LAGRANGIAN SYSTEMS
WITH LIPSCHITZ OBSTACLE ON MANIFOLDS

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Abstract. Lagrangian systems constrained on the closure of an open subset with Lipschitz boundary in a manifold are considered. Under suitable assumptions, the existence of infinitely many periodic solutions is proved.

1. Introduction

The study of Lagrangian functionals of the form

\[ f(\gamma) = \int_0^1 L(s, \gamma(s), \gamma'(s)) \, ds \]

on a manifold \( M \), where \( L(s, (q, v)) : \mathbb{R} \times TM \to \mathbb{R} \), constitutes a well studied topic in Mechanics and Global analysis. In particular, about the existence and multiplicity of periodic solutions \( \gamma \) of the associated Euler equation, we refer the reader to [1], where the case in which \( M \) is a compact manifold without boundary is considered. Starting from [1], some extensions have been considered in the literature, when \( M \) is embedded in an Euclidean space. In [3] the case where \( M \) is a compact submanifold with boundary in \( \mathbb{R}^n \) has been considered.

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In such a case, the associated Euler equation has the form

\[
\frac{d}{ds}(D_vL(s, \gamma, \gamma')) - D_qL(s, \gamma, \gamma') \in N_q M,
\]

where \(N_q M\) is the outer normal cone to \(M\) at \(q\). The main feature is that the natural domain of the functional (1.1) is

\[
X = \{ \gamma \in W^{1,2}(0,1; \mathbb{R}^n) : \gamma(0) = \gamma(1), \gamma(s) \in M \text{ for all } s \}
\]

which is naturally a metric space, but not a smooth manifold (even with boundary). Moreover, solutions \(\gamma\) of (1.2) are not of class \(C^2\), but only \(W^{2,\infty}\) and satisfy (1.2) almost everywhere. In the same direction, the case in which \(M\) is a compact \(p\)-convex subset of \(\mathbb{R}^n\) has been considered in [4]. The class of \(p\)-convex subsets [8] includes in particular subsets with corners of convex type and concave parts of class \(C^2\). This direction of research was started by [18], where the case of an \(n\)-dimensional submanifold with boundary of class \(C^2\) in \(\mathbb{R}^n\) had been considered.

Another development has been started more recently in [11], [19], considering the case in which \(M\) is the closure of a bounded open subset of \(\mathbb{R}^n\) with Lipschitz boundary. Also in this case the set \(X\) is naturally only a metric space. Moreover, since in this case we cannot expect the solution \(\gamma\) of (1.2) to be of class \(C^1\), the Euler equation itself requires a reformulation.

The purpose of this paper is to consider the intrinsic case in which \(M\) is the closure of a bounded open subset of a differentiable manifold \(N\), instead of \(\mathbb{R}^n\), and also to relax the convexity condition on \(L\), which was in [19] of uniform quadratic type, to the mere convexity with coercivity of order \(p > 1\).

Our approach follows the lines of [19], but it is completely intrinsic. Of course the lack of strict convexity in \(L\) causes new technical difficulties.

The paper is organized as follows: in Section 2 we state our main results, while Section 3 is devoted to some recalls of nonsmooth analysis. Finally, in Section 4 we prove the main results.

### 2. Statement of the main results

Let \(N\) be a differentiable manifold without boundary of class \(C^2\) and \(M \subseteq N\). In the sequel, each \(\gamma \in W^{1,p}(a,b; N)\) will be identified with its continuous representative \(\tilde{\gamma} : [a,b] \to N\). We set

\[
W^{1,p}(a,b; M) := \{ \gamma \in W^{1,p}(a,b; N) : \gamma(s) \in M \text{ for each } s \in [a,b] \}.
\]

**Remark 2.1.** Let \(g\) and \(\tilde{g}\) be two Riemannian structures on \(N\) and let \(d\) and \(\tilde{d}\) be the induced distances on \(N\). Then there exists a continuous function \(c : N \to [0, \infty]\) such that, for all \(q \in N\) and all \(v \in T_q N\),

\[
g(q)(v,v) \leq c(q)\tilde{g}(q)(v,v), \quad \tilde{g}(q)(v,v) \leq c(q)g(q)(v,v).
\]
In particular, for every compact subset $K \subseteq N$ there exists $C > 0$ such that, for all $q_1, q_2 \in K$,
\[ d(q_1, q_2) \leq C \tilde{d}(q_1, q_2), \quad \tilde{d}(q_1, q_2) \leq Cd(q_1, q_2). \]

Let $1 < p < \infty$ and $L: \mathbb{R} \times TN \to \mathbb{R}$ be a function of class $C^1$ such that there exist two continuous functions $c, k: M \to [0, \infty]$ and $d \in \mathbb{R}$ such that for every $s \in \mathbb{R}$ and $q \in M$ one has
\begin{align*}
(2.1) & \quad k(q)|v|^p - d \leq L(s, q, v) \leq c(q)(1 + |v|^p) \quad \text{for all } v \in T_qN, \\
(2.2) & \quad |D_q L(s, q, v)| \leq c(q)(1 + |v|^p) \quad \text{for all } v \in T_qN, \\
(2.3) & \quad L(s, q, \cdot) \text{ is convex} \quad \text{on } T_qN,
\end{align*}

where $|v| = \sqrt{g(q)(v,v)}$.

In (2.1), (2.2) we mean that $N$ is provisionally endowed with a Riemannian structure. By Remark 2.1 the above conditions do not depend on the Riemannian structure chosen on $N$.

In charts, (2.1), (2.2) mean that for every $s \in \mathbb{R}$ and $q \in M$ it is
\begin{align*}
(2.1) & \quad k(q)|v|^p - d \leq L(s, q, v) \leq c(q)(1 + |v|^p) \quad \text{for all } v \in T_qN, \\
(2.2) & \quad |D_q L(s, q, v)| \leq c(q)(1 + |v|^p) \quad \text{for all } v \in T_qN, \\
(2.3) & \quad |D_v L(s, q, v)| \leq c(q)(1 + |v|^p) \quad \text{for all } v \in T_qN.
\end{align*}

Let us remark that (2.1), (2.3) imply that for every $s \in \mathbb{R}$, $q \in M$ and any $v, w \in T_qN$ we have
\[ |D_v L(s, q, v) w| \leq \hat{c}(q)(1 + |v|^{p-1})|w| \]

namely, in charts,
\[ |D_v L(s, q, v)| \leq \hat{c}(q)(1 + |v|^{p-1}), \]

where $\hat{c}: M \to [0, \infty]$ is continuous.

Define a continuous functional $f_{a,b}: W^{1,p}(a,b; M) \to \mathbb{R}$ by
\[ f_{a,b}(\gamma) = \int_a^b L(s, \gamma(s), \gamma'(s)) \, ds. \]

Given a Riemannian structure on $N$, for every $\gamma, \eta \in W^{1,p}(a,b; M)$ we set
\begin{align*}
& d_1(\gamma, \eta) = \int_a^b d(\gamma(s), \eta(s)) \, ds, \\
& d_\infty(\gamma, \eta) = \max\{d(\gamma(s), \eta(s)) : a \leq s \leq b\},
\end{align*}

where $d$ is the distance on $N$ associated with the Riemannian structure.
DEFINITION 2.2. We say that $\gamma \in W^{1,p}(a, b; M)$ is $L$-stationary, if it is not possible to find $r, c, \sigma > 0$ and a map

$$\mathcal{H}: \{ \eta \in W^{1,p}(a, b; M) : d_\infty(\eta, \gamma) < r, \ f_{a,b}(\eta) < f_{a,b}(\gamma) + r \} \times [0, r] \rightarrow W^{1,p}(a, b; M)$$

such that:

(a) $\mathcal{H}$ is continuous from the product of the topology of the uniform convergence and that of $\mathbb{R}$ to that of the uniform convergence;

(b) for every $\eta \in W^{1,p}(a, b; M)$ with $d_\infty(\eta, \gamma) < r, f_{a,b}(\eta) < f_{a,b}(\gamma) + r$ and $t \in [0, r]$ we have

$$d_1(\mathcal{H}(\eta, t)(a) = \eta(a), \quad \mathcal{H}(\eta, t)(b) = \eta(b), \quad f_{a,b}(\mathcal{H}(\eta, t)) \leq f_{a,b}(\eta) - \sigma t.$$ 

Again we mean that the assertion holds after introducing a Riemannian structure on $N$. By Remark 2.1 this definition does not depend on the choice of the Riemannian structure itself.

PROPOSITION 2.3. Let $\gamma \in W^{1,p}(a, b; M)$ be $L$-stationary. Then for every $[\alpha, \beta] \subseteq [a, b]$ the restriction $\gamma|_{[\alpha, \beta]}$ is $L$-stationary.

PROOF. Set $\tilde{\gamma} = \gamma|_{[\alpha, \beta]}$. By contradiction, assume that there exist $r, c, \sigma > 0$ and

$$\mathcal{H}: \{ \eta \in W^{1,p}(\alpha, \beta; M) : d_\infty(\eta, \tilde{\gamma}) < r, \ f_{a,b}(\eta) f_{a,b}(\tilde{\gamma}) + r \} \times [0, r] \rightarrow W^{1,p}(\alpha, \beta; M)$$

according to Definition 2.2.

We claim that there exists $r' \in [0, r]$ such that if $\eta \in W^{1,p}(a, b; M)$ with $d_\infty(\eta, \gamma) < r'$ and $f_{a,b}(\eta) < f_{a,b}(\gamma) + r'$, then $f_{a,b}(\eta) < f_{a,b}(\gamma) + r$, where $\eta = \eta|_{[a, b]}$.

