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AN EIGENVALUE SEMICLASSICAL PROBLEM FOR THE SCHRÖDINGER OPERATOR WITH AN ELECTROSTATIC FIELD

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Abstract. We consider the following system of Schrödinger–Maxwell equations in the unit ball B_1 of $\mathbb{R}^3$

$$-\frac{\hbar^2}{2m}\Delta v + e\phi v = \omega v, \quad -\Delta\phi = 4\pi e v^2$$

with the boundary conditions $u = 0$, $\phi = g$ on ∂B_1, where $\hbar$, m, e, $\omega > 0$, v, $\phi\colon B_1 \to \mathbb{R}$, $g\colon \partial B_1 \to \mathbb{R}$. Such system describes the interaction of a particle constrained to move in B_1 with its own electrostatic field. We exhibit a family of positive solutions $(v_\hbar, \phi_\hbar)$ corresponding to eigenvalues $\omega_\hbar$ such that $v_\hbar$ concentrates around some points of the boundary ∂B_1 which are minima for g when $\hbar \to 0$.

1. Introduction

According to the mathematical formalism of Quantum Mechanics the dynamical state of a particle constrained to move in a 3-dimensional region Ω is completely defined, at a given instant, by a definite (in general complex) function $\psi(x,t)$: $|\psi(x,t)|^2\,dx$ gives the probability of finding the particle in the element dx at the instant t. The function ψ is called the *wave function* associated to the particle. In order to make this statistical interpretation consistent, the following

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normalization equation has to be fulfilled at each moment:

$$\int_\Omega |\psi(x,t)|^2\,dx = 1.$$

In the case of a charged particle with mass m and charge e in an electromagnetic field derived from a vector potential $\mathbf{A}$ and a scalar potential ϕ, the wave ψ satisfies the following Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t}(x,t) = \frac{1}{2m}\bigg(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}(x,t)\bigg)^2\psi(x,t) + e\phi(x,t)\psi(x,t), \tag{1.1}$$

for $x \in \Omega$, $t \in \mathbb{R}$. In (1.1) i is the imaginary unit while $\hbar$ is the *Planck's constant* and c denotes the velocity of light in vacuo. On the right hand side of equation (1.1) the operator $(\hbar\nabla/i - e\mathbf{A}/c)^2$ designates the *formal* scalar product of the vector operator $\hbar\nabla/i - e\mathbf{A}/c$ by itself, i.e.

$$\bigg(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\bigg)^2\psi = -\hbar^2\Delta\psi + \bigg(\frac{e^2}{c^2}|\mathbf{A}|^2 - \frac{e\hbar}{ic}\operatorname{div}\mathbf{A}\bigg)\psi - \frac{e\hbar}{ic}(\mathbf{A}\cdot\nabla\psi).$$

Let us assume that the potentials $\mathbf{A}$ and ϕ do not depend on the time and that the particle is represented by a wave of the type

$$\psi(x,t) = v(x)\exp\bigg(-i\,\frac{\omega}{\hbar}\,t\bigg)$$

where $\omega \in \mathbb{R}$ and $v\colon\Omega \to \mathbb{R}$. Such a wave is said to be a stationary wave. With this *ansatz* equation (1.1) reduces to the following eigenvalue equation:

$$\frac{1}{2m}\bigg(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}(x)\bigg)^2 v(x) + e\phi(x)v(x) = \omega v(x), \quad x \in \Omega. \tag{1.2}$$

Many papers are concerned with the eigenvalue problem (1.2) in the case of assigned external potentials $\mathbf{A}$, ϕ. For fixed $\hbar > 0$ the spectral theory for this type of operator has been studied in detail, particularly by Avron, Herbst and Simon ([4]–[5]). The papers [7], [22] and [23] deal with the effect of the magnetic field on the spectrum of the Schrödinger operator in the semiclassical limit, i.e. as $\hbar \to 0$, and especially it is studied the influence on the bottom of the essential spectrum and on the decay of the eigenfunctions.

Following the same idea of [6] (later developed in [11]), in this paper we consider the case of a charged particle interacting with its own electromagnetic field. Hence, since we do not assume that the electromagnetic field is assigned, we have to solve a system whose unknowns are the wave function ψ associated to the particle and the potentials $\mathbf{A}$ and ϕ. More precisely, considering the stationary case, we are reduced to solve equation (1.2) coupled with the following Maxwell equations:

$$-\Delta\phi = 4\pi e v^2, \quad \nabla\times\nabla\times\mathbf{A} = \frac{4\pi e^2}{mc^2}\mathbf{A}v^2, \quad x \in \Omega. \tag{1.3}$$

Let B_1 denote the unit ball on $\mathbb{R}^3$:

$$B_1 = \{x \in \mathbb{R}^3 \mid |x| < 1\}.$$

We shall investigate the case $\Omega = B_1$, $\mathbf{A} = 0$ and non-trivial electric potential ϕ, so that the second equation of (1.3) is identically satisfied and then the Schrödinger-Maxwell system takes the form:

$$\begin{aligned} -\hbar^2\Delta v + e\phi v &= \omega v && \text{in } B_1, \\ -\Delta\phi &= 4\pi e v^2 && \text{in } B_1. \end{aligned} \tag{1.4}$$

(where we have set, for sake of simplicity, $2m = 1$) with the boundary and the normalization conditions

$$v = 0 \quad \text{on } \partial B_1, \qquad \phi = g \quad \text{on } \partial B_1, \qquad \int_{B_1} |v|^2\,dx = 1 \tag{1.6}$$

where $g\colon \partial B_1 \to \mathbb{R}$ is an assigned function.

The system (1.4)–(1.5) has been studied in [6] in the case in which the charged particle lies in a bounded space region Ω and in [8] in all $\mathbb{R}^3$ where the action of an external nonzero potential is considered. In both cases, for fixed $\hbar > 0$, the authors prove the existence of infinitely many solutions $\{v_k, \phi_k, \omega_k\}$. This paper deals with the *semiclassical limit* of the system (1.4)–(1.5), i.e. it is concerned with the problem of finding nontrivial solutions and studying their asymptotic behaviour when $\hbar \to 0^+$; hence such solutions are usually referred to as *semiclassical* ones.

The analysis of the spectrum and eigenfunctions of Schrödinger operators in the semiclassical limit $\hbar \to 0$ is not only a challenging mathematical task, but also of some relevance for the understanding of a wide class of quantum phenomena. Indeed, according to the *correspondence principle*, letting $\hbar$ go to zero in the Schrödinger equation formally describes the transition from Quantum Mechanics to Classical Mechanics; it is therefore interesting the problem of finding nontrivial solutions and studying their asymptotic behaviour as $\hbar \to 0^+$.

While there is a wide literature concerning semiclassical states for linear and nonlinear Schrödinger equation in an assigned potential ϕ (we recall, among many others, [1]–[3], [9], [14]–[17], [20], [21], [24]–[28], [30], [31]), there are few papers dealing with the case of an unknown potential. The first time the semiclassical limit for a Schrödinger–Maxwell system has been considered seems to be in [12], [13], [29]. In such papers a nonlinear perturbation of the system (1.4)–(1.5) is studied and it is proved that the solutions exhibit some kind of notable behaviour in the semiclassical limit, a concentration behaviour: their form consists of very sharp peaks which become highly concentrated when $\hbar$ is small. More precisely in [12] and [29] the authors construct a family of radially symmetric positive waves concentrating around a sphere. In [13] it is shown that for any integer K there exists a solution of the system exhibiting exactly K spikes. However, at our knowledge, except for [10], we are unaware of semiclassical phenomena for the system (1.4)–(1.5); this paper and [10] seem to be the first results in this line. Notice that the nature of the problem is still nonlinear, but the nonlinearity is merely internal to the system, being given only by the coupling, i.e. by the interaction of the particle with its own electrostatic field.

The purpose of this paper is to show the existence of solutions exhibiting a concentration behaviour at one or more points of the boundary ∂B_1 which are proved to be minima for g. In order to state the exact result we enumerate the assumptions on the function g that will be steadily assumed.

(g1) $g \in \mathcal{C}(\partial B_1)$,

(g2) The set $Z(g) = \{x \in \partial B_1 \mid g(x) = \min_{\partial B_1} g\}$ is finite and, setting $Z(g) = \{z_0, \dots, z_\ell\}$ ($\ell \geq 0$), for every $i \in \{0, \dots, \ell\}$:

$$A_i = \int_{\partial B_1} \frac{g(y) - g(z_0)}{|y - z_i|^3} \, dS < \infty. \tag{1.7}$$

We notice that (1.7) is certainly verified if g is of class $C^{1,\sigma}$ ($\sigma > 0$) in a neighbourhood of the point z_i.

Finally we order the points in $Z(g)$ in such a way that the following holds:

(g3) there exist $0 \leq \ell' \leq \ell$ and $\overline{q} \in (1, \infty]$ such that

$$\begin{aligned} &\lim_{y \to z_i} \frac{g(y) - g(z_0)}{|y - z_i|^{\overline{q}}} = 0 && \text{for all } i \in \{0, \dots, \ell'\}, \\ &\liminf_{y \to z_i} \frac{g(y) - g(z_0)}{|y - z_i|^{\overline{q}}} \in (0, \infty] && \text{for all } i \in \{\ell' + 1, \dots, \ell\}. \end{aligned}$$

In other words we separate the minima having higher order from the others. More precisely we assume that there is a power function which divides the graphic of g near a points z_i from that around z_j for every couple $(i, j) \in \{0, \dots, \ell'\} \times \{\ell' + 1, \dots, \ell\}$. For example, this situation occurs z_i, for $i \in \{0, \dots, \ell'\}$, are minima of order $k_i > \overline{q}$ and z_j, for $j \in \{\ell' + 1, \dots, \ell\}$, are nondegenerate minima of order $k_j \leq \overline{q}$. Roughly speaking, hypothesis (g3) means that the function g is "smaller" near the first $\ell' + 1$ points rather than near the remaining $\ell - \ell'$.

Under similar hypotheses on g, concentrated solutions for the system (1.4)–(1.5) are produced in [10]. However, while in [10] the vertexes of the spikes are located near those points z_i which minimize the numbers A_i, in this paper the location of the peaks is determined by the highest order of the minima. More precisely the result can be summarized as follows. For small values of the parameter $\hbar$, we prove the existence of a positive wave $v_\hbar$ and a potential $\phi_\hbar$ satisfying schmaxone–schmaxtwo. Furthermore, in the limit when $\hbar \to 0$, $v_\hbar$ concentrates around the points $z_0, \dots, z_{\ell'}$; roughly speaking, the analysis reveals that the limit form of $v_\hbar$ resembles the sum of bumps located around the points $z_0,\ldots,z_{\ell'}$ which become highly concentrated as $\hbar \to 0^+$. Now we proceed to provide the exact formulation of our main result.

Theorem 1.1. *Let* (g1)–(g3) *hold. Then for every* $\hbar > 0$ *the system* (1.4)–(1.5) *with the conditions* (1.6) *has a solution* $(v_\hbar, \phi_\hbar, \omega_\hbar)$ *such that*

(a) $v_\hbar,\ \phi_\hbar \in H^1(B_1)$,
(b) $\omega_\hbar \to eg(z_0)$ *as* $\hbar \to 0$,
(c) $\phi_\hbar \to ((1-|x|^2)/4\pi)\int_{\partial B_1}(g(y)/|x-y|^3)\,dS$ *in* $L^1(B_1)$.

Furthermore, for each sequence $\hbar_n \to 0^+$, *possibly passing to a subsequence, there exist* $\alpha_0,\ldots,\alpha_{\ell'} \geq 0$ *such that* $\alpha_0+\ldots+\alpha_{\ell'} = 1$ *and*

(d) $|v_{\hbar_n}|^2 \to \alpha_0\delta_{z_0}+\ldots+\alpha_{\ell'}\delta_{z_{\ell'}}$ *in the sense of distributions* (δ_{z_i} *denoting the Dirac measure on* $\mathbb{R}^3$ *giving unit mass to the point* z_i).

