Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 27, 2006, 195–228

ON THE FUČÍK SPECTRUM FOR ELLIPTIC SYSTEMS

Eugenio Massa — Bernhard Ruf

Abstract. We propose an extension of the concept of Fučík spectrum to the case of coupled systems of two elliptic equations, we study its structure and some applications. We show that near a simple eigenvalue of the system, the Fučík spectrum consists (after a suitable reparametrization) of two (maybe coincident) 2-dimensional surfaces. Furthermore, by variational methods, parts of the Fučík spectrum which lie far away from the diagonal (i.e. from the eigenvalues) are found. As application, some existence, non-existence and multiplicity results to systems with eigenvalue crossing ("jumping") nonlinearities are proved.

1. Introduction

In this work we propose an extension of the concept of Fučík spectrum for the case of coupled systems of two equations, we study its structure and some applications.

The notion of Fučík spectrum was introduced for the scalar Laplace problem in [10] and [4]; it is defined as the set $\Sigma_{\mathrm{scal}} \subseteq \mathbb{R}^2$ of the points (λ^+, λ^-) for which there exists a non trivial solution of the problem

$$(1.1)\qquad \begin{cases} -\Delta u = \lambda^+ u^+ - \lambda^- u^- & \text{in } \Omega, \\ Bu = 0 & \text{in } \partial\Omega, \end{cases}$$

2000 *Mathematics Subject Classification.* 35J50, 35J55, 35J60.

Key words and phrases. Elliptic system, Fučík spectrum, variational methods, topological degree.

The first named author was supported by Fapesp/Brasil.

where Ω is a bounded domain in $\mathbb{R}^n$, $u^\pm(x) = \max\{0, \pm u(x)\}$ and $Bu = 0$ represents Dirichlet or Neumann boundary conditions.

We will consider here the following generalization to the case of coupled systems: we will call *Fučík problem* the system

$$
\begin{cases} -\Delta u = \lambda^+ v^+ - \lambda^- v^- & \text{in } \Omega, \\ -\Delta v = \mu^+ u^+ - \mu^- u^- & \text{in } \Omega, \\ Bu = Bv = 0 & \text{in } \partial\Omega, \end{cases} \tag{1.2}
$$

and we define the *Fučík spectrum* as the set

$$
\Sigma = \{(\lambda^+, \lambda^-, \mu^+, \mu^-) \in \mathbb{R}^4 \text{ such that (1.2) has nontrivial solutions}\}.
$$

For the scalar problem, in dimension $n = 1$ the Fučík spectrum Σ_{scal} is explicitly known and consists of curves in $\mathbb{R}^2$ containing the diagonal points (λ_k, λ_k). In dimension $n \geq 2$, the Fučík spectrum is only partially known; we recall some important known cases:

- The so-called trivial part of the spectrum, corresponding to positive or negative solutions.
- If λ_k is a simple eigenvalue, then the Fučík spectrum in $(\lambda_{k-1}, \lambda_{k+1})^2$ consists of two curves (maybe coincident) which pass through the point (λ_k, λ_k), see [11], [18]; for the case of multiple eigenvalues, see [15].
- The first nontrivial curve, passing through the point (λ_2, λ_2), see [8].

The knowledge of the Fučík spectrum is important in many applications such as oscillations of suspension bridges (see e.g. [12], [17]), motions of ships (floating beam) in water ([14]), stationary solutions for the equation of competing species ([5]), etc. We believe that also the systems considered here will prove useful in applications.

Also, we recall that if a variational characterization of the Fučík spectrum is known, then other interesting results can be obtained, cf. [2], [8], [9] and [3].

In this paper we will first deduce some properties of the set Σ and of the corresponding non trivial solutions for the Fučík system (1.2). This will serve to obtain the main results, i.e. to find nontrivial points in the Fučík spectrum. In Section 5 we will characterize the Fučík spectrum for points near a simple eigenvalue of the system: it consists, after a suitable reparametrization of the Fučík spectrum which in fact reduces the parameters to three, of exactly two (maybe coincident) spectral surfaces of dimension 2. Furthermore, in Section 6 we will use a variational characterization to obtain also points of the Fučík spectrum which are "far away" from the diagonal points.

More precisely, we will prove the following results (here H is $H_0^1(\Omega)$ for the Dirichlet problem and $H^1(\Omega)$ for the Neumann problem):

Theorem 1.1. *Let λ_k, $k \geq 2$, be a simple eigenvalue of the Laplacian with corresponding eigenfunction ϕ_k, and let $d = \min(\lambda_{k+1} - \lambda_k, \lambda_k - \lambda_{k-1})$. Then for every fixed pair (γ, δ) satisfying $0 \leq |\delta| \leq \gamma < d/6$, there exists a unique $\lambda_{k+} \in [\lambda_k - 3\gamma, \lambda_k + 3\gamma]$ such that system* (1.2) *with coefficients*

$$\lambda^+ = \mu^+ = \lambda_{k+}, \quad \lambda^- = \lambda_{k+} - \gamma, \quad \mu^- = \lambda_{k+} - \delta,$$

has a (unique) solution $(u, v) \in H \times H$ with $(u, v) \cdot (\phi_k, \phi_k) = 2$ ($[L^2(\Omega)]^2$ scalar product). Moreover, a second value λ_{k-} is obtained by imposing $(u, v) \cdot (\phi_k, \phi_k) = -2$, and no solution exists with $(u, v) \cdot (\phi_k, \phi_k) = 0$.

Theorem 1.2. *For any $r, s \in (0, \infty)$, we can find and characterize variationally one point in Σ of the form*

$$\lambda^+ = \mu^+ = \lambda_1 + \xi, \quad \lambda^- = \lambda_1 + s\xi, \quad \mu^- = \lambda_1 + r\xi,$$

for some $\xi > 0$, where λ_1 is the first eigenvalue of the Laplacian.

Observe that this result implies the existence of points in Σ which are "far" from the diagonal $\lambda^+ = \mu^+ = \lambda^- = \mu^-$, since we are not asking r, s to be near to 1.

We will also see that the points found in Theorem 1.2 form a continuum in Σ which contains the second eigenvalue of the Laplacian.

As a consequence of the Theorems 1.1 and 1.2 we will also see that the Fučík spectrum for the system is much richer than that for the scalar problem. Actually, it is known that the linear spectrum for the system consists of λ_k and $-\lambda_k$, where λ_k, $k \in \mathbb{N}$, are the eigenvalues of the scalar problem, and the corresponding eigenfunctions are always composed by a pair of eigenfunctions (in fact, the same eigenfunction) of the scalar problem. In contrast with this, we will prove in Section 7 the existence of nontrivial solutions of the Fučík problem for the system that have no relation with any solution of the scalar Fučík problem and for which at least three of the four products u^+v^+, u^+v^-, u^-v^+ and u^-v^- are not identically zero.

Finally, as an application we will consider the problem of existence of solutions for sublinear perturbations of system (1.2), i.e. we consider systems with "eigenvalue crossing" (or "jumping") nonlinearities:

Let $f_1, f_2 \in C(\overline{\Omega} \times \mathbb{R})$ such that (uniformly for $x \in \Omega$)

$$\lim_{s \to \infty} \frac{f_1(x,s)}{s} = \lambda, \qquad \lim_{s \to -\infty} \frac{f_1(x,s)}{s} = \lambda - \gamma,$$
$$\lim_{s \to \infty} \frac{f_2(x,s)}{s} = \lambda, \qquad \lim_{s \to -\infty} \frac{f_2(x,s)}{s} = \lambda - \delta.$$

Then, in dependence on the location of λ with respect to the Fučík spectrum, the system

$$\begin{cases} -\Delta u = f_1(x,v) + h_1(x) & \text{in } \Omega, \\ -\Delta v = f_2(x,u) + h_2(x) & \text{in } \Omega, \\ Bu = Bv = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

has either a solution for all $h_1, h_2 \in L^2(\Omega)$, or it has no solution for some $h_1, h_2 \in L^2(\Omega)$, and at least two solutions for other $h_1, h_2 \in L^2(\Omega)$. More precisely, we will show:

Theorem 1.3. *Let γ, δ be as in Theorem* 1.1 *and $\lambda \in [\lambda_k - 3\gamma, \lambda_k + 3\gamma]$. Then, problem* (1.3) *has a solution for any $h_1, h_2 \in L^2(\Omega)$ provided that $\lambda < \lambda_{k-}$ or $\lambda > \lambda_{k+}$. If instead $\lambda_{k-} < \lambda < \lambda_{k+}$, let $(h_1, h_2)^\perp$ denote the component of (h_1, h_2) orthogonal (in the $[L^2(\Omega)]^2$ scalar product) to (ϕ_k, ϕ_k) and set $2s = \int_\Omega (h_1 + h_2)\phi_k$; then there exist two real numbers S and S_1, which depend on $(h_1, h_2)^\perp$, such that problem* (1.3) *has no solution for $s < S$ and at least two solutions for $s > S_1$.*

The above three theorems will be proven in the Sections 5, 6 and 5.4, respectively.

2. Some properties of Σ and of the corresponding nontrivial solutions

In this section we will derive some properties of Σ and of the corresponding nontrivial solutions. We begin with

2.1. Some useful identities. By testing the first equation of (1.2) against v and the second against u one gets

$$\int_\Omega \nabla u \nabla v = \lambda^+ \int_\Omega (v^+)^2 + \lambda^- \int_\Omega (v^-)^2 = \mu^+ \int_\Omega (u^+)^2 + \mu^- \int_\Omega (u^-)^2. \tag{2.1}$$

By testing the first with u and the second with v one gets

$$\begin{aligned} \int_\Omega |\nabla u|^2 &= \lambda^+ \int_\Omega v^+ u - \lambda^- \int_\Omega v^- u, \\ \int_\Omega |\nabla v|^2 &= \mu^+ \int_\Omega u^+ v - \mu^- \int_\Omega u^- v, \end{aligned}$$

while by using only the positive and negative parts the following relations are obtained:

$$\int_\Omega |\nabla u^+|^2 = \lambda^+ \int_\Omega v^+ u^+ - \lambda^- \int_\Omega v^- u^+, \tag{2.2}$$

$$\int_\Omega |\nabla u^-|^2 = -\lambda^+ \int_\Omega v^+ u^- + \lambda^- \int_\Omega v^- u^-, \tag{2.3}$$

$$\int_\Omega |\nabla v^+|^2 = \mu^+ \int_\Omega u^+ v^+ - \mu^- \int_\Omega u^- v^+, \tag{2.4}$$

$$\int_\Omega |\nabla v^-|^2 = -\mu^+ \int_\Omega u^+ v^- + \mu^- \int_\Omega u^- v^-. \tag{2.5}$$

For the Neumann problem ($\phi_1 = \text{const.}$), by testing both equations with ϕ_1 one gets

$$\lambda^+ \int_\Omega v^+ = \lambda^- \int_\Omega v^- \quad \text{and} \quad \mu^+ \int_\Omega u^+ = \mu^- \int_\Omega u^-. \tag{2.6}$$

For the Dirichlet problem, instead, one cannot uncouple the equations and then obtains, by testing both with ϕ_1 and first summing them and then subtracting:

$$\begin{aligned} (\lambda^+ - \lambda_1) \int_\Omega v^+ \phi_1 + (\mu^+ - \lambda_1) \int_\Omega u^+ \phi_1 \\ = (\lambda^- - \lambda_1) \int_\Omega v^- \phi_1 + (\mu^- - \lambda_1) \int_\Omega u^- \phi_1, \end{aligned} \tag{2.7}$$

$$\begin{aligned} (\lambda^+ + \lambda_1) \int_\Omega v^+ \phi_1 - (\mu^+ + \lambda_1) \int_\Omega u^+ \phi_1 \\ = (\lambda^- + \lambda_1) \int_\Omega v^- \phi_1 - (\mu^- + \lambda_1) \int_\Omega u^- \phi_1. \end{aligned} \tag{2.8}$$

2.2. Symmetries of the Fučík spectrum. The following properties may be easily verified:

Lemma 2.1. *If* $(u, v, \lambda^+, \lambda^-, \mu^+, \mu^-)$ *satisfy* (1.2) *then*

(a) $(u, \delta v, \lambda^+/\delta, \lambda^-/\delta, \delta\mu^+, \delta\mu^-)$ *satisfy* (1.2) *for any* $\delta > 0$,
(b) $(u, -v, -\lambda^-, -\lambda^+, -\mu^+, -\mu^-)$ *satisfy* (1.2),
(c) $(v, u, \mu^+, \mu^-, \lambda^+, \lambda^-)$ *satisfy* (1.2),
(d) $(-u, -v, \lambda^-, \lambda^+, \mu^-, \mu^+)$ *satisfy* (1.2).

2.3. Solutions which change sign or not. Now we deduce some properties of the nontrivial solutions of (1.2) corresponding to a point in Σ; these properties will help us to understand better the structure of Σ.

We first consider the Dirichlet case:

Proposition 2.2. *With Dirichlet boundary conditions, let* (u, v) *be a solution of* (1.2) *with coefficients* $\lambda^\pm, \mu^\pm$, *then:*

(a) *Both* u *and* v *change sign or none of the two.*
(b) *If both* u *and* v *change sign then all the coefficients have the same sign* (*and no one is zero*)*; in fact, when they are positive* $\sqrt{\lambda^+\mu^+} > \lambda_1$ *and* $\sqrt{\lambda^-\mu^-} > \lambda_1$ *and when they are negative* $\sqrt{\lambda^+\mu^-} > \lambda_1$, $\sqrt{\lambda^-\mu^+} > \lambda_1$.
(c) *If* u *and* v *do not change sign then they are both non zero multiples of* ϕ_1 *and two of the coefficients are respectively* $\delta\lambda_1$ *and* λ_1/δ *for some real*

$\delta \neq 0$, while the others may be any real. In particular if we normalize by imposing the two coefficients to be equal we have the cases

$$\begin{aligned}
u &= v = \phi_1 && \textit{and} \quad \lambda^+ = \mu^+ = \lambda_1, \\
u &= v = -\phi_1 && \textit{and} \quad \lambda^- = \mu^- = \lambda_1, \\
u &= -v = \phi_1 && \textit{and} \quad \lambda^- = \mu^+ = -\lambda_1, \\
u &= -v = -\phi_1 && \textit{and} \quad \lambda^+ = \mu^- = -\lambda_1.
\end{aligned}$$

Proof. (a) Let (without loss of generality) $u \geq 0$ and $\mu^+ \geq 0$, then equation (2.5) gives $\int_\Omega |\nabla v^-|^2 \leq 0$.