Again by contradiction, let $(\eta_h) \subseteq W^{1,p}(a, b; M)$ with $\eta_h$ convergent to $\gamma$ with respect to the uniform convergence and $\lim \sup_h f_{a,b}(\eta_h) \leq f_{a,b}(\gamma)$ such that $f_{a,b}(\eta_h) \geq f_{a,b}(\gamma) + r$. By (2.1) and (2.3) we have

$$\lim \sup_h f_{a,b}(\eta_h) \leq \lim \sup_h f_{a,b}(\eta_h) - \lim \inf_h \int_{[a,b] \setminus [\alpha, \beta]} L(s, \eta_h, \eta_h') \ ds \leq f_{a,b}(\gamma) - \int_{[a,b] \setminus [\alpha, \beta]} L(s, \gamma, \gamma') \ ds = f_{a,b}(\tilde{\gamma}),$$

whence a contradiction. Then, for any $\eta \in W^{1,p}(a, b; M)$ define

$$\mathcal{K}: \{ \eta \in W^{1,p}(a, b; M) : d_\infty(\eta, \gamma) < r', \ f_{a,b}(\eta) < f_{a,b}(\gamma) + r' \} \times [0, r'] \rightarrow W^{1,p}(a, b; M)$$
by
\[ K(\eta, t)(s) = \begin{cases} \mathcal{H}(\hat{\eta}, t)(s) & \text{if } s \in [\alpha, \beta], \\ \eta(s) & \text{if } s \notin [\alpha, \beta]. \end{cases} \]

It is readily seen that \( K \) has all the properties required in Definition 2.2. It follows that \( \gamma \) is not \( L \)-stationary, which is absurd. \( \square \)

**Definition 2.4.** Let \( I \) be an interval in \( \mathbb{R} \) with \( \text{int}(I) \neq \emptyset \). A continuous map \( \gamma: I \to M \) is said to be a **generalized solution** of the Lagrangian system associated to \( L \) on \( M \), if every \( s \in \text{int}(I) \) admits a neighbourhood \( [a, b] \) in \( I \) such that \( \gamma \mid [a, b] \) belongs to \( W^{1,p}(a, b; M) \) and is \( L \)-stationary.

**Definition 2.5.** Given \( T > 0 \), a **\( T \)-periodic generalized solution** of the Lagrangian system associated to \( L \) on \( M \) is a generalized solution \( \gamma: \mathbb{R} \to M \) which is periodic of period \( T \).

We now state our main existence result.

**Theorem 2.6.** Assume that \( M \) is the closure of an open subset of \( N \) with locally Lipschitz boundary. Suppose also that \( M \) is compact, 1-connected and non-contractible in itself and that
\[ L(s + 1, q, v) = L(s, q, v) \quad \text{for all } s \in \mathbb{R} \text{ and all } (q, v) \in TN. \]

Then there exists a sequence \( (\gamma_h) \) of \( 1 \)-periodic generalized solutions of the Lagrangian system associated to \( L \) on \( M \) with
\[ \lim_{h \to \infty} \int_0^1 L(s, \gamma_h(s), \gamma_h'(s)) \, ds = +\infty. \]

The notion of generalized solution we have introduced follows the approach of [11, Definition 3.3] and [19, Definition 2.6] and has the advantage to be intrinsically connected to \( M \), although quite indirect. However, at least in the particular case \( p = 2 \), it is possible to deduce further informations on the generalized solutions.

For every \( q \in M \), denote by \( N_qM \) the normal cone to \( M \) at \( q \) (see e.g. Definition 3.2 below).

**Theorem 2.7.** Let \( p = 2 \) and assume that there exists a continuous function \( \omega: N \to [0, \infty) \) such that for every \( s \in \mathbb{R}, q \in M \) it is
\[ D_v L(s, q, v)(v - w) - D_w L(s, q, w)(v - w) \geq \omega(q) |v - w|^2 \quad \text{for all } v, w \in T_qN. \]

Let \( \gamma \in W^{1,2}(a, b; M) \) be \( L \)-stationary. Then \( \gamma \in W^{1,\infty}(a, b; M), D_{(q,v)} L(s, \gamma, \gamma') \in L^\infty(a, b; T^*(TN)) \) and there exist a finite Borel measure \( \mu \) on \( [a, b] \) and a bounded Borel function \( \nu: [a, b] \to T^*N \) such that \( \nu(s) \in N_{\gamma(s)}M \) for \( \mu \)-a.e. \( s \in [a, b] \) and
\[ \int_a^b D_{(q,v)} L(s, \gamma, \gamma')(\delta, \delta') \, ds = -\int_a^b \nu(\delta) \, d\mu. \]
for any $\delta \in W^{1,1}_0(a,b; TN)$ with $\delta(s) \in T_{\gamma(s)}N$ for every $s \in [a,b]$.

Also in this assertion we mean that $N$ is provisionally endowed with a Riemannian structure. Since $\gamma$ is continuous, by Remark 2.1 the assertion is independent of the choice of the structure.

**Proof of Theorem 2.7.** By Proposition 2.3, we may assume that $\gamma([a,b])$ is contained in a coordinated neighbourhood. Then the assertion follows from [19, Theorem 2.10]. $\square$

### 3. Some relevant results of nonsmooth analysis

In the first part of this section let $N$ be a differentiable manifold of class $C^2$ and $M$ be the closure of an open set in $N$ with locally Lipschitz boundary.

If $X$ is a Banach space, $E \subseteq X$ and $x \in E$, we denote by $T_x E$ the tangent cone to $E$ at $x$, according to [6]. We also denote by $B_r(x)$ the open ball of center $x$ and radius $r$.

**Definition 3.1.** Let $x \in E$ and $v \in X$. We say that $v$ is hypertangent to $E$ at $x$ if there exists $\delta > 0$ such that $B_\delta(x) + [0, \delta]B_\delta(v) \subseteq E$. Let us denote by $\text{Hyp}_x E$ the set of the $v$'s hypertangent to $E$ at $x$.

**Definition 3.2.** Let $q \in M$ and $v \in T_q N$. We say that $v$ is tangent to $M$ at $q$ if there exists a chart $(U, \varphi)$ at $q$ such that $d\varphi(q)v \in T_{\varphi(q)}(U \cap M)$. The set of the $v$'s tangent to $M$ at $q$ is denoted by $T_q M$ and is called the tangent cone to $M$ at $q$.

We say that $v$ is hypertangent to $M$ at $q$ if there exists a chart $(U, \varphi)$ at $q$ such that $d\varphi(q)v$ is hypertangent to $\varphi(U \cap M)$ at $\varphi(q)$. The set of the $v$'s hypertangent to $M$ at $q$ is denoted by $\text{Hyp}_q M$ and is called the hypertangent cone to $M$ at $q$. Finally, we set $N_q M = \{ \varphi \in T_q^* N : \varphi(v) \leq 0 \text{ for all } v \in T_q M \}$. $N_q M$ is called the normal cone to $M$ at $q$.

**Remark 3.3.** For every $q \in M$ it is $\text{Hyp}_q M \neq \emptyset$ (see [6]) and $\text{Hyp}_q M \subseteq T_q M$.

**Theorem 3.4.** There exists a section $\nu: N \rightarrow TN$ of class $C^1$ such that $\nu(q) \in \text{Hyp}_q M$ for all $q \in M$.

**Proof.** For all $q \in N$, let

$$\Psi(q) = \begin{cases} \text{Hyp}_q M & \text{if } q \in M, \\ T_q N & \text{if } q \in N \setminus M. \end{cases}$$

Then for every $q \in N$, $\Psi(q)$ is convex in $T_q N$ and for every $q \in N$ there exists a chart $(U, \varphi)$ at $q$ such that

$$\bigcap_{\xi \in U} (d\varphi(\xi)(\Psi(\xi))) \neq \emptyset.$$
It follows that there exists \( \nu: N \to TN \) of class \( C^1 \) with \( \nu(q) \in \Psi(q) \) for every \( q \in N \), hence the assertion. \( \square \)

**Lemma 3.5.** Let \( \tilde{N} \) be a submanifold of class \( C^2 \) of \( \mathbb{R}^n \), \( \tilde{M} \) be the closure of an open subset of \( \tilde{N} \) with locally Lipschitz boundary, \( A \) be an open subset of \( \mathbb{R}^n \) with \( \tilde{N} \subseteq A \) and \( \pi: A \to \tilde{N} \) be a retraction of class \( C^2 \) such that \( \pi \) is Lipschitz continuous of constant 2. Then there exists a map \( \nu: \tilde{N} \to \mathbb{R}^n \) of class \( C^1 \) such that the following facts hold:

(a) for any \( q \in \tilde{N} \) we have \( \nu(q) \in T_q \tilde{N} \);
(b) for any \( q \in \tilde{M} \) there exists \( \delta > 0 \) such that

\[
\begin{align*}
\text{if} & \quad \begin{cases}
\xi \in B_\delta(q), \\
\pi(\xi) \in \tilde{M}, \\
0 < t \leq \delta,
\end{cases}
\text{then} & \quad \pi(\xi + tv) \in \text{int}(\tilde{M}); \\
v \in B_\delta(\nu(q)),
\end{align*}
\]

(c) for every compact subset \( K \subseteq \tilde{M} \) there exist \( \tilde{\tau}, \tilde{c} > 0 \) satisfying

\[
\pi((1-t)q + t\pi(\xi + \rho \nu(\xi))) \in \tilde{M}
\]

whenever \( q \in \tilde{M} \), \( \xi \in K \), \( \tilde{\tau}|q - \xi| \leq \rho \leq \tilde{\tau} \) and \( t \in [0,1] \).

**Proof.** By Theorem 3.4 there exists a map \( \nu: \tilde{N} \to \mathbb{R}^n \) of class \( C^1 \) such that for any \( q \in \tilde{N} \) it is \( \nu(q) \in T_q \tilde{N} \).

To prove (b), assume by contradiction that \( q \in \tilde{M} \), \( \xi_h \to q \), \( t_h \to 0^+ \) and \( v_h \to \nu(q) \) with \( \pi(\xi_h) \in \tilde{M} \) and \( \pi(\xi_h + t_h v_h) \notin \text{int}(\tilde{M}) \).

Let \( (U, \varphi) \) be the chart at \( q \) such that \( \varphi: U \to T_q \tilde{N} \), \( \varphi(q) = 0 \) and \( \pi(q + \varphi(\xi)) = \xi \) for any \( \xi \in U \); in particular, \( \nu(q) \in \text{Hyp}_0(\varphi(U \cap \tilde{M})) \).

Then we have

\[
\varphi(\pi(\xi_h + t_h v_h)) \notin \text{int}(\varphi(U \cap \tilde{M})).
\]

Since

\[
\varphi(\pi(\xi_h + t_h v_h)) = \varphi(\pi(\xi_h)) + t_h d[\varphi \circ \pi](\xi_h)v_h + \varepsilon_h
\]

with \( \varepsilon_h \to 0 \) in \( T_q \tilde{N} \), it follows that \( d[\varphi \circ \pi](\xi_h)v_h + \varepsilon_h \in T_q \tilde{N} \) and

\[
\varphi(\pi(\xi_h + t_h v_h)) \notin \text{int}(\varphi(U \cap \tilde{M}))
\]

for large \( h \), which is absurd.