Remark 1.2. Notice that if $\ell' = 0$, then all the family $v_\hbar$ concentrates at the point z_0, i.e. the part *iv*) of the theorem becomes $|v_\hbar|^2 \to \delta_{z_0}$ as $\hbar \to 0$ in the sense of distributions.

We point out that such concentration phenomena have an interesting physical interpretation. Indeed the appearance of such type of notable behaviour in the semiclassical limit for the system (1.4)–(1.5) may be looked at as a model describing particle-like matter: indeed the existence of solutions exhibiting a "spike-layer" pattern provides some examples of spatially localised functions which resemble as closely as possible classical particles.

The proof of Theorem conc relies on a variational approach and is based on the study of the behaviour of sequences with bounded energy, in the spirit of the concentration compactness principle. Since equations (1.4)–(1.5) have a variational structure, we capture our solutions by a constrained minimization method; hence $v_\hbar$ is obtained as a minimum point of a suitable functional $J_\hbar$ on the constraint $\|v\|_{L^2} = 1$. The constraint causes a Lagrange multiplier $\omega = \omega_\hbar$ to appear. Once we have found the solutions, we want to investigate their asymptotic behaviour. First we prove that as $\hbar \to 0$ the waves $v_\hbar$ vanish outside the set $Z(g)$, hence their form consists of $\ell+1$ peaks, each of them converging to a Dirac delta centered at the points z_i having weight $\alpha_i \geq 0$ and (according to (1.6)) $\alpha_0+\ldots+\alpha_\ell = 1$. Once we have split the solutions, the crucial step consists in proving that $\alpha_j = 0$ for $j = \ell'+1,\ldots,\ell$; this is the most technical and lengthy part of this paper. The basic idea, however, is simple. Roughly speaking, if $\alpha_j \neq 0$, then we could isolate the peak centered at z_j and move it near the point z_0. Hence we apply a suitable rescalation in the coordinates, and, after this process, we obtain that the resulting new bump provides a function still lying in the unit ball of L^2 which makes the functional $J_\hbar$ lower than $J_\hbar(v_\hbar)$, in contradiction with the minimizing property of the original $v_\hbar$. We briefly outline the organization of the contents of this paper. Section 2 is devoted to the description of the functional setting in which we work. In Section 3 we provide some preliminary results which will play a key role in the rest of the arguments. In Section 4 we construct the solutions and establish some asymptotic estimates which will be useful in order to locate their peaks. Finally the proof of Theorem 1.1 is completed in Section 5.

Notations.

- For any $\Omega \subset \mathbb{R}^3$, $\partial\Omega$ is its boundary and χ_Ω denotes the characteristic function of Ω.
- Given $\Omega \subset \mathbb{R}^3$ and $u\colon \Omega \to \mathbb{R}$, $\operatorname{supp} u$ denotes the set $\{x \in \Omega \mid u(x) \neq 0\}$. Furthermore we will continue to denote by u the function defined by using polar coordinates: $(r, \theta, \varphi) \mapsto u(x)$ with x having polar coordinates (r, θ, φ).
- Given $\Omega \subset \mathbb{R}^N$ a measurable set, $L^p(\Omega)$ is the usual Lebesgue space endowed with the norm
$$\|u\|_p^p := \int_\Omega |u|^p \, dx \quad \text{for } 1 \le p < \infty, \quad \|u\|_\infty = \sup_{x \in \Omega} |u(x)|.$$
- We will often use the symbols $C, C_1, C_2, \ldots$ for denoting a positive constant independent on $\hbar$. Their values are allowed to vary from line to line (and also in the same formula).
- $o(1)$ denotes quantities that tends to zero as $\hbar \to 0^+$.
- Given $\{a_\hbar\}_{\hbar>0}$ and $\{b_\hbar\}_{\hbar>0}$ two family of numbers, we write $a_\hbar = o(b_\hbar)$ (resp. $a_\hbar = O(b_\hbar)$) to mean that $a_\hbar / b_\hbar \to 0$ (resp. $|a_\hbar| \le C|b_\hbar|$) as $\hbar \to 0^+$.

2. Abstract setting

In order to obtain solutions of (1.4)–(1.5) we choose a suitable functional setting. For every $\hbar > 0$ set $B_\hbar := \{x \in \mathbb{R}^3 \mid |x| < 1/\hbar\}$. Then in the sequel we will work in the Sobolev space $H_0^1(B_\hbar)$ endowed with the norm
$$\|u\|^2_{H_0^1(B_\hbar)} := \|\nabla u\|^2_{L^2(B_\hbar)} = \int_{B_\hbar} |\nabla u|^2 \, dx.$$

First we provide the following two propositions.

Proposition 2.1. *For every $\hbar > 0$ the function*
$$\begin{aligned} f_\hbar(x) &= \frac{1 - |\hbar x|^2}{4\pi} \int_{\partial B_1} \frac{g(y) - g(z_0)}{|y - \hbar x|^3} \, dS \quad && \textit{if } x \in B_\hbar, \\ f_\hbar(x) &= g(\hbar x) && \textit{if } x \in \partial B_\hbar, \end{aligned}$$

is the unique solution in $\mathcal{C}^2(B_\hbar)\cap\mathcal{C}(\overline{B}_\hbar)$ *of*

$$-\Delta f(x)=0 \quad \text{in } B_\hbar, \qquad f(x)=g(\hbar x)-g(z_0) \quad \text{on } \partial B_\hbar. \tag{2.1}$$

See [18, Theorems 5 and 15] for the proof.

REMARK 2.2. Notice that, if we assume $g\in\mathcal{C}(\partial B_1)\cap H^{1/2}(\partial B_1)$, then the variational method applies and gives that $f_\hbar$ is also the unique weak solution in $H^1(B_\hbar)$ of (2.1).

Before going on with the second proposition we recall that the Green's function $G_\hbar$ for the ball $B_\hbar$ is given by

$$\begin{aligned} G_\hbar(x,y) &= \frac{1}{|y-x|}-\frac{|\hbar x|}{|y|\hbar x|^2-x|} \\ &= \frac{|x|^2(1-|\hbar x|)(1-|\hbar y|)}{|y-x||y|\hbar x|^2-x|(|y|\hbar x|^2-x|+|\hbar x||y-x|)} \end{aligned} \tag{2.2}$$

(see [18, p. 40]). It is immediate that

$$G_\hbar(x,y)=G_\hbar(y,x), \quad G_\hbar(x,y)\geq 0 \quad \text{for every } x,y\in B_\hbar,\ x\neq y.$$

PROPOSITION 2.3. *For every* $\gamma\in L^2(B_\hbar)$ *the function* $\Phi_\hbar[\gamma]$ *defined by*

$$\Phi_\hbar[\gamma](x)=\frac{e}{\hbar}\int_{B_\hbar} G_\hbar(x,y)\gamma(y)\,dy \tag{2.3}$$

is the unique solution in $H_0^1(B_\hbar)$ *of*

$$-\Delta\phi=\frac{4\pi e}{\hbar}\gamma \quad \text{in } B_\hbar, \qquad \phi=0 \quad \text{on } \partial B_\hbar. \tag{2.4}$$

Furthermore the following properties hold:

(a) *if* $\gamma\in\mathcal{C}(\overline{B}_\hbar)$, *then* $\Phi_\hbar[\gamma]\in\mathcal{C}^2(\overline{B}_\hbar)\cap\mathcal{C}(\overline{B}_\hbar)$;

(b) *if* γ *is radially symmetric, then* $\Phi_\hbar[\gamma]$ *is radial too and*

$$\Phi_\hbar[\gamma](x)=\frac{4\pi e}{|\hbar x|}\int_0^{1/\hbar} r(\min\{|x|,r\}-|\hbar x|r)\gamma(r)\,dr;$$

(c) *the functional* $F_\hbar\colon u\in H_0^1(B_\hbar)\mapsto\int_{B_\hbar}u^2\Phi_\hbar[u^2]\,dx$ *is compact and* $\mathcal{C}^1$ *and*

$$F_\hbar'(u)[w]=4\int_{B_\hbar}uw\Phi_\hbar[u^2]\,dx \quad \textit{for all } u,w\in H_0^1(B_\hbar).$$

PROOF. By Lax–Milgram's Lemma for every $\gamma\in L^2(B_\hbar)$ we get the existence of a unique $\Phi=\Phi_\hbar[\gamma]\in H_0^1(B_\hbar)$ which solves (2.4). If $\gamma\in\mathcal{C}(\overline{B}_\hbar)$, then by standard results $\Phi_\hbar[\gamma]\in\mathcal{C}^2(B_\hbar)\cap\mathcal{C}(\overline{B}_\hbar)$ and the representation formula (2.3) holds; by density (2.3) can be extended to any $\gamma\in L^2(B_\hbar)$.

Now assume that γ is radial; it is immediate that $\Phi_\hbar[\gamma]$ is radial too. Then, by using the spherical coordinates in the space we can write

$$\begin{aligned}\Phi_\hbar[\gamma](x) &= \Phi_\hbar[\gamma]((0,0,|x|))\\ &= \frac{e}{\hbar}\int_{B_\hbar}\bigg(\frac{1}{\sqrt{|y|^2+|x|^2-2|x|y_3}} - \frac{|\hbar x|}{\sqrt{|y|^2|\hbar x|^4+|x|^2-2y_3|\hbar x|^2|x|}}\bigg)\gamma(y)\,dy\\ &= \frac{2\pi e}{\hbar}\int_0^{1/\hbar} r^2\gamma(r)\,dr\\ &\quad\cdot\int_0^\pi\bigg(\frac{\sin\theta}{\sqrt{r^2+|x|^2-2|x|r\cos\theta}} - \frac{\sin\theta}{\sqrt{r^2|\hbar x|^2+\hbar^{-2}-2|x|r\cos\theta}}\bigg)\,d\theta,\end{aligned}$$

and a direct integration leads to (b). An immediate computation shows that $F_\hbar$ is differentiable and

$$\begin{aligned}F_\hbar'(u)[w] &= \frac{2e}{\hbar}\int_{B_\hbar} uw\,dx\int_{B_\hbar} G_\hbar(x,y)u^2\,dy + \frac{2e}{\hbar}\int_{B_\hbar} u^2\,dx\int_{B_\hbar} uwG_\hbar(x,y)\,dy\\ &= 4\int_{B_\hbar} uw\Phi_\hbar[u^2]\,dx. \qquad\square\end{aligned}$$

Let us define the functional $J_\hbar: H_0^1(B_\hbar)\to\mathbb{R}$ given by

$$J_\hbar(u) = \frac{1}{2}\int_{B_\hbar}|\nabla u|^2\,dx + \frac{e}{2}\int_{B_\hbar} f_\hbar|u|^2\,dx + \frac{e}{4}\int_{B_\hbar}|u|^2\Phi_\hbar[u^2]\,dx.$$

According to Propositions 2.1 and 2.3 $J_\hbar\in\mathcal{C}^1(H_0^1(B_\hbar),\mathbb{R})$. Let us consider the manifold

$$\mathcal{M}_\hbar = \bigg\{u\in H_0^1(B_\hbar)\ \bigg|\ \int_{B_\hbar}|u|^2\,dx = 1\bigg\}.$$

Our aim is to capture solutions of schmaxone–schmaxtwo as critical points of the functional $J_\hbar$ constrained on the manifold $\mathcal{M}_\hbar$. Then for every $\hbar>0$ we define the infimum value $J_\hbar^*$ as follows:

$$J_\hbar^* = \inf_{u\in\mathcal{M}_\hbar} J_\hbar(u).$$

3. Preliminaries

In this section we collect some preliminary facts which will be used in the following for the proof of Theorem 1.1. We begin with the following elementary lemma.

