f_k(x, u_1, \ldots, u_\omega) \geq \eta_k(u_1 + \ldots + u_\omega)$ for all nonnegative numbers $u_1, \ldots, u_\omega$ and $x \in G$ which satisfy $u_1 + \ldots + u_\omega \geq R_{f_k}$ and some $\eta_k > 0$. In view of (H1), there exists $\delta_k > 0$ such that
There exists a sequence \( \{ x^{(n)} \} \subset G(t), t \in G, \) such that \( \phi_k^{(n)}(x^{(n)}) = \max_{x \in G(t), t \in G} \phi_k^{(n)}(x) \) by the periodicity and differentiability of \( \phi_k^{(n)}(x) \). Thus, we have
\[
\| \phi_k^{(n)} \| = \phi_k^{(n)}(x^{(n)}) = A \lambda^{(n)}(\phi^{(n)})(x^{(n)}) \geq \lambda_k^{(n)} m_k \beta_k \alpha^* \| \phi^{(n)} \| \cdot \mu G(x^{(n)}) \geq \lambda_k^{(n)} m_k \beta_k \alpha^* \| \phi^{(n)} \| \cdot \mu G(x^{(n)}) > \| \phi_k^{(n)} \|.
\]
But this is a contradiction. The proof is complete. \( \square \)

3. Main theorem

We may now show that there exists a continuous surface \( \Gamma \) separating \( \Xi \) into two disjoint subsets \( \Theta_1 \) and \( \Theta_2 \) such that \( (0, \ldots, 0) \) is a boundary point of \( \Theta_1 \) and \( (1.3) \) has at least one positive \( T \)-periodic solution for \( \lambda \in \Theta_1 \cup \Gamma \) and no positive \( T \)-periodic solution for \( \lambda \in \Theta_2 \). First let \( e^{(1)}, \ldots, e^{(\omega)} \) be the standard orthonormal vectors in \( \mathbb{R}^\omega \). Let \( \Lambda \) be the set of all convex combinations of \( e^{(1)}, \ldots, e^{(\omega)} \), that is, \( \Lambda \) is the \( (\omega - 1) \)-simplex in \( \mathbb{R}^\omega \). For each \( \mu \in \Lambda \), the half ray
\[
L_\mu = \{ \lambda \in \Xi : \lambda = t \mu, \ t > 0 \}
\]
has points which belong to \( \Pi \), defined by (2.2) and points outside \( \Pi \) (in view of Lemma 2.3). Thus the set \( \{ t > 0 : t \mu \in \Pi \} \) is nonempty and bounded above. Let
\[
t^*_\mu = \sup\{ t > 0 : t \mu \in \Pi \} \quad \text{and} \quad \lambda^*_\mu = t^*_\mu \mu.
\]
Then for each \( \mu \in \Lambda \), \( \lambda^*_\mu \in \Pi \). Indeed, let \( \{ \lambda^{(n)} \}_{n=1}^{\infty} \) be a sequence which satisfies \( \lambda^{(n)} < \lambda^{(n+1)} \) for \( n \geq 1 \) and converges to \( \lambda^*_\mu \). For each \( n \), let \( \phi^{(n)} \) be a positive \( T \)-periodic solution of (1.3) associated with \( \lambda^{(n)} \). In view of Lemma 2.1, we know that the set \( \{ \phi^{(n)} \} \) is uniformly bounded in \( X^\omega \). Thus, the sequence \( \{ \phi^{(n)} \} \) has a subsequence converging to \( \phi \in X^\omega \). Then we can easily show, by the Lebesgue dominated convergence theorem, that \( \phi \) is a positive \( T \)-periodic solution of (1.3) at \( \lambda^*_\mu \).

Next, we let \( \rho : \Lambda \rightarrow (0, \infty) \) be defined by
\[
\rho(\mu) = t^*_\mu > 0.
\]
Then we may assert that \( \rho \) is continuous. In order to see this, we will assume for the sake of simplicity that \( \omega = 2 \) and that \( \zeta = (\zeta_1, \zeta_2) \in \Lambda \) such that \( \zeta_1, \zeta_2 > 0 \). Let \( \lambda = (\lambda_1, \lambda_2) \) be a neighbouring vector of \( \zeta \) in \( \Lambda \) such that \( \lambda_1, \lambda_2 > 0 \). Consider first the case \( \lambda_1 < \zeta_1 \) and \( \lambda_2 > \zeta_2 \). We will compare the vectors \( t^*_\zeta \zeta = (\zeta_1^*, \zeta_2^*) \) and \( t^*_\lambda = (\lambda_1^*, \lambda_2^*) \). Since Lemma 2.2 asserts that for each \( \xi \) inside
\[
\{ \xi \in \Xi : \xi \leq t^*_\zeta \zeta \},
\]
there is a positive $T$-periodic solution of (1.3) associated with $\xi$, we see that
\[
\frac{\lambda_1\zeta_2}{\zeta_1\lambda_1} \leq \lambda^*_1 \quad \text{and} \quad \frac{\lambda_2\zeta_2}{\zeta_1\lambda_2} \leq \frac{\zeta_1\lambda_2}{\lambda_1\zeta_2}.
\]
If $\lambda_1 > \zeta_1$ and $\lambda_2 < \zeta_2$, by similar arguments, we may also show that
\[
\lambda^*_1 \leq \frac{\zeta_2\lambda_1}{\lambda_2\zeta_1} \quad \text{and} \quad \lambda^*_2 \geq \frac{\zeta_1\lambda_2}{\lambda_1\zeta_2}.
\]
In either cases, if $(\lambda_1, \lambda_2) \to (\zeta_1, \zeta_2)$, then $(\lambda^*_1, \lambda^*_2) \to (\zeta^*_1, \zeta^*_2)$ as required.

