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EIGENVALUE CRITERIA FOR EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS OF LOCAL AND NONLOCAL TYPE

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Abstract. New criteria are established for the existence of multiple positive solutions of a Hammerstein integral equation of the form

$$u(t) = \int_0^1 k(t,s)g(s)f(s,u(s))\,ds \equiv Au(t)$$

where k can have discontinuities in its second variable and $g \in L^1$.

These criteria are determined by the relationship between the behaviour of $f(t,u)/u$ as u tends to 0^+ or ∞ and the principal (positive) eigenvalue of the linear Hammerstein integral operator

$$Lu(t) = \int_0^1 k(t,s)g(s)u(s)\,ds.$$

We obtain new results on the existence of multiple positive solutions of a second order differential equation of the form

$$u''(t) + g(t)f(t,u(t)) = 0 \quad \text{a.e. on } [0,1],$$

subject to general separated boundary conditions and also to nonlocal m-point boundary conditions. Our results are optimal in some cases. This work contains several new ideas, and gives a *unified* approach applicable to many BVPs.

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1. Introduction

We are interested in the existence of (multiple) positive solutions of differential equations of the form

$$-u''(t) = g(t)f(t, u(t)) \quad \text{for a.e. } t \in (0,1)$$

under a variety of boundary conditions (BCs) which include separated BCs and non-local BCs known as m-point BCs. Here $g \in L^1(0,1)$ and f satisfies Carathéodory conditions so the problem can be called weakly singular. We seek solutions via fixed points of the Hammerstein integral operator

$$Au(t) := \int_0^1 k(t,s)g(s)f(s,u(s))\,ds \tag{1.1}$$

where the kernel k is the Green's function for the differential operator $-u''$ with the given BCs. Our method is to apply the theory of fixed point index to the compact operator A defined on some cone K in the Banach space $C[0,1]$. We impose fairly weak conditions on the kernel k, which is not symmetric in general. To show that nonzero solutions exist, one needs to give conditions under which the fixed point index on some open set in K equals 1, and other conditions which give the index on some other open set in K to be 0. Such conditions have previously been given. For example, if f depends only on u and if

$$\lim_{u\to 0+} f(u)/u > M, \qquad \lim_{u\to\infty} f(u)/u < m, \tag{1.2}$$

where $m < M$ are computable constants defined in terms of integrals of $k(t,s)g(s)$ (see later in the paper for precise definitions) then there is at least one positive (nontrivial) solution. For results for integral equations see [12, §45.4], for separated BCs see [4], [14], for 3-point BCs see [23]. Some other authors have made the stronger assumptions that f is either sub- or super-linear, for example [19], [20].

This approach to obtaining multiple solutions of BVPs can be traced back to work of Krasnosel'skiĭ and others, using the well-known theorem on compression and expansion of a cone, and may be found in [12, §45.4]; note that some monotonicity assumption is assumed on $f(t,u)$ and that there is a misprinted inequality sign in Theorem 45.8 of [12]. Our results are sharper: in the present paper we show that (1.2) can be replaced by an optimal condition

$$\lim_{u\to 0+} f(u)/u > \mu_1, \qquad \lim_{u\to\infty} f(u)/u < \mu_1, \tag{1.3}$$

with $\mu_1 = 1/r(L)$, where $r(L)$ is the spectral radius of the compact linear operator

$$Lu(t) := \int_0^1 k(t,s)g(s)u(s)\,ds.$$

Under our hypotheses, $r(L)$ is an eigenvalue of L with a positive eigenfunction. We show that $m \leq \mu_1 \leq M$ and that these inequalities are often strict, in fact $1/m = \|L\| \geq r(L)$, so our results are definite improvements on earlier ones.

To give our existence results we have to prove some new results on the fixed point index of A which involve the eigenvalue $r(L)$. These new results apply to very general kernels. When $g \equiv 1$ and k is symmetric, Liu and Li [18] and Erbe [4] have proved results involving eigenvalues which are of the same type but by using entirely different methods. We do not need k to be symmetric and we allow the term g which can have singularities at arbitrary points in $[0,1]$.

In one case we have to use an eigenvalue $r(\widetilde{L})$ where

$$\widetilde{L}u(t) := \int_a^b k(t,s)g(s)u(s)\,ds,$$

for a given subset $[a,b]$ of $[0,1]$. We prove results which show that, for many BVPs, $r(\widetilde{L})$ can be replaced by $r(L)$, giving a stronger, best possible result. We make use of the permanence property of index to obtain the stronger result. In particular this is so if k is symmetric, corresponding to separated BCs. But the stronger result also applies in other nonsymmetric cases. In particular, our results apply to so called multi-point (or m-point) BVPs, studied in [2], [3], [7], [8], [19], [20] and elsewhere, and enable us to give multiplicity results for these problems. This establishes the existence of *multiple* positive solutions for these m-point BVPs under better conditions than have been previously employed.