Lemma 3.1. *Let $A\subset\mathbb{R}^N$ a measurable set, $\{f_n\}, f\subset L^\infty(A,\mathbb{R})$ and $g_n: A\to\mathbb{R}$ two sequences of functions verifying:*

(a) *g_n is measurable, $g_n(x)\ge 0$ a.e. in A,*
(b) *$\inf_A f>0$, $(f_n-f)\chi_{\operatorname{supp} g_n}\to 0$ uniformly in A as $n\to\infty$.*

Then the following holds

$$\int_A f_n g_n\,dx = (1+o(1))\int_A f\,g_n\,dx.$$

The proof is an easy computation.

Before going on we fix some notations. In the remaining part of the paper we assume, for sake of simplicity, $z_0 = (0,0,1)$ and we use the spherical coordinates $x = x(r,\theta,\varphi)$:

$$x_1 = r\sin\theta\cos\varphi,\quad x_2 = r\sin\theta\sin\varphi,\quad x_3 = r\cos\theta, \tag{3.1}$$

for $r>0$, $\theta\in[0,\pi]$, $\varphi\in[0,2\pi]$. For every $i\in\{1,\dots,\ell\}$ let M_i denote a rotation matrix in $\mathbb{R}^3$ such that $M_i z_0 = z_i$.

A crucial step in the proof of Theorem 1.1 are next lemmas which describe the behaviour of the functional $J_\hbar$ with respect to suitable sequences of functions in $H_0^1(B_\hbar)$.

Lemma 3.2. *Consider $i\in\{0,\dots,\ell\}$, $\hbar_n\to 0$ an arbitrary sequence and let $w_n\in L^2(B_{\hbar_n})$ be such that $w_n\equiv 0$ if $|\hbar_n x - z_i|\ge \delta_n$ for some $\delta_n\to 0^+$. Then the following holds*

(a) *if $i\in\{0,\dots,\ell'\}$, then*

$$\begin{aligned}(1+o(1))\int_{B_{\hbar_n}} |w_n|^2 f_{\hbar_n}\,dx &= \frac{A_i}{2\pi}\int_{B_{\hbar_n}} |w_n|^2(1-|\hbar_n x|)\,dx\\ &\qquad + o\bigg(\int_{B_{\hbar_n}} |w_n(M_i x)|^2\theta^{\overline{q}}(x)\,dx\bigg);\end{aligned}$$

(b) *if $i\in\{\ell'+1,\dots,\ell\}$ then, for some $C>0$,*

$$(1+o(1))\int_{B_{\hbar_n}} |w_n|^2 f_{\hbar_n}\,dx \ge C\int_{B_{\hbar_n}} |w_n(M_i x)|^2\theta^{\overline{q}}(x)\,dx$$

If, in addition, if $i=0$ and $w_n\in H_0^1(B_{\hbar_n})$, then

$$\begin{aligned}&(1+o(1))\|\nabla w_n\|_2^2\\ &\quad = \int_0^{1/\hbar_n} dr\int_0^\pi d\theta\int_0^{2\pi}\bigg(\frac{\theta}{\hbar_n^2}\bigg|\frac{\partial w_n}{\partial r}\bigg|^2 + \theta\bigg|\frac{\partial w_n}{\partial \theta}\bigg|^2 + \frac{1}{\theta}\bigg|\frac{\partial w_n}{\partial \varphi}\bigg|^2\bigg)\,d\varphi.\end{aligned} \tag{3.2}$$

Proof. Fix $x\in B_{\hbar_n}$ with $|\hbar_n x - z_i|\le\delta_n$. Notice that, since $|a^{-3}-b^{-3}|\le 3(\min\{a,b\})^{-2}|a-b|$ for $a,b>0$, then for every $y\in\partial B_1$ with $|y-z_i|\ge\sqrt[4]{\delta_n}$ we have

$$\bigg|\frac{1}{|y-z_i|^3}-\frac{1}{|y-\hbar_n x|^3}\bigg| \le \frac{3}{(\sqrt[4]{\delta_n}-\delta_n)^2}|\hbar_n x - z_i| \le \frac{3\delta_n}{(\sqrt[4]{\delta_n}-\delta_n)^2}$$

and, consequently,

$$\begin{aligned}\Big|\int_{|y-z_i|\geq\sqrt[4]{\delta_n}}&\frac{g(y)-g(z_0)}{|y-\hbar_n x|^3}\,dS-A_i\Big|\\&\leq\int_{|y-z_i|\geq\sqrt[4]{\delta_n}}(g(y)-g(z_0))\Big|\frac{1}{|y-\hbar_n x|^3}-\frac{1}{|y-z_i|^3}\Big|\,dS\\&\quad+\int_{|y-z_i|\leq\sqrt[4]{\delta_n}}\frac{g(y)-g(z_0)}{|y-z_i|^3}\,dS\\&\leq\frac{3\delta_n}{(\sqrt[4]{\delta_n}-\delta_n)^2}\int_{\partial B_1}(g(y)-g(z_0))\,dS\\&\quad+\int_{|y-z_i|\leq\sqrt[4]{\delta_n}}\frac{g(y)-g(z_0)}{|y-z_i|^3}\,dS\to 0\end{aligned}$$

by (1.7); hence we deduce

$$(1+|\hbar_n x|)\int_{|y-z_i|\geq\sqrt[4]{\delta_n}}\frac{g(y)-g(z_0)}{|y-\hbar_n x|^3}\,dS\to 2A_i$$

uniformly for $|\hbar_n x-z_i|\leq\delta_n$. Lemma 3.1 gives

$$\begin{aligned}\int_{B_{\hbar_n}}|w_n|^2\frac{1-|\hbar_n x|^2}{4\pi}\,dx\int_{|y-z_i|\geq\sqrt[4]{\delta_n}}&\frac{g(y)-g(z_0)}{|y-\hbar_n x|^3}\,dS\\&=\frac{A_i+o(1)}{2\pi}\int_{B_{\hbar_n}}|w_n|^2(1-|\hbar_n x|)\,dx.\end{aligned}$$

It remains to examine the term

$$\begin{aligned}\int_{B_{\hbar_n}}|w_n|^2(1-|\hbar_n x|)^2\,dx&\int_{|y-z_i|\leq\sqrt[4]{\delta_n}}\frac{g(y)-g(z_0)}{|y-\hbar_n x|^3}\,dS\\&=\int_{B_{\hbar_n}}|w_n(M_i x)|^2(1-|\hbar_n x|)^2\,dx\int_{|y-z_0|\leq\sqrt[4]{\delta_n}}\frac{g(M_i y)-g(z_0)}{|y-\hbar_n x|^3}\,dS.\end{aligned}\tag{3.3}$$

Then for every $x=x(|x|,\theta,\varphi)\in B_{\hbar_n}$ consider the rotation matrix

$$M_x\equiv\begin{pmatrix}\cos\theta\cos\varphi & -\sin\varphi & \sin\theta\cos\varphi\\ \cos\theta\sin\varphi & \sin\theta & \sin\theta\sin\varphi\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}.$$

It is obvious that $M_x(0,0,|x|)=x$ and $M_x(-\sin\theta,0,\cos\theta)=z_0$.

First assume $i\in\{0,\dots,\ell'\}$. Fix $x=x(|x|,\theta,\varphi)\in B_{\hbar_n}$ with $|\hbar_n x-z_0|\leq\delta_n$; then $\sin\theta\leq\delta_n$ and, for every $y=y(1,\theta',\varphi')\in\partial B_1$ with $|M_x y-z_0|\leq\sqrt[4]{\delta_n}$, we have

$$\begin{aligned}|y-z_0|&\leq|y-M_x^{-1}z_0|+|z_0-M_x^{-1}z_0|\\&\leq\sqrt[4]{\delta_n}+\sqrt{2-2\cos\theta}\leq\sqrt[4]{\delta_n}+2\delta_n\leq 2\sqrt[4]{\delta_n},\end{aligned}$$

at least for large n. Then, for such x compute

$$\begin{aligned}
(3.4)\quad & \int_{|y-z_0|\le \sqrt[4]{\delta_n}} \frac{|y-z_0|^{\overline{q}}}{|y-\hbar_n x|^3}\, dS = \int_{|M_x y-z_0|\le \sqrt[4]{\delta_n}} \frac{|M_x y-z_0|^{\overline{q}}}{|M_x y-\hbar_n x|^3}\, dS \\
&\le \int_{|y-z_0|\le 2\sqrt[4]{\delta_n}} \frac{|M_x y-z_0|^{\overline{q}}}{|M_x y-\hbar_n x|^3}\, dS = \int_{|y-z_0|\le 2\sqrt[4]{\delta_n}} \frac{|y-M_x^{-1} z_0|^{\overline{q}}}{|y-(0,0,|\hbar_n x|)|^3}\, dS \\
&\le 2^{\overline{q}/2}\int_0^{2\pi} d\varphi' \int_0^{\arcsin(2\sqrt[4]{\delta_n})} \frac{(1+\sin\theta\sin\theta'\cos\varphi'-\cos\theta'\cos\theta)^{\overline{q}/2}}{(1+|\hbar_n x|^2-2|\hbar_n x|\cos\theta')^{3/2}}\sin\theta'\, d\theta'.
\end{aligned}$$

Note that for every $\theta,\theta'\in(0,\pi/2)$ and $\varphi'\in(0,2\pi)$

$$\begin{aligned}
(3.5)\quad & (1+\sin\theta\sin\theta'\cos\varphi'-\cos\theta'\cos\theta)^{\overline{q}/2} \\
&\le C(1-\cos\theta\cos\theta')^{\overline{q}/2} + C\sin^{\overline{q}/2}\theta\sin^{\overline{q}/2}\theta' \\
&\le C(1-\cos\theta)^{\overline{q}/2} + C(1-\cos\theta')^{\overline{q}/2} + C\sin^{\overline{q}/2}\theta\sin^{\overline{q}/2}\theta' \\
&\le C\sin^{\overline{q}}\theta + C\sin^{\overline{q}}\theta'.
\end{aligned}$$

A direct computation shows that

$$\begin{aligned}
&\int_0^{\arcsin(2\sqrt[4]{\delta_n})} \frac{\sin\theta' d\theta'}{(1+|\hbar_n x|^2-2|\hbar_n x|\cos\theta')^{3/2}} \\
&\quad = \frac{1}{|\hbar_n x|}\left(\frac{1}{1-|\hbar_n x|} - \frac{1}{(1+|\hbar_n x|^2-2|\hbar_n x|\cos\arcsin(2\sqrt[4]{\delta_n}))^{1/2}}\right) \\
&\quad \le \frac{2}{1-|\hbar_n x|}
\end{aligned}$$

for $|\hbar_n x - z_0| \le \delta_n$; furthermore, since $1+x^2-2x\cos\theta' \ge \sin^2\theta'$ for every $x\in[0,1]$,

$$\begin{aligned}
&\int_0^{\arcsin(2\sqrt[4]{\delta_n})} \frac{(\sin\theta')^{\overline{q}+1}}{(1+|\hbar_n x|^2-2|\hbar_n x|\cos\theta')^{3/2}}\, d\theta' \\
&\quad \le \int_0^{\arcsin(2\sqrt[4]{\delta_n})} \frac{1}{(\sin\theta')^{2-\overline{q}}}\, d\theta' = o(1)
\end{aligned}$$

uniformly for $|\hbar_n x - z_0| \le \delta_n$. Combining last two inequality with (3.4), and using (3.5), one deduces that

$$\int_{|y-z_0|\le \sqrt[4]{\delta_n}} \frac{|y-z_0|^{\overline{q}}}{|y-\hbar_n x|^3}\, dS \le C\frac{\theta^{\overline{q}}(x)}{1-|\hbar_n x|} + o(1)$$

uniformly for $|\hbar_n x - z_0| \le \delta_n$. Since by assumption (g3)

$$\sup_{|y-z_0|\le \sqrt[4]{\delta_n}} \frac{g(M_i y)-g(z_0)}{|y-z_0|^{\overline{q}}} = o(1),$$

using (3.3) we conclude

$$\int_{B_{\hbar_n}} |w_n|^2(1-|\hbar_n x|^2)\,dx \int_{|y-z_i|\le\sqrt[4]{\delta_n}} \frac{g(y)-g(z_0)}{|y-\hbar_n x|^3}\,dS = o(1)\int_{B_{\hbar_n}} |w_n(M_i x)|^2\theta^{\overline{q}}(x)\,dx + o(1)\int_{B_{\hbar_n}} |w_n|^2(1-|\hbar_n x|)\,dx$$

and part (a) of the lemma follows.