(b) To prove that no coefficient may be zero, suppose without loss of generality $\lambda^+ = 0$ and $\lambda^- \geq 0$: then (2.2) gives $\int_\Omega |\nabla u^+|^2 \leq 0$, contradiction.

Let now, without loss of generality, $\lambda^+ > 0$. This implies $\lambda^- > 0$, otherwise (2.3) would give $\int_\Omega |\nabla u^-|^2 \leq 0$, contradiction. Then by equation (2.1) at least one of the other coefficients must be positive and reasoning as above the last one is too.

Finally, let all coefficients be positive (the argument for the case in which they are all negative is analogous) and deduce from equations (2.2)–(2.5), Poincaré and Hölder inequalities that

$$\begin{aligned}
\lambda_1 \|u^+\|_{L^2}^2 &< \|\nabla u^+\|_{L^2}^2 \leq \lambda^+ \|u^+\|_{L^2} \|v^+\|_{L^2}, \\
\lambda_1 \|v^+\|_{L^2}^2 &< \|\nabla v^+\|_{L^2}^2 \leq \mu^+ \|u^+\|_{L^2} \|v^+\|_{L^2}, \\
\lambda_1 \|u^-\|_{L^2}^2 &< \|\nabla u^-\|_{L^2}^2 \leq \lambda^- \|u^-\|_{L^2} \|v^-\|_{L^2}, \\
\lambda_1 \|v^-\|_{L^2}^2 &< \|\nabla v^-\|_{L^2}^2 \leq \mu^- \|u^-\|_{L^2} \|v^-\|_{L^2},
\end{aligned}$$

(the inequalities on the left are strict since equality holds only for multiples of ϕ_1, which is not the case); by multiplying the first two and the second two and taking the square root, we deduce

$$\begin{aligned}
\lambda_1 \|u^+\|_{L^2} \|v^+\|_{L^2} &< \sqrt{\lambda^+ \mu^+} \|u^+\|_{L^2} \|v^+\|_{L^2}, \\
\lambda_1 \|u^-\|_{L^2} \|v^-\|_{L^2} &< \sqrt{\lambda^- \mu^-} \|u^-\|_{L^2} \|v^-\|_{L^2},
\end{aligned}$$

which imply the result since $\|u^\pm\|_{L^2} \|v^\pm\|_{L^2} > 0$.

(c) Consider the case $u, v \geq 0$: then, from (2.2) and (2.4), $\lambda^+, \mu^+ > 0$ while λ^- and μ^- may be any real. Now, if (λ^+, μ^+) were not of the form $(\delta\lambda_1, \lambda_1/\delta)$ for some real $\delta > 0$, then, by using the symmetry (a) in Lemma 2.1, we could obtain the existence of a point $(\widetilde{\lambda}^+, \widetilde{\lambda}^-, \widetilde{\mu}^+, \widetilde{\mu}^-)$ in Σ with $(\widetilde{\lambda}^+ - \lambda_1)(\widetilde{\mu}^+ - \lambda_1) > 0$ and such that the corresponding nontrivial solutions are positive multiples of u and v; then equation (2.7) would give a contradiction. Finally, with such coefficients u and v result to be multiples of ϕ_1.

The other cases may be proven by a similar argument. $\square$

For the Neumann problem we obtain the corresponding result:

Proposition 2.3. *With Neumann boundary conditions, let* (u, v) *be a solution of* (1.2) *with coefficients* $\lambda^{\pm}$, $\mu^{\pm}$:

(a) *Both* u *and* v *change sign or none of the two.*
(b) *If both* u *and* v *change sign then all the coefficients have the same sign* (*and no one is zero*).
(c) *If* u *and* v *do not change sign then they are both multiples of* ϕ_1 (*one of the two may be zero*) *and one of the coefficients is* $\lambda_1 = 0$ *while the others may be any real. If both* u *and* v *are nonzero, then two of the coefficients must be* $\lambda_1 = 0$. *In particular if we normalize the eigenfunctions we have the cases*

$$
\begin{aligned}
&u = v = \phi_1 && \text{and} \quad \lambda^+ = \mu^+ = 0,\\
&u = v = -\phi_1 && \text{and} \quad \lambda^- = \mu^- = 0,\\
&u = -v = \phi_1 && \text{and} \quad \lambda^- = \mu^+ = 0,\\
&u = -v = -\phi_1 && \text{and} \quad \lambda^+ = \mu^- = 0,\\
&u = \phi_1 \ (\textit{resp. } u = -\phi_1),\ v = 0 && \text{and} \quad \mu^+ = 0 \ (\textit{resp. } \mu^- = 0),\\
&u = 0, v = \phi_1 \ (\textit{resp. } v = -\phi_1) && \text{and} \quad \lambda^+ = 0 \ (\textit{resp. } \lambda^- = 0).
\end{aligned}
$$

Proof. (a) Let $u \geq 0$, then by (2.6) $\mu^+ = 0$ and then v is a constant, that is it does not change sign.

(b) If at least one coefficient is not zero then the result follows from equations (2.1) and (2.6). If one is zero then in the same way one would obtain that they must be all zero, but in this case the solutions would be constants and so would not change sign.

(c) By the same argument as in the previous point. □

3. A more concise definition of the Fučík spectrum

From Propositions 2.2 and 2.3 we have already a complete description of the Fučík spectrum when the nontrivial solutions do not change sign. Thus we may concentrate on the case in which the solutions change sign. Moreover, the points in Σ with (all) negative coefficients may be obtained from points with positive coefficients by the symmetry (b) in Lemma 2.1.

But we may do more: by the symmetries of the spectrum (see Lemma 2.1), we see that four parameters are redundant to describe this non trivial part of the spectrum. More precisely, if we restrict to the case in which the solutions change sign and the coefficients are positive, we may always make a change of the unknown functions (that is, to exploit the symmetry (a) in Lemma 2.1), in order to obtain one single point that represents the whole curve generated by this symmetry for $\delta \in \mathbb{R}^+$. In particular, we may choose δ such that $\delta\mu^+ = \lambda^+/\delta$.

Thus, we may consider the problem

$$
\begin{cases} -\Delta u = \lambda^+ v^+ - \lambda^- v^- & \text{in } \Omega, \\ -\Delta v = \lambda^+ u^+ - \mu^- u^- & \text{in } \Omega, \\ Bu = Bv = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}
$$

In view of this we define what we will call Fučík spectrum from now on:

$$
\widehat{\Sigma} = \widehat{\Sigma}_t \cup \widehat{\Sigma}_{nt}; \tag{3.2}
$$

here $\widehat{\Sigma}_t$ denotes the trivial part

$$
\begin{aligned} \widehat{\Sigma}_t = \{(\lambda^+, \lambda^-, \mu^-) \in \mathbb{R}^3 : \lambda^\pm, \mu^- \geq 0 \\ \text{and (3.1) has nontrivial solutions which do not change sign}\}, \end{aligned} \tag{3.3}
$$

and $\widehat{\Sigma}_{nt}$ the non trivial part

$$
\begin{aligned} \widehat{\Sigma}_{nt} = \{(\lambda^+, \lambda^-, \mu^-) \in \mathbb{R}^3 : \lambda^\pm, \mu^- > 0 \\ \text{and (3.1) has nontrivial solutions which (both) change sign}\}. \end{aligned} \tag{3.4}
$$

With this new definition we see that $\widehat{\Sigma}_{nt}$ still has the following symmetries:

Proposition 3.1. *If* $(u, v, \lambda^+, \lambda^-, \lambda^+, \mu^-)$ *satisfy* (3.1) *then*

(a) $(v, u, \lambda^+, \mu^-, \lambda^+, \lambda^-)$ *satisfy* (3.1),
(b) $(-u, -\sqrt{\lambda^-/\mu^-}v, \sqrt{\lambda^-\mu^-}, \lambda^+\sqrt{\mu^-/\lambda^-}, \sqrt{\lambda^-\mu^-}, \lambda^+\sqrt{\lambda^-/\mu^-})$ *satisfy* (3.1).

That is

$$
\begin{gathered} (\lambda^+, \lambda^-, \mu^-) \in \widehat{\Sigma} \Rightarrow (\lambda^+, \mu^-, \lambda^-) \in \widehat{\Sigma}, \\ (\lambda^+, \lambda^-, \mu^-) \in \widehat{\Sigma}_{nt} \Rightarrow \left(\sqrt{\lambda^-\mu^-}, \lambda^+\sqrt{\frac{\mu^-}{\lambda^-}}, \lambda^+\sqrt{\frac{\lambda^-}{\mu^-}}\right) \in \widehat{\Sigma}_{nt}. \end{gathered}
$$

Moreover, the set $\widehat{\Sigma}_t$ may be explicitly calculated from Propositions 2.2 and 2.3:

- for the Dirichlet case
$$
\widehat{\Sigma}_t = \{\lambda^+ = \lambda_1\} \cup \{\lambda^-, \mu^- > 0,\ \lambda^-\mu^- = \lambda_1^2\};
$$
where the plane $\{\lambda^+ = \lambda_1\}$ corresponds to the family of solutions $u = v = k\phi_1$, $k > 0$, while the surface $\{\lambda^-, \mu^- > 0,\ \lambda^-\mu^- = \lambda_1^2\}$ corresponds to the family $u = \sqrt{\lambda^-/\mu^-}v = -h\phi_1$, $h > 0$;
- for the Neumann problem
$$
\widehat{\Sigma}_t = \{\lambda^+ = 0\} \cup \{\lambda^- = 0\} \cup \{\mu^- = 0\};
$$

where the tree planes $\{\lambda^+ = 0\}$, $\{\lambda^- = 0\}$ and $\{\mu^- = 0\}$ correspond respectively to the solutions ($u = k\phi_1,\ v = h\phi_1 : h, k \geq 0$), ($v = -\phi_1$, $u = 0$) and ($u = -\phi_1$, $v = 0$).

Finally, we can prove that $\widehat{\Sigma}_{nt}$ lies completely in one of the regions bounded by $\widehat{\Sigma}_t$: in fact

PROPOSITION 3.2. *$(\lambda^+, \lambda^-, \mu^-) \in \widehat{\Sigma}_{nt}$ implies $\lambda^+ > \lambda_1$ and $\sqrt{\lambda^-\mu^-} > \lambda_1$.*

PROOF. For the Neumann case ($\phi_1 = 0$) this is trivial by the definition of $\widehat{\Sigma}_{nt}$. For the Dirichlet case this follows straightforward from point (b) in Proposition 2.2. $\square$

Up to this point we have given a complete description of $\widehat{\Sigma}_t$ and we exhibited regions where we may guarantee the absence of points in $\widehat{\Sigma}$; now we will present some points in $\widehat{\Sigma}_{nt}$, and we will give some properties of the corresponding nontrivial solutions of problem (3.1).

A first set of points in $\widehat{\Sigma}_{nt}$ may be obtained from the Fučík spectrum for the scalar problem Σ_{scal}: in fact

LEMMA 3.3. *If $(\lambda^+, \lambda^-) \in \Sigma_{\mathrm{scal}}$ with $\lambda^\pm > \lambda_1$, then $(\lambda^+, \lambda^-, \lambda^-) \in \widehat{\Sigma}_{nt}$.*

PROOF. Let u be the non trivial solution for the scalar problem corresponding to (λ^+, λ^-); then the pair (u, u) satisfies problem (3.1) with coefficients $(\lambda^+, \lambda^-, \lambda^-)$; moreover, since $\lambda^\pm > \lambda_1$, it is known that u changes sign, and then $(\lambda^+, \lambda^-, \lambda^-) \in \widehat{\Sigma}_{nt}$. $\square$

A general property of the nontrivial solutions corresponding to points in $\widehat{\Sigma}_{nt}$ is given in the following proposition:

PROPOSITION 3.4. *Let $(\lambda^+, \lambda^-, \mu^-) \in \widehat{\Sigma}_{nt}$ and (u, v) be a corresponding nontrivial solution: then $u^+v^+ \not\equiv 0$ and $u^-v^- \not\equiv 0$.*

PROOF. In the equations (2.2)–(2.5), we have that the left hand sides must be all strictly positive since u and v both change sign; then $\int_\Omega u^+v^+ > 0$ and $\int_\Omega u^-v^- > 0$. $\square$

REMARK 3.5. The above proposition depends on our choice to consider $\lambda^+ > 0$: obviously, by the symmetry (b) in Lemma 2.1, one sees that for $\lambda^+ < 0$ one finds $u^+v^- \not\equiv 0$ and $u^-v^+ \not\equiv 0$.

4. Extension of some classical results

In the introduction we have briefly recalled some properties of the Fučík spectrum for the scalar case. In this section we show how we can recover some of these results in the case of the system.

4.1. The linear spectrum of the system. Consider first the eigenvalue problem

$$
\begin{cases} -\Delta u = \lambda v & \text{in } \Omega, \\ -\Delta v = \lambda u & \text{in } \Omega, \\ Bu = Bv = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}
$$

Let $E = H \times H$. If we denote by $0 \leq \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_k \leq \ldots$ the eigenvalues of $-\Delta$ in H and with $(\phi_k,\, k = 1, 2, \ldots)$ the corresponding eigenfunctions, taken orthogonal and normalized with $\|\phi_k\|_{L^2} = 1$ and $\phi_1 > 0$, then it is known that the eigenvalues of problem (4.1) are:

- λ_k, $k = 1, 2, \ldots$ (with corresponding eigenfunctions the pairs (ϕ_k, ϕ_k)),
- $-\lambda_k$, for $k = 1, 2, \ldots$ (with corresponding eigenfunctions the pairs $(\phi_k, -\phi_k)$).