The only *multiplicity* results we have seen in the literature for multi-point BVPs are those of Karakostas and Tsamatos [10], [11], and of Bai and Fang [2], [3]. Karakostas and Tsamatos treat a nonlocal BVP with very general BCs given by Riemann–Stieltjes integrals. They have results on the existence of two or three positive solutions for integral equations with more restrictions on their kernel than we have, and under conditions on f such as sub- or super-linear behaviour near 0 or ∞. These conditions on f are more restrictive than allowing constants such as m, M; their results are extensions of results of [20]. Bai and Fang have given multiplicity results for the 1-dimensional p-Laplacian, using the methodology of Lan [14] and obtaining generalisations of his results. Other previous results for multi-point problems have only treated the existence of one positive solution when f is either sub- or super-linear, for example [19], [20].

For two types of 3-point BVPs, existence of multiple positive solutions has been done in [23]; the results here improve on those of [23]. None of these earlier works use eigenvalues.

We firstly obtain results for the Hammerstein integral operator in (1.1) under rather weak hypotheses on the kernel k. We define a suitable cone K and show that the linear operator L has an eigenvalue $r(L)$ with an eigenfunction in this cone K. We then obtain our results on fixed point index and apply them firstly to existence of positive fixed points of the Hammerstein integral operator and then to BVPs with separated and some nonlocal BCs.

After this paper was completed we saw two very recent papers. Zhang and Sun, [25] also use index methods related to eigenvalues but only for the m-point BVPs. They consider the problem

$$u''(t) + g(t)f(u(t)) = 0,$$

when $g \in L^1$ and is continuous and positive on $(0,1)$. They prove some index results similar to ours but do not use the subinterval $[a,b]$ (see (C_2) below) they do not consider the general integral equation, and they never discuss conditions when optimal results can be obtained. We impose weaker conditions, give general multiplicity results and obtain some optimal results.

In the second recent paper, Y. Li [16] essentially uses fixed point index results involving the first eigenvalue but he works in the space L^2, and, in an essential way, requires that the linear operator is normal. This type of calculation was discarded by us in favour of the calculations we give which do not need a normal operator, see Remark 3.9. Li only gives results for the existence of one positive solution. He applies the results to Sturm-Liouville BVPs, which is similar to but more complicated than the separated BCs we include. Our method applies to these problems too but we do not give details for all possibilities.

Our method involves several new ideas and gives a *unified* method of attack for many BVPs. Previous papers dealt with one problem at a time whereas our method allows us to discuss all problems at once.

2. Integral equations and linear eigenvalue problems

Motivated by BVPs for a differential equation of the form

$$u''(t) + g(t)f(t,u(t)) = 0$$

we shall consider the existence of (multiple) positive solutions of a Hammerstein equation of the form

$$u(t) = Au(t) := \int_0^1 k(t,s)g(s)f(s,u(s))\,ds, \quad t \in [0,1].$$

Our methodology will involve the fixed point index of compact operators and our conditions will involve eigenvalues of a related compact linear integral operator L, that is, study of the equation

$$\lambda u(t) = Lu(t) := \int_0^1 k(t,s)g(s)u(s)\,ds.$$

We want our integral operators to be well defined and compact in the space $C[0,1]$ of continuous functions endowed with the usual supremum norm. We make the following hypotheses on g, k, and f.

(C_1) $k \geq 0$ is measurable, and for every $\tau \in [0,1]$ we have

$$\lim_{t\to\tau} |k(t,s) - k(\tau,s)| = 0 \quad \text{for a.e. } s \in [0,1].$$

(C_2) There exist a subinterval $[a,b] \subseteq [0,1]$, a function $\Phi \in L^\infty[0,1]$, and a constant $c \in (0,1]$ such that

$$\begin{aligned} k(t,s) &\leq \Phi(s) \quad \text{for } t \in [0,1] \text{ and almost every } s \in [0,1], \\ k(t,s) &\geq c\Phi(s) \quad \text{for } t \in [a,b] \text{ and almost every } s \in [0,1]. \end{aligned}$$

(C_2^*) (C_2) holds for an arbitrary choice of a, b with $[a,b] \subset (0,1)$.

(C_3) $g\,\Phi \in L^1[0,1]$, $g \geq 0$ a.e. and $\int_a^b \Phi(s)g(s)\,ds > 0$.

(C_4) $f\colon [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies Carathéodory conditions, that is, $f(\,\cdot\,,u)$ is measurable for each fixed $u \in \mathbb{R}_+$ and $f(t,\,\cdot\,)$ is continuous for almost every $t \in [0,1]$, and for each $r > 0$, there exists $\phi_r \in L^\infty[0,1]$ such that

$$0 \leq f(t,u) \leq \phi_r(t) \quad \text{for all } u \in [0,r] \text{ and almost all } t \in [0,1].$$

Note that (C_1) allows some discontinuity in the kernel which occurs, for example, in the study of the nonlocal BVP

$$-u'' = f(t,u) \quad u(0) = 0,\ u(1) = \alpha u'(\eta)$$

studied in [9]. This is yet another BVP that can be treated by our methods.