Now assume that $i\in\{\ell'+1,\dots,\ell\}$. Fix $x=x(|x|,\theta,\varphi)\in B_{\hbar_n}$ with $|\hbar_n x - z_0|\le\delta_n$; we have $\sin\theta\le\delta_n$; then for every $y\in\partial B_1$ with $|y-z_0|\le 1-|\hbar_n x|$

$$\begin{aligned}|M_x y - z_0| &\le |y-z_0| + |z_0 - M_x z_0|\\ &\le 1-|\hbar_n x| + \sqrt{2(1-\cos\theta)} \le \delta_n + 2\delta_n \le \sqrt[4]{\delta_n}\end{aligned}$$

at least for large n. Then for such x compute

$$\begin{aligned}(3.6)\quad \int_{|y-z_0|\le\sqrt[4]{\delta_n}} \frac{|y-z_0|^{\overline{q}}}{|y-\hbar_n x|^3}\,dS &= \int_{|M_x y - z_0|\le\sqrt[4]{\delta_n}} \frac{|M_x y - z_0|^{\overline{q}}}{|M_x y-\hbar_n x|^3}\,dS\\ &\ge \int_{|y-z_0|\le 1-|\hbar_n x|} \frac{|M_x y - z_0|^{\overline{q}}}{|M_x y-\hbar_n x|^3}\,dS = \int_{|y-z_0|\le 1-|\hbar_n x|} \frac{|y-M_x^{-1}z_0|^{\overline{q}}}{|y-(0,0,|\hbar_n x|)|^3}\,dS\\ &\ge 2^{\overline{q}/2}\int_0^{2\pi} d\varphi' \int_0^{1-|\hbar_n x|} \frac{(1+\sin\theta\sin\theta'\cos\varphi' - \cos\theta\cos\theta')^{\overline{q}/2}}{(1+|\hbar_n x|^2 - 2|\hbar_n x|\cos\theta')^{3/2}}\sin\theta'\,d\theta'.\end{aligned}$$

We recall the following well known inequality:

$$|a-b|^{\overline{q}/2} \ge C_1|a|^{\overline{q}/2} - C_2|b|^{\overline{q}/2} \quad \text{for all } a,b\in\mathbb{R},$$

for some positive constants C_1, C_2. Then for every $\theta,\theta'\in[0,\pi/2]$, $\varphi'\in[0,2\pi]$, $\varepsilon>0$ (using Young's inequality):

$$\begin{aligned}|1+\sin\theta\sin\theta'\cos\varphi' &- \cos\theta\cos\theta'|^{\overline{q}/2}\\ &\ge C_1(1-\cos\theta\cos\theta')^{\overline{q}/2} - C_2\sin^{\overline{q}/2}\theta\sin^{\overline{q}/2}\theta'\\ &\ge C_1(1-\cos\theta)^{\overline{q}/2} - \frac{C_2\varepsilon}{2}\sin^{\overline{q}}\theta - \frac{C_2}{2\varepsilon}\sin^{\overline{q}}\theta'\\ &\ge \frac{C_1}{2^{\overline{q}/2}}\sin^{\overline{q}}\theta - \frac{C_2}{2}\varepsilon\sin^{\overline{q}}\theta - \frac{C_2}{2\varepsilon}\sin^{\overline{q}}\theta',\end{aligned}$$

by which, choosing $\varepsilon = C_1/2^{\overline{q}/2}C_2$,

$$(3.7)\qquad |1+\sin\theta\sin\theta'\cos\varphi' - \cos\theta\cos\theta'|^{\overline{q}/2} \ge C_3\sin^{\overline{q}}\theta - C_4\sin^{\overline{q}}\theta'.$$

Taking into account that $1+t^2-2t\cos(1-t)\ge 16(1-t)^2/9$ as $t\to 1^-$, we easily compute

$$\begin{aligned}&\int_0^{1-|\hbar_n x|} \frac{\sin\theta'\,d\theta'}{(1+|\hbar_n x|^2-2|\hbar_n x|\cos\theta')^{3/2}}\\ &= \frac{1}{|\hbar_n x|}\left(\frac{1}{1-|\hbar_n x|} - \frac{1}{(1+|\hbar_n x|^2 - 2|\hbar_n x|\cos(1-|\hbar_n x|))^{1/2}}\right) \ge \frac{1}{4}\cdot\frac{1}{1-|\hbar_n x|}\end{aligned}$$

for $|\hbar_n x - z_0| \le \delta_n$. Proceeding as in the previous case

$$\int_0^{1-|\hbar_n x|} \frac{(\sin\theta')^{\overline{q}+1}}{(1+|\hbar_n x|^2 - 2|\hbar_n x|\cos\theta')^{3/2}}\,d\theta' \le \int_0^{\delta_n} \frac{1}{(\sin\theta')^{2-\overline{q}}}\,d\theta' = o(1)$$

uniformly for $|\hbar_n x - z_0| \le \delta_n$. Combining last two inequality with (3.6) and using (3.7), one deduces

$$\int_{|y-x_i|\le\sqrt[4]{\delta_n}} \frac{|y-z_0|^{\overline{q}}}{|y-\hbar_n x|^3}\,dS \ge C\frac{\sin^{\overline{q}}\theta(x)}{1-|\hbar_n x|} + o(1) \ge C\frac{\theta^{\overline{q}}(x)}{1-|\hbar_n x|} + o(1)$$

uniformly for $|\hbar_n x - z_0| \le \delta_n$. Since according to assumption (g3)

$$\inf_{|y-z_0|\le\sqrt{[4]}\delta_n} \frac{g(M_i y) - g(z_0)}{|y-z_0|^{\overline{q}}} \ge C,$$

by (3.3) we conclude

$$\begin{aligned}\int_{B_{\hbar_n}} |w_n|^2(1-|\hbar_n x|^2)\,dx \int_{|y-z_i|\le\sqrt{[4]}\delta_n} \frac{g(y)-g(z_0)}{|y-\hbar_n x|^3}\,dS \\ \ge C\int_{B_{\hbar_n}} |w_n(M_i x)|^2\theta^{\overline{q}}(x)\,dx + o(1)\int_{B_{\hbar_n}} |w_n|^2(1-|\hbar_n x|)\,dx\end{aligned}$$

and part (b) follows. In order to prove (3.2) it is sufficient to compute:

$$\begin{aligned}\int_{B_{\hbar_n}} |\nabla w_n|^2 = \int_0^{1/\hbar_n} dr \int_0^{\pi} d\theta \\ \cdot\int_0^{2\pi} r^2\sin\theta\left(\left|\frac{\partial w_n}{\partial r}\right|^2 + \frac{1}{r^2}\left|\frac{\partial w_n}{\partial\theta}\right|^2 + \frac{1}{r^2\sin^2\theta}\left|\frac{\partial w_n}{\partial\varphi}\right|^2\right)d\varphi.\end{aligned}$$

Now, since $\hbar_n^2 r^2 \to 1$ and $(\sin\theta)/\theta \to 1$ uniformly on supp w_n, then Lemma 3.1 gives (3.2). $\square$

We are now in position to provide the following result.

Corollary 3.3. $\lim_{\hbar\to 0^+} J_\hbar^* = 0$.

Proof. Fix $\xi \in \mathcal{C}_0^\infty(\mathbb{R})$ with supp $\xi \subset (0,1)$; for every $\hbar > 0$ and $x \in B_\hbar$ set $\varphi_\hbar(x) = \xi(\sqrt{\hbar}|x| + 1 - 1/\sqrt{\hbar})$. Then $\varphi_\hbar \in \mathcal{C}_0^\infty(B_\hbar)$ and

$$\varphi_\hbar \equiv 0 \quad \text{if } |x| \le \frac{1}{\hbar} - \frac{1}{\sqrt{\hbar}}.$$

For every $\hbar > 0$ take $\eta_\hbar \in \mathcal{C}^\infty([0,\pi])$ satisfying

$$\eta_\hbar = 1 \quad \text{if } \theta \le \hbar^{1/6}, \qquad \eta_\hbar = 0 \quad \text{if } \theta \ge \hbar^{1/6} + \hbar^{1/5}, \qquad |\eta_\hbar'| \le 2\hbar^{-1/5}.$$

Define

$$w_\hbar(x) = \lambda_\hbar\varphi_\hbar(x)\eta_\hbar(\theta(x)) \quad \text{where } \theta(x) = \arctan\frac{x_3}{\sqrt{x_1^2+x_2^2}}$$

(according to (3.1)) and $\lambda_\hbar > 0$ is such that $\int_{B_\hbar} |w_\hbar|^2\,dx = 1$. More precisely

$$\begin{aligned}(3.8)\qquad \lambda_\hbar^{-2} &= \int_{B_\hbar} |\eta_\hbar(\theta(x))\varphi_\hbar(x)|^2\,dx \geq 2\pi \int_0^{1/\hbar} r^2|\varphi_\hbar|^2\,dr \int_0^{\hbar^{1/6}} \sin\theta\,d\theta \\ &\geq 2\pi(1-\cos(\hbar^{1/6}))\Big(\frac{1}{\hbar}-\frac{1}{\sqrt{\hbar}}\Big)^2 \int_0^{1/\hbar} |\varphi_\hbar|^2\,dr \\ &\geq \pi(1-\cos(\hbar^{1/6}))\frac{1}{\hbar^2\sqrt{\hbar}}\int_0^1 |\xi|^2\,ds\end{aligned}$$

at least for small $\hbar$.

Now for every $x = x(|x|,\theta,\varphi) \in B_\hbar$, since

$$|\nabla(\varphi_\hbar\eta_\hbar(\theta))|^2 \leq 2|\varphi_\hbar'(|x|)\eta_\hbar(\theta)|^2 + \frac{2}{|x|^2}|\eta_\hbar'(\theta)\varphi_\hbar(|x|)|^2,$$

we compute

$$\begin{aligned}(3.9)\qquad &\int_{B_\hbar} |\nabla w_\hbar|^2\,dx \\ &\leq 4\pi\lambda_\hbar^2 \int_0^{\hbar^{1/6}+\hbar^{1/5}} \sin\theta\,d\theta \int_0^{1/\hbar} (r^2|\varphi_\hbar'(r)|^2 + \hbar^{-2/5}|\varphi_\hbar(r)|^2)\,dr \\ &\leq 4\pi\frac{\lambda_\hbar^2}{\hbar\sqrt{\hbar}}(1-\cos(\hbar^{1/6}+\hbar^{1/5}))\int_0^1 \big(|\xi'|^2+|\xi|^2\big)\,ds \\ &\leq 4\hbar\frac{1-\cos(\hbar^{1/6}+\hbar^{1/5})}{1-\cos(\hbar^{1/6})}\int_0^1 (|\xi'|^2+|\xi|^2)\,ds\Big(\int_0^1 |\xi|^2\,ds\Big)^{-1} \to 0,\end{aligned}$$

as $\hbar \to 0$, where, in the last inequality, we have used (3.8).