In view of the above structure, we define

$$
\begin{aligned}
E^+ &= \{(u,v) \in E : u = v\}, \qquad E^- = \{(u,v) \in E : u = -v\}, \\
E_n^+ &= \{(u,v) \in E : u = v \in \operatorname{span}\{\phi_1, \ldots, \phi_n\}\}, \\
E_n^- &= \{(u,v) \in E : u = -v \in \operatorname{span}\{\phi_1, \ldots, \phi_n\}\},
\end{aligned}
$$

and finally one notes that

$$
E = E^+ \oplus E^-, \qquad E_n = E_n^+ \oplus E_n^-.
$$

4.2. Regions void of spectral points. Consider now the problem

$$
\begin{cases} -\Delta u = \lambda v + f & \text{in } \Omega, \\ -\Delta v = \lambda u + g & \text{in } \Omega, \\ Bu = Bv = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}
$$

with $f, g \in L^2(\Omega)$; summing and subtracting the equations one gets, with straightforward computations,

$$
\begin{cases} (-\Delta - \lambda)(u + v) = f + g & \text{in } \Omega, \\ (-\Delta + \lambda)(u - v) = f - g & \text{in } \Omega, \\ Bu = Bv = 0 & \text{on } \partial\Omega. \end{cases}
$$

Then, if $\pm\lambda \notin \sigma(-\Delta)$, one may invert the operators. Let

$$
u = \sum_{i=1}^{\infty} u_i \phi_i, \quad v = \sum_{i=1}^{\infty} v_i \phi_i, \quad f = \sum_{i=1}^{\infty} f_i \phi_i \quad \text{and} \quad g = \sum_{i=1}^{\infty} g_i \phi_i,
$$

then

$$
\begin{aligned}
u_i &= \frac{1}{\lambda_i - \lambda} \frac{f_i + g_i}{2} + \frac{1}{\lambda_i + \lambda} \frac{f_i - g_i}{2} = \frac{\lambda_i f_i + \lambda g_i}{\lambda_i^2 - \lambda^2}, \\
v_i &= \frac{1}{\lambda_i - \lambda} \frac{f_i + g_i}{2} - \frac{1}{\lambda_i + \lambda} \frac{f_i - g_i}{2} = \frac{\lambda f_i + \lambda_i g_i}{\lambda_i^2 - \lambda^2}.
\end{aligned}
$$

Thus, we may consider the operator (well defined for $\pm\lambda \notin \sigma(-\Delta)$)

$$T_\lambda\colon [L^2(\Omega)]^2 \to [L^2(\Omega)]^2, \qquad (f,g) \mapsto (u,v)$$

where (u,v) is the (unique) solution of problem (4.2); simple computations give

$$\|T_\lambda\| = \frac{1}{d_\lambda},$$

where $d_\lambda = \operatorname{dist}(\{\pm\lambda\}, \sigma(-\Delta))$.

Let now $p = (\lambda^+, \lambda^-, \mu^+, \mu^-) \in \mathbb{R}^4$, and define

$$\begin{gathered} M_\lambda^p\colon [L^2(\Omega)]^2 \to [L^2(\Omega)]^2, \\ (u,v) \mapsto ((\lambda^+ - \lambda)v^+ - (\lambda^- - \lambda)v^-, (\mu^+ - \lambda)u^+ - (\mu^- - \lambda)u^-). \end{gathered} \tag{4.3}$$

It is clear that M_λ^p is Lipschitz with constant m_λ^p

$$m_\lambda^p = \max\{|\lambda^+ - \lambda|, |\lambda^- - \lambda|, |\mu^+ - \lambda|, |\mu^- - \lambda|\}.$$

Making use of these operators we may prove the following result:

Proposition 4.1. *If $\lambda^\pm, \mu^- > 0$ are such that $\lambda_k < \lambda^+/\delta, \lambda^-/\delta, \delta\lambda^+, \delta\mu^- < \lambda_{k+1}$ for some $\delta > 0$, $k \geq 1$, then $(\lambda^+, \lambda^-, \mu^-) \notin \widehat{\Sigma}$.*

The analogue of this result for the scalar problem asserts that if both coefficients are between two consecutive eigenvalues, then (λ^+, λ^-) is not in the Fučík spectrum (see for example [4]).

We remark that by Proposition 3.2 we already knew a region without points in $\widehat{\Sigma}$; proposition 4.1 provides more of such regions.

Proof. Let $p = (\lambda^+, \lambda^-, \mu^+, \mu^-) \in \mathbb{R}^4$: problem (1.2) may be written as

$$(u,v) = T_\lambda M_\lambda^p(u,v); \tag{4.4}$$

so we have, for $\mathbf{u} = (u,v)$, $\mathbf{y} = (y,z)$

$$\|T_\lambda M_\lambda^p \mathbf{u} - T_\lambda M_\lambda^p \mathbf{y}\|_{[L^2]^2} \leq \frac{m_\lambda^p}{\operatorname{dist}(\pm\lambda, \sigma(-\Delta))}\|\mathbf{u} - \mathbf{y}\|_{[L^2]^2}.$$

Thus, if we consider two consecutive eigenvalues $\lambda_k < \lambda_{k+1}$ and we suppose $\lambda_k < \lambda^\pm, \mu^\pm < \lambda_{k+1}$, we may set

$$\lambda = \frac{\max\{\lambda^+, \lambda^-, \mu^+, \mu^-\} + \min\{\lambda^+, \lambda^-, \mu^+, \mu^-\}}{2}$$

and we obtain that $T_\lambda M_\lambda^p$ is a contraction and then problem (4.4) (and so problem (1.2)) admits a unique solution: the trivial one.

Finally, considering the symmetries in Lemma 2.1, we obtain the claim. □

4.3. Some implications for the degree. Another useful known property of the Fučík spectrum which may be recovered in our case is the following:

LEMMA 4.2. *Let* $p = (\lambda^+, \lambda^-, \lambda^+, \mu^-)$, $\widetilde{p} = (\widetilde{\lambda}^+, \widetilde{\lambda}^-, \widetilde{\lambda}^+, \widetilde{\mu}^-)$ *with all components positive, and* λ *such that* $\pm\lambda \notin \sigma(-\Delta)$; *if the line*

$$S = \{t(\widetilde{\lambda}^+, \widetilde{\lambda}^-, \widetilde{\mu}^-) + (1-t)(\lambda^+, \lambda^-, \mu^-),\ t \in [0,1]\}$$

never intersects $\widehat{\Sigma}$, *then*

$$\operatorname{Deg}(\mathrm{id} - T_\lambda M_\lambda^p, B_1, 0) = \operatorname{Deg}(\mathrm{id} - T_\lambda M_\lambda^{\widetilde{p}}, B_1, 0),$$

where Deg *is the Leray–Schauder degree, and* $B_1 = B_1(0)$ *the unit ball in* $L^2 \times L^2$. *Moreover, if* $p = (\lambda, \lambda, \lambda, \lambda)$ *then* $\operatorname{Deg}(\mathrm{id} - T_\lambda M_\lambda^p, B_1, 0) \neq 0$.

PROOF. Consider $\operatorname{Deg}(\mathrm{id} - T_\lambda(tM_\lambda^{\widetilde{p}} + (1-t)M_\lambda^p), B_1, 0)$: it is well defined since T_λ is compact, M_λ^p and $M_\lambda^{\widetilde{p}}$ are continuous and the hypothesis that S never intersects $\widehat{\Sigma}$ avoids the existence of solutions in ∂B_1; then it is constant.

Finally, for $p = (\lambda, \lambda, \lambda, \lambda)$, $M_\lambda^p = 0$ and so

$$\operatorname{Deg}(\mathrm{id} - T_\lambda M_\lambda^p, B_1, 0) = \operatorname{Deg}(\mathrm{id}, B_1, 0) \neq 0. \qquad \square$$

COROLLARY 4.3. $\operatorname{Deg}(\mathrm{id} - T_\lambda M_\lambda^p, B_1, 0)$ *is constant if the coefficients in the vector* p *are such that* $\lambda^+ = \mu^+$ *and* $(\lambda^+, \lambda^-, \mu^-)$ *is in a path connected component of* $\{\lambda^+, \lambda^-, \mu^- > 0\} \setminus \widehat{\Sigma}$.

REMARK 4.4. Since for $\lambda^+ < \lambda_1$ and $\lambda^-, \mu^- > 0$ no solution of (3.1) exists, we may assert, by the same argument, that $\operatorname{Deg}(\mathrm{id} - T_\lambda M_\lambda^p, B_1, 0)$ is the same for all such points (also for $\lambda^+ < 0$); in fact, it is zero, as will be proved below.

The above result may be applied to the following perturbed nonlinear problem:

$$\begin{cases} -\Delta u = \lambda^+ v^+ - \lambda^- v^- + g_1(x,u,v) + h_1(x) & \text{in } \Omega, \\ -\Delta v = \mu^+ u^+ - \mu^- u^- + g_2(x,u,v) + h_2(x) & \text{in } \Omega, \\ Bu = Bv = 0 & \text{in } \partial\Omega, \end{cases} \tag{4.5}$$

where we assume $h_{1,2} \in L^2(\Omega)$, $g_{1,2} \in \mathcal{C}^0(\overline{\Omega} \times \mathbb{R}^2)$ with

$$\lim_{s \to \pm\infty} \frac{g_{1,2}(x,s,t)}{s} = 0, \qquad \lim_{t \to \pm\infty} \frac{g_{1,2}(x,s,t)}{t} = 0 \tag{4.6}$$

uniformly with respect to $x \in \overline{\Omega}$ and to t (resp. s) in $\mathbb{R}$. Then we have

THEOREM 4.5. *If* $\lambda^+ = \mu^+$ *and* $(\lambda^+, \lambda^-, \mu^-)$ *belongs to a path connected component of the set* $\{\lambda^+, \lambda^-, \mu^- > 0\} \setminus \widehat{\Sigma}$ *containing points with* $\widetilde{\lambda}^+ = \widetilde{\lambda}^- = \widetilde{\mu}^-$, *then problem* (4.5) *has a solution for any* $h_1, h_2 \in L^2(\Omega)$.

PROOF. We sketch the proof, which is analogous to that in [10]. The idea is to find a zero of the map

$$S\colon [L^2(\Omega)]^2 \to [L^2(\Omega)]^2, \quad \mathbf{u} \mapsto \mathbf{u} - T_\lambda(M_\lambda^p \mathbf{u} + G(\mathbf{u}) + \mathbf{h}),$$

where T_λ and M_λ^p are defined as before with $\lambda = \widetilde{\lambda}^+$ and $p = (\lambda^+, \lambda^-, \mu^+, \mu^-)$; $G: \mathbf{u} \mapsto (g_1(x,u,v), g_2(x,u,v))$ is the Nemytskiĭ operator for the nonlinearities g_1 and g_2, and $\mathbf{h} = (h_1, h_2)$. Since $T_\lambda M_\lambda^p$ is compact and $p \notin \Sigma$, one obtains the estimate

$$\|\mathbf{u} - T_\lambda M_\lambda^p \mathbf{u}\|_{[L^2]^2} \geq C\|\mathbf{u}\|_{[L^2]^2}.$$

One then obtains an a priori estimate for the solutions of $0 = \mathbf{u} - T_\lambda(M_\lambda^p \mathbf{u} + t(G(\mathbf{u}) + \mathbf{h}))$ for $t \in [0,1]$: first estimate

$$\begin{aligned} C\|\mathbf{u}\|_{[L^2]^2} &\leq \|\mathbf{u} - T_\lambda M_\lambda^p \mathbf{u}\|_{[L^2]^2} \\ &= t\|T_\lambda(G(\mathbf{u}) + \mathbf{h})\|_{[L^2]^2} \leq t\|T_\lambda\|(\|G(\mathbf{u}) + \mathbf{h}\|_{[L^2]^2}); \end{aligned} \tag{4.7}$$

then, from hypothesis (4.6) one gets, for any $\varepsilon > 0$, $\|G(\mathbf{u})\|_{[L^2]^2} \leq \varepsilon\|\mathbf{u}\|_{[L^2]^2} + C_\varepsilon$, and so, from (4.7)

$$(C - t\varepsilon\|T_\lambda\|)\|\mathbf{u}\|_{[L^2]^2} \leq t\|T_\lambda\|(C_\varepsilon + \|\mathbf{h}\|_{[L^2]^2}).$$

This last estimate allows to use degree theory (on a sufficiently large ball B_R) and Lemma 4.2 to assert that

$$\begin{aligned} \operatorname{Deg}(\mathrm{id} - T_\lambda(M_\lambda^p + G(\,\cdot\,) + \mathbf{h}), B_R, 0) &= \operatorname{Deg}(\mathrm{id} - T_\lambda M_\lambda), B_R, 0) \\ &= \operatorname{Deg}(\mathrm{id}, B_R, 0) \neq 0; \end{aligned}$$

then there exists a solution. □

Since multiplying the unknown functions by a constant does not affect the hypotheses we have on $g_{1,2}$ and $h_{1,2}$, we may generalize the above result as follows:

Corollary 4.6. *The same result of Theorem* 4.5 *holds for* $(\lambda^+, \lambda^-, \mu^+, \mu^-)$ *with* $\lambda^+ \neq \mu^+$, *if it may be transformed into a point as in the hypotheses of Theorem* 4.5 *through one of the symmetries in Lemma* 2.1.

Observe that for the Dirichlet problem we obtained (see Section 3) that the region $\{\lambda^+, \lambda^-, \mu^- > 0, \lambda^+ < \lambda_1,\ \text{or}\ \sqrt{\lambda^-\mu^-} < \lambda_1\}$ contains no point in $\widehat{\Sigma}_{nt}$, and so it is divided by $\widehat{\Sigma}_t$ into the three subregions

$$\begin{aligned} R_1 &= \{0 < \lambda^+ < \lambda_1,\ 0 < \sqrt{\lambda^-\mu^-} < \lambda_1\}, \\ R_2 &= \{0 < \lambda^+ < \lambda_1,\ \sqrt{\lambda^-\mu^-} > \lambda_1\}, \\ R_3 &= \{\lambda^+ > \lambda_1,\ 0 < \sqrt{\lambda^-\mu^-} < \lambda_1\}. \end{aligned}$$

It follows from Lemma 4.2 that $\operatorname{Deg}(\mathrm{id} - T_\lambda M_\lambda^p, B_1, 0) \neq 0$ if the coefficients in the vector p are such that $\mu^+ = \lambda^+$ and $(\lambda^+, \lambda^-, \mu^-)$ is in R_1, since this region contains points of the form $(\lambda, \lambda, \lambda)$ with $\lambda \notin \sigma(-\Delta)$.

The following theorem is concerned with the remaining two regions:

THEOREM 4.7. *If the coefficients in the vector* p *are such that* $\mu^+ = \lambda^+$ *and* $(\lambda^+, \lambda^-, \mu^-)$ *is in* R_2 *or* R_3*, then* $\mathrm{Deg}(\mathrm{id} - T_\lambda M^p_\lambda, B_1, 0) = 0$.

PROOF. By the proof of theorem 4.5, it is sufficient to provide two examples of systems like (4.5) with coefficients in each of the two regions, for which there exists no solution.

Consider the case $\lambda^+ = \mu^+ = \lambda_1 - \varepsilon$ and $\lambda^- = \mu^- = \lambda_1 + \varepsilon$ for a suitable $\varepsilon > 0$ small enough, with $g_1 = g_2 = 0$: by multiplying the two equations by ϕ_1, integrating, then integrating by parts and summing one gets

$$-\varepsilon \int_\Omega (|v| + |u|)\phi_1 + \int_\Omega (h_1 + h_2)\phi_1 = 0,$$

which gives rise to a contradiction if $\int_\Omega (h_1 + h_2)\phi_1 < 0$.