Hence by defining
\[(3.1) \quad \Gamma = \{ \lambda : \lambda = \rho(\mu), \mu \in \Lambda \},\]
we see that $\Gamma$ is the desired continuous surface described above.

We intend to show that there are at least one more solution for each $\lambda$ in $\Theta_1$. To this end, we first recall the following lemmas for arguments involving the topological degree. One may refer to Guo and Lakshmikantham [1] for proofs and further discussion of the topological degree.

**Lemma 3.1.** Let $X$ be a Banach space with cone $K$. Let $\Omega$ be a bounded and open subset in $X$. Let $0 \in \Omega$ and $T : K \cap \Omega \to K$ be condensing (or completely continuous). Suppose that $Tx \notin \xi x$ for all $x \in K \cap \partial \Omega$ and all $\xi \geq 1$. Then $i(T, K \cap \Omega, K) = 1$.

**Lemma 3.2.** Let $X$ be a Banach space and $K$ a cone in $X$. For $r > 0$, define $K_r = \{ x \in K : \|x\| < r \}$. Assume that $T : K_r \to K$ is a compact map such that $Tx \notin x$ for $x \in \partial K_r$. If $\|x\| \leq \|Tx\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 0$.

Let $\phi^*$ be a positive $T$-periodic solution of (1.3) associated with $\lambda^* \in \Gamma$. Then for $\lambda < \lambda^*$ and $\lambda \in \Xi$, by the uniform continuity of $f_j$ on compact sets, there exists $\varepsilon_0 > 0$ such that
\[
\frac{f_j(s, 0, \ldots, 0)(\lambda^*_j - \lambda_j)}{\lambda_j} > f_j(s, \phi^*_1(s - \tau_{j1}(s)) + \varepsilon, \ldots, \phi^*_\omega(s - \tau_{j\omega}(s)) + \varepsilon) - f_j(s, \phi^*_1(s - \tau_{j1}(s)), \ldots, \phi^*_\omega(s - \tau_{j\omega}(s)))
\]
for $j \in \{1, \ldots, \omega\}$, $s \in G$ and $0 < \varepsilon \leq \varepsilon_0$. Thus, we have
\[
\lambda_j \int_{G(x)} K_j(x, s)f_j(s, \phi^*_1(s - \tau_{j1}(s)) + \varepsilon, \ldots, \phi^*_\omega(s - \tau_{j\omega}(s)) + \varepsilon) \, ds
\]
\[- \lambda_j^* \int_{G(x)} K_j(x, s)f_j(s, \phi^*_1(s - \tau_{j1}(s)), \ldots, \phi^*_\omega(s - \tau_{j\omega}(s)) + \varepsilon) \, ds
\]
\[= \lambda_j \int_{G(x)} K_j(x, s)f_j(s, \phi^*_1(s - \tau_{j1}(s)) + \varepsilon, \ldots, \phi^*_\omega(s - \tau_{j\omega}(s)) + \varepsilon) - f_j(s, \phi^*_1(s - \tau_{j1}(s)), \ldots, \phi^*_\omega(s - \tau_{j\omega}(s))) \, ds
\]
\[ - (\lambda^*_j - \lambda_j) \int_{G(x)} K_j(x, s)f_j(s, \phi^*_1(s - \tau_{j1}(s)), \ldots, \phi^*_\omega(s - \tau_{j\omega}(s))) \, ds
\]
Then there exists

\[ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial y} \]

\[ \text{condensing (since it is completely continuous).} \]

Let \( \phi \) for all \( \xi \)

\[ \text{properties of the fixed point index (see Lemma 3.1), we have} \]

\[ \text{Let} \]

\[ \text{Let} \]

\[ \text{Then} \]

\[ \text{and} \]

\[ \lambda_j \int_{G(x)} K_j(x, s) f_j(s, \phi^*_j(s - \tau_{j1}(s)), \ldots, \phi^*_j(s - \tau_{j\omega}(s))) ds \]

\[ \leq \lambda_j \int_{G(x)} K_j(x, s) f_j(s, \phi^*_j(s - \tau_{j1}(s)), \ldots, \phi^*_j(s - \tau_{j\omega}(s))) ds \]

\[ \text{and} \]

\[ \Psi = \{ (\phi_1, \ldots, \phi_\omega) \in X^\omega : -\varepsilon < \phi_j(x) < \tilde{\phi}_j^*(x), j = 1, \ldots, \omega, x \in G \}. \]

Then \( \Psi \) is bounded and open in \( X^\omega \), \( (0, \ldots, 0) \in \Psi \) and \( \mathbf{T}_\lambda : \Omega \cap \overline{\Psi} \to \Omega \) is condensing (since it is completely continuous). Let \( \phi = (\phi_1, \ldots, \phi_\omega) \in \Omega \cap \partial\Psi \).