By Lemma 3.2, considering a sequence $\hbar_n \to 0$,

$$\begin{aligned}(3.10)\qquad &(1+o(1))\int_{B_{\hbar_n}} |w_{\hbar_n}|^2 f_{\hbar_n}\,dx \\ &\quad= \frac{A_0}{2\pi}\int_{B_{\hbar_n}} |w_{\hbar_n}|^2(1-|\hbar_n x|)\,dx + o(\hbar_n^{1/6}) \leq \frac{A_0}{2\pi}\sqrt{\hbar_n} + o(\hbar_n^{1/6}).\end{aligned}$$

Finally by Proposition 2.3, since

$$r\frac{\min\{|x|,r\}-\hbar|x|r}{|\hbar x|} \leq \frac{\sqrt{\hbar}}{\hbar^2} \quad \text{for } |x|,r \in \Big(\frac{1}{\hbar}-\frac{1}{\sqrt{\hbar}},\frac{1}{\hbar}\Big),$$

we get

$$\begin{aligned}(3.11)\qquad \int_{B_\hbar} |w_\hbar|^2\Phi_\hbar[w_\hbar^2]dx &\leq \int_{B_\hbar} |w_\hbar|^2\Phi_\hbar[\lambda_\hbar^2\varphi_\hbar^2]\,dx \\ &\leq 4\pi e\frac{\lambda_\hbar^2\sqrt{\hbar}}{\hbar^2}\int_{B_\hbar}|w_\hbar|^2\,dx\int_0^{1/\hbar}|\varphi_\hbar(r)|^2\,dr \\ &= 4\pi e\frac{\lambda_\hbar^2}{\hbar^2}\int_0^1 |\xi|^2\,ds \leq 4e\frac{\sqrt{\hbar}}{1-\cos(\hbar^{1/6})} \to 0\end{aligned}$$

as $\hbar \to 0$. Combining (3.9)–(3.11) we achieve the thesis. □

Before going on with the second lemma we need the following result concerning the symmetrization of functions on a ball.

PROPOSITION 3.4. *Consider $\hbar > 0$. For every $\gamma \in L^2(B_\hbar)$ we can associate a function $\gamma^* \in L^2(B_\hbar)$ such that γ^* is radial and the following properties hold*

$$\int_{B_\hbar} \Phi_\hbar[\gamma]\gamma\,dx \geq \int_{B_\hbar} \Phi_\hbar[\gamma^*]\gamma^*\,dx, \qquad \int_{B_\hbar} f(|x|)\gamma^*\,dx = \int_{B_\hbar} f(|x|)\gamma\,dx$$

for every $f \in L^\infty([0,1/\hbar])$. Furthermore the operator $\gamma \in L^2(B_\hbar) \mapsto \gamma^* \in L^2(B_\hbar)$ is linear and $(\gamma \circ M)^* = \gamma^*$ for every rotation matrix M in $\mathbb{R}^3$.

PROOF. First assume $\gamma \in \mathcal{C}(\overline{B}_\hbar)$. For every $x \in B_\hbar$ define

$$\gamma^*(x) = \frac{1}{4\pi|x|^2}\int_{\partial B(0,|x|)} \gamma(y)\,dS = \frac{1}{4\pi}\int_{\partial B_1} \gamma(|x|y)\,dS.$$

It is immediate to prove that $\gamma^* \in C(\overline{B}_\hbar)$ and for every $f \in L^\infty([0,1/\hbar])$,

$$\begin{aligned}\int_{B_\hbar} f(|x|)\gamma\,dx &= \int_0^{1/\hbar} f(r)\,dr \int_{\partial B(0,r)} \gamma\,dS \\ &= 4\pi\int_0^{1/\hbar} r^2 f(r)\gamma^*\,dr = \int_{B_\hbar} f(|x|)\gamma^*\,dx.\end{aligned}$$

Furthermore, since by Proposition 2.3 $\Phi_\hbar[\gamma] \in \mathcal{C}(\overline{B}_\hbar)$, we can compute

$$\Phi_\hbar[\gamma]^*(r) = \frac{e}{4\pi\hbar r^2}\int_{B_\hbar} \gamma(y)\,dy \int_{\partial B(0,r)} G_\hbar(x,y)\,dS(x).$$

Notice that for every $r > 0$

$$\int_{\partial B(0,r)} G_\hbar(x,y)\,dS(x) = \frac{4\pi r}{|y|}(\min\{r,|y|\} - \hbar r|y|),$$

then the function $y \in B_\hbar \mapsto \int_{\partial B(0,r)} G_\hbar(x,y)\,dS(x)$ is radial and belongs to $L^\infty([0,1/\hbar])$, then (3.12) applies and gives

$$\begin{aligned}\Phi_\hbar[\gamma]^*(r) &= \frac{e}{4\pi\hbar r^2}\int_{B_\hbar} \gamma^*(y)\,dy \int_{\partial B(0,r)} G_\hbar(x,y)\,dS(x) \\ &= \frac{1}{4\pi r^2}\int_{\partial B(0,r)} \Phi_\hbar[\gamma^*]\,dS = \Phi_\hbar[\gamma^*](r)\end{aligned}$$

by Proposition 2.3(b). Setting $f = \Phi_\hbar[\gamma]^* = \Phi_\hbar[\gamma^*]$ and $f = \gamma^*$, from (3.12) we deduce respectively

$$\int_{B_\hbar} (\gamma - \gamma^*)\Phi_\hbar[\gamma^*]\,dx = 0, \qquad \int_{B_\hbar} \gamma^*(\Phi_\hbar[\gamma] - \Phi_\hbar[\gamma^*])\,dx = 0. \tag{3.13}$$

Then, using (3.13) and equation (2.4), we compute

$$\begin{aligned}
&\int_{B_\hbar} \gamma\Phi_\hbar[\gamma]\,dx - \int_{B_\hbar} \gamma^*\Phi_\hbar[\gamma^*]\,dx \\
&\qquad = \int_{B_\hbar} \gamma(\Phi_\hbar[\gamma] - \Phi_\hbar[\gamma^*])\,dx - \int_{B_\hbar} \gamma^*(\Phi_\hbar[\gamma] - \Phi_\hbar[\gamma^*])\,dx \\
&\qquad = \frac{\hbar}{4\pi e}\int_{B_\hbar} \nabla\Phi_\hbar[\gamma]\nabla(\Phi_\hbar[\gamma] - \Phi_\hbar[\gamma^*])\,dx \\
&\qquad\quad - \frac{\hbar}{4\pi e}\int_{B_\hbar} \nabla\Phi_\hbar[\gamma^*]\nabla(\Phi_\hbar[\gamma] - \Phi_\hbar[\gamma^*])\,dx \\
&\qquad = \frac{\hbar}{4\pi e}\int_{B_\hbar} |\nabla\Phi_\hbar[\gamma] - \nabla\Phi_\hbar[\gamma^*]|^2\,dx \geq 0.
\end{aligned}$$

In order to conclude we use the density of $\mathcal{C}(\overline{B}_\hbar)$ in $L^2(B_\hbar)$. □

Finally we derive an estimates regarding the nonlinear energy term

$$\int_{B_\hbar} |u|^2\Phi_\hbar[u^2].$$

LEMMA 3.5. *Consider $N \geq 100$, $\hbar_n \to 0$ an arbitrary sequence, $y_n^1, \ldots, y_n^N$, $y_0 \in \partial B_1$, $v_n \in L^4(B_{\hbar_n})$ such that*

$$v_n = 0 \quad \text{for } |\hbar_n x - y_0| \geq \delta_n, \quad \delta_n \to 0^+, \quad \delta_n \leq \frac{1}{N}\min_{i\neq j} |y_n^i - y_n^j|.$$

Then, considering $Q_n^1, \ldots, Q_n^N$ rotation matrixes in $\mathbb{R}^3$ such that $Q_n^i y_0 = y_n^i$ and setting $w_n(x) = \sum_{i=1}^N v_n((Q_n^i)^{-1}x)$, the following holds

$$\int_{B_{\hbar_n}} w_n^2\Phi_{\hbar_n}[w_n^2]dx \leq 2N\int_{B_{\hbar_n}} v_n^2\Phi_{\hbar_n}[v_n^2]\,dx.$$

PROOF. First note that by construction for $i \neq j$ the functions $v_n((Q_n^i)^{-1}x)$ and $v_n((Q_n^j)^{-1}x)$ have disjoint supports. Then we compute

$$\begin{aligned}
\int_{B_{\hbar_n}} w_n^2\Phi_{\hbar_n}[w_n^2]\,dx &= \sum_{i=1}^N\sum_{j=1}^N \frac{e}{\hbar_n}\int_{B_{\hbar_n}} |v_n(x)|^2\,dx\int_{B_{\hbar_n}} G_{\hbar_n}(Q_n^i x, Q_n^j y)|v_n(y)|^2\,dy \\
&= \sum_{i=1}^N \int_{B_{\hbar_n}} |v_n|^2\Phi_{\hbar_n}[v_n^2]\,dx \\
&\quad + \sum_{i=1}^N\sum_{j\neq i} \frac{e}{\hbar_n}\int_{B_{\hbar_n}} |v_n(x)|^2\,dx\int_{B_{\hbar_n}} G_{\hbar_n}(Q_n^i x, Q_n^j y)|v_n(y)|^2\,dy \\
&= N\int_{B_{\hbar_n}} |v_n|^2\Phi_{\hbar_n}[v_n^2]\,dx \\
&\quad + \sum_{i=1}^N\sum_{j\neq i} \frac{e}{\hbar_n}\int_{B_{\hbar_n}} |v_n(x)|^2\,dx\int_{B_{\hbar_n}} G_{\hbar_n}(Q_n^i x, Q_n^j y)|v_n(y)|^2\,dy.
\end{aligned}$$

By (2.2) for large n and for every $x, y \in B(y_0/\hbar_n, \delta_n/\hbar_n)$ (since $\hbar_n|x-y| \le 2\delta_n$ and $\hbar_n|y|\hbar_n x|^2 - x| \le |\hbar_n x|^2\hbar_n|y-x| + |\hbar_n x|(1-|\hbar_n x|^2) \le 4\delta_n$)

$$\frac{G_{\hbar_n}(x,y)}{\hbar_n^3} \ge \frac{|x|^2(1-|\hbar_n x|)(1-|\hbar_n y|)}{48\delta_n^3}$$

and, if $i \ne j$ (since $\hbar_n|x-y| \ge (N-2)\delta_n$ and $\hbar_n|y|\hbar_n x|^2 - x| \ge |\hbar_n x|^2\hbar_n|y - x| - |\hbar_n x|(1-|\hbar_n x|^2) \ge (N-5)\delta_n$),

$$\frac{G_{\hbar_n}(Q_n^i x, Q_n^j y)}{\hbar_n^3} \le \frac{|x|^2(1-|\hbar_n x|)(1-|\hbar_n y|)}{(N-5)^3\delta_n^3}.$$

Hence combining last two inequalities we can write

$$\begin{aligned}\int_{B_{\hbar_n}} |w_n|^2\Phi_{\hbar_n}[w_n^2]\,dx &\le N\int_{B_{\hbar_n}} v_n^2\Phi_{\hbar_n}[v_n^2]\,dx \\ &\quad + \sum_{i=1}^N\sum_{j\ne i}\frac{48e}{\hbar_n(N-5)^3}\int_{B_{\hbar_n}} v_n^2\,dx\int_{B_{\hbar_n}} G_{\hbar_n}(x,y)v_n^2\,dy \\ &= N\int_{B_{\hbar_n}} |v_n|^2\Phi_{\hbar_n}[v_n^2]\,dx + \frac{48N(N-1)}{(N-5)^3}\int_{B_{\hbar_n}} |v_n|^2\Phi_{\hbar_n}[v_n^2]\,dx.\end{aligned}$$

Since $48N(N-1)/(N-5)^3 \le N$ for $N \ge 100$ we obtain the thesis. □

4. Asymptotic estimates

The purpose in now to provide some suitable *asymptotic estimates* which will be useful for analyzing the behaviour of our solutions in the limit when $\hbar \to 0^+$. We begin with the following lemma which establishes the existence of a minimizer for the infimum $J_\hbar^*$ and proves that its L^2-norm is concentrated around the points z_i.