The case $\lambda^+ = \mu^+ = \lambda_1 + \varepsilon$ and $\lambda^- = \mu^- = \lambda_1 - \varepsilon$ is analogous. $\square$

5. The Fučík spectrum in the neighborhood of a simple eigenvalue

In this section we show that in the neighborhood of each point in $\widehat{\Sigma}$ corresponding to a simple eigenvalue of the Laplacian, we can find a continuum of points in $\widehat{\Sigma}$. Indeed, we show that through each such point there pass exactly two (maybe coincident) 2-dimensional surfaces belonging to $\widehat{\Sigma}$.

The techniques we use are inspired from [18].

5.1. Statement of the result. We introduce the following notation: assume (without restriction) $\lambda^+ > \lambda^-$, and set

$$\lambda = \lambda^+, \quad \gamma = \lambda - \lambda^-, \quad \delta = \lambda - \mu^-.$$

Then the system (3.1) becomes

$$\begin{cases} -\Delta u = \lambda v + \gamma v^- & \text{in } \Omega, \\ -\Delta v = \lambda u + \delta u^- & \text{in } \Omega, \\ Bu = Bv = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

which we will consider in the vectorial form:

$$-\Delta \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} v \\ u \end{pmatrix} + \begin{pmatrix} \gamma(v)^- \\ \delta(u)^- \end{pmatrix}. \tag{5.2}$$

Assume that λ_k, $k \geq 1$, is a simple eigenvalue of the Laplacian on H, and let

$$d = \begin{cases} \min\{\lambda_{k+1} - \lambda_k, \lambda_k - \lambda_{k-1}\} & \text{if } k \geq 2, \\ \lambda_2 - \lambda_1 & \text{if } k = 1. \end{cases}$$

We assume throughout this section that

$$0 \leq |\delta| \leq \gamma < \frac{d}{6}. \tag{5.3}$$

We denote by $\langle\,\cdot\,,\,\cdot\,\rangle$ the scalar product and by $\|\cdot\|$ the norm in L^2, and by $\cdot$ the scalar product in $[L^2]^2$.

Theorem 1.1 will be proved in the following form:

THEOREM 5.1. *For every fixed pair* (γ,δ) *satisfying* (5.3) *there exists a unique* $\lambda_{k+}\in[\lambda_k-3\gamma,\lambda_k+3\gamma]$ *such that system* (5.1) *has a* (*unique*) *solution* $(u,v)\in H\times H$ *with* $(u,v)\cdot(\phi_k,\phi_k)=2$*; in the same way, there exists a unique* $\lambda_{k-}\in[\lambda_k-3\gamma,\lambda_k+3\gamma]$ *such that system* (5.1) *has a* (*unique*) *solution* $(u,v)\in H\times H$ *with* $(u,v)\cdot(\phi_k,\phi_k)=-2$*. On the other hand, there exist no nontrivial solutions* (u,v) *with* $(u,v)\cdot(\phi_k,\phi_k)=0$*, for* $\lambda\in[\lambda_k-3\gamma,\lambda_k+3\gamma]$.

Thus, we obtain

THEOREM 5.2. *Through each point* $(\lambda_k,\lambda_k,\lambda_k)\in\widehat{\Sigma}$ *corresponding to a simple eigenvalue of the linear system there pass exactly two* (*maybe coincident*) *"Fučík surfaces", parametrized by* $\lambda=\lambda_{k+}(\gamma,\delta)$ *and* $\lambda=\lambda_{k-}(\gamma,\delta)$*, for which system* (5.1) (*and hence system* (3.1)) *admits nontrival solutions.*

REMARK 5.3. We will always suppose $\lambda_{k-}<\lambda_{k+}$, since this may always be obtained by taking $-\phi_k$ in place of ϕ_k.

REMARK 5.4. We observe that by Lemma 3.3 we already know some parts of the surfaces obtained in Theorem 5.2: the curves coming from the scalar Fučík spectrum. This provides also examples of both, the case in which the two points $\lambda_{k+},\lambda_{k-}$ coincide, and the case in which they are distinct (for example, in the one dimensional Dirichlet problem they coincide for k even and are distinct for k odd).

5.2. Lyapunov–Schmidt reduction. We introduce the following notations: let E_1, E_2 and F be subspaces of E such that

$$E_1=E^-\oplus E^+_{k-1},\quad F=[(\phi_k,\phi_k)]^\perp=E_1\oplus E_2,\quad E_1\perp E_2,$$
$$P\colon E\to[(\phi_k,\phi_k)]^\perp,\quad P_1\colon E\to E_1,\quad P_2\colon E\to E_2\quad \textit{orthogonal projections.}$$

We want to apply a Lyapunov–Schmidt procedure. Thus, we consider for fixed $s\in\mathbb{R}$ the system

$$-\Delta\begin{pmatrix}u\\ v\end{pmatrix}=\lambda\begin{pmatrix}v\\ u\end{pmatrix}+P\begin{pmatrix}\gamma(s\phi_k+v)^-\\ \delta(s\phi_k+u)^-\end{pmatrix},\tag{5.4}$$

with $(u,v)\in F$.

Observe that, since the considered subspaces are orthogonal and invariant with respect to the operator $-\Delta$, the projections commute with this operator and one has, for every $\mathbf{u},\widetilde{\mathbf{u}}\in E$, $\mathbf{u}\cdot P_i\widetilde{\mathbf{u}}=P_i\mathbf{u}\cdot P_i\widetilde{\mathbf{u}}$ $(i=1,2)$ in the scalar product of E.

We prove several lemmas:

Lemma 5.5. *Assume that* (5.3) *holds, and that*

$$\lambda \in [\lambda_k - 3\gamma, \lambda_k + 3\gamma].$$

Assume that $(u,v) = (u_1,v_1) + (u_2,v_2) \in E_1 \oplus E_2$ *solves system* (5.4)*. Then we have*

$$\|u_2\|^2 + \|v_2\|^2 \le c_1 |s|^2 \frac{\gamma}{d} \quad \text{with } c_1 \le 4, \tag{5.5}$$

$$\|u_1\|^2 + \|v_1\|^2 \le c_2 |s|^2 \frac{\gamma}{d} \quad \text{with } c_2 \le 4. \tag{5.6}$$

In particular, we conclude that $u_2 = v_2 = u_1 = v_1 = 0$*, if* $s = 0$*, that is, no nontrivial solution of (5.4) exists in* F*.*

Proof. Multiplying the vectorial equation (5.4) by $(v_1,u_1) \in E_1$ and integrating over Ω yields

$$2\int_\Omega \nabla u_1 \nabla v_1 = \lambda(\|v_1\|^2 + \|u_1\|^2) + \gamma \int_\Omega (v_1+v_2+s\phi_k)^- v_1 + \delta \int_\Omega (u_1+u_2+s\phi_k)^- u_1$$

and thus, setting $\Omega_v^- := \{x \in \Omega | v + s\phi_k < 0\}$, and similarly for Ω_u^-

$$\begin{aligned} \lambda(\|v_1\|^2 + \|u_1\|^2) &- \lambda_{k-1}(\|v_1\|^2 + \|u_1\|^2) \\ &\le \gamma \int_{\Omega_v^-} \bigg[\frac{1}{2}|v_1|^2 + \frac{1}{2}(|v_1|^2 + |v_2|^2 + |s\phi_k|^2)\bigg] \\ &\quad + |\delta| \int_{\Omega_u^-} \bigg[\frac{1}{2}|u_1|^2 + \frac{1}{2}(|u_1|^2 + |u_2|^2 + |s\phi_k|^2)\bigg] \end{aligned}$$

and then, using $\lambda - \lambda_{k-1} \ge \lambda_k - \lambda_{k-1} - |\lambda - \lambda_k| \ge d - 3\gamma$

$$\begin{aligned} (d - 4\gamma)(\|v_1\|^2 + \|u_1\|^2) &\le (\lambda - \lambda_{k-1} - \gamma)(\|v_1\|^2 + \|u_1\|^2) \\ &\le \frac{\gamma}{2}\|v_2\|^2 + \frac{|\delta|}{2}\|u_2\|^2 + \frac{1}{2}(\gamma + |\delta|)|s|^2 \\ &\le \gamma\bigg[\frac{1}{2}(\|v_2\|^2 + \|u_2\|^2) + |s|^2\bigg]. \end{aligned} \tag{5.7}$$

Next, multiplying the vectorial equation (5.4) by $(u_2,v_2) \in E_2$ and integrating, gives

$$\begin{aligned} 0 &= \int_\Omega (|\nabla u_2|^2 + |\nabla v_2|^2) - 2\lambda \int_\Omega u_2 v_2 \\ &\quad + \gamma \int_{\Omega_v^-} (v_1 + v_2 + s\phi_k) u_2 + \delta \int_{\Omega_u^-} (u_1 + u_2 + s\phi_k) v_2 \\ &\ge (\lambda_{k+1} - \lambda)(\|u_2\|^2 + \|v_2\|^2) - \frac{\gamma}{2}\{(\|v_1\|^2 + \|v_2\|^2 + |s|^2) + \|u_2\|^2\} \\ &\quad - \frac{|\delta|}{2}\{(\|u_1\|^2 + \|u_2\|^2 + |s|^2) + \|v_2\|^2\} \end{aligned}$$

and hence, using that $(\lambda_{k+1}-\lambda-\gamma/2-|\delta|/2) \geq [d-|\lambda-\lambda_k|-\gamma/2-|\delta|/2] \geq d-4\gamma$, and relation (5.7)

$$\begin{aligned}(d-4\gamma)(\|u_2\|^2+\|v_2\|^2) &\leq \frac{\gamma}{2}\|v_1\|^2+\frac{|\delta|}{2}\|u_1\|^2+\frac{1}{2}(\gamma+|\delta|)|s|^2 \\ &\leq \frac{\gamma}{2}(\|v_1\|^2+\|u_1\|^2)+\gamma|s|^2 \\ &\leq \frac{\gamma}{2}\frac{\gamma}{d-4\gamma}\left[\frac{1}{2}(\|v_2\|^2+\|u_2\|^2)+|s|^2\right]+\gamma|s|^2.\end{aligned}$$

Thus

$$\left[d-4\gamma-\frac{\gamma^2}{4(d-4\gamma)}\right](\|v_2\|^2+\|u_2\|^2) \leq \gamma\left(1+\frac{\gamma}{2(d-4\gamma)}\right)|s|^2,$$

which yields

$$\|v_2\|^2+\|u_2\|^2 \leq \frac{\gamma}{d}\frac{1+\frac{\gamma}{2(d-4\gamma)}}{1-\frac{4\gamma}{d}\left(1+\frac{\gamma}{16(d-4\gamma)}\right)}|s|^2 =: \frac{\gamma}{d}c_1|s|^2.$$

This gives (5.5). An easy calculation shows that $c_1 \leq 4$. Relation (5.6) now follows by combining (5.7) with (5.5):

$$\|v_1\|^2+\|u_1\|^2 \leq \frac{\gamma}{d}\frac{d}{d-4\gamma}\left[\frac{\gamma}{2d}c_1+1\right]|s|^2 =: \frac{\gamma}{d}c_2|s|^2.$$

One checks that also $c_2 \leq 4$. □

Proposition 5.6. *Let γ, δ be as in Lemma* 5.5. *Then, once fixed $s=1$, for every λ as in Lemma* 5.5 *there exists a unique $(u,v) \in F$ which solves system* (5.4). *Furthermore, (u,v) depends continuously on λ.*

Proof. Write (5.4) as

$$A\begin{pmatrix}u\\v\end{pmatrix} := \begin{pmatrix}-\Delta & -\lambda\\ -\lambda & -\Delta\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix} = P\begin{pmatrix}\gamma(s\phi_k+v)^-\\ \delta(s\phi_k+u)^-\end{pmatrix} =: Q\begin{pmatrix}u\\v\end{pmatrix}.$$

One notes that $A\colon F \to F$ is invertible, and so we can write equivalently

$$\begin{pmatrix}u\\v\end{pmatrix} = A^{-1}Q\begin{pmatrix}u\\v\end{pmatrix}. \tag{5.8}$$

One shows as in [18], using Lemma 5.5, that there exists $K>0$ such that

$$\begin{pmatrix}u\\v\end{pmatrix} \neq \tau A^{-1}Q\begin{pmatrix}u\\v\end{pmatrix}, \quad \text{for all } \begin{pmatrix}u\\v\end{pmatrix} \text{ with } \left\|\begin{pmatrix}u\\v\end{pmatrix}\right\| \geq K \text{ and for all } \tau \in [0,1].$$

Thus, by the Leray–Schauder principle there exists a solution $(u,v) \in F \cap B_K(0)$ of (5.8), and thus of system (5.4).

The uniqueness is a special case of Lemma 5.7 below. Finally, the continuous dependence on the parameters λ is easily seen, see also [18]. □

Lemma 5.7. *Let γ, δ be as in Lemma* 5.5, *and suppose that there exist two values λ and $\widetilde{\lambda}$ in $[\lambda_k - 3\gamma, \lambda_k + 3\gamma]$ such that system* (5.4) *with $s = 1$ has nontrivial solutions (u, v) and $(\widetilde{u}, \widetilde{v})$ in F. Then, for $0 \le |\delta| \le \gamma < d/6$, we have*

$$\|u - \widetilde{u}\|^2 + \|v - \widetilde{v}\|^2 \le 4c_0^2(|\widetilde{\lambda} - \lambda|^2)$$

where $c_0 = (c_1\gamma/d)^{1/2}/(d - 9\gamma/2)$, c_1 as in (5.5).