Then there exists \( x_0 \) such that either \( \phi_k(x_0) = \tilde{\phi}_k^*(x_0) \) for some \( k \in \{1, 2, \ldots, \omega\} \). Then, by (H1),

\[ A_{\lambda_k}(\phi)(x_0) = \lambda_k \int_{G(x_0)} K_k(x_0, s) f_k(s, \phi_1(s - \tau_{k1}(s)), \ldots, \phi_k(s - \tau_{k\omega}(s))) ds \]

\[ \leq \lambda_k \int_{G(x_0)} K_k(x_0, s) f_k(s, \tilde{\phi}_1^*(s - \tau_{k1}(s)), \ldots, \tilde{\phi}_k^*(s - \tau_{k\omega}(s))) ds \]

\[ < \tilde{\phi}_k^*(x_0) = \phi_k(x_0) \leq \xi \phi_k(x_0) \]

for all \( \xi \geq 1 \). Thus \( \mathbf{T}_\lambda(\phi) \neq \xi \phi \) for \( \phi \in \Omega \cap \partial\Psi \) and \( \xi \geq 1 \). In view of the properties of the fixed point index (see Lemma 3.1), we have \( i(\mathbf{T}_\lambda, \Omega \cap \Psi, \Omega) = 1 \).

By (H2), we may choose \( R_{f_k} > 0 \) such that \( f_k(x, u_1, \ldots, u_\omega) \geq \eta_k(u_1 + \ldots + u_\omega) \) for all \( u_1 + \ldots + u_\omega \geq R_{f_k} \), where \( \eta_k \) satisfies

\[ \alpha^* \eta_k m_k \lambda_k \cdot \mu G(x) > 1. \]

Let \( R_k = \max\{b_D, R_{f_k}/\alpha^* \| (\tilde{\phi}_1^*, \ldots, \tilde{\phi}_\omega^*) \| \} \), where \( b_D \) is given in Lemma 2.1 with \( D \) a closed rectangle in \( \Xi \) containing \( \lambda \). Let \( \Omega_{R_k} = \{ \phi \in \Omega : \| \phi \| < R_k \} \).
Then in view of Lemma 2.1, \( \phi \neq T_\lambda(\phi) \) for \( \phi \in \partial \Omega_{R_k} \). Furthermore, if \( \phi \in \partial \Omega_{R_k} \), then \( \phi_1(x) + \ldots + \phi_\omega(x) \geq \alpha^* \|\phi\| \geq R_{f_k} \). Thus, we have

\[
A_{\lambda_k}(\phi)(x) = \lambda_k \int_{G(x)} K_k(x, s) f_k(s, \phi_1(s - \tau_1(s)), \ldots, \phi_\omega(s - \tau_\omega(s))) \, ds \\
\geq \alpha^* \eta_k m_k \lambda_k \cdot \mu G(x) \|\phi\| > \|\phi\|.
\]

Therefore \( \|T_\lambda(\phi)\| \geq \|A_{\lambda_k}(\phi)\| > \|\phi\| \) and Lemma 3.2 then implies

\[
i(T_\lambda, \Omega_{R_k}, \Omega) = 0.
\]

Consequently, by the additivity of the fixed point index,

\[
0 = i(T_\lambda, \Omega_{R_k}, \Omega) = i(T_\lambda, \Omega \cap \Psi, \Omega) + i(T_\lambda, \Omega_{R_k} \setminus \Omega \cap \Psi, \Omega).
\]

Since \( i(T_\lambda, \Omega \cap \Psi, \Omega) = 1 \), \( i(T_\lambda, \Omega_{R_k} \setminus \Omega \cap \Psi, \Omega) = -1 \) and \( T_\lambda \) has a fixed point in \( \Omega \cap \Psi \) and another in \( \Omega_{R_k} \setminus \Omega \cap \Psi \). Thus, we have the following result.

**Theorem 3.3.** There exists a continuous surface \( \Gamma \) of the form (3.1) separating \( \Xi \) into two disjoint subsets \( \Theta_1 \) (which is bounded) and \( \Theta_2 \) (which is unbounded) such that (1.3) has at least two positive \( T \)-periodic solutions for \( \lambda \in \Theta_1 \), at least one positive \( T \)-periodic solution for \( \lambda \in \Gamma \), and no positive \( T \)-periodic solution for \( \lambda \in \Theta_2 \).

As our final remark, note that the surface \( \Gamma \) is defined by the shooting method. Therefore, numerical methods can be applied to calculate this surface.

**References**


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