Now let us prove (c). By contradiction, let \( (q_h) \) in \( \tilde{M} \), \( (\xi_h) \) in \( K \), \( (t_h) \) in \( [0,1] \), \( \rho_h \to 0 \) with \( h|q_h - \xi_h| \leq \rho_h \leq 1/h \) and

\[
\pi((1-t_h)q_h + t_h \pi(\xi_h + \rho_h \nu(\xi_h))) \notin \tilde{M}.
\]

Up to a subsequence \( \xi_h \to \xi \) in \( K \), \( q_h \to \xi \) in \( \tilde{M} \) and \( t_h \to t \) in \( [0,1] \). It is

\[
\pi((1-t_h)q_h + t_h \pi(\xi_h + \rho_h \nu(\xi_h))) = \pi\left(q_h + t_h \rho_h \left(\frac{\pi(\xi_h + \rho_h \nu(\xi_h)) - q_h}{\rho_h}\right)\right).
\]
On the other hand,

$$\frac{\pi(\xi_h + \rho_h \nu(\xi_h)) - q_h - \nu(\xi)}{\rho_h} = \frac{\pi(\xi_h + \rho_h \nu(\xi)) - \xi_h - \rho_h \nu(\xi)}{\rho_h} + \frac{\xi_h - q_h}{\rho_h} + \frac{\pi(\xi_h + \rho_h \nu(\xi_h)) - \pi(\xi_h + \rho_h \nu(\xi))}{\rho_h}.$$

By [11, Theorem 4.4], it is

$$\lim_{h} \frac{\pi(\xi_h + \rho_h \nu(\xi_h)) - \xi_h - \rho_h \nu(\xi)}{\rho_h} = 0.$$

Moreover, by the lipschitzianity of $\pi$ it is also

$$\left| \frac{\pi(\xi_h + \rho_h \nu(\xi_h)) - \pi(\xi_h + \rho_h \nu(\xi))}{\rho_h} \right| \leq 2|\nu(\xi_h) - \nu(\xi)|.$$ 

It follows that

$$\lim_{h} \frac{\pi(\xi_h + \rho_h \nu(\xi_h)) - q_h}{\rho_h} = \nu(\xi),$$

hence by (a) it is

$$\pi \left( q_h + t_h \rho_h \frac{\pi(\xi_h + \rho_h \nu(\xi_h)) - q_h}{\rho_h} \right) \in \tilde{M}$$

for large $h$, which is a contradiction. \(\square\)

**Definition 3.6.** A subset $E$ of $N$ is said to be a LNR in $N$ if there exists an open neighbourhood $U$ of $E$ in $N$ and a locally Lipschitzian retraction $r: U \to E$.

**Theorem 3.7.** The set $M$ is a LNR in $N$. 

**Proof.** By [14, §2, Theorems 2.10 and 2.14], we may assume that $N$ is a smooth submanifold of $\mathbb{R}^n$. By [14, §4, Theorem 5.1], there exist an open subset $A$ of $\mathbb{R}^n$ with $N \subseteq A$ and a retraction $\pi: A \to N$ of class $C^\infty$ such that $\pi$ is Lipschitz continuous of constant 2. Let $\nu: N \to \mathbb{R}^n$ be as in Lemma 3.5. By (b) of Lemma 3.5, for every $q \in M$ there exists $\delta_q > 0$ such that

<table>
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<tr>
<th>$\xi \in B_{\delta_q}(q)$,</th>
<th>$\pi(\xi) \in M$,</th>
<th>$0 &lt; t \leq \delta_q$,</th>
<th>then</th>
<th>$\pi(\xi + tv) \in \text{int}(M)$,</th>
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<tr>
<td>$v \in B_{\delta_q}(\nu(q))$,</td>
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Let $\delta_q' \in [0, \delta_q]$ be such that

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<th>then</th>
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<tr>
<td>$\xi + tv(\xi) \in B_{\delta_q}(q)$,</td>
<td>$v(\xi) \in B_{\delta_q/2}(\nu(q))$,</td>
<td>$</td>
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</table>
For every $q \in M$, define
\[ U_q = \{ \xi \in B_{\delta_q}(q) : \pi(\xi + T(\xi)\nu(\xi)) \in \text{int}(M) \}, \quad U = \bigcup_{q \in M} U_q. \]

For every $\xi \in U$, let $T(\xi) = \min\{ t \geq 0 : \pi(\xi + t\nu(\xi)) \in M \}$. It is easy to see that, if $q \in M$ and $\xi \in U_q$, then
\[ T(\xi) < \delta_q, \quad \xi + T(\xi)\nu(\xi) \in B_{\delta_q}(q), \quad \pi(\xi + T(\xi)\nu(\xi)) \in M \]
and
\[ (3.1) \quad \text{if} \quad \begin{cases} 0 \leq t \leq \delta_q, \\ v \in B_{\delta_q}(\nu(q)), \end{cases} \quad \pi(\xi + T(\xi)\nu(\xi) + tv) \in M. \]

Let now $q \in M$ and $\xi_1, \xi_2 \in U_q$ with $\xi_1 \neq \xi_2$. We set
\[ s = \frac{2}{\delta_q}(|\xi_1 - \xi_2| + T(\xi_1)|\nu(\xi_1) - \nu(\xi_2)|) \]
and
\[ v = \nu(\xi_2) - \frac{1}{s}(\xi_1 - \xi_2 + T(\xi_1)(\nu(\xi_1) - \nu(\xi_2))). \]
We have $s \in [0, \delta_q]$ and $v \in B_{\delta_q}(\nu(q))$. If we consider $t = T(\xi_1) + s$, an easy calculation shows that
\[ \xi_2 + tv(\xi_2) = \xi_1 + T(\xi_1)\nu(\xi_1) + sv. \]

By (3.1) it follows that $\pi(\xi_2 + tv(\xi_2)) \in M$, hence $T(\xi_2) \leq t$. Therefore we get
\[ T(\xi_2) \leq T(\xi_1) + s \leq T(\xi_1) + \frac{2}{\delta_q}(|\xi_1 - \xi_2| + \delta_q|\nu(\xi_1) - \nu(\xi_2)|); \]

exchanging the role of $\xi_1$ and $\xi_2$ we have
\[ |T(\xi_1) - T(\xi_2)| \leq \frac{2}{\delta_q}(|\xi_1 - \xi_2| + \delta_q|\nu(\xi_1) - \nu(\xi_2)|), \]
hence $T$ is locally Lipschitzian. It follows that the map $\pi: U \to M$ defined by $\pi(\xi) = \pi(\xi + T(\xi)\nu(\xi))$ is a locally Lipschitzian retraction. Therefore $M$ is an LNR in $\mathbb{R}^n$, in particular in $N$. $\square$

In the second part of this section, we recall some abstract notions and results of nonsmooth analysis.

Let $Y$ be a metric space endowed with the metric $d$ and let $f: Y \to \mathbb{R}$ be a function. We set
\[ \text{epi}(f) = \{ (u, \lambda) \in Y \times \mathbb{R} : f(u) \leq \lambda \}. \]

In the following, $Y \times \mathbb{R}$ will be endowed with the metric
\[ d((u, \lambda), (v, \mu)) = (d(u, v)^2 + (\lambda - \mu)^2)^{1/2} \]
and $\text{epi}(f)$ with the induced metric.
Definition 3.8. For every $u \in Y$ with $f(u) \in \mathbb{R}$, we denote by $|df|(u)$ the supremum of the $\sigma$’s in $[0, \infty]$ such that there exist $r > 0$ and a continuous map

$$\mathcal{H}: (B_r(u, f(u)) \cap \text{epi}(f)) \times [0, r] \to Y$$

satisfying

$$d(\mathcal{H}((v, \mu), t), v) \leq t, \quad f(\mathcal{H}((v, \mu), t)) \leq \mu - \sigma t,$$

whenever $(v, \mu) \in B_r(u, f(u)) \cap \text{epi}(f)$ and $t \in [0, r]$.

The extended real number $|df|(u)$ is called the weak slope of $f$ at $u$.

The above notion has been introduced in [9], following an equivalent approach. When $f$ is continuous, it has been independently introduced also in [17], while a variant appears in [15], [16]. The version we have recalled here is taken from [2].

Proposition 3.9. Let $u \in Y$ with $f(u) \in \mathbb{R}$. Assume there exist $r, c, \sigma > 0$ and a continuous map

$$\mathcal{H}: \{v \in B_r(u) : f(v) < f(u) + r\} \times [0, r] \to Y$$

such that for any $v \in B_r(u)$ with $f(v) < f(u) + r$ and any $t \in [0, r]$ it is

$$d(\mathcal{H}(v, t), v) \leq ct, \quad f(\mathcal{H}(v, t)) \leq f(v) - \sigma t.$$

Then we have $|df|(u) \geq \sigma/c$.

Proof. See [11, Proposition 2.3]. □

Now, according to [8], we define a function $\mathcal{G}_f: \text{epi}(f) \to \mathbb{R}$ by $\mathcal{G}_f(u, \lambda) = \lambda$. Of course, $\mathcal{G}_f$ is Lipschitzian of constant 1.

Proposition 3.10. For every $u \in Y$ with $f(u) \in \mathbb{R}$, we have $f(u) = \mathcal{G}_f(u, f(u))$ and

$$|df|(u) = \begin{cases} 
\frac{|d\mathcal{G}_f|(u, f(u))}{\sqrt{1 - |d\mathcal{G}_f|(u, f(u))^2}} & \text{if } |d\mathcal{G}_f|(u, f(u)) < 1, \\
\infty & \text{if } |d\mathcal{G}_f|(u, f(u)) = 1.
\end{cases}$$

Proof. See [2, Proposition 2.3]. □

The previous proposition allows us to reduce, at some extent, the study of the general function $f$ to that of the continuous function $\mathcal{G}_f$. For this purpose, the next result will be useful.