Proof. As before, we write $(u, v) = (u_1, v_1) + (\phi_k, \phi_k) + (u_2, v_2)$, and similarly for $(\widetilde{u}, \widetilde{v})$. Subtracting the two vectorial systems (5.2), multiplying the resulting equation by $(v_1 - \widetilde{v}_1, u_1 - \widetilde{u}_1) \in E_1$ and integrating gives

$$\begin{aligned} 0 = 2\int_\Omega &\nabla(u_1 - \widetilde{u}_1)\nabla(v_1 - \widetilde{v}_1) - \langle\lambda v_1 - \widetilde{\lambda}\widetilde{v}_1, v_1 - \widetilde{v}_1\rangle - \langle\lambda u_1 - \widetilde{\lambda}\widetilde{u}_1, u_1 - \widetilde{u}_1\rangle \\ &- \gamma\langle(v_1 + \phi_k + v_2)^- - (\widetilde{v}_1 + \phi_k + \widetilde{v}_2)^-, v_1 - \widetilde{v}_1\rangle \\ &- \delta\langle(u_1 + \phi_k + u_2)^- - (\widetilde{u}_1 + \phi_k + \widetilde{u}_2)^-, u_1 - \widetilde{u}_1\rangle. \end{aligned}$$

We estimate (using $\langle\lambda v_1 - \widetilde{\lambda}\widetilde{v}_1, v_1 - \widetilde{v}_1\rangle = \lambda\|v_1 - \widetilde{v}_1\|^2 + (\lambda - \widetilde{\lambda})\langle\widetilde{v}_1, v_1 - \widetilde{v}_1\rangle$, the analogue for $u_1, \widetilde{u}_1$ and $|a^- - b^-| \le |a - b|$)

$$\begin{aligned} (\lambda - \lambda_{k-1})(\|v_1 - \widetilde{v}_1\|^2 + \|u_1 - \widetilde{u}_1\|^2) \\ \le |\widetilde{\lambda} - \lambda|\ &(|\langle\widetilde{v}_1, v_1 - \widetilde{v}_1\rangle| + |\langle\widetilde{u}_1, u_1 - \widetilde{u}_1\rangle|) \\ &+ \gamma\|v_1 + v_2 - (\widetilde{v}_1 + \widetilde{v}_2)\|\|v_1 - \widetilde{v}_1\| \\ &+ |\delta|\,\|u_1 + u_2 - (\widetilde{u}_1 + \widetilde{u}_2)\|\|u_1 - \widetilde{u}_1\| \\ \le |\widetilde{\lambda} - \lambda|\ &(\|\widetilde{v}_1\|\|v_1 - \widetilde{v}_1\| + \|\widetilde{u}_1\|\|u_1 - \widetilde{u}_1\|) \\ &+ \gamma\|v_1 - \widetilde{v}_1\|^2 + \frac{\gamma}{2}\|v_2 - \widetilde{v}_2\|^2 + |\delta|\,\|u_1 - \widetilde{u}_1\|^2 + \frac{|\delta|}{2}\|u_2 - \widetilde{u}_2\|^2 \end{aligned}$$

and hence, arguing as in Lemma 5.5, and using (5.6)

$$\begin{aligned} &(d - 3\gamma - \gamma)(\|v_1 - \widetilde{v}_1\|^2 + \|u_1 - \widetilde{u}_1\|^2) \\ &\quad\le |\widetilde{\lambda} - \lambda|\sqrt{c_2\,\frac{\gamma}{d}}(\|v_1 - \widetilde{v}_1\| + \|u_1 - \widetilde{u}_1\|) + \frac{\gamma}{2}(\|v_2 - \widetilde{v}_2\|^2 + \|u_2 - \widetilde{u}_2\|^2). \end{aligned} \tag{5.9}$$

Next, subtract again the two vectorial systems, multiply the resulting equation by $(u_2 - \widetilde{u}_2, v_2 - \widetilde{v}_2) \in E_2$, integrate and estimate as before

$$\begin{aligned} \int_\Omega |\nabla(u_2 - \widetilde{u}_2)|^2 + \int_\Omega |\nabla(v_2 - \widetilde{v}_2)|^2 - \lambda\langle v_2 - \widetilde{v}_2, u_2 - \widetilde{u}_2\rangle - \lambda\langle u_2 - \widetilde{u}_2, v_2 - \widetilde{v}_2\rangle \\ \le |\lambda - \widetilde{\lambda}|\ &(|\langle\widetilde{v}_2, u_2 - \widetilde{u}_2\rangle| + |\langle\widetilde{u}_2, v_2 - \widetilde{v}_2\rangle|) \\ &+ \gamma\|v_1 + v_2 - (\widetilde{v}_1 + \widetilde{v}_2)\|\|u_2 - \widetilde{u}_2\| \\ &+ |\delta|\,\|u_1 + u_2 - (\widetilde{u}_1 + \widetilde{u}_2)\|\|v_2 - \widetilde{v}_2\| \end{aligned}$$

and hence, using Lemma 5.5 and estimating $\|v_1+v_2-(\widetilde v_1+\widetilde v_2)\|\|u_2-\widetilde u_2\| \le (\|v_1-\widetilde v_1\|^2+\|v_2-\widetilde v_2\|^2+\|u_2-\widetilde u_2\|^2)/2$,

$$\begin{aligned}(\lambda_{k+1}-\lambda)(\|u_2-\widetilde u_2\|^2+\|v_2-\widetilde v_2\|^2) &\le |\lambda-\widetilde\lambda|\sqrt{c_1\frac{\gamma}{d}}(\|u_2-\widetilde u_2\|+\|v_2-\widetilde v_2\|)\\ &+\frac{\gamma+|\delta|}{2}(\|v_2-\widetilde v_2\|^2+\|u_2-\widetilde u_2\|^2)+\frac{\gamma}{2}\|v_1-\widetilde v_1\|^2+\frac{|\delta|}{2}\|u_1-\widetilde u_1\|^2\end{aligned}$$

which gives, arguing as in Lemma 5.5,

$$\begin{aligned}&(d-3\gamma-\gamma)(\|u_2-\widetilde u_2\|^2+\|v_2-\widetilde v_2\|^2)\\ &\le|\lambda-\widetilde\lambda|\sqrt{c_1\,\frac{\gamma}{d}}(\|u_2-\widetilde u_2\|+\|v_2-\widetilde v_2\|)+\frac{\gamma}{2}(\|v_1-\widetilde v_1\|^2+\|u_1-\widetilde u_1\|^2).\end{aligned}\tag{5.10}$$

Adding (5.9) and (5.10) gives

$$\begin{aligned}&\left(d-4\gamma-\frac{\gamma}{2}\right)(\|v-\widetilde v\|^2+\|u-\widetilde u\|^2)\\ &\le|\widetilde\lambda-\lambda|\sqrt{c_2\frac{\gamma}{d}}(\|v_1-\widetilde v_1\|+\|u_1-\widetilde u_1\|)+|\lambda-\widetilde\lambda|\sqrt{c_1\frac{\gamma}{d}}(\|u_2-\widetilde u_2\|+\|v_2-\widetilde v_2\|).\end{aligned}$$

Using $c_1=c_2$, and setting $c_0=(c_1\gamma/d)^{1/2}/(d-9\gamma/2)$, we get

$$\begin{aligned}\|v-\widetilde v\|^2&+\|u-\widetilde u\|^2\\ &\le c_0|\widetilde\lambda-\lambda|\,(\|\widetilde v_1-v_1\|+\|\widetilde u_1-u_1\|+\|\widetilde u_2-u_2\|+\|\widetilde v_2-v_2\|)\\ &\le\frac12|\widetilde\lambda-\lambda|^2c_0^2+\frac12\|\widetilde v_1-v_1\|^2+\frac12|\widetilde\lambda-\lambda|^2c_0^2+\frac12\|\widetilde u_1-u_1\|^2\\ &\quad+\frac12|\widetilde\lambda-\lambda|^2c_0^2+\frac12\|\widetilde u_2-u_2\|^2+\frac12|\widetilde\lambda-\lambda|^2c_0^2+\frac12\|\widetilde v_2-v_2\|^2\end{aligned}$$

and then

$$\|\widetilde u-u\|^2+\|\widetilde v-v\|^2\le 4c_0^2|\widetilde\lambda-\lambda|^2. \qquad\square$$

5.3. The reduced equation. Multiplying the vectorial equation (5.2) by (ϕ_k,ϕ_k) and integrating over Ω yields (with $s=1$)

$$\begin{aligned}\lambda_k[1+t(\lambda)]-\lambda[1-t(\lambda)]-\gamma\int_\Omega[(1-t(\lambda))\phi_k+v(\lambda)]^-\phi_k&\\ +\lambda_k[1-t(\lambda)]-\lambda[1+t(\lambda)]-\delta\int_\Omega[(1+t(\lambda))\phi_k+u(\lambda)]^-\phi_k&=0.\end{aligned}$$

Here $t(\lambda)\phi_k+u(\lambda)$ and $-t(\lambda)\phi_k+v(\lambda)$ denote the unique solutions in $F=[(\phi_k,\phi_k)]^\perp$ given by Proposition 5.6, where for convenience we wrote separately the component $t(\phi_k,-\phi_k)$.

So, consider the map $\Gamma_+\colon[\lambda_k-3\gamma,\lambda_k+3\gamma]\subset\mathbb R\to\mathbb R$ defined by

$$\begin{aligned}\lambda\mapsto\Gamma_+(\lambda)=2(\lambda_k-\lambda)-\gamma\int_\Omega[(1-t(\lambda))\phi_k+v(\lambda)]^-\phi_k&\\ -\delta\int_\Omega[(1+t(\lambda))\phi_k+u(\lambda)]^-\phi_k&.\end{aligned}\tag{5.11}$$

We show

Proposition 5.8. *For $0 \le |\delta| \le \gamma < d/6$ there exists a unique point $\lambda_{k+} \in (\lambda_k - 3\gamma, \lambda_k + 3\gamma)$ such that $\Gamma_+(\lambda_{k+}) = 0$.*

Proof. First we estimate, using Lemma 5.5 (we will omit in the notation the dependence on λ in u, v and t)

$$\begin{aligned}
&\bigg|\gamma \int_\Omega [(1-t)\phi_k + v]^- \phi_k + \delta \int_\Omega [(1+t)\phi_k + u]^- \phi_k\bigg| \\
&\quad\le \gamma \int_{[(1-t)\phi_k + v<0]} \phi_k^2 + \gamma\| - t\phi_k + v\| + |\delta| \int_{[(1+t)\phi_k+u<0]} \phi_k^2 + |\delta|\|t\phi_k + u\| \\
&\quad\le \gamma(2 + \|t\phi_k + u\| + \| - t\phi_k + v\|) \\
&\quad\le \gamma\bigg(2 + \frac{1}{2}\|t\phi_k + u\|^2 + \frac{1}{2}\| - t\phi_k + v\|^2 + 1\bigg) \\
&\quad\le \gamma\bigg(3 + \frac{1}{2}(c_1 + c_2)\frac{\gamma}{d}\bigg) \le 4\gamma.
\end{aligned}$$

To conclude we prove that $\Gamma_+(\lambda_k - 3\gamma) > 0$ and $\Gamma_+(\lambda_k + 3\gamma) < 0$:

$$\begin{aligned}
\Gamma_+(\lambda_k - 3\gamma) &= 6\gamma - \gamma \int_\Omega [(1-t)\phi_k + v]^- \phi_k - \delta \int_\Omega [(1+t)\phi_k + u]^- \phi_k \\
&\ge 6\gamma - \gamma\bigg(3 + \frac{1}{2}(c_1 + c_2)\frac{\gamma}{d}\bigg) > 0
\end{aligned}$$

and

$$\begin{aligned}
\Gamma_+(\lambda_k + 3\gamma) &= -6\gamma - \gamma \int_\Omega [(1-t)\phi_k + v]^- \phi_k - \delta \int_\Omega [(1+t)\phi_k + u]^- \phi_k \\
&\le -6\gamma + \gamma\bigg(3 + \frac{1}{2}(c_1 + c_2)\frac{\gamma}{d}\bigg) < 0.
\end{aligned}$$

Hence, the proposition is proved, except for the uniqueness, which follows from the following lemma. □

Lemma 5.9. *The mapping Γ_+ is strictly decreasing, in the sense that*

$$(\Gamma_+(\lambda) - \Gamma_+(\widetilde{\lambda}))(\lambda - \widetilde{\lambda}) < 0 \quad \text{for } \lambda \ne \widetilde{\lambda}.$$

Proof. We have, assuming again $0 \le |\delta| \le \gamma$, and using Lemma 5.7

$$\begin{aligned}
&\bigg\{2\widetilde{\lambda} - 2\lambda - \gamma \int_\Omega [((1-t)\phi_k + v)^- - ((1-\widetilde{t})\phi_k + \widetilde{v})^-]\phi_k \\
&\qquad - \delta \int_\Omega [((1+t)\phi_k + u)^- - ((1+\widetilde{t})\phi_k + \widetilde{u})^-]\phi_k\bigg\}(\lambda - \widetilde{\lambda}) \\
&\le -2|\lambda - \widetilde{\lambda}|^2 + \gamma(\|(\widetilde{t} - t)\phi_k + v - \widetilde{v}\| + \|(t - \widetilde{t})\phi_k + u - \widetilde{u}\|)|\lambda - \widetilde{\lambda}| \\
&\le -2|\lambda - \widetilde{\lambda}|^2 + \frac{\gamma^2}{2}\|(\widetilde{t} - t)\phi_k + v - \widetilde{v}\|^2 + \frac{\gamma^2}{2}\|(t - \widetilde{t})\phi_k + u - \widetilde{u}\|^2 + |\widetilde{\lambda} - \lambda|^2 \\
&= -|\widetilde{\lambda} - \lambda|^2 + \gamma^2 2c_0^2|\widetilde{\lambda} - \lambda|^2;
\end{aligned}$$

the last expression is negative provided that $\gamma^2 2c_0^2 < 1$; this is the case, since

$$\gamma^2 2c_0^2 = \gamma^2 2\frac{c_1\gamma/d}{(d-9\gamma/2)^2} < \gamma^2 2\frac{c_1}{6(3\gamma/2)^2} = \frac{4c_1}{27} < 1. \qquad \square$$

Proof of Theorems 5.1 and 1.1. The case $(u,v)\cdot(\phi_k,\phi_k)=2$ follows from Propositions 5.6 and 5.8; the same procedure provides the result for the case $(u,v)\cdot(\phi_k,\phi_k)=-2$. In fact, we get a unique solution for the system

$$A\begin{pmatrix} u \\ v \end{pmatrix} = P\begin{pmatrix} \gamma(-s\phi_k+v)^- \\ \delta(-s\phi_k+u)^- \end{pmatrix},$$

as in proposition 5.6, and then we find a zero λ_{k-} of the increasing function

$$\Gamma_-\colon [\lambda_k-3\gamma,\lambda_k+3\gamma]\subset\mathbb{R}\to\mathbb{R},$$

$$\begin{aligned}\lambda\mapsto\widetilde{\Gamma}_-(\lambda) = &-2(\lambda_k-\lambda)-\gamma\int_\Omega[(-1-t(\lambda))\phi_k+v(\lambda)]^-\phi_k \\ &-\delta\int_\Omega[(-1+t(\lambda))\phi_k+u(\lambda)]^-\phi_k.\end{aligned}\tag{5.12}$$