Lemma 4.1. *For every $\hbar > 0$ the infimum $J_\hbar^*$ is attained in the set $\mathcal{M}_\hbar$ by a function $u_\hbar \ge 0$. Furthermore there exists $\delta_\hbar \to 0^+$ such that*

$$\lim_{\hbar\to 0^+}\int_{B_\hbar\setminus\bigcup_{i=0}^{\ell} B(z_i/\hbar,\delta_\hbar/\hbar)} |u_\hbar|^2\,dx = 0. \tag{4.1}$$

Proof. Fix $\hbar > 0$ and consider $\{u_n\} \subset \mathcal{M}_\hbar$ a minimizing sequence. Since $\{u_n\}$ is bounded in the $H_0^1(B_\hbar)$-norm, up to a subsequence we have

$$u_n \rightharpoonup u_\hbar \quad \text{weakly in } H_0^1(B_\hbar), \qquad u_n \to u_\hbar \quad \text{in } L^2(B_\hbar) \text{ and a.e. as } n\to\infty,$$

for some $u_\hbar \in \mathcal{M}_\hbar$. By using the weakly lower semicontinuity and Fatou's lemma, we obtain $J_\hbar^* = \lim_n J_\hbar(u_n) \ge J_\hbar(u_\hbar)$, i.e. $u_\hbar$ is a minimizing function. Since $J_\hbar(u) = J_\hbar(|u|)$, we may assume $u_\hbar \ge 0$. In order to prove (4.1) fix $a > 0$ and set

$D = \{y \in \partial B_1 \mid |y - z_i| \geq a/2 \text{ for all } i = 0, \dots, \ell\}$, $d = \inf_D (g(y) - g(z_0)) > 0$. We compute

$$\begin{aligned} J_\hbar^* &\geq \frac{e}{2} \int_{B_\hbar \setminus \bigcup_{i=0}^{\ell} B(z_i/\hbar, a/\hbar)} |u_\hbar|^2 \, dx \frac{1 - |\hbar x|^2}{4\pi} \int_D \frac{d}{|y - \hbar x|^3} \, dS(y) \\ &= \frac{e}{2} d \int_{B_\hbar \setminus \bigcup_{i=0}^{\ell} B(z_i/\hbar, a/\hbar)} |u_\hbar|^2 \, dx \\ &\quad - \frac{e}{2} \int_{B_\hbar \setminus \bigcup_{i=0}^{\ell} B(z_i/\hbar, a/\hbar)} |u_\hbar|^2 \, dx \frac{1 - |\hbar x|^2}{4\pi} \int_{\partial B_1 \setminus D} \frac{d}{|y - \hbar x|^3} \, dS(y). \end{aligned}$$

Notice that if $x \in B_\hbar \setminus \bigcup_{i=0}^{\ell} B(z_i/\hbar, a/\hbar)$ and $y \in \partial B_1 \setminus D$, then $|y - \hbar x| \geq a/2$, by which

$$\begin{aligned} J_\hbar^* \geq \frac{e}{2} d &\int_{B_\hbar \setminus \bigcup_{i=0}^{\ell} B(z_i/\hbar, a/\hbar)} |u_\hbar|^2 \, dx \\ &- \frac{16\pi e}{a^3} d \int_{B_\hbar \setminus \bigcup_{i=0}^{\ell} B(z_i/\hbar, a/\hbar)} |u_\hbar|^2 \frac{1 - |\hbar x|^2}{4\pi} \, dx. \end{aligned}$$

Since $(1 - |\hbar x|^2)/4\pi \leq 8 f_\hbar(x) (\int_{\partial B_1} (g(y) - g(z_0)) \, dS)^{-1}$ we get

$$\frac{e}{2} d \int_{B_\hbar \setminus \bigcup_{i=0}^{\ell} B(x_i/\hbar, a/\hbar)} |u_\hbar|^2 \, dx \leq J_\hbar^* + \frac{256\pi d}{a^3 \int_{\partial B_1} (g(y) - g(z_0)) \, dS} J_\hbar^* \to 0$$

by Corollary 3.1. The arbitrariness of a gives the thesis. □

Next we go further in the analysis of the distance of the minimizers $u_\hbar$ from the boundary ∂B_1 and, as a corollary, provide an asymptotic estimate for the values $J_\hbar^*$.

Lemma 4.2. *There exist numbers $\varepsilon_\hbar \to 0^+$ such that*

$$\lim_{\hbar \to 0^+} \int_{|\hbar x| \leq 1 - \varepsilon_\hbar J_\hbar^*} |u_\hbar|^2 dx = 0 \tag{4.2}$$

and

$$\liminf_{\hbar \to 0^+} \hbar^{-2/3} \varepsilon_\hbar^{2/3} J_\hbar^* > 0. \tag{4.3}$$

Proof.. In order to prove (4.2), we proceed by contradiction. Taking into account of Lemma 4.1, assume the existence of $i \in \{0, \dots, \ell\}$, a sequence $\hbar_n \to 0$ and $a > 0$ such that, setting $v_n = u_{\hbar_n} \chi_{B(z_i/\hbar_n, \delta_{\hbar_n}/\hbar_n)}$ and $J_n = J_{\hbar_n}^*$,

$$\int_{|\hbar_n x| \leq 1 - a \ J_n^*} |v_n|^2 \, dx > 0.$$

In this situation we can choose $b_n > a$ such that

$$\int_{|\hbar_n x| \leq 1 - b_n J_n^*} |v_n|^2 \, dx \geq c_1 > 0, \qquad \int_{1 - b_n J_n^* \leq |\hbar_n x| \leq 1 - a \ J_n^*} |v_n|^2 \, dx \geq c_2 > 0.$$

Fix $N \in \mathbb{N}$ and take N points $y_1, \dots, y_N \in \partial B_1$ with $y_h \neq y_k$ for $h \neq k$. Denote by Q_k a rotation matrix in $\mathbb{R}^3$ such that $Q_k z_i = y_k$ and consider the sequence

$$w_n = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} v_n(Q_k^{-1} x).$$

By Proposition 3.4 we deduce

$$(w_n^2)^* = \frac{1}{N} \sum_{k=1}^{N} ((v_n \circ Q_k^{-1})^2)^* = \frac{1}{N} \sum_{k=1}^{N} (v_n^2 \circ Q_k^{-1})^* = \frac{1}{N} \sum_{k=1}^{N} (v_n^2)^* = (v_n^2)^*. \tag{4.4}$$

Hence, combining Lemma 3.5 with Proposition 3.4 and (4.4) we obtain

$$\begin{aligned}
\frac{2}{N} \int_{B_{\hbar_n}} & v_n^2 \Phi_{\hbar_n}[v_n^2]\, dx \geq \int_{B_{\hbar_n}} w_n^2 \Phi_{\hbar_n}[w_n^2]\, dx \geq \int_{B_{\hbar_n}} (v_n^2)^* \Phi_{\hbar_n}[(v_n^2)^*]\, dx \\
&= 16\pi^2 e \int_0^{1/\hbar_n} dr \int_0^{1/\hbar_n} \frac{r\rho}{\hbar_n} (v_n^2)^*(r)(v_n^2)^*(\rho)(\min\{r,\rho\} - \hbar_n r\rho)\, d\rho \\
&\geq 16\pi^2 e \int_{(1-b_n J_n^*)/\hbar_n}^{(1-aJ_n^*)/\hbar_n} dr \int_0^{(1-b_n J_n^*)/\hbar_n} \frac{r\rho^2}{\hbar_n} (v_n^2)^*(r)(v_n^2)^*(\rho)(1-\hbar_n r)\, d\rho \\
&\geq 16\pi^2 e a J_n^* \int_{(1-b_n J_n^*)/\hbar_n}^{(1-aJ_n^*)/\hbar_n} dr \int_0^{(1-b_n J_n^*)/\hbar_n} r^2\rho^2 (v_n^2)^*(r)(v_n^2)^*(\rho)\, d\rho \\
&\geq aeJ_n^* \int_{aJ_n^* \leq 1-|\hbar_n x| \leq b_n J_n^*} (v_n^2)^*\, dx \int_{|\hbar_n x| \leq 1-b_n J_n^*} (v_n^2)^*\, dx \geq aeJ_n^* c_1 c_2.
\end{aligned}$$

Last inequality follows from Proposition 3.4 by taking

$$f(r) = \chi_{\{aJ_{\hbar_n}^* \leq 1-\hbar_n r \leq b_n J_{\hbar_n}^*\}} \quad \text{and} \quad f(r) = \chi_{\{\hbar_n r \leq 1-b_n J_{\hbar_n}^*\}},$$

respectively. The arbitrariness of N gives

$$\limsup_{n\to\infty} (J_n^*)^{-1} \int_{B_{\hbar_n}} v_n^2 \Phi_{\hbar_n}[v_n^2]\, dx = \infty.$$

On the other hand

$$\frac{e}{4} \int_{B_{\hbar_n}} v_n^2 \Phi_{\hbar_n}[v_n^2]\, dx \leq J_{\hbar_n}(u_{\hbar_n}) = J_n^*$$

and the contradiction follows.

To prove (4.3) consider $\xi \in \mathcal{C}_0^\infty(B_\hbar)$ and for every $x = x(r,\theta,\varphi) \in B_\hbar$ compute

$$|\xi(x)|^2 = \left| \int_r^{1/\hbar} \frac{\partial \xi}{\partial r}(\rho,\theta,\varphi)\, d\rho \right|^2 \leq \left(\frac{1}{\hbar} - r\right) \int_r^{1/\hbar} \left| \frac{\partial \xi}{\partial r}(\rho,\theta,\varphi) \right|^2 d\rho,$$

by which

$$\int_{|\hbar x|\geq 1-\varepsilon_\hbar J^*_\hbar} |\xi|^2\,dx \leq \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\,d\theta \int_{(1-\varepsilon_\hbar J^*_\hbar)/\hbar}^{1/\hbar} r^2\left(\frac{1}{\hbar}-r\right) \cdot \int_r^{1/\hbar} \left|\frac{\partial \xi}{\partial r}(\rho,\theta,\varphi)\right|^2 d\rho dr$$

$$\leq \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\,d\theta \int_{(1-\varepsilon_\hbar J^*_\hbar)/\hbar}^{1/\hbar} \left(\frac{1}{\hbar}-r\right) \cdot \int_r^{1/\hbar} \rho^2\left|\frac{\partial \xi}{\partial r}(\rho,\theta,\varphi)\right|^2 d\rho\,dr = \frac{(\varepsilon_\hbar J^*_\hbar)^2}{2\hbar^2}\int_{B_\hbar} |\nabla \xi|^2\,dx.$$

By density we deduce

$$\int_{|\hbar x|\geq 1-\varepsilon_\hbar J^*_\hbar} |u_\hbar|^2\,dx \leq \frac{(\varepsilon_\hbar J^*_\hbar)^2}{2\hbar^2}\int_{B_\hbar} |\nabla u_\hbar|^2\,dx \leq \frac{(\varepsilon_\hbar J^*_\hbar)^2}{\hbar^2} J_\hbar(u_\hbar) = \frac{\varepsilon_\hbar^2 (J^*_\hbar)^3}{\hbar^2}.$$

Using (4.2) we obtain (4.3). □

5. Proof of Theorem 1.1

We are now in position to complete the proof of Theorem 1.1. In what follows we will always assume $z_0 = (0,0,1)$ and we will make use of the spherical coordinates (r,θ,φ) defined in (3.1). We begin with the following elementary lemma.