Proposition 3.11. Let $(u, \lambda) \in \text{epi}(f)$ with $f(u) < \lambda$. Assume that for every $\varepsilon > 0$ there exist $r > 0$ and a continuous map

$$\mathcal{H}: \{v \in B_r(u) : f(v) < \lambda + r\} \times [0, r] \to Y$$
such that for any \( v \in B_r(u) \) with \( f(v) < \lambda + r \) and any \( t \in [0, r] \) it is
\[
d(\mathcal{H}(v, t), v) \leq \varepsilon t,
\]
\[
f(\mathcal{H}(v, t)) \leq (1 - t)f(v) + t(f(u) + \varepsilon).
\]
Then we have \(|dG_f|(u, \lambda) = 1|.

**Proof.** See [10, Corollary 2.11]. \( \Box \)

Definition 3.8 may be simplified, when \( f \) is continuous.

**Proposition 3.12.** Let \( f: Y \to \mathbb{R} \) be continuous. Then \(|df|(u)\) is the supremum of the \( \sigma \)'s in \([0, +\infty[ \) such that there exist \( r > 0 \) and a continuous map
\[
\mathcal{H}: B_r(u) \times [0, r] \to Y
\]
satisfying
\[
d(\mathcal{H}(v, t), v) \leq t, \quad f(\mathcal{H}(v, t)) \leq f(v) - \sigma t,
\]
whenever \( v \in B_r(u) \) and \( t \in [0, r] \).

**Proof.** See [2, Proposition 2.2]. \( \Box \)

By means of the weak slope, we can now introduce the two main notions of critical point theory.

**Definition 3.13.** We say that \( u \in Y \) is a (lower) critical point of \( f \), if \( f(u) \in \mathbb{R} \) and \(|df|(u) = 0\). We say that \( c \in \mathbb{R} \) is a (lower) critical value of \( f \), if there exists a (lower) critical point \( u \in Y \) of \( f \) with \( f(u) = c \).

**Remark 3.14.** Let \( \tilde{d} \) be another metric on \( Y \) and let \( u \in Y \). Assume that there exist a neighbourhood \( U \) of \( u \) and \( c > 0 \) such that, for all \( v, w \in U \),
\[
d(v, w) \leq c\tilde{d}(v, w), \quad \tilde{d}(v, w) \leq cd(v, w).
\]
Then one has \(|df|(u) = 0\) if and only if \(|\tilde{d}f|(u) = 0\), where \(|\tilde{d}f|(u)\) is the weak slope with respect to \( \tilde{d} \).

**Definition 3.15.** Let \( c \in \mathbb{R} \). A sequence \((u_k)\) in \( Y \) is said to be a Palais–Smale sequence at level \( c \) (\((PS)_c\)-sequence, for short) for \( f \), if \( f(u_k) \to c \) and \(|df|(u_k) \to 0\).

We say that \( f \) satisfies the Palais–Smale condition at level \( c \) (\((PS)_c\), for short), if every \((PS)_c\)-sequence \((u_k)\) for \( f \) admits a convergent subsequence \((u_{k_j})\) in \( Y \).

**Definition 3.16.** A topological space \( Z \) is said to be weakly locally contractible, if every \( u \in Z \) admits a neighbourhood \( U \) which is contractible in \( Z \).
Theorem 3.17. Let \( Y \) be weakly locally contractible with \( \text{cat} Y = \infty \), let \( f: Y \to \mathbb{R} \) be continuous and bounded from below and assume that \( \{ u \in Y : f(u) \leq c \} \) is complete and (PS)\(_c\) hold for every \( c \in \mathbb{R} \). Then there exists a sequence \((u_n)\) of critical points of \( f \) with \( f(u_n) \to \infty \).

Proof. See [7, Theorem 3.6] and [5, Theorem 1.4.13]. □

Corollary 3.18. Let \( Z \) be a metrizable tolopogical space and \( f: Z \to \mathbb{R} \) a continuous function. Assume that

(a) \( Z \) is weakly locally contractible and \( \text{cat} Z = \infty \);
(b) for every \( c \in \mathbb{R} \), the set \( \{ u \in Z : f(u) \leq c \} \) is compact.

Then, for every compatible metric on \( Z \), there exists a sequence \((u_n)\) of critical points of \( f \) with \( f(u_n) \to \infty \).

4. Proof of the main results

In the first part of this section, let \( N \) be a differentable manifold of class \( C^2 \) and \( M \) be a LNR in \( N \). Let us consider

\[
\Lambda(M) = \{ \gamma \in C([0, 1]; M) : \gamma(0) = \gamma(1) \}
\]

endowed with the uniform topology (\( \Lambda(M) \) is called the free loop space of \( M \)) and

\[
X = \{ \gamma \in W^{1,p}(0, 1; M) : \gamma(0) = \gamma(1) \}.
\]

Let \( L: \mathbb{R} \times TN \to \mathbb{R} \) be a function of class \( C^1 \) satisfying (2.1)–(2.4) and define a lower semicontinuous functional \( f: \Lambda(M) \to \mathbb{R} \cup \{ \infty \} \) by

\[
f(\gamma) = \begin{cases} 
\int_0^1 L(s, \gamma(s), \gamma'(s)) \, ds & \text{if } \gamma \in X, \\
\infty & \text{if } \gamma \in \Lambda(M) \setminus X.
\end{cases}
\]

In the following, we will consider the metrizible topological space \( \text{epi}(f) \), endowed with the topology induced by \( \Lambda(M) \times \mathbb{R} \), and the continuous function \( \mathcal{G}_f: \text{epi}(f) \to \mathbb{R} \).

Given a Riemannian structure on \( N \), for every \( \gamma, \eta \in W^{1,p}(0, 1; M) \), we set as before

\[
\begin{align*}
    d_1(\gamma, \eta) &= \int_0^1 d(\gamma(s), \eta(s)) \, ds, \\
    d_\infty(\gamma, \eta) &= \max \{ d(\gamma(s), \eta(s)) : 0 \leq s \leq 1 \},
\end{align*}
\]

where \( d \) is the distance on \( N \) associated with the Riemannian structure.
LEMMA 4.1. Consider a Riemannian structure on $N$. Let $(\gamma_h)$ be a sequence in $W^{1,p}(0,1; M)$ convergent to $\gamma \in W^{1,p}(0,1; M)$ with respect to the topology induced by $d_1$ and such that $(f(\gamma_h))$ is bounded. Then $(\gamma_h)$ is convergent to $\gamma$ with respect to the uniform convergence.

PROOF. Let $U$ be an open subset of $M$ with $\overline{U}$ compact such that $\gamma([0,1]) \subseteq U$. First of all we claim that $\gamma_h([0,1]) \subseteq U$ for $h$ large enough. By contradiction, let $h_k \to \infty$ and $(s_k) \subseteq [0,1]$ such that $\gamma_{h_k}(s_k) \notin U$. Up to a subsequence we have that $s_k \to s \in [0,1]$ and $\gamma_{h_k} \to \gamma$ a.e. in $[0,1]$. Let $a \in [0,1]$ be such that $\gamma_{h_k}(a) \to \gamma(a)$. Assume that $a < s$. It follows that, for $k$ large enough, there exists $b_k \in [a,s_k]$ such that $\gamma_{h_k}([a,b_k]) \subseteq \overline{U}$ and $\gamma_{h_k}(b_k) \notin U$. Since $\overline{U}$ is compact, there exists $C > 0$ such that, by (2.1),

$$\int_a^{b_k} L(s, \gamma_{h_k}, \gamma'_{h_k}) \, ds \geq \int_a^{b_k} \left(k(\gamma_{h_k})|\gamma'_{h_k}|^p - d\right) \, ds \geq \int_a^{b_k} (C|\gamma'_{h_k}|^p - d) \, ds.$$ 

Moreover, again by (2.1), we have

$$\int_0^a L(s, \gamma_{h_k}, \gamma'_{h_k}) \, ds + \int_{b_k}^1 L(s, \gamma_{h_k}, \gamma'_{h_k}) \, ds \geq -(1 - b_k + a).$$

It follows that

$$f(\gamma_{h_k}) = \int_0^1 L(s, \gamma_{h_k}, \gamma'_{h_k}) \, ds \geq C \int_a^{b_k} |\gamma'_{h_k}|^p \, ds - d.$$ 

Hence for every $\sigma, \tau \in [a,b_k]$ with $\tau \leq \sigma$ we have

$$d(\gamma_{h_k}(\sigma), \gamma_{h_k}(\tau)) \leq \int_{\tau}^{\sigma} |\gamma'_{h_k}(t)| \, dt \leq \left(\int_{\tau}^{\sigma} |\gamma'_{h_k}(t)|^p \, dt\right)^{1/p} |\sigma - \tau|^{1/p'} \leq \left(\int_a^{b_k} |\gamma'_{h_k}(t)|^p \, dt\right)^{1/p} |\sigma - \tau|^{1/p'}. $$

It follows that $(\gamma_{h_k})$ is equi-uniformly continuous on $[a,b_k]$. Up to a further subsequence we have that $\gamma_{h_k}(b_k) \to x \in \partial U$. Since $\inf \{d(\gamma(a), y) : y \in \partial U\} > 0$, if $a$ is sufficiently close to $s$ a contradiction follows.

Arguing as above, for any $s, t \in [0,1]$ we have that

$$d(\gamma_h(s), \gamma_h(t)) \leq \left(\frac{f(\gamma_h) + d}{C}\right)^{1/p} |s - t|^{1/p'}.$$ 

Since $(f(\gamma_h))$ is bounded, we deduce that $(\gamma_h)$ is equi-uniformly continuous on $[0,1]$. Therefore it is easy to see that $(\gamma_h)$ is convergent to $\gamma$ with respect to the uniform convergence. \[\square\]
Theorem 4.2. Consider any Riemannian structure on $N$ and define on $\text{epi}(f)$ the metric
\begin{align}
(4.1) \quad d((\gamma, \lambda), (\eta, \mu)) = \sqrt{d_1(\gamma, \eta)^2 + |\lambda - \mu|^2}.
\end{align}
Then the following facts hold:
(a) the metric $d$ is compatible with the topology of $\text{epi}(f)$;
(b) the set of critical points of $G_f: \text{epi}(f) \to \mathbb{R}$ does not depend on the Riemannian structure;
(c) if $(\gamma, \lambda) \in \text{epi}(f)$ is a critical point of $G_f$ with $f(\gamma) = \lambda$, then $\gamma$ is the restriction to $[0,1]$ of a 1-periodic generalized solution of the Lagrangian system associated to $L$ on $M$.