Finally, in the case $(u,v)\cdot(\phi_k,\phi_k)=0$ (which means $(u,v)\in F$ and so $s=0$), a solution of (5.2) would also be a solution of (5.4), but this was excluded in the last claim of Lemma 5.5. $\square$

5.4. The inhomogeneous problem. We will consider in this section the system (1.3), written in the form

$$\begin{cases} -\Delta u = \lambda v+\gamma v^- + g_1(x,v)+h_1 & \text{in } \Omega, \\ -\Delta v = \lambda u+\delta u^- + g_2(x,u)+h_2 & \text{in } \Omega, \\ Bu=Bv=0 & \text{on } \partial\Omega, \end{cases}\tag{5.13}$$

so that one has $g_{1,2}\in C(\overline{\Omega}\times\mathbb{R})$, $\lim_{s\to\pm\infty} g_{1,2}(x,s)/s=0$ and $h_{1,2}\in L^2$; we will consider this system in the vectorial form:

$$-\Delta\begin{pmatrix} u \\ v \end{pmatrix} = \lambda\begin{pmatrix} v \\ u \end{pmatrix} + \begin{pmatrix} \gamma v^- \\ \delta u^- \end{pmatrix} + \begin{pmatrix} g_1(x,v)+h_1 \\ g_2(x,u)+h_2 \end{pmatrix}.\tag{5.14}$$

We will now prove Theorem 1.3: we consider $\lambda\in[\lambda_k-3\gamma,\lambda_k+3\gamma]$ fixed and prove

Proposition 5.10. *Let γ, δ, λ be as in Theorem* 1.3. *Then, for every $s\in\mathbb{R}$ there exists a* (*not necessarily unique*) $(u,v)\in F$ *which solves*

$$-\Delta\begin{pmatrix} u \\ v \end{pmatrix} = \lambda\begin{pmatrix} v \\ u \end{pmatrix} + P\begin{pmatrix} \gamma(s\phi_k+v)^- \\ \delta(s\phi_k+u)^- \end{pmatrix} + P\begin{pmatrix} g_1(x,s\phi_k+v)+h_1 \\ g_2(x,s\phi_k+u)+h_2 \end{pmatrix}.\tag{5.15}$$

PROOF. Proceeding as in Proposition 5.6 and denoting by $Q_s\binom{u}{v}$ and $G_s\binom{u}{v}$ the last two terms in (5.15), one needs to show that there exists $K > 0$ such that

$$(5.16)\qquad \begin{pmatrix} u \\ v \end{pmatrix} \neq \tau A^{-1}\left[Q_s\begin{pmatrix} u \\ v \end{pmatrix} + G_s\begin{pmatrix} u \\ v \end{pmatrix}\right], \qquad \text{if } \left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\| \geq K \quad \text{and} \quad \text{for all } \tau \in [0,1],$$

so that, by the Leray–Schauder principle, there exists a solution $(u,v) \in F \cap B_K(0)$ of (5.15).

To prove this, one supposes the existence of sequences $(u_n, v_n) \in F$ with $\|(u_n,v_n)\| \to \infty$ and $\tau_n \in [0,1]$ for which inequality (5.16) does not hold; then one divides the system by $\|(u_n,v_n)\|$, defines $(U_n,V_n) = (u_n,v_n)/\|(u_n,v_n)\|$ and obtains

$$\begin{aligned} A\begin{pmatrix} U_n \\ V_n \end{pmatrix} &= \tau_n \frac{1}{\|(u_n,v_n)\|}\left[Q_s\begin{pmatrix} u_n \\ v_n \end{pmatrix} + G_s\begin{pmatrix} u_n \\ v_n \end{pmatrix}\right] \\ &= \tau_n P\begin{pmatrix} \gamma\Big(\dfrac{s}{\|(u_n,v_n)\|}\phi_k + V_n\Big)^- \\ \delta\Big(\dfrac{s}{\|(u_n,v_n)\|}\phi_k + U_n\Big)^- \end{pmatrix} + \tau_n P\begin{pmatrix} \dfrac{g_1(x, s\phi_k + v_n) + h_1}{\|(u_n,v_n)\|} \\ \dfrac{g_2(x, s\phi_k + u_n) + h_2}{\|(u_n,v_n)\|} \end{pmatrix}, \end{aligned}$$

where the right hand side is bounded in $[L^2]^2$ and then (since A^{-1} is compact) we may assume $(U_n,V_n) \to (U,V) \in F$ in the $[L^2]^2$ norm (then (U,V) is not trivial) and $\tau_n \to \widetilde{\tau} \in [0,1]$. Then

$$\frac{1}{\|(u_n,v_n)\|}Q_s\begin{pmatrix} u_n \\ v_n \end{pmatrix} \to P\begin{pmatrix} \gamma V^- \\ \delta U^- \end{pmatrix}, \qquad \frac{1}{\|(u_n,v_n)\|}G_s\begin{pmatrix} u_n \\ v_n \end{pmatrix} \to 0,$$

and then one concludes

$$\begin{pmatrix} U \\ V \end{pmatrix} = A^{-1}P\begin{pmatrix} \widetilde{\tau}\gamma V^- \\ \widetilde{\tau}\delta U^- \end{pmatrix}.$$

Thus (U,V) is a nontrivial solution in F of (5.4) with $s = 0$ (and coefficients $\widetilde{\tau}\delta$, $\widetilde{\tau}\gamma$, with $\widetilde{\tau} \in [0,1]$), which is excluded by the last claim in Lemma 5.5. $\square$

Observe that we do not know if the solution obtained in the above lemma is unique; however, by the topological properties of the topological degree, we may assert as in [18] that there exists a continuum of solutions connecting two arbitrary values of s.

Now we consider on the space $\mathbb{R} \times F$ the (continuous) map $\Phi\colon \mathbb{R} \times F \to \mathbb{R}$, defined by

$$(5.17)\qquad \begin{aligned} (s,(u,v)) \mapsto 2(\lambda_k - \lambda)s &- \gamma\int_\Omega [s\phi_k + v]^-\phi_k - \delta\int_\Omega [s\phi_k + u]^-\phi_k \\ &- \int_\Omega [g_1(s\phi_k + v)]\phi_k - \int_\Omega [g_2(s\phi_k + u)]\phi_k, \end{aligned}$$

and the set

$$T = \{(s,(u,v)) \in \mathbb{R} \times F : (s,(u,v)) \text{ satisfies system (5.15)}\}.$$

Lemma 5.11. *Given $H > 0$, there exists $C_H > 0$ such that*

$$|\Phi(s,(u,v))| \le C_H, \quad \text{for } (s,(u,v)) \in T \text{ and } |s| \le H.$$

Moreover, we have:

$$\begin{aligned}
\text{for } s \to \infty: \quad & \lim \Phi(s,(u,v)) = -\infty && \text{if } (s,(u,v)) \in T \text{ and } \lambda > \lambda_{k+},\\
& \lim \Phi(s,(u,v)) = \infty && \text{if } (s,(u,v)) \in T \text{ and } \lambda < \lambda_{k+},\\
\text{for } s \to -\infty: \quad & \lim \Phi(s,(u,v)) = \infty && \text{if } (s,(u,v)) \in T \text{ and } \lambda > \lambda_{k-},\\
& \lim \Phi(s,(u,v)) = -\infty && \text{if } (s,(u,v)) \in T \text{ and } \lambda < \lambda_{k-}.
\end{aligned}$$

Proof. The first part of the claim follows as in Proposition 5.10: indeed, if we assume that the sequence (u_n,v_n) considered in the proof gives equality in equation (5.16) for $\tau = 1$ but together with a bounded sequence s_n instead of considering a fixed $s \in \mathbb{R}$, then we get the same contradiction, furnishing an estimate for any solution with s in a bounded set, and then for Φ.

Now let $\chi = \pm 1$ and suppose $s_n \to \chi\infty$ and $(s_n,(u_n,v_n)) \in T$. We first claim that $\|(u_n,v_n)\|/|s_n|$ is bounded: otherwise suppose $\|(u_n,v_n)\|/|s_n| \to \infty$ and proceed as in Proposition 5.10, dividing the system by $\|(u_n,v_n)\|$ and considering the sequence $(U_n,V_n) = (u_n,v_n)/\|(u_n,v_n)\|$:

$$\begin{aligned}
A\begin{pmatrix} U_n \\ V_n \end{pmatrix} &= \frac{1}{\|(u_n,v_n)\|}\left[Q_{s_n}\begin{pmatrix} u_n \\ v_n \end{pmatrix} + G_{s_n}\begin{pmatrix} u_n \\ v_n \end{pmatrix}\right]\\
&= P\begin{pmatrix} \gamma\Big(\dfrac{s_n}{\|(u_n,v_n)\|}\phi_k + V_n\Big)^- \\ \delta\Big(\dfrac{s_n}{\|(u_n,v_n)\|}\phi_k + U_n\Big)^- \end{pmatrix} + P\begin{pmatrix} \dfrac{g_1(x,s_n\phi_k+v_n)+h_1}{\|(u_n,v_n)\|} \\ \dfrac{g_2(x,s_n\phi_k+u_n)+h_2}{\|(u_n,v_n)\|} \end{pmatrix};
\end{aligned}$$

again the right hand side is bounded in $[L^2]^2$ since we are assuming $\|(u_n,v_n)\|/|s_n| \to \infty$, and then we may assume $(U_n,V_n) \to (U,V) \in F$ in the $[L^2]^2$ norm, with (U,V) nontrivial, and again deduce

$$\frac{1}{\|(u_n,v_n)\|}Q_{s_n}\begin{pmatrix} u_n \\ v_n \end{pmatrix} \to P\begin{pmatrix} \gamma V^- \\ \delta U^- \end{pmatrix}, \qquad \frac{1}{\|(u_n,v_n)\|}G_{s_n}\begin{pmatrix} u_n \\ v_n \end{pmatrix} \to 0,$$

and then

$$\begin{pmatrix} U \\ V \end{pmatrix} = A^{-1}P\begin{pmatrix} \gamma V^- \\ \delta U^- \end{pmatrix},$$

which is impossible.

Now divide the system by $|s_n|$, let $(U_n, V_n) = (u_n, v_n)/|s_n|$ (observe that this is bounded by the previous claim) and obtain

$$\begin{aligned} A \begin{pmatrix} U_n \\ V_n \end{pmatrix} &= \frac{1}{|s_n|}\left[Q_{s_n}\begin{pmatrix} u_n \\ v_n \end{pmatrix} + G_{s_n}\begin{pmatrix} u_n \\ v_n \end{pmatrix}\right] \\ &= P\begin{pmatrix} \gamma(\chi\phi_k + V_n)^- \\ \delta(\chi\phi_k + U_n)^- \end{pmatrix} + P\begin{pmatrix} \dfrac{g_1(x, s_n\phi_k + v_n) + h_1}{|s_n|} \\ \dfrac{g_2(x, s_n\phi_k + u_n) + h_2}{|s_n|} \end{pmatrix}, \end{aligned}$$

so that again $(U_n, V_n) \to (U, V) \in F$ in the $[L^2]^2$ and now

$$\frac{1}{|s_n|}Q_{s_n}\begin{pmatrix} u_n \\ v_n \end{pmatrix} \to P\begin{pmatrix} \gamma(\chi\phi_k + V)^- \\ \delta(\chi\phi_k + U)^- \end{pmatrix}, \qquad \frac{1}{|s_n|}G_{s_n}\begin{pmatrix} u_n \\ v_n \end{pmatrix} \to 0.$$

Then

$$\begin{pmatrix} U \\ V \end{pmatrix} = A^{-1}P\begin{pmatrix} \gamma(\chi\phi_k + V)^- \\ \delta(\chi\phi_k + U)^- \end{pmatrix},$$

and then it is the unique solution obtained in Proposition 5.6 (or the analogue in the case $\chi = -1$).

Now divide also equation (5.17) by $|s_n|$:

$$\begin{aligned} \frac{\Phi(s_n, (u_n, v_n))}{|s_n|} &= 2(\lambda_k - \lambda)\chi - \gamma\int_\Omega [\chi\phi_k + V_n]^-\phi_k - \delta\int_\Omega [\chi\phi_k + U_n]^-\phi_k \\ &\quad - \int_\Omega \frac{[g_1(s_n\phi_k + v_n)]}{|s_n|}\phi_k - \int_\Omega \frac{[g_2(s_n\phi_k + u_n)]}{|s_n|}\phi_k, \end{aligned}$$

where again the last two terms go to zero and so we have (compare with (5.11) and (5.12) and use the fact that (U, V) is the unique solution of Proposition 5.6)

$$\begin{aligned} \lim \frac{\Phi(s_n, (u_n, v_n))}{|s_n|} &= 2(\lambda_k - \lambda)\chi - \gamma\int_\Omega [\chi\phi_k + V]^-\phi_k - \delta\int_\Omega [\chi\phi_k + U]^-\phi_k \\ &= \begin{cases} \Gamma_+(\lambda) & \text{for } \chi = 1, \\ \Gamma_-(\lambda) & \text{for } \chi = -1. \end{cases} \end{aligned}$$

Finally, we know by Proposition 5.8, Lemma 5.9 and the analogues for Γ_-, that $\Gamma_+(\lambda)(\lambda - \lambda_{k+}) < 0$ and $\Gamma_-(\lambda)(\lambda - \lambda_{k-}) > 0$ and this concludes the proof of the lemma. □

Proof of Theorem 1.3. The equation which still has to be satisfied in order to obtain a solution of problem ((5.13)) is

$$\Phi(s, (u, v)) = \int_\Omega (h_1 + h_2)\phi_k, \quad \text{with } (s, (u, v)) \in T.$$

By Lemma 5.11 and the existence of a connected component in T going from $s \to -\infty$ to $s \to \infty$, one deduces that there exists a solution for any value of $\int_\Omega (h_1 + h_2)\phi_k$ when the two limits have opposite sign, while one deduces the existence of at least two solutions for $\int_\Omega (h_1 + h_2)\phi_k$ sufficiently (in dependence of the other components of (h_1, h_2)) positive when both limits are positive. The existence of no solution for sufficiently negative values of the integral follows from the same limits and the boundedness of $\Phi(s, (u, v))$ in T when s is bounded. $\square$

REMARK 5.12. As done in Section 4.3 for system (4.5), the result in Theorem 1.3 may be extended to the case in which we have different coefficients for u and v, provided the new system may be transformed into system 5.13 through one of the changes of unknowns used in Lemma 2.1.