Lemma 5.1. *Consider the function* $P\colon [0,\pi]^2 \times [0,2\pi]^2 \to \mathbb{R}$ *defined by*

$$P(\theta,\theta',\varphi,\varphi') = 1 - \sin\theta\sin\theta'\cos(\varphi-\varphi') - \cos\theta\cos\theta'.$$

Then $P(\theta,\theta',\varphi,\varphi') \geq 0$ *and for every* $x = (r,\theta,\varphi)$, $y = (r',\theta',\varphi') \in B_\hbar$ *the following holds*

$$|x-y|^2 = |r-r'|^2 + 2rr'P(\theta,\theta',\varphi,\varphi').$$

The proof is an easy computation.

Next we give a result showing the behaviour of the Green function $G_\hbar$ with respect to suitable dilatations in the directions (r,θ).

Lemma 5.2. *Fix* $\lambda \geq 1$, $r \in (\lambda/((\lambda+1)\hbar), 1/\hbar)$, $\theta,\theta' \in [0,\pi/(2\lambda)]$, $\varphi,\varphi' \in [0,2\pi]$. *Then, setting* $r_\lambda = \lambda(r-1/\hbar)+1/\hbar$ *and* $r'_\lambda = \lambda(r'-1/\hbar)+1/\hbar$, *the following holds*

$$G((r_\lambda,\lambda\theta,\varphi),(r'_\lambda,\lambda\theta',\varphi')) \leq \lambda^{13} G((r,\theta,\varphi),(r',\theta',\varphi')).$$

Proof. We begin by proving that, choosing $\lambda,\theta,\theta',\varphi,\varphi'$ as in the statement of the lemma,

$$P(\lambda\theta,\lambda\theta',\varphi,\varphi') \geq P(\theta,\theta',\varphi,\varphi'). \tag{5.1}$$

Setting $a(\lambda) = P(\lambda\theta, \lambda\theta', \varphi, \varphi')$, an easy computation shows that

$$\begin{aligned}a'(\lambda) = &(-\theta\cos\lambda\theta\sin\lambda\theta' - \theta'\sin\lambda\theta\cos\lambda\theta')\cos(\varphi-\varphi')\\ &+ \theta\sin\lambda\theta\cos\lambda\theta' + \theta'\cos\lambda\theta\sin\lambda\theta'.\end{aligned}$$

If $\cos(\varphi-\varphi') \le 0$, then all the terms in the sum of $a'(\lambda)$ are positive. Assume $\cos(\varphi-\varphi') \ge 0$; then

$$\begin{aligned}(a(\lambda))' \ge &-\theta\cos\lambda\theta\sin\lambda\theta' - \theta'\sin\lambda\theta\cos\lambda\theta' + \theta\sin\lambda\theta\cos\lambda\theta' + \theta'\cos\lambda\theta\sin\lambda\theta'\\ &= \theta\sin(\lambda\theta-\lambda\theta') + \theta'\sin(\lambda\theta'-\lambda\theta) = \sin(\lambda(\theta-\theta'))(\theta-\theta') \ge 0.\end{aligned}$$

Hence (5.1) holds.

Now denote by $x_\lambda, x, y_\lambda, y$ the points of $B_\hbar$ having spherical coordinates $x = (r,\theta,\varphi)$, $x_\lambda = (r_\lambda, \lambda\theta, \varphi)$, $y = (r', \theta', \varphi')$, $y_\lambda = (r'_\lambda, \lambda\theta', \varphi')$. Then, from the choice of r we have $|x|/\lambda \le |x_\lambda| \le |x|$; using (2.2), compute

$$\begin{aligned}G(x_\lambda, y_\lambda) &= \frac{\lambda^2|x_\lambda|^2(1-|\hbar x|)(1-|\hbar y|)}{|y_\lambda - x_\lambda|\,|y_\lambda|\hbar x_\lambda|^2 - x_\lambda|(|y_\lambda|\hbar x_\lambda|^2 - x_\lambda| + |\hbar x_\lambda||y_\lambda - x_\lambda|)}\\ &\le \frac{\lambda^3|x|^2(1-|\hbar x|)(1-|\hbar y|)}{|y_\lambda - x_\lambda|\,|y_\lambda|\hbar x_\lambda|^2 - x_\lambda|(|y_\lambda|\hbar x_\lambda|^2 - x_\lambda| + |\hbar x||y_\lambda - x_\lambda|)}.\end{aligned}\tag{5.2}$$

Now we analyze separately the terms $|x_\lambda - y_\lambda|$ and $|y_\lambda|\hbar x_\lambda|^2 - x_\lambda|$. By using Lemma 5.1 and (5.1) we get

$$\begin{aligned}|x_\lambda - y_\lambda|^2 &= \lambda^2|r-r'|^2 + 2r_\lambda r'_\lambda P(\lambda\theta, \lambda\theta', \varphi, \varphi')\\ &\ge \lambda^2|r-r'|^2 + 2r_\lambda r'_\lambda P(\theta, \theta', \varphi, \varphi')\\ &\ge \lambda^2|r-r'|^2 + 2\frac{rr'}{\lambda^2}P(\theta, \theta', \varphi, \varphi')\\ &\ge \frac{1}{\lambda^2}(|r-r'|^2 + 2rr'P(\theta, \theta', \varphi, \varphi')) = \frac{1}{\lambda^2}|x-y|^2;\end{aligned}\tag{5.3}$$

$$\begin{aligned}|y_\lambda|\hbar x_\lambda|^2 - x_\lambda|^2 &\ge r_\lambda^2|\hbar^2 r'_\lambda r_\lambda - 1|^2 + 2|\hbar r_\lambda|^2 r'_\lambda r_\lambda P(\theta, \theta', \varphi, \varphi')\\ &\ge \frac{r^2}{\lambda^2}|\hbar^2 r' r - 1|^2 + \frac{1}{\lambda^4}2|\hbar r|^2 r' r P(\theta, \theta', \varphi, \varphi')\\ &\ge \frac{1}{\lambda^4}(|r'|\hbar r|^2 - r|^2 + 2|\hbar r|^2 r' r P(\theta, \theta', \varphi, \varphi'))\\ &= \frac{1}{\lambda^4}|y|\hbar x|^2 - x|^2.\end{aligned}\tag{5.4}$$

Combining (5.2)–(5.4) we obtain the thesis. □

The object is now to analyse the asymptotic behaviour of the minimizers $\{u_\hbar\}$ when $\hbar \to 0$. From now on we focus on a generic sequence $\hbar_n \to 0^+$. For sake of simplicity we set $u_n \equiv u_{\hbar_n}$, $\mathcal{M}_n \equiv \mathcal{M}_{\hbar_n}$, … According to Lemma 4.1, up to a subsequence we may assume

$$\int_{|\hbar_n x - z_i| \le \delta_n} |u_n|^2 = \alpha_n^i, \quad \alpha_n^i \to \alpha_i, \quad \sum_{i=0}^{\ell} \alpha_i = 1.\tag{5.5}$$

We divide the remaining part into 4 steps.

Step 1. There exists a constant $\tau > 0$ such that for every $i \in \{\ell' + 1, \dots, \ell\}$ with $\alpha_i \neq 0$

$$\limsup_{n\to+\infty} \int_{\theta(x)\geq(\tau J_n^*)^{1/\overline{q}}} |u_n(M_i x)\chi_{B(z_0/\hbar_n,\delta_n/\hbar_n)}|^2\,dx \leq \frac{\alpha_i}{2}.$$

Assume by contradiction the existence of $i \in \{\ell' + 1, \dots, \ell\}$ and a sequence $\tau_n \to \infty$ such that, up to a subsequence,

$$\int_{\theta(x)\geq(\tau_n J_n^*)^{1/\overline{q}}} |u_n(M_i x)\chi_{B(z_0/\hbar_n,\delta_n/\hbar_n)}|^2\,dx \geq \frac{\alpha_i}{4} > 0.$$

We apply (b) of Lemma 3.2 and obtain

$$\begin{aligned}(1+o(1))J_n^* &= (1+o(1))J_n(u_n)\\ &\geq C\tau_n J_n^* \int_{\theta(x)\geq(\tau_n J_n^*)^{1/\overline{q}}} |u_n(M_i x)\chi_{B(z_0/\hbar_n,\delta_n/\hbar_n)}|^2\,dx \geq C\frac{\alpha_i}{4}\tau_n J_n^*\end{aligned}$$

and the contradiction follows.

For every $n \in \mathbb{N}$ consider $\eta_n \in C^\infty[0,\pi]$, $\xi_n \in C^\infty[0,1/\hbar]$ such that

$$\begin{gathered}\eta_n \equiv 1 \quad \text{if } \theta \leq (\tau J_n^*)^{1/\overline{q}}, \qquad \eta_n \equiv 0 \quad \text{if } \theta \geq 2(\tau J_n^*)^{1/\overline{q}},\\ 0 \leq \eta_n \leq 1, \quad |\eta_n'| \leq \frac{2}{(\tau J_n^*)^{1/\overline{q}}},\\ \xi_n \leq 1 \quad \text{if } t \geq \frac{1-\sqrt{\varepsilon_n}J_n^*}{\hbar_n}, \quad \xi_n \equiv 0 \quad \text{if } t \leq \frac{1-2\sqrt{\varepsilon_n}J_n^*}{\hbar_n},\\ 0 \leq \xi_n \leq 1, \quad |\xi_n'| \leq \frac{2\hbar_n}{\sqrt{\varepsilon_n}J_n^*}.\end{gathered}$$

Then for every $i \in \{\ell' + 1, \dots, \ell\}$ set

$$u_n^i(x) = \eta_n(\theta(x))\xi_n(x)u_n(M_i x).$$

Step 2. For every $i \in \{\ell' + 1, \dots, \ell\}$: $J_n(u_n^i) \leq J_n^* + o(J_n^*)$.

First compute

$$\begin{aligned}|\nabla[\xi_n(\eta_n\circ\theta)]|^2 &\leq \frac{2}{|x|^2}|\eta_n'(\theta)|^2\xi_n^2 + 2|\xi_n'|^2|\eta_n(\theta)|^2\\ &\leq C\frac{\hbar_n^2}{(2J_n^*)^{2/q}} + C\frac{\hbar_n^2}{\varepsilon_n(J_n^*)^2} \leq C\frac{\hbar_n^2}{\varepsilon_n(J_n^*)^2}\end{aligned}$$

at least for large n. By using Hölder's inequality we obtain

$$\begin{aligned}\int_{B_n} |\nabla u_n^i|^2\,dx &\le \int_{B_n} |\nabla u_n|^2\,dx \\ &\quad + C\frac{\hbar_n^2}{\varepsilon_n (J_n^*)^2}\int_{B_n} |u_n|^2\,dx + C\frac{\hbar_n}{\sqrt{\varepsilon_n}J_n^*}\int_{B_n} |\nabla u_n| u_n\,dx \\ &\le \int_{B_n} |\nabla u_n|^2\,dx + C\frac{\hbar_n^2}{\varepsilon_n (J_n^*)^2} + C\frac{\hbar_n}{\sqrt{\varepsilon_n}J_n^*}\bigg(\int_{B_n} |\nabla u_n|^2 dx\bigg)^{1/2} \\ &\le \int_{B_n} |\nabla u_n|^2 dx + C\frac{\hbar_n^2}{\varepsilon_n (J_n^*)^2} + C\frac{\hbar_n}{\sqrt{\varepsilon_n J_n^*}}.\end{aligned}$$

Since by (4.3) $\hbar^2 \le C\varepsilon^2 (J_\hbar^*)^3$, we deduce

$$\int_{B_n} |\nabla u_n^i|^2\,dx \le \int_{B_n} |\nabla u_n|^2\,dx + o(J_n^*). \tag{5.6}$$

It is immediate that

$$\begin{aligned}\int_{B_n} |u_n^i|^2 \Phi_n[|u_n^i|^2]\,dx \ &\le \int_{B_n} |u_n(M_i x)|^2 \Phi_n[|u_n(M_i x)|^2]\,dx \\ &= \int_{B_n} |u_n|^2 \Phi_n[|u_n|^2]\,dx.\end{aligned} \tag{5.7}$$

Finally, by Lemma 3.2(a),

$$\begin{aligned}(1+o(1))\int_{B_n} f_n |u_n^i|^2\,dx \ &= \frac{A_0}{2\pi}\int_{B_n} |u_n^i|^2 (1-|\hbar_n x|)\,dx + o(J_n^*) \\ &\le \sqrt{\varepsilon_n} J_n^* \frac{A_0}{\pi} + o(J_n^*) = o(J_n^*).\end{aligned} \tag{5.8}$$

Combining (5.6)–(5.8) we achieve the thesis of the step.