Proof. (a) is an easy consequence of Lemma 4.1; (b) follows from Remarks 2.1 and 3.14. Let us consider property (c). First, let us prove that $\gamma$ is $L$-stationary on $[0,1]$. By contradiction, assume that there exist $r, c, \sigma > 0$ and $\mathcal{H}: \{\eta \in W^{1,p}(0,1; M) : d_\infty(\eta, \gamma) < r, f(\eta) < f(\gamma) + r\} \times [0,r] \to W^{1,p}(0,1; M)$ continuous from the product of the uniform convergence and that of $\mathbb{R}$ to that of the uniform convergence such that
\begin{align}
\mathcal{H}(\eta, t)(0) = \eta(0), \quad \mathcal{H}(\eta, t)(1) = \eta(1),
\end{align}
\begin{align}
d_1(\mathcal{H}(\eta, t), \eta) \leq ct, \quad f(\mathcal{H}(\eta, t)) \leq f(\eta) - \sigma t.
\end{align}
If $r' \in [0,r]$ is such that if $\eta \in W^{1,p}(0,1; M)$ with $d_1(\eta, \gamma) < r'$ and $f(\eta) < f(\gamma) + r'$, then $d_\infty(\eta, \gamma) < r$. Then the restriction of $\mathcal{H}$ to
\begin{align}
\{\eta \in W^{1,p}(0,1; M) : d_1(\eta, \gamma) < r', f(\eta) < f(\gamma) + r'\} \times [0,r']
\end{align}
satisfies the assumptions of Proposition 3.9. It follows that $\gamma$ is not a critical point of $f$, a contradiction.

Finally, if we define
\begin{align}
\hat{\gamma}(s) = \begin{cases}
\gamma \left(s + \frac{1}{2}\right) & \text{if } 0 \leq s \leq \frac{1}{2}, \\
\gamma \left(s - \frac{1}{2}\right) & \text{if } \frac{1}{2} \leq s \leq 1,
\end{cases}
\end{align}
it turns out that also $\hat{\gamma}$ is $L$-stationary on $[0,1]$, whence the assertion. \hfill $\blacksquare$

Lemma 4.3. Define $E: \Lambda([0,1]; N) \to \mathbb{R} \cup \{\infty\}$ by
\begin{align}
E(\gamma) = \begin{cases}
\int_0^1 |\gamma'(s)|^p \, ds & \text{if } \gamma \in X, \\
\infty & \text{if } \gamma \in \Lambda([0,1]; N) \setminus X.
\end{cases}
\end{align}
Then $\text{epi}(f)$ is homotopically equivalent to $\text{epi}(E)$.

**Proof.** By (2.1), for every $\gamma \in X$ we have

$$E(\gamma) \leq \left\| \frac{1}{k \circ \gamma} \right\|_\infty (f(\gamma) + d), \quad f(\gamma) \leq \|c \circ \gamma\|_\infty (E(\gamma) + 1).$$

Define $\Phi: \text{epi}(f) \to \text{epi}(E)$ and $\Psi: \text{epi}(E) \to \text{epi}(f)$ by

$$\Phi(\gamma, \lambda) = \left( \gamma, \left\| \frac{1}{k \circ \gamma} \right\|_\infty (\lambda + d) \right), \quad \Psi(\gamma, \lambda) = \left( \gamma, \|c \circ \gamma\|_\infty (\lambda + 1) \right).$$

Then $\Psi$ and, by Lemma 4.1, $\Phi$ are continuous and it is readily seen that $\Psi \circ \Phi$ is homotopic to the identity of $\text{epi}(f)$ while $\Phi \circ \Psi$ is homotopic to the identity of $\text{epi}(E)$.

□

**Lemma 4.4.** Let $U$ be an open subset of $\mathbb{R}^n$ and let

$$\Lambda^1(U) = \{ \gamma \in W^{1,p}(0,1; U) : \gamma(0) = \gamma(1) \}$$

drafted with the $W^{1,p}$-metric. Then there exists a continuous map

$$\mathcal{K}: \Lambda(U) \times [0,1] \to \Lambda(U)$$

such that

$$\mathcal{K}(\gamma, 0) = \gamma, \quad \mathcal{K}(\gamma, 1) \in \Lambda^1(U) \quad \text{for all } \gamma \in \Lambda(U),$$

$$\mathcal{K}(\cdot, 1): \Lambda(U) \to \Lambda^1(U) \text{ is continuous},$$

$$\mathcal{K}(\Lambda^1(U) \times [0,1]) \subseteq \Lambda^1(U),$$

$$\left\| [\mathcal{K}(\gamma, t)]' \right\|_p \leq \|\gamma'\|_p \quad \text{for all } \gamma \in \Lambda^1(U) \text{ and all } t \in [0,1].$$

**Proof.** Let $(\rho_\varepsilon)$ be a sequence of mollifiers of class $C^\infty_c$ on $\mathbb{R}^n$. Let $R_0 \gamma = \gamma$ and for every $\varepsilon > 0$ let

$$R_\varepsilon \gamma(s) = \int \rho_\varepsilon(s - t) \tau(t) \, dt,$$

where $\tau: \mathbb{R} \to U$ is 1-periodic such that $\tau_{|[0,1]} = \gamma$. It turns out that there exists a continuous function $\lambda: \Lambda(U) \to [0,1]$ such that for every $\gamma \in \Lambda(U)$ it is

$$R_\varepsilon \gamma(s) \in U \quad \text{for all } \varepsilon \in [0,\lambda(\gamma)], \text{ and all } s \in [0,1].$$

Let $\mathcal{K}: \Lambda(U) \times [0,1] \to \Lambda(U)$ defined by $\mathcal{K}(\gamma, t) = R_{t \lambda(\gamma)} \gamma$. It is readily seen that $\mathcal{K}$ satisfies all the properties required and the assertion follows. □
Lemma 4.5. The map \( \tilde{\pi}: \text{epi}(E) \to \Lambda(M) \) defined by \( \tilde{\pi}(\gamma, \lambda) = \gamma \) is a homotopy equivalence (\( \text{epi}(E) \) is endowed with the product of the uniform topology and that of \( \mathbb{R} \)).

Proof. Arguing as in the proof of Theorem 3.7, we may assume that \( N \) is a smooth submanifold of \( \mathbb{R}^n \) and we may consider an open subset \( A \) of \( \mathbb{R}^n \) with \( N \subseteq A \) and a retraction \( \pi: A \to N \) of class \( C^\infty \) such that \( \pi \) is Lipschitz continuous of constant 2. Since \( M \) is a LNR in \( N \), there exists an open neighbourhood \( U \) of \( M \) in \( N \) and a locally Lipschitzian retraction \( r: U \to M \). Since \( r \circ \pi: \pi^{-1}(U) \to M \) and \( U \) is a locally Lipschitzian retraction, then \( M \) is also a LNR in \( \mathbb{R}^n \). Now taking into account Lemma 4.4 the proof follows the same argument of [11, Theorem 5.3].

Theorem 4.6. The map \( \hat{\pi}: \text{epi}(f) \to \Lambda(M) \) defined by \( \hat{\pi}(\gamma, \lambda) = \gamma \) is a homotopy equivalence (\( \text{epi}(f) \) is endowed with the product of the uniform topology and that of \( \mathbb{R} \)).

Proof. Combining Lemmas 4.3 and 4.5 the assertion follows.

From now on, we assume that \( M \) is the closure of an open subset in \( N \) with locally Lipschitz boundary. By Theorem 3.7, \( M \) is a LNR in \( N \).

Theorem 4.7. Consider a Riemannian structure on \( N \) and the metric defined in (4.1). Let \( (\gamma, \lambda) \) be in \( \text{epi}(f) \) such that \( f(\gamma) < \lambda \). Then

\[
|dGf(\gamma, \lambda)| = 1.
\]

Proof. Arguing as in the proof of Theorem 3.7, we may assume that \( N \) is a smooth submanifold of \( \mathbb{R}^n \) and we may consider an open subset \( A \) of \( \mathbb{R}^n \) with \( N \subseteq A \) and a retraction \( \pi: A \to N \) of class \( C^\infty \) such that \( \pi \) is Lipschitz continuous of constant 2. Therefore we may also consider the function \( \tilde{L}: \mathbb{R} \times A \times \mathbb{R}^n \to \mathbb{R} \) such that \( \tilde{L} \) is a \( C^1 \)-extension of \( L \) to \( \mathbb{R} \times A \times \mathbb{R}^n \) and such that there exist two continuous functions \( \tilde{c}, \tilde{k}: A \to [0, \infty[ \) and \( d \in \mathbb{R} \) such that for every \( (s, q, v) \in \mathbb{R} \times A \times \mathbb{R}^n \) one has

\begin{align}
|D_q \tilde{L}(s, q, v)| &\leq \tilde{c}(q)(1 + |v|^p), \\
|D_s \tilde{L}(s, q, v)| &\leq \tilde{c}(q)(1 + |v|^{p-1}), \\
\tilde{L}(s, q, v) &\geq \tilde{k}(q)|v|^p - d, \\
\tilde{L}(s, q, \cdot) &\text{ is convex.}
\end{align}