6. Variational characterization of points in the Fučík spectrum above the first positive eigenvalue

In this section we take a variational approach to finding points in the Fučík spectrum. We will find a variational characterization of a continuum of points in $\widehat{\Sigma}_{nt}$ with arbitrary ratios $(\lambda^- - \lambda_1)/(\lambda^+ - \lambda_1)$ and $(\mu^- - \lambda_1)/(\lambda^+ - \lambda_1)$, which passes through the point $(\lambda_2, \lambda_2, \lambda_2)$.

In particular we will prove Theorem 1.2 in the form:

THEOREM 6.1. *For any $r, s \in (0, \infty)$, we can find and characterize one intersection of the halfline $\{(\lambda_1 + t, \lambda_1 + st, \lambda_1 + rt),\ t > 0\}$ with $\widehat{\Sigma}_{nt}$.*

The possibility to obtain a continuum in $\widehat{\Sigma}_{nt}$ passing through $(\lambda_2, \lambda_2, \lambda_2)$ will be proved in Proposition 6.8 below.

6.1. A strongly indefinite functional. To obtain the variational characterization for our problem, we will proceed basically as in [16] (see also in [8], [2] for similar procedures), that is, we will find critical points of the functional

$$J_1 : E = H \times H \to \mathbb{R},$$
$$\mathbf{u} = (u, v) \mapsto J_1(\mathbf{u}) = \int_\Omega 2\nabla u \nabla v - \lambda_1 \int_\Omega (u^2 + v^2), \tag{6.1}$$

constrained to the set in equation (6.6) below: the constrained critical points of J_1 will be nontrivial solutions of problem (3.1).

However, here we encounter the difficulty that the principal part of the functional is strongly indefinite, that is, there exist two infinite dimensional subspaces of E on which it is unbounded from above and from below, respectively. This follows from the following lemma (see Section 4.1 for the definitions of the considered spaces)

LEMMA 6.2.

$$\int_\Omega 2\nabla u \nabla v \geq \lambda_{k+1} \int_\Omega (u^2 + v^2) \qquad \textit{for } \mathbf{u} = (u, v) \in (E^- \oplus E_k^+)^\perp, \tag{6.2}$$

$$\int_\Omega 2\nabla u \nabla v \leq -\lambda_{k+1} \int_\Omega (u^2 + v^2) \quad \textit{for } \mathbf{u} = (u, v) \in (E_k^- \oplus E^+)^\perp, \tag{6.3}$$

$$(6.4)\qquad \int_\Omega 2\nabla u\nabla v \le \lambda_k \int_\Omega (u^2+v^2) \qquad \textit{for } \mathbf{u}=(u,v)\in E^-\oplus E_k^+,$$

$$(6.5)\qquad \int_\Omega 2\nabla u\nabla v \ge -\lambda_k \int_\Omega (u^2+v^2) \qquad \textit{for } \mathbf{u}=(u,v)\in E_k^-\oplus E^+.$$

Proof. In $(E^-\oplus E_k^+)^\perp$ one has $u=v$ and then

$$\int_\Omega 2\nabla u\nabla v = 2\int_\Omega |\nabla u|^2 \ge 2\lambda_{k+1}\int_\Omega u^2 = \lambda_{k+1}\int_\Omega (u^2+v^2),$$

proving (6.2). Then observe that

$$\int_\Omega 2\nabla u\nabla v = \frac{1}{2}\int_\Omega |\nabla(u+v)|^2 - |\nabla(u-v)|^2$$

and that for $\mathbf{u}\in E^-\oplus E_k^+$ one has $(u+v,u+v)\in E_k^+$, then

$$\int_\Omega 2\nabla u\nabla v \le \frac{1}{2}\int_\Omega |\nabla(u+v)|^2 \le \lambda_k\frac{1}{2}\int_\Omega (u^2+v^2+2uv) \le \lambda_k\int_\Omega (u^2+v^2),$$

proving (6.4).

The same argument gives the other two estimates. □

6.2. Finite dimensional approximation. As a consequence of the strong indefiniteness of the functional J_1 the classical linking theorems may not be applied. Some of the techniques used in approaching this kind of problems may be seen in [1], [7], [13], [6].

In particular, we will use here an approximation technique, that is, we will restrict the functional to a sequence of finite dimensional subspaces of E, in order to obtain (with classical linking theorems) critical points for these restrictions; then we will prove that from this sequence one can deduce the existence of a critical point for the functional in the whole space E.

This technique is known as Galerkin approximation procedure; one example of its application to this kind of problem may be seen in [6].

Let $r,s\in(0,\infty)$, $n>1$ and define

$$(6.6)\qquad Q_{r,s} = \left\{\mathbf{u}\in E : \int_\Omega (u^+)^2 + r(u^-)^2 + (v^+)^2 + s(v^-)^2 = 1\right\},$$

$$(6.7)\qquad Q_{r,s,n} = Q_{r,s}\cap E_n,$$

$$(6.8)\qquad L_n = Q_{r,s}\cap(E_n^-\oplus E_1^+).$$

Then consider

$$(6.9)\qquad d_n = \inf_{\gamma\in\Gamma_n}\ \sup_{\mathbf{u}\in\gamma(B^{n+1})} J_1(\mathbf{u})$$

where

$$(6.10)\qquad \Gamma_n = \{\gamma\in\mathcal{C}^0(B^{n+1},Q_{r,s,n}) : \gamma|_{\partial B^{n+1}} \text{ is a homeomorphism onto } L_n\},$$

and $B^h = \{(x_1, \dots, x_h) \in \mathbb{R}^h : \sum_{i=1}^h x_i^2 \le 1\}$.

We claim that

Proposition 6.3. *The values d_n are critical values constrained to $Q_{r,s,n}$ for the restriction to E_n of the functional J_1. Moreover, up to a subsequence, $d_n \to d > 0$ for $n \to \infty$, and the critical points corresponding to the values d_n converge to a nontrivial solution of problem* (3.1), *with coefficients $(\lambda_1+d, \lambda_1+sd, \lambda_1+rd)$, which then is a point in $\widehat{\Sigma}_{nt}$.*

We first prove some lemmas:

Lemma 6.4. *For $\mathbf{u} = (u, v) \in Q_{r,s}$ we have*

$$\frac{1}{\max\{r, s, 1\}} \le \int_\Omega u^2 + v^2 \le \frac{1}{\min\{r, s, 1\}}.$$

Proof.

$$\begin{aligned}\frac{1}{\max\{r, s, 1\}} &= \frac{\int_\Omega (u^+)^2 + (v^+)^2 + r(u^-)^2 + s(v^-)^2}{\max\{r, s, 1\}} \\ &\le \int_\Omega (u^+)^2 + (v^+)^2 + (u^-)^2 + (v^-)^2 = \int_\Omega u^2 + v^2 \\ &\le \frac{\int_\Omega (u^+)^2 + (v^+)^2 + r(u^-)^2 + s(v^-)^2}{\min\{r, s, 1\}} = \frac{1}{\min\{r, s, 1\}}. \qquad \square\end{aligned}$$

The following lemma establishes a kind of Palais–Smale property for the functional J_1 (or more general ones) constrained to $Q_{r,s}$.

Lemma 6.5. *Let $(\alpha^+, \alpha^-, \beta^+, \beta^-) \in \mathbb{R}^4$ and the sequence $\{\mathbf{u}_n\} = \{(u_n, v_n)\} \subseteq Q_{r,s}$ with $\mathbf{u}_n \in E_n$, be such that*

$$\left| \int_\Omega 2\nabla u_n \nabla v_n - \int_\Omega \beta^+(u_n^+)^2 + \beta^-(u_n^-)^2 + \alpha^+(v_n^+)^2 + \alpha^-(v_n^-)^2 \right| \le C \tag{6.11}$$

and, for all $(\psi, \phi) \in E_n$,

$$\begin{aligned}\int_\Omega \nabla u_n \nabla \phi + \nabla v_n \nabla \psi - \int_\Omega (\beta^+ u_n^+ - \beta^- u_n^-)\psi + (\alpha^+ v_n^+ - \alpha^- v_n^-)\phi \\ = \delta_n \int_\Omega (u_n^+ - r u_n^-)\psi + (v_n^+ - s v_n^-)\phi,\end{aligned} \tag{6.12}$$

for a suitable sequence $\{\delta_n\} \subseteq \mathbb{R}$. Then, up to a subsequence, $\mathbf{u}_n \xrightarrow{E} \mathbf{u} = (u, v) \in Q_{r,s}$ and $\delta_n \to \delta \in \mathbb{R}$, such that for all $(\psi, \phi) \in E$,

$$\begin{aligned}\int_\Omega \nabla u \nabla \phi + \nabla v \nabla \psi - \int_\Omega (\beta^+ u^+ - \beta^- u^-)\psi + (\alpha^+ v^+ - \alpha^- v^-)\phi \\ = \delta \int_\Omega (u^+ - r u^-)\psi + (v^+ - s v^-)\phi.\end{aligned} \tag{6.13}$$

Proof. Since $\{(u_n, v_n)\} \subseteq Q_{r,s}$, it is a bounded sequence in $[L^2(\Omega)]^2$. By choosing $\phi = v_n$ and $\psi = u_n$ in (6.12) one gets

$$\int_\Omega 2\nabla u_n \nabla v_n - \int_\Omega \beta^+(u_n^+)^2 + \beta^-(u_n^-)^2 + \alpha^+(v_n^+)^2 + \alpha^-(v_n^-)^2 \\ = \delta_n \int_\Omega (u_n^+)^2 + r(u_n^-)^2 + (v_n^+)^2 + s(v_n^-)^2 = \delta_n$$

and then δ_n is bounded by (6.11).

By choosing $\phi = u_n$ and $\psi = v_n$ in (6.12) one gets, by the boundedness of δ_n in $\mathbb{R}$ and of (u_n, v_n) in $[L^2]^2$, that $\int_\Omega |\nabla u_n|^2 + |\nabla v_n|^2$ is bounded, which implies that u_n and v_n are also bounded in H. Then, up to a subsequence, $\delta_n \to \delta \in \mathbb{R}$ and $\mathbf{u}_n \to \mathbf{u} \in E$, weakly in E and strongly in $[L^2]^2$. The strong convergence in $[L^2]^2$ implies that $\mathbf{u} \in Q_{r,s}$. Then we obtain (6.13) for a given $(\psi, \phi) \in E_h$ by taking limit in (6.12) and, since $\bigcup_{h\in\mathbb{N}} E_h$ is dense in E, this remains true for arbitrary $(\psi, \phi) \in E$.

Finally, we show that the convergence of $\mathbf{u}_n$ to $\mathbf{u} = (u, v)$ is in fact strong. Let $P_n\colon H \to \operatorname{span}\{\phi_1, \ldots, \phi_n\}$ be the orthogonal projection map, then $P_n u \to u$ and $P_n v \to v$ in H and so $P_n u - u_n \to 0$ and $P_n v - v_n \to 0$ in L^2. Then consider equation (6.12) with $\phi = u_n - P_n u$ and $\psi = 0$:

$$\int_\Omega \nabla u_n \nabla(u_n - P_n u) = \int_\Omega (\alpha^+ v_n^+ - \alpha^- v_n^-)(u_n - P_n u) + \delta_n \int_\Omega (v_n^+ - s v_n^-)(u_n - P_n u);$$

the right hand side tends to zero and then

$$\int_\Omega \nabla u_n \nabla(u_n - u + u - P_n u) \to 0,$$

that is

$$\int_\Omega |\nabla u_n|^2 - \int_\Omega \nabla u \nabla u_n + \int_\Omega \nabla u_n \nabla(u - P_n u) \to 0$$

and since also the last term tends to zero, we conclude that $\|\nabla u_n\|_{L^2} \to \|\nabla u\|_{L^2}$ and then $u_n \to u$ strongly in H.

The same argument gives $v_n \to v$ strongly in H. □

Finally, we estimate the functional J_1 on $\gamma(\partial B^{n+1})$ and $\gamma(B^{n+1})$, respectively, in order to apply the linking theorem:

Lemma 6.6. *$J_1|_{\gamma(\partial B^{n+1})} \leq 0$ for any $\gamma \in \Gamma_n$.*

Proof. Since $\gamma(\partial B^{n+1}) \subseteq Q_{r,s} \cap (E_n^- \oplus E_1^+)$ we have, by Lemma 6.2,

$$J_1(\mathbf{u}) \leq (\lambda_1 - \lambda_1) \int_\Omega u^2 + v^2 = 0, \quad \text{for all } \mathbf{u} \in \gamma(\partial B^{n+1}). \qquad \square$$

Lemma 6.7. *There exist* $\zeta, \eta > 0$ *such that* $\zeta > \inf_{\gamma \in \Gamma_n} \sup_{\mathbf{u} \in \gamma(B^{n+1})} J_1(\mathbf{u}) > \eta > 0$*, for any* $n > 1$.

Proof. By a topological degree argument one sees that for all $\gamma \in \Gamma_n$, there exists a point $\mathbf{u} \in \gamma(B^{n+1}) \cap (E_n^- \oplus E_1^+)^\perp$, the orthogonal being intended in E_n; for such $\mathbf{u}$ one has (we use Lemmas 6.4 and 6.2)

$$J_1(\mathbf{u}) \geq (\lambda_2 - \lambda_1) \int_\Omega u^2 + v^2 \geq \frac{\lambda_2 - \lambda_1}{\max\{r, s, 1\}} > 0.$$

Moreover, one may easily build a map $\widetilde{\gamma} \in \Gamma_n$ such that $\widetilde{\gamma}(B^{n+1}) \subseteq E_n^- \oplus E_2^+$, so that (again using Lemmas 6.4 and 6.2)

$$\sup_{\mathbf{u} \in \widetilde{\gamma}(B^{n+1})} J_1(\mathbf{u}) \leq (\lambda_2 - \lambda_1) \int_\Omega u^2 + v^2 \leq \frac{\lambda_2 - \lambda_1}{\min\{r, s, 1\}}. \qquad \square$$

Now we are in the position to give the

Proof of Proposition 6.3 and Theorems 6.1 and 1.2. The above lemmas allow to apply a linking theorem to obtain that the levels d_n are critical values constrained to $Q_{r,s,n}$ for the restriction to E_n of the functional J_1, that is there exist, for $n > 1$, $\mathbf{u}_n = (u_n, v_n) \in Q_{r,s,n}$ and $\delta_n \in \mathbb{R}$ such that $J_1(\mathbf{u}_n) = d_n$ and

$$\int_\Omega \nabla u_n \nabla \phi + \nabla v_n \nabla \psi - \lambda_1 \int_\Omega u_n \psi + v_n \phi = \delta_n \int_\Omega (u_n^+ - r u_n^-)\psi + (v_n^+ - s v_n^-)\phi,$$

for all $(\psi, \phi) \in E_n$. By choosing $\phi = v_n$ and $\psi = u_n$ one gets

$$\int_\Omega 2 \nabla u_n \nabla v_n - \lambda_1 \int_\Omega u_n^2 + v_n^2 = d_n = \delta_n.$$

Moreover, we have the estimates $\zeta > d_n > \eta > 0$ and then we may apply Lemma 6.5 to obtain that (up to a subsequence) $\delta_n = d_n \to d \in [\eta, \zeta]$ (then $d > 0$) and $\mathbf{u}_n \xrightarrow{E} \mathbf{u} = (u, v) \in Q_{r,s}$ such that, for all $(\psi, \phi) \in E$,

$$\int_\Omega \nabla u \nabla \phi + \nabla v \nabla \psi = \int_\Omega ((\lambda_1 + d)u^+ - (\lambda_1 + rd)u^-)\psi + ((\lambda_1 + d)v^+ - (\lambda_1 + sd)v^-)\phi$$

implying that $(\lambda_1 + d, \lambda_1 + sd, \lambda_1 + rd) \in \widehat{\Sigma}_{nt}$ and $\mathbf{u}$ is the corresponding nontrivial (being in $Q_{r,s}$) solution: indeed, this point is in $\widehat{\Sigma}$, and since $d > 0$ it does not belong to $\widehat{\Sigma}_t$ (which is explicitly known).