Step 3. $\alpha_i = 0$ for every $i \in \{\ell'+1,\dots,\ell\}$.

This is the most technical and lengthy part of this paper. Assume by contradiction that there exists $j \in \{\ell'+1,\dots,\ell\}$ such that $\alpha_j \ne 0$. Taking into account of (4.2), (5.5) and Step 1, letting $\beta_n \ge 0$ be such that $\int_{B_n} |u_n^j|^2 dx = \beta_n$, up to a subsequence

$$\beta_n \to \beta \ge \alpha_j/2.$$

Roughly speaking, we will make a suitable dilatation of the functions u_n^j in the direction ρ and θ in such a way to construct a new sequence $\widetilde{w}_n \in \mathcal{M}_n$ which makes the functional J_n lower then $J_n(u_n)$, which contradicts the minimizing property of u_n. Fix $N \ge 100$ and take $y_n^1,\dots,y_n^N \in \partial B_1$ such that

$$|y_n^k - z_0| \le 4N^2 (\tau J_n^*)^{1/\overline{q}}, \qquad |y_n^h - y_n^k| \ge 4N (\tau J_n^*)^{1/\overline{q}} \quad \text{for } h \ne k.$$

Denote by Q_n^k a rotation matrix in $\mathbb{R}^3$ such that $Q_n^k z_0 = y_n^k$ and consider the new sequence

$$w_n(x) \equiv \frac{1}{\sqrt{N}} \sum_{k=1}^{N} u_n^j((Q_n^k)^{-1}x).$$

By construction at least for large n we have $u_n^j = 0$ for $|\hbar_n x - z_0| \geq 4(\tau J_n^*)^{1/\overline{q}}$; hence Lemma 3.5 applies and gives

$$\int_{B_n} w_n^2 \Phi_n[w_n^2]\, dx \leq \frac{2}{N} \int_{B_n} |u_n^j|^2 \Phi_n[(u_n^j)^2]\, dx \leq \frac{8}{N} J_n^* + o(J_n^*),$$

where, in the last equality, we have used Step 2. In particular the functions $u_n^j((Q_n^k)^{-1}x)$ have disjoint support, by which we compute

$$\int_{B_n} |w_n|^2 = \frac{1}{N} \sum_{k=1}^{N} \int_{B_n} |u_n^j|^2\, dx = \beta_n.$$

Analogously, using again Step 2,

$$\int_{B_n} |\nabla w_n|^2\, dx = \int_{B_n} |\nabla u_n^j|^2\, dx \leq 2J_n^* + o(J_n^*). \tag{5.10}$$

Next fix $\lambda > 1$ arbitrarily and set

$$\widetilde{w}_n(r, \theta, \varphi) = \frac{\gamma_n}{\lambda\sqrt{\lambda}} w_n\bigg(\frac{(\hbar_n r - 1)/\lambda + 1}{\hbar_n}, \frac{1}{\lambda}\theta, \varphi\bigg)$$

where $\gamma_n > 0$ is such that $\widetilde{w}_n \in \mathcal{M}_n$. Since $\hbar_n r \to 1$ and $(\sin\theta)/\theta \to 1$ as $n \to \infty$ uniformly on supp $\widetilde{w}_n$ and supp w_n, by using Lemma 3.1, we compute

$$\begin{aligned} 1 &= \int_{B_n} |\widetilde{w}_n|^2\, dx = \frac{1}{\hbar_n^2} \int_0^{1/\hbar_n} dr \int_0^{2\pi} d\varphi \int_0^{\pi} \theta |\widetilde{w}_n|^2\, d\theta + o(1) \\ &= \frac{\gamma_n^2}{\hbar_n^2} \int_0^{1/\hbar_n} dr \int_0^{2\pi} d\varphi \int_0^{\pi} \theta |w_n|^2\, d\theta + o(1) \\ &= \gamma_n^2 \int_{B_n} |w_n|^2\, dx + o(1) = \gamma_n^2 \beta_n + o(1), \end{aligned}$$

i.e. $\gamma_n \to \beta^{-1/2}$. By (3.2) and (5.10),

$$\begin{aligned} &(1 + o(1))\|\nabla \widetilde{w}_n\|_2^2 \\ &= \frac{\gamma_n^2}{\lambda^2} \int_0^{1/\hbar_n} dr \int_0^{\pi} d\theta \int_0^{2\pi} \left(\frac{\theta}{\hbar_n^2} \left| \frac{\partial w_n}{\partial r} \right|^2 + \theta \left| \frac{\partial w_n}{\partial \theta} \right|^2 + \frac{1}{\theta} \left| \frac{\partial w_n}{\partial \varphi} \right|^2 \right) d\varphi \\ &\leq \frac{1 + o(1)}{\lambda^2 \beta} \|\nabla w_n\|_2^2 \leq 2 \frac{1 + o(1)}{\lambda^2 \beta} J_n^*. \end{aligned} \tag{5.11}$$

By Lemma 3.2(a), since $\widetilde{w}_n = 0$ for $|\hbar_n x| \leq 1 - 2\lambda\sqrt{\varepsilon_n} J_n^*$ or $\theta \geq 2\lambda(\tau J_n^*)^{1/\overline{q}}$, we get

$$(1 + o(1)) \int_{B_n} |\widetilde{w}_n|^2 f_n\, dx \leq \frac{A_0}{\pi} \lambda \sqrt{\varepsilon_n} J_n^* + o(J_n^*) = o(J_n^*). \tag{5.12}$$

Finally by using Lemmas 3.1 and 5.2

$$
\begin{aligned}
(5.13)\quad (1+o(1))&\int_{B_n}\widetilde{w}_n^2\int_{B_n}\Phi_n[\widetilde{w}_n^2]\,dx\\
=&\frac{e}{\hbar_n}\int_0^{1/\hbar_n}dr\int_0^{\pi}d\theta\int_0^{2\pi}\frac{\theta}{\hbar_n^2}\widetilde{w}_n^2\,d\varphi\\
&\cdot\int_0^{1/\hbar_n}dr'\int_0^{\pi}d\theta'\int_0^{2\pi}\frac{\theta'}{\hbar_n^2}G(r,\theta,\varphi,r',\theta',\varphi')\widetilde{w}_n^2\,d\varphi'\\
=&\frac{e}{\hbar_n}\gamma_n^4\int_0^{1/\hbar_n}dr\int_0^{\pi}d\theta\int_0^{2\pi}\frac{\theta}{\hbar_n^2}w_n^2\,d\varphi\\
&\cdot\int_0^{1/\hbar_n}dr'\int_0^{\pi}d\theta'\int_0^{2\pi}\frac{\theta'}{\hbar_n^2}G(r_\lambda,\lambda\theta,\varphi,r'_\lambda,\lambda\theta',\varphi')w_n^2\,d\varphi'\\
\le& e\gamma_n^4\lambda^{13}(1+o(1))\int_{B_n}w_n^2\int_{B_n}\Phi_n[w_n^2]\,dx\le 8\lambda^{13}\frac{1+o(1)}{\beta^2N}J_n^*,
\end{aligned}
$$

where in the last inequality we have used (5.9). Now choose $\lambda>1$ and $N\in\mathbb{N}$ such that

$$
\frac{2}{\lambda^2\beta}<\frac{1}{2}\quad\text{and}\quad\frac{8\lambda^{13}}{\beta^2N}<1.
$$

Then, combining (5.11)–(5.13) we achieve $J_n(\widetilde{w}_n)\le J_n^*/2+o(J_n^*)$ which is a contradiction since $\widetilde{w}_n\in\mathcal{M}_n$ and consequently $J_n(\widetilde{w}_n)\ge J_n^*$.

Step 4. End of the proof of Theorem 1.1.

According to Propositions 2.1 and 2.3 and Lemma 4.1, by applying the Lagrange multiplier rule, for every $\hbar>0$ there exists $\lambda_\hbar\in\mathbb{R}$ such that $u_\hbar$ solves

$$
-\Delta u_\hbar+ef_\hbar u_\hbar+e\Phi_\hbar[u_\hbar^2]u_\hbar=\lambda_\hbar u_\hbar.
$$

By multiplying both members by $u_\hbar$ and integrating by parts we deduce

$$
0\le\lambda_\hbar\le 4J_\hbar(u_\hbar),
$$

by which, using Corollary 3.1, $\lambda_\hbar\to 0$ as $\hbar\to 0^+$. Now put

$$
v_\hbar(x)=\hbar^{-3/2}u_\hbar\left(\frac{x}{\hbar}\right)\quad\text{and}\quad\phi_\hbar(x)=f_\hbar\left(\frac{x}{\hbar}\right)+\Phi_\hbar[u_\hbar^2]\left(\frac{x}{\hbar}\right)+g(z_0);
$$

then an easy computation shows that $\int_{B_\hbar}|v_\hbar|^2\,dx=1$ and $(v_\hbar,\phi_\hbar)$ solves the system (1.4)–(1.5) with the conditions (1.6) and

$$
\omega=\omega_\hbar=\lambda_\hbar+eg(z_0)\to eg(z_0)
$$

as $\hbar\to 0$. Furthermore Theorem 1.1(d) follows directly from Lemma 4.1, (5.5) and Step 3. Notice that by (2.2) (since $|y|x|^2-x|\ge|x|(1-|x|)$) we obtain

$G_1(x,y) \le (1-|y|)/(|y-x|^2)$. Now, fixed $\varepsilon > 0$,

$$\begin{aligned}\int_{B_1} \Phi_\hbar[u_\hbar^2]\bigg(\frac{x}{\hbar}\bigg)\,dx &= \int_{B_1} v_\hbar^2\,dy \int_{B_1} G_1(x,y)\,dx \\ &\le \int_{|y|\le 1-\varepsilon} v_\hbar^2(y)\,dy \int_{B_1} \frac{1}{|y-x|}\,dx + \int_{|y|\ge 1-\varepsilon} v_\hbar^2(y)\,dy \int_{B_1} \frac{1-|y|}{|y-x|^2}\,dx \\ &= \int_{|y|\le 1-\varepsilon} v_\hbar^2 dy \int_{|x|\le 2} \frac{1}{|x|}dx + \varepsilon \int_{|x|\le 2} \frac{1}{|x|^2}\,dx = o(1) + \varepsilon \int_{|x|\le 2} \frac{1}{|x|^2}\,dx.\end{aligned}$$

Furthermore an immediate computation shows that

$$f_\hbar\bigg(\frac{x}{\hbar}\bigg) = \frac{1-|x|^2}{4\pi} \int_{\partial B_1} \frac{g(y)}{|y-x|^3}\,dS - g(z_0),$$

then Theorem 1.1(c) is proved. □

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