First of all we claim that there exist \( \varepsilon > 0 \) and \( \overline{C} > 0 \) such that for every \( \eta_1, \eta_2 \in X \) with \( \|\eta_1 - \gamma\|_\infty \leq \varepsilon \) and for every \( t \in [0, 1] \) it is

\[
\left| \int_0^1 \tilde{L}(s, \pi(\eta_1 + (\eta_2 - \eta_1)), \pi'(\eta_1 + t(\eta_2 - \eta_1))\eta_1') - \tilde{L}(s, \eta_1, \eta_1') \, ds \right| \\
\leq \overline{C}t \left( 1 + \int_0^1 \tilde{L}(s, \eta_1, \eta_1') \, ds + \int_0^1 \tilde{L}(s, \eta_2, \eta_2') \, ds \right) \|\eta_1 - \eta_2\|_\infty.
\]
Let $\varepsilon > 0$ be such that if $\eta \in W^{1,p}(0,1; \mathbb{R}^n)$ with $\|\eta - \gamma\|_\infty \leq \varepsilon$ then $\eta \in W^{1,p}(0,1; A)$. Since $\pi$ is of class $C^\infty$ and Lipschitz continuous of constant 2, there exists $\tau \in [0, \varepsilon]$ and $\tilde{C} \geq 2$ such that for every $\eta_1, \eta_2 \in W^{1,p}(0,1; A)$ with $\|\eta - \gamma\|_\infty \leq \tau$ and for every $\xi \in \mathbb{R}^n$ it is

$$\|\pi'(\eta_1) - \pi'(\eta_2)\|_\xi \leq \tilde{C}|\eta_1 - \eta_2|.$$ 

Now let $\eta_1, \eta_2 \in X$ with $\|\eta - \gamma\|_\infty \leq \tau$ and let $\theta \in [0, 1]$. For every $\vartheta \in [0, 1]$ we have

$$\|\vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \eta'_1)| = \|\vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \eta'_1)| \\
\leq \|\vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \eta'_1)| \\
\leq \tilde{C}|\eta_1 - \eta_2||\eta'_1 - \eta'_1|.$$ 

for some $\tilde{C} > 0$. Unless reducing $\tau$, we may suppose that $\tilde{C}, \tilde{k}$ are constants on $\{\eta \in W^{1,p}(0,1; A) : d_\infty(\eta, \gamma) < \tau\}$. Furthermore, applying Lagrange’s Theorem, (4.2), (4.3) and (4.6) it is, for some $\vartheta \in [0, 1],$

$$\tilde{L}(s, \pi(\eta_1 + t(\eta_2 - \eta_1)), \pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1) - \tilde{L}(s, \eta_1, \eta'_1) \\
= D_\vartheta \tilde{L}(s, \eta_1 + \vartheta(\pi(\eta_1 + t(\eta_2 - \eta_1)) - \eta_1), \eta'_1) \\
+ \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \eta'_1)(\pi(\eta_1 + t(\eta_2 - \eta_1)) - \eta_1) \\
+ D_\vartheta \tilde{L}(s, \eta_1 + \vartheta(\pi(\eta_1 + t(\eta_2 - \eta_1)) - \eta_1), \eta'_1) \\
+ \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \eta'_1)(\pi(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \eta'_1) \\
\leq C(1 + |\eta'_1 + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \eta'_1)|p|\pi(\eta_1 + t(\eta_2 - \eta_1)) - \pi(\eta_1)| \\
\cdot |\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \pi'(\eta_1)\eta'_1| \\
\leq C_3 t(1 + |\eta'_1 + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \eta'_1)|p)|\eta_1 - \eta_2| \\
\cdot |\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1 - \pi'(\eta_1)\eta'_1| \\
\leq C_3 t(1 + |\eta'_1|p + |\eta'_2|p)|\eta_1 - \eta_2| + C_4 t(1 + |\eta'_1|p - 1 + |\eta'_2|p - 1)|\eta'_1||\eta_1 - \eta_2| \\
= C_3 t(1 + |\eta'_1|p + |\eta'_2|p)|\eta_1 - \eta_2| + C_4 t(|\eta'_1| + |\eta'_1|p + |\eta'_2|p - 1)|\eta_1 - \eta_2|$$

for some $C_3 > 0$. It follows that

$$\int_0^1 \left| \tilde{L}(s, \pi(\eta_1 + t(\eta_2 - \eta_1)), \pi'(\eta_1 + t(\eta_2 - \eta_1))\eta'_1) - \tilde{L}(s, \eta_1, \eta'_1) \right| ds \\
\leq C_3 (1 + 2||\eta'_1|p + ||\eta'_2|p + ||\eta'_1||1 + ||\eta'_1|p||p-1||\eta_1 - \eta_2| \\
\leq C_4 (1 + ||\eta'_1|p + ||\eta'_2|p)|\eta_1 - \eta_2|.$$
for some \( C_4 > 0 \). Finally, applying (4.4) we may find \( \overline{C} > 0 \) such that

\[
\left| \int_0^1 \left[ \tilde{L}(s, \pi(\eta_1 + t(\eta_2 - \eta_1)), \pi'(\eta_1 + t(\eta_2 - \eta_1))\eta_1') - \tilde{L}(s, \eta_1, \eta_1') \right] ds \right|
\leq \overline{C} t \left( 1 + \int_0^1 \tilde{L}(s, \eta_1, \eta_1') ds + \int_0^1 \tilde{L}(s, \eta_2, \eta_2') ds \right) \| \eta_1 - \eta_2 \|_\infty
\]

and the claim follows. Let \( \varepsilon > 0 \), \( K = \gamma([0,1]) \) and let \( \overline{\varepsilon}, \overline{C} > 0 \) be as before. Let \( C_2 = \overline{C}(1 + 2\lambda + \varepsilon) \). Let now \( \widehat{r} \) and \( \widehat{c} \) be as in (c) of Lemma 3.5, and let

\[
\widehat{\gamma}(s) = \gamma(s) + \rho \nu(\gamma(s)),
\]

where \( \rho \in \]0, \widehat{r}[\) is such that

\[
\| \pi(\widehat{\gamma}) - \gamma \|_\infty \leq \min \left\{ \frac{\varepsilon}{4}, \frac{\varepsilon}{8C_2}, \overline{\varepsilon} \right\}, \quad f(\pi \circ \widehat{\gamma}) \leq f(\gamma) + \frac{\varepsilon}{4}.
\]

Let \( r \in \]0, \varepsilon/2[\) be such that if \( \| \eta - \gamma \|_1 < r \) with \( f(\eta) < \lambda + r \), then \( \| \eta - \gamma \|_\infty \leq \min\{\rho/\widehat{c}, \varepsilon/4, \varepsilon/8C_2, \overline{\varepsilon}\} \). Then, again by (c) of Lemma 3.5 it is possible to define a continuous map

\[
\mathcal{H}: \{ \eta \in X : \| \eta - \gamma \|_1 < r, \ f(\eta) < \lambda + r \} \times [0,r] \to X
\]

by

\[
\mathcal{H}(\eta, t) = \pi((1-t)\eta + t\pi(\widehat{\gamma})).
\]

It is

\[
\| \mathcal{H}(\eta, t) - \eta \|_\infty \leq 2t\| \pi(\widehat{\gamma}) - \eta \|_\infty \leq 2t(\| \pi(\widehat{\gamma}) - \gamma \|_\infty + \| \gamma - \eta \|_\infty) \leq \varepsilon t
\]

and hence also

\[
\| \mathcal{H}(\eta, t) - \eta \|_1 \leq \varepsilon t.
\]

Since \( \tilde{L} \) is convex with respect to the third variable, we get

\[
f(\mathcal{H}(\eta, t))
= \int_0^1 \tilde{L}(s, \pi(\eta + t(\pi(\widehat{\gamma}) - \eta)), \pi'(\eta + t(\pi(\widehat{\gamma}) - \eta)))(\eta' + t((\pi \circ \widehat{\gamma})' - \eta')) ds
\leq \int_0^1 \tilde{L}(s, \pi(\eta + t(\pi(\widehat{\gamma}) - \eta)), \pi'(\eta + t(\pi(\widehat{\gamma}) - \eta))\eta') ds
+ t \left[ \int_0^1 \tilde{L}(s, \pi(\eta + t(\pi(\widehat{\gamma}) - \eta)), \pi'(\eta + t(\pi(\widehat{\gamma}) - \eta))(\pi \circ \widehat{\gamma})' ds
- \int_0^1 \tilde{L}(s, \pi(\eta + t(\pi(\widehat{\gamma}) - \eta)), \pi'(\eta + t(\pi(\widehat{\gamma}) - \eta))\eta') ds \right].
\]
Furthermore, it is
\[
\left| \int_0^1 [\tilde{L}(s, \pi(\eta + t(\tilde{\gamma}) - \eta)), \pi'(\eta + t(\tilde{\gamma}) - \eta)] ds \right|
\]
\[
\leq Ct(1 + f(\eta) + f(\pi(\tilde{\gamma})))\|\pi(\tilde{\gamma}) - \eta\|\infty
\]
\[
< Ct(1 + 2\lambda + \varepsilon)(\|\pi(\tilde{\gamma}) - \gamma\|\infty + \|\gamma - \eta\|\infty) \leq \frac{\varepsilon}{4} t
\]
and
\[
\left| \int_0^1 [\tilde{L}(s, \pi(\eta + t(\tilde{\gamma}) - \eta)), \pi'(\eta + t(\tilde{\gamma}) - \eta)(\pi(\tilde{\gamma}))'] \right|
\]
\[
\leq Ct(1 + f(\eta) + f(\pi(\tilde{\gamma})))\|\pi(\tilde{\gamma}) - \eta\|\infty
\]
\[
< Ct(1 + 2\lambda + \varepsilon)(\|\pi(\tilde{\gamma}) - \gamma\|\infty + \|\gamma - \eta\|\infty) \leq \frac{\varepsilon}{4} t.
\]
Therefore we finally get
\[
f(\mathcal{H}(\eta, t)) \leq f(\eta) + \frac{\varepsilon}{4} t + \left( f(\pi(\tilde{\gamma}) - f(\eta) + \frac{\varepsilon}{2} \right) t \leq f(\eta) + t(f(\gamma) - f(\eta) + \varepsilon)
\]
and the assertion follows from Proposition 3.11. \(\square\)

Finally, we can prove Theorem 2.6.

Proof. Now assume also that \(M\) is compact, 1-connected and non-contractible in itself. By Theorem 3.7, we have that \(M\) is a LNR in \(N\), in particular an absolute neighbourhood retract. From [13, Corollary 1.4] it follows that \(\text{cat} \Lambda(M) = \infty\). Moreover, \(\Lambda(M)\) also is an absolute neighbourhood retract, hence weakly locally contractible. On the other hand, by Theorem 4.6 \(\Lambda(M)\) is homotopically equivalent to \(\text{epi}(f)\). Therefore \(\text{cat} \text{epi}(f) = \infty\) and \(\text{epi}(f)\) is weakly locally contractible.