Theorems 6.1 and 1.2 follow immediately from Proposition 6.3. $\square$

6.3. A continuum in $\widehat{\Sigma}_{nt}$ passing through $(\lambda_2, \lambda_2, \lambda_2)$. In this section we make more precise the notation introduced in equations (6.9) and (6.10): in particular we will denote d_n, d by $d_n(r, s)$, $d(r, s)$ and Γ_n by $\Gamma_{n,r,s}$, in order to make explicit the parameters r, s we are considering.

We will prove:

Proposition 6.8. *Fixed a compact interval $[a,b] \subseteq (0,\infty)$ with $a<1<b$, it is possible to choose the subsequence in Proposition 6.3 in such a way that the limit points $(\lambda_1 + d(r,s), \lambda_1 + sd(r,s), \lambda_1 + rd(r,s)) \in \widehat{\Sigma}_{nt}$ describe, for $(r,s) \in [a,b]^2$, a continuum which passes through the point $(\lambda_2,\lambda_2,\lambda_2)$.*

First we need

Lemma 6.9. *Fixed a compact interval $[a,b] \subseteq (0,\infty)$, the sequence of functions of r, s: $d_n(r,s)\colon [a,b]^2 \to \mathbb{R}$ is (uniformly) equicontinuous.*

Proof. The claim would follow if we proved

$$|d_n(r,s) - d_n(\rho,\sigma)| \le C(|r-\rho| + |s-\sigma|) \quad \text{for } |r-\rho| + |s-\sigma| < D, \tag{6.14}$$

where C and D do not depend neither on n nor on $(r,s) \in [a,b]^2$.

First note that, looking at the definitions in equations (6.6) and (clGammanone), it is clear that the projection map:

$$P_{(r,s)}^{(\rho,\sigma)}\colon Q_{r,s} \to Q_{\rho,\sigma},$$
$$(u,v) \mapsto \frac{(u,v)}{(\int_\Omega (u^+)^2 + \int_\Omega (v^+)^2 + \rho\int_\Omega (u^-)^2 + \sigma\int_\Omega (v^-)^2)^{1/2}}$$

gives a one to one relation between the elements of the two families $\Gamma_{n,r,s}$ and $\Gamma_{n,\rho,\sigma}$, for any $n \in \mathbb{N}$:

$$\widetilde{P}_{(r,s)}^{(\rho,\sigma)}\colon \Gamma_{n,r,s} \to \Gamma_{n,\rho,\sigma}, \quad \gamma \mapsto P_{(r,s)}^{(\rho,\sigma)} \circ \gamma.$$

By the inf sup characterization (6.9), fixed $\varepsilon > 0$ there exists $\gamma \in \Gamma_{n,r,s}$ such that

$$x := \sup_{\mathbf{u}\in\gamma(B^{n+1})} J_1(\mathbf{u}) < d_n(r,s) + \varepsilon.$$

Then consider

$$\begin{aligned} y &:= \sup_{\mathbf{u}\in P_{(r,s)}^{(\rho,\sigma)}\circ\gamma(B^{n+1})} J_1(\mathbf{u}) \\ &= \sup_{\mathbf{u}\in\gamma(B^{n+1})} \frac{J_1(\mathbf{u})}{\int_\Omega (u^+)^2 + \int_\Omega (v^+)^2 + \rho\int_\Omega (u^-)^2 + \sigma\int_\Omega (v^-)^2}. \end{aligned} \tag{6.15}$$

Since $(u,v) \in Q_{r,s}$, we have $\int_\Omega (u^-)^2 \le 1/r$ and $\int_\Omega (v^-)^2 \le 1/s$, so that

$$\int_\Omega (u^-)^2, \quad \int_\Omega (v^-)^2 \le \frac{1}{a},$$

and the denominator in (6.15) is $1 + (\rho - r)\int_\Omega (u^-)^2 + (\sigma - s)\int_\Omega (v^-)^2$. By using $1/(1+\xi+\eta) \le 1 + 2|\xi| + 2|\eta|$, for $|\xi| + |\eta| < 1/2$, we get

$$\frac{1}{1 + (\rho - r)\int_\Omega (u^-)^2 + (\sigma - s)\int_\Omega (v^-)^2} \le 1 + \frac{2}{a}(|r-\rho| + |s-\sigma|)$$

for $(|r-\rho|+|s-\sigma|)<a/2$. Then, by (6.9), since $P_{(r,s)}^{(\rho,\sigma)}\circ\gamma\in\Gamma_{n,\rho,\sigma}$, we have

$$\begin{aligned} d_n(\rho,\sigma) \le y &\le x\Big(1+\frac{2}{a}(|r-\rho|+|s-\sigma|)\Big) \\ &< (d_n(r,s)+\varepsilon)\Big(1+\frac{2}{a}(|r-\rho|+|s-\sigma|)\Big); \end{aligned}$$

this provides

$$d_n(\rho,\sigma)-d_n(r,s)\le d_n(r,s)\frac{2}{a}(|r-\rho|+|s-\sigma|)+\varepsilon\Big(1+\frac{2}{a}(|r-\rho|+|s-\sigma|)\Big).$$

To conclude, we know by the proof of Lemma 6.7 that

$$\frac{\lambda_2-\lambda_1}{\max\{b,1\}}\le\frac{\lambda_2-\lambda_1}{\max\{r,s,1\}}\le d_n(r,s)\le\frac{\lambda_2-\lambda_1}{\min\{r,s,1\}}\le\frac{\lambda_2-\lambda_1}{\min\{a,1\}},\tag{6.16}$$

and then

$$d_n(\rho,\sigma)-d_n(r,s)\le\frac{\lambda_2-\lambda_1}{\min\{a,1\}}\frac{2}{a}(|r-\rho|+|s-\sigma|)+\varepsilon\bigg(1+\frac{2}{a}(|r-\rho|+|s-\sigma|)\bigg),$$

valid for $|r-\rho|+|s-\sigma|<a/2$.

Since we may estimate analogously the difference $d_n(r,s)-d_n(\rho,\sigma)$, and ε is arbitrary, we get the claimed estimate (6.14). $\square$

Proof of Proposition 6.8. The equicontinuous and equibounded (by equation (6.16)) sequence $d_n(r,s)$ admits a subsequence converging uniformly to a continuous function $\widetilde{d}(r,s)$, by the Ascoli Arzelá theorem.

Observe that in Proposition 6.3 we obtained the levels d and the nontrivial solutions (u,v) but we had no idea whether they were unique. However, if we apply the procedure which proves Proposition 6.3 to the subsequence obtained here, we are sure to converge to the levels $\widetilde{d}(r,s)$ above, that is to a continuous function.

To conclude, we just need to see that for $r=s=1$ the resulting point in $\widehat{\Sigma}_{nt}$ is indeed $(\lambda_2,\lambda_2,\lambda_2)$: in fact, it follows from (6.16) that $d_n(1,1)=\lambda_2-\lambda_1$ for any $n>1$. $\square$

7. Behaviour of the nontrivial solutions

As mentioned in the introduction, we will prove now that there exist points in $\widehat{\Sigma}_{nt}$ which are not related to any point in the Fučík spectrum of the scalar problem.

It is interesting to remark that in the case of the linear spectrum of systems, the eigenvalues and the corresponding nontrivial solutions are always related to those of the scalar problem; actually, as seen in Section 4.1, the latter consist of pairs of eigenfunctions (in fact, the same eigenfunction) of the scalar problem, and so at least two of the four products u^+v^+, u^+v^-, u^-v^+, and u^-v^- are identically zero.

We prove here (we restrict to Dirichlet boundary conditions) that for points of the Fučík spectrum of the form $(\lambda^+, \mu^-, \lambda^-)$ with $\mu^- \neq \lambda^-$ (i.e. if Lemma 3.3 does not apply), the corresponding nontrivial solutions u and v have opposite sign is some region. More precisely, we prove the following

Proposition 7.1. *Consider the Dirichlet problem with $\partial\Omega$ sufficiently regular. If $(\lambda^+, \mu^-, \lambda^-) \in \widehat{\Sigma}_{nt}$ with $\mu^- \neq \lambda^-$ then for the corresponding nontrivial solutions u, v at least three of the products u^+v^+, u^+v^-, u^-v^+, u^-v^- are not identically zero.*

Proof. First we claim that u, v are in fact classical solutions: since weak solutions of problem (3.1) are by definition in $H^1(\Omega)$, then the right hand sides in the equations are in L^{p_1} for some $p_1 > 2$, which gives, by a boot strap argument, $u, v \in W^{2,p_1}$ and so also $u, v \in L^{p_2}$ for a $p_2 > p_1$; by iterating this procedure one obtains $u, v \in W^{2,p}$ for any $p > 2$, but then the right hand sides are at least $W^{1,p}$, giving (again by boot strap) $u, v \in W^{3,p}$ for any $p > 2$, which finally implies $u, v \in \mathcal{C}^2(\Omega) \cup \mathcal{C}^(\overline{\Omega})$, and then they are classical solutions, as claimed.

This regularity allows us to consider the nodal domains of u and v, that is those maximal connected opens subsets of $\{x \in \Omega : u(x) \neq 0\}$ and of $\{x \in \Omega : v(x) \neq 0\}$, and also to apply, later, the strong maximum principle.

Suppose now, for the sake of contradiction, that the positive and negative nodal domains of u and v coincide (observe that by the definition of $\widehat{\Sigma}_{nt}$ there exist at least one positive and one negative nodal domain).

In any positive nodal domain ω^+, the functions (u, v) satisfy

$$\begin{cases} -\Delta u = \lambda^+ v & \text{in } \omega^+, \\ -\Delta v = \lambda^+ u & \text{in } \omega^+, \\ u = v = 0 & \text{on } \partial\omega^+, \end{cases}$$

and then $u = v = \phi_{1,\omega^+}$, where ϕ_{1,ω^+} is a positive multiple of the first eigenfunction of the Laplacian with Dirichlet boundary conditions in ω^+.

In a similar way, we have that in any negative nodal domain ω^-, the functions (u, v) satisfy

$$\begin{cases} -\Delta u = \lambda^- v & \text{in } \omega^-, \\ -\Delta v = \mu^- u & \text{in } \omega^-, \\ u = v = 0 & \text{on } \partial\omega^-. \end{cases}$$

By replacing the function u with the scaled function $U = u\sqrt{\mu^-/\lambda^-}$ one gets

$$\begin{cases} -\Delta U = \sqrt{\lambda^-\mu^-}\, v & \text{in } \omega^-, \\ -\Delta v = \sqrt{\lambda^-\mu^-}\, U & \text{in } \omega^-, \\ U = v = 0 & \text{on } \partial\omega^-, \end{cases}$$

and then $U = u\sqrt{\mu^-/\lambda^-} = v = -\phi_{1,\omega^-}$ where ϕ_{1,ω^-} is a positive multiple of the first eigenfunction of the Laplacian with Dirichlet boundary conditions in ω^-.

Now, by subtracting the equations in (3.1) (remember that we are assuming that $u^+v^- \equiv u^-v^+ \equiv 0$) we obtain

$$\begin{cases} -\Delta(u-v) = \lambda^+(v^+ - u^+) + \mu^- u^- - \lambda^- v^- & \text{in } \Omega, \\ u - v = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.1}$$

but, by the computations above, $u^+ - v^+ \equiv 0$ while $\mu^- u^- - \lambda^- v^- = \mu^- u^- - \lambda^- \sqrt{\mu^-/\lambda^-} u^- = (\mu^- - \sqrt{\mu^-\lambda^-})u^- \neq 0$.

This means that the right hand side in (7.1) does not change sign and then, by the strong maximum principle, the $\mathcal{C}^2(\Omega) \cup \mathcal{C}^0(\overline{\Omega})$ function $u - v$ may not be zero in the sets ω^+: contradiction. □

A remark on the one dimensional case. If we concentrate on the one dimensional case, where everything is simpler, it is known that the eigenfunctions of the scalar problem (and then also of the system) are of the form $\sin(\sqrt{|\lambda|}x)$, while the nontrivial solutions of the scalar Fučík problem are built by suitably gluing bumps of the form $\sin(\sqrt{\lambda^+}x)$ and $-\sin(\sqrt{\lambda^-}x)$.

What appears new in the Fučík problem for the system is the existence of regions where u and v have opposite sign even if the coefficients are all positive (observe that the proof in Proposition 7.1 is given for the Dirichlet problem, but in dimension one it is simple to extend it also to the Neumann problem): in these regions we then have $\operatorname{sign}(-u'') = -\operatorname{sign}(u)$ and $\operatorname{sign}(-v'') = -\operatorname{sign}(v)$ and then u and v have also a component of the form of the hyperbolic functions sinh and cosh.

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Manuscript received March 31, 2005

Eugenio Massa
IMECC
Universidade Estadual de Campinas
13081-970, Campinas, SP, BRAZIL

E-mail address: eugenio@ime.unicamp.br

Bernhard Ruf
Dip. di Matematica
Università degli Studi
Via Saldini 50
20133 Milano, ITALY

E-mail address: ruf@mat.unimi.it