Let now \(c \in \mathbb{R}\) and consider the sublevel
\[
\mathcal{G}_f^c = \{ (\gamma, \lambda) \in \Lambda(M) \times \mathbb{R} : f(\gamma) \leq \lambda \leq c \}.
\]
Since \(M\) is compact, from (2.1) and Ascoli’s theorem we deduce that \(\mathcal{G}_f^c\) is compact. By Corollary 3.18, there exists a sequence \((\gamma_h, \lambda_h)\) of critical points of \(\mathcal{G}_f^c\) with respect to the metric (4.1) with \(\lambda_h \to \infty\). By Theorem 4.7 we have that \(\lambda_h = f(\gamma_h)\). From (c) of Theorem 4.2 the assertion follows. \(\square\)

The next two results correspond to the well-known equation \(d/ds H = -D_s L\), where \(H\) is the Hamiltonian function associated with \(L\).

Theorem 4.8. Let \(\gamma \in W^{1,p}(a, b; M)\) be \(L\)-stationary. Assume that \(L\) does not depend on \(s\). Then the map \(s \mapsto D_s L(\gamma, \gamma')\gamma' - L(\gamma, \gamma')\) is constant a.e.

Proof. Arguing as in the proof of Theorem 4.7, we may assume that \(N\) is a smooth submanifold of \(\mathbb{R}^n\), \(A\) is an open subset of \(\mathbb{R}^n\) with \(N \subseteq A\) and
\[ \widetilde{L} : A \times \mathbb{R}^n \to \mathbb{R} \] is a \( C^1 \)-extension of \( L \) to \( A \times \mathbb{R}^n \) satisfying (4.2)–(4.5). Assume, for a contradiction, that there exists \( \varphi \in C^{\infty}_c(a, b) \) such that

\[ \sigma := \frac{1}{2} \int_a^b \| [D_v \widetilde{L}(\gamma, \gamma')] \cdot \gamma' - \widetilde{L}(\gamma, \gamma')] \varphi' \| \, ds > 0. \]

Let \( r > 0 \) be such that \( r \| \varphi' \|_\infty < 1 \) and let \( \psi : [a, b] \times [0, r] \to [a, b] \) be the smooth function such that

\[ \lambda = \psi(\lambda, t) - t \varphi(\psi(\lambda, t)) \quad \text{for all } \lambda \in [a, b] \text{ and all } t \in [0, r]. \]

Unless reducing \( r \) we may suppose that the functions \( c, k \) in (4.2)–(4.4) are constants on \( \{ \eta \in W^{1,p}(a, b; M) : d_{\infty}(\eta, \gamma) < r \} \). Define \( \mathcal{H} : \{ \eta \in W^{1,p}(a, b; M) : d_{\infty}(\eta, \gamma) < r \} \to W^{1,p}(a, b; M) \) by

\[ \mathcal{H}(\eta, t)(\mu) = \eta(\mu - t \varphi(\mu)) \ . \]

It is easy to see that \( \mathcal{H} \) is continuous from the product topology of the uniform convergence and of \( \mathbb{R} \) to that of the uniform convergence and that

\[ \mathcal{H}(\eta, t)(a) = \eta(a), \quad \mathcal{H}(\eta, t)(b) = \eta(b). \]

Moreover, by (4.4)

\[
\begin{align*}
&d_1(\mathcal{H}(\eta, t), \eta) = \int_a^b |\eta(\mu - t \varphi(\mu)) - \eta(\mu)| \, d\mu \\
&= t \int_a^b |\eta'(\mu) - t \varphi'(\mu)| \, d\mu \\
&\leq t \left( \int_a^b |\eta'(\lambda)|^p \, d\lambda \right)^{1/p} \left( \int_a^b |1 - t \varphi'(\mu)|^{p'} \, d\mu \right)^{1/p'} \\
&\leq \left( (1 - \theta\| \varphi' \|_\infty)^p \left( \int_a^b |\eta'(\lambda)|^p \, d\lambda \right)^{1/p} \left( \int_a^b |1 - t \varphi'(\mu)|^{p'} \, d\mu \right)^{1/p'} \right) \\
&\leq C t \left( \int_a^b (L(\eta(\lambda), \eta'(\lambda)) + d) \, d\lambda \right)^{1/p} < C t (f_{a,b}(\gamma) + r + d(b - a))^{1/p},
\end{align*}
\]

for some \( C > 0 \). Following the same argument of the proof of [19, Theorem 5.10] we also have

\[ f_{a,b}(\mathcal{H}(\eta, t)) = f_{a,b}(\eta) + t \Theta(\eta, t) \]

where

\[
\Theta(\eta, t) = \int_a^b \left[ - D_v \widetilde{L}(\eta(\lambda), (1 - t \varphi'(\psi(\lambda, t))) \eta'(\lambda)) \cdot \eta'(\lambda) \varphi'(\psi(\lambda, t)) \\
+ \widetilde{L}(\eta(\lambda), (1 - t \varphi'(\psi(\lambda, t))) \eta'(\lambda)) \cdot \frac{\varphi'(\psi(\lambda, t))}{1 - t \varphi'(\psi(\lambda, t))} \right] d\lambda.
\]
We claim that, for \( r \) sufficiently small, we have \( \Theta(\eta, t) \leq -\sigma \) for any \( \eta \in W^{1,p}(a, b; M) \) with \( d_\infty(\eta, \gamma) < r, f_{a,b}(\eta) < f_{a,b}(\gamma) + r \) and \( 0 \leq t \leq r \). By contradiction, let \( (\eta_h) \) be a sequence in \( W^{1,p}(a, b; M) \) uniformly convergent to \( \gamma \) with \( f_{a,b}(\eta_h) < f_{a,b}(\gamma) + 1/h \) and \( (\eta_h) \) be a non-negative sequence convergent to 0 such that \( \Theta(\eta_h, t_h) > -\sigma \). Because of (4.4) and \( f_{a,b} \) is lower semicontinuous, we have that \( f_{a,b}(\eta_h) \to f_{a,b}(\gamma) \). Again by (4.4) \( (\eta_h) \) is bounded in \( W^{1,p}(a, b; M) \) and up to a subsequence \( \eta_h' \to \gamma' \) in \( L^p(a, b; M) \). Therefore \( \left[ 1 - t_h \varphi'(\psi(\cdot, t_h)) \right] \eta_h' \to \gamma' \) in \( L^p(a, b; M) \). We have that

\[
\int_a^b \left[ \tilde{L}(\gamma, \lambda, [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta_h')(\lambda) - \tilde{L}(\gamma(\lambda), \gamma' (\lambda)) \right] \lambda \\
= \int_a^b D_\gamma \tilde{L}(\gamma(\lambda), (1 - \tau) \gamma'(\lambda) + \tau \eta_h' (\lambda)) \cdot (\eta_h' (\lambda) - \gamma'(\lambda)) \lambda \\
+ t_h \int_a^b \varphi' (\psi(\lambda, t_h)) D_\gamma \tilde{L}(\gamma(\lambda), (1 - \vartheta) \eta_h' (\lambda) \\
+ \vartheta [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta_h' (\lambda) \cdot \eta_h' (\lambda) \lambda.
\]

By (4.3) we have that \( D_\gamma \tilde{L}(\gamma, (1 - \tau) \gamma' + \tau \eta_h') \in L^p(a, b; M) \) and hence

\[
\int_a^b D_\gamma \tilde{L}(\gamma(\lambda), (1 - \tau) \gamma'(\lambda) + \tau \eta_h'(\lambda)) \cdot (\eta_h'(\lambda) - \gamma'(\lambda)) \lambda \to 0.
\]

Again by (4.3) we have that

\[
\int_a^b \varphi' (\psi(\lambda, t_h)) D_\gamma \tilde{L}(\gamma(\lambda), (1 - \vartheta) \eta_h'(\lambda) + \vartheta [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta_h'(\lambda) \cdot \eta_h'(\lambda) \lambda 
\]

is bounded. Therefore we have that

\[
\int_a^b \tilde{L}(\gamma, [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta_h'(\lambda)) \lambda \to \int_a^b \tilde{L}(\gamma(\lambda), \gamma'(\lambda)) \lambda.
\]

By [12, Lemma 3.1] applied to the function \( F(\lambda, \xi) = \tilde{L}(\gamma(\lambda), \xi) \) we obtain that

\[
\tilde{L}(\gamma, [1 - t_h \varphi'(\psi(\cdot, t_h))] \eta_h'(\lambda)) \to \tilde{L}(\gamma, \gamma') \quad \text{in } L^1(a, b; M),
\]

\[
D_\gamma \tilde{L}(\gamma, [1 - t_h \varphi'(\psi(\cdot, t_h))] \eta_h'(\lambda)) \to D_\gamma \tilde{L}(\gamma, \gamma') \quad \text{in } L^p(a, b; M)
\]

and there exists \( \Psi \in L^1(a, b; M) \) such that \( |\eta_h'|^p \leq \Psi \). For some \( t \in ]0, 1[ \) we have that

\[
\tilde{L}(\eta_h(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta_h'(\lambda)) - \tilde{L}(\gamma(\lambda), [1 - t_h \varphi'(\psi(\lambda, t_h))] \eta_h'(\lambda)) = D_\gamma \tilde{L}((1 - t) \gamma(\lambda) + t \eta_h(\lambda), [1 - t_h \varphi'(\psi(\psi(\lambda, t_h))] \eta_h'(\lambda)) \cdot (\eta_h(\lambda) - \gamma(\lambda))
\]

By (4.2) we deduce that \( D_\gamma \tilde{L}((1 - t) \gamma + t \eta_h, [1 - t_h \varphi'(\psi(\cdot, t_h))] \eta_h') \in L^p(a, b; M) \) and hence

\[
[\tilde{L}(\eta_h, [1 - t_h \varphi'(\psi(\cdot, t_h))] \eta_h') - \tilde{L}(\gamma, [1 - t_h \varphi'(\psi(\cdot, t_h))] \eta_h')] \to 0 \quad \text{in } L^1(a, b; M).
\]
It follows that
\[ \tilde{L}(\eta, [1 - t_h \varphi'(\psi(\cdot, t_h))]\eta'_h) \to \tilde{L}(\gamma, \gamma') \] in \( L^1(a, b; M) \).

Fix \( \varepsilon > 0 \), let \( \delta > 0 \) such that for any \( L^1 \)-measurable subset \( \Omega \subseteq [a, b] \) with \( L^1(\Omega) < \delta \) we have
\[ \int_{\Omega} \Phi(\lambda) \, d\lambda < \frac{\varepsilon}{2} \] for all \( \Phi \in L^1(a, b; M) \).

Let \( R > 0 \) be such that \( L^1([\lambda \in [a, b] : |\eta'_h(\lambda)| > R]) < \delta \). Let \( \Omega_h = \{ \lambda \in [a, b] : |\eta'_h(\lambda)| > R \} \) and \( \Omega'_h = \{ \lambda \in [a, b] : |\eta'_h(\lambda)| \leq R \} \). By (4.3) we have
\[ \int_a^b |D_v \tilde{L}(\eta, [1 - t_h \varphi'(\psi(\lambda, t_h))]\eta'_h(\lambda)) - D_v \tilde{L}(\gamma, [1 - t_h \varphi'(\psi(\lambda, t_h))]\eta'_h(\lambda))| \, d\lambda \]
\[ \leq \int_{\Omega_h} C(1 + \Psi(\lambda)) \, d\lambda + \int_{\Omega'_h} |D_v \tilde{L}(\eta, [1 - t_h \varphi'(\psi(\lambda, t_h))]\eta'_h(\lambda)) - D_v \tilde{L}(\gamma, [1 - t_h \varphi'(\psi(\lambda, t_h))]\eta'_h(\lambda))| \, d\lambda \]
\[ \leq \frac{\varepsilon}{2} + \int_{\Omega'_h} |D_v \tilde{L}(\eta, [1 - t_h \varphi'(\psi(\lambda, t_h))]\eta'_h(\lambda)) - D_v \tilde{L}(\gamma, [1 - t_h \varphi'(\psi(\lambda, t_h))]\eta'_h(\lambda))| \, d\lambda. \]

Since the map
\[ \{ \lambda \to [D_v \tilde{L}(\eta, [1 - t_h \varphi'(\psi(\lambda, t_h))]\eta'_h(\lambda)), [1 - t_h \varphi'(\psi(\lambda, t_h))]\eta'_h(\lambda)] \} \]
is uniformly continuous on \( \Omega'_h \), for \( h \) sufficiently large we have
\[ \int_{\Omega'_h} |D_v \tilde{L}(\eta, [1 - t_h \varphi'(\psi(\lambda, t_h))]\eta'_h(\lambda)) - D_v \tilde{L}(\gamma, [1 - t_h \varphi'(\psi(\lambda, t_h))]\eta'_h(\lambda))| \, d\lambda < \frac{\varepsilon}{2}. \]

It follows that
\[ \| D_v \tilde{L}(\eta, [1 - t_h \varphi'(\psi(\cdot, t_h))]\eta'_h) - D_v \tilde{L}(\gamma, [1 - t_h \varphi'(\psi(\cdot, t_h))]\eta'_h) \|_{L^p} \to 0. \]

Therefore
\[ D_v \tilde{L}(\eta, [1 - t_h \varphi'(\psi(\cdot, t_h))]\eta'_h) \to D_v \tilde{L}(\gamma, \gamma') \] in \( L^p(a, b; M) \)
and we deduce that
\[ \Theta(\eta, t_h) = \int_a^b \{ -D_v \tilde{L}(\gamma, \gamma') \cdot \gamma' + \tilde{L}(\gamma, \gamma') \} \, d\lambda = -2\sigma, \]
a contradiction. Finally, we have \( f_{a,b}(\mathcal{H}(\eta, t)) \leq f_{a,b}(\eta) - \sigma t \). It follows that \( \gamma \) is not \( L \)-stationary, a contradiction. \( \square \)
Theorem 4.9. Let $\gamma \in W^{1,p}(a, b; M)$ be $L$-stationary. Assume that for every $s \in \mathbb{R}$ and $q \in M$ one has

$$\left| D_s L(s, q, v) \right| \leq c(q)(1 + |v|^p), \quad \text{for all } v \in T_q N,$$

and

$$L(s, q, \cdot) \text{ is strictly convex on } T_q N.$$

Then the map $\{ s \mapsto D_s L(s, \gamma, \gamma') \gamma' - L(s, \gamma, \gamma') \}$ belongs to $W^{1,1}(a, b)$ and we have

$$\int_a^b \left[ D_s L(s, \gamma, \gamma') \gamma' - L(s, \gamma, \gamma') \right] \varphi' \, ds = \int_a^b D_s L(s, \gamma, \gamma') \varphi \, ds$$

for all $\varphi \in C_c^\infty(a, b)$.

Proof. Arguing as in the proof of Theorem 4.7, we may assume that $N$ is a smooth submanifold of $\mathbb{R}^n$, $A$ is an open subset of $\mathbb{R}^n$ with $N \subseteq A$ and $\tilde{L} : \mathbb{R} \times A \times \mathbb{R}^n \to \mathbb{R}$ is a $C^1$-extension of $L$ to $\mathbb{R} \times A \times \mathbb{R}^n$ satisfying (4.2)–(4.4) and such that for every $(s, q, v) \in \mathbb{R} \times A \times \mathbb{R}^n$ one has

$$|D_s \tilde{L}(s, q, v)| \leq c(q)(1 + |v|^p),$$

and

$$\tilde{L}(s, q, \cdot) \text{ is strictly convex.}$$

Assume, for a contradiction, that there exists $\varphi \in C_c^\infty(a, b)$ such that

$$\sigma := \frac{1}{2} \int_a^b \{ [D_s \tilde{L}(s, \gamma, \gamma') \gamma' - \tilde{L}(s, \gamma, \gamma')] \varphi' - D_s \tilde{L}(s, \gamma, \gamma') \varphi \} \, ds > 0.$$

Arguing as in the proof of Theorem 4.8 we may introduce the continuous map

$$\mathcal{H} : \{ \eta \in W^{1,p}(a, b; M) : d_\infty(\eta, \gamma) < r, \ f_{a,b}(\eta) < f_{a,b}(\gamma) + r \} \times [0, r] \to W^{1,p}(a, b; M)$$

defined by

$$\mathcal{H}(\eta, t)(\mu) = \eta(\mu - t \varphi(\mu))$$

satisfying the following facts:

$$\mathcal{H}(\eta, t)(a) = \eta(a), \quad \mathcal{H}(\eta, t)(b) = \eta(b),$$

$$d_1(\mathcal{H}(\eta, t), \eta) < \tilde{C} t (f_{a,b}(\gamma) + r + (b-a))^{1/p},$$

$$f_{a,b}(\mathcal{H}(\eta, t)) \leq f_{a,b}(\eta) + t \Theta(\eta, t)$$

where $\tilde{C} > 0$,

$$\Theta(\eta, t) = \int_a^b \left[ D_s \tilde{L}(\lambda + t \varphi(\lambda, t), \varphi(\lambda, t)) \eta(1 - t \varphi(\lambda, t) \varphi'(\psi(\lambda, t))) \eta' \varphi'(\psi(\lambda, t)) \right. \left. - D_s \tilde{L}(\lambda + t \varphi(\lambda, t), \eta(\lambda), (1 - t \varphi'(\psi(\lambda, t))) \eta(\lambda)) \eta'(\lambda) \varphi'(\psi(\lambda, t)) + \tilde{L}(\varphi(\lambda, t), \eta(\lambda), (1 - t \varphi'(\psi(\lambda, t))) \eta'(\lambda)) \frac{\varphi'(\psi(\lambda, t))}{1 - t \varphi'(\psi(\lambda, t))} \right] d\lambda$$

and $0 < \varphi(\lambda, t) < 1$. 

We claim that, for $r$ sufficiently small, we have $\Theta(\eta, t) \leq -\sigma$ for any $\eta \in W^{1,p}(a,b;M)$ with $d_\infty(\eta, \gamma) < r$, $f_{a,b}(\eta) < f_{a,b}(\gamma) + r$ and $0 \leq t \leq r$. By contradiction, let $(\eta_h)$ be a sequence in $W^{1,p}(a,b;M)$ uniformly convergent to $\gamma$ with $f_{a,b}(\eta_h) < f_{a,b}(\gamma) + \frac{1}{h}$ and $(t_h)$ be a non-negative sequence convergent to 0 such that $\Theta(\eta_h, t_h) > -\sigma$. Because of (4.4) and $f_{a,b}$ is lower semicontinuous, we have that $f_{a,b}(\eta_h) \to f_{a,b}(\gamma)$. Again by (4.4) $(\eta_h)$ is bounded in $W^{1,p}(a,b;M)$ and up to a subsequence $\eta_h \to \gamma$ in $W^{1,p}(a,b;M)$. On the other hand, we have

$$\int_a^b \tilde{L}(\lambda, \gamma(\lambda), \eta'_h(\lambda))\,d\lambda - \int_a^b \tilde{L}(\lambda, \gamma(\lambda), \gamma'(\lambda))\,d\lambda
\geq f_{a,b}(\eta_h) - f_{a,b}(\gamma) - \int_a^b \tilde{L}(\lambda, \eta_h(\lambda), \eta'_h(\lambda))\,d\lambda + \int_a^b \tilde{L}(\lambda, \gamma(\lambda), \eta'_h(\lambda))\,d\lambda.$$

Taking into account (4.2), we get that

$$\int_a^b \tilde{L}(\lambda, \gamma(\lambda), \eta'_h(\lambda))\,d\lambda \to \int_a^b \tilde{L}(\lambda, \gamma(\lambda), \gamma'(\lambda))\,d\lambda.$$

By [20, Theorem 3] applied to the function $\Phi(\lambda, \xi) = \tilde{L}(\lambda, \gamma(\lambda), \xi)$ it follows that $\eta'_h$ is strongly convergent to $\gamma'$ in $L^p(a,b;M)$; hence $\eta_h \to \gamma$ in $W^{1,p}(a,b;M)$. Because of (4.2), (4.3) and (4.9), we have that

$$\Theta(\eta_h, t_h) \to \int_a^b \{[-D_\sigma \tilde{L}(\lambda, \gamma, \gamma') \cdot \gamma' + \tilde{L}(\lambda, \gamma, \gamma')]\varphi' + D_\sigma \tilde{L}(\lambda, \gamma, \gamma')\varphi\}\,d\lambda = -2\sigma,$$

a contradiction. Finally, we have $f_{a,b}(H(\eta, t)) \leq f_{a,b}(\eta) - \sigma t$. It follows that $\gamma$ is not $L$-stationary, a contradiction. \hfill $\Box$

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