The condition (C_4) means that the singular behaviour of the nonlinearity is captured by the term g, a typical example being when the nonlinearity is $g(t)f(u)$ with f continuous. Note that (C_3) implies that $g(s) > 0$ on a subset of $[a,b]$ of positive measure but, in general, g could be identically zero on some subinterval of $[0,1]$ and its singularities can occur at arbitrary points of $[0,1]$. Also, (C_1)–(C_3) and (C_4) together imply that

$$\lim_{t\to\tau} \int_0^1 |k(t,s) - k(\tau,s)| g(s)\,ds = 0, \tag{2.3}$$

$$\lim_{t\to\tau} \int_0^1 |k(t,s) - k(\tau,s)| g(s)\phi_r(s)\,ds = 0, \tag{2.4}$$

because the integrand is dominated by (a constant times) $2\Phi(s)g(s)$. We could replace the pointwise assumption (C_1) by the integral properties (2.3), (2.4) but use the pointwise assumption for simplicity.

Let $P = \{u \in C[0,1] : u \geq 0\}$ denote the standard cone of nonnegative functions. To obtain multiplicity results it is convenient to work in a smaller cone than P. The above hypotheses allow us to work in such a cone. Let $q\colon C[0,1]\to\mathbb{R}$ denote the continuous function

$$q(u)=\min\{u(t): t\in[a,b]\}, \tag{2.5}$$

and, with c as in (C_2), let

$$K=\{u\in P,\ q(u)\ge c\|u\|\}. \tag{2.6}$$

This type of cone has been used by, for example, D. Guo and Lakshmikantham [6], Krasnosel'skiĭ and Zabreĭko [12], and more recently by Lan [14], Ma [19], and Bai and Fang [2], [3], and many authors not mentioned in our bibliography.

Remark 2.1. When k is continuous on $[0,1]\times[0,1]$ and $k(t,s)>0$ for $t\in(0,1)$, $s\in[0,1]$ then (C_2) holds for an arbitrary $[a,b]\subset(0,1)$. In fact we can take

$$\Phi=\max_{(t,s)\in[0,1]\times[0,1]}k(t,s)\quad\text{and}\quad c=\min_{t\in[a,b],\,s\in[0,1]}k(t,s)/\Phi.$$

The interval $[a,b]$ occurring in (C_2) is not unique for if (C_2) holds for some interval $[a_0,b_0]$ then it clearly also holds for any smaller interval.

Note that K, q and c depend on the choice of $[a,b]$. Obviously if one c is valid in (C_2) then so is any smaller c, but the largest possible choice of c in (C_2) is optimal in requiring weaker conditions on f elsewhere. When there is a choice of interval $[a,b]$, there may be some choice which leads to weaker conditions on f, see for example [24] for a discussion of these matters for 3-point BVPs.

For much of this work a,b remain fixed and when this is the case we simply write K, q and c rather than $K_{a,b}$, $q_{a,b}$ and $c_{a,b}$.

Our integral operators are compact and leave K invariant, in fact they map P into K.

Lemma 2.2. *Under the hypotheses* (C_1)–(C_4) *the map* $A\colon P\to C[0,1]$ *defined in* (2.1) *maps* P *into* K *and is compact.*

Lemma 2.3. *Assume that* (C_1)–(C_3) *hold. Then* $L\colon C[0,1]\to C[0,1]$ *defined in* (2.2) *is compact and maps* P *into* K.

Proof. The compactness of A, L follows from Proposition 3.1 of [21, p. 164] since, as $[0,1]$ is compact, the limit in each of (2.3), (2.4) is readily shown to be uniform in $\tau\in[0,1]$. To see that $A\colon P\to K$, for $u\in P$ and $t\in[0,1]$, we have,

$$|Au(t)|\le\int_0^1 k(t,s)g(s)f(s,u(s))\,ds$$

so

$$\|Au\|\le\int_0^1 \Phi(s)g(s)f(s,u(s))\,ds.$$

Also, for $t \in [a,b]$,

$$Au(t) \geq c \int_0^1 \Phi(s)g(s)f(s,u(s))\,ds.$$

Hence $Au \in K$ for every $u \in P$. The same calculation works for L. $\square$

Recall that a cone K in a Banach space X is said to be *reproducing* if $X = K - K$ and is a *total* cone if $X = \overline{K - K}$. Writing $x(t) = x^+(t) - x^-(t)$ shows that P is reproducing.

We shall use the Krein–Rutman theorem, using a special case of some more general results of Nussbaum [22]. We recall that λ is an eigenvalue of L with corresponding eigenfunction φ if $\varphi \neq 0$ and $\lambda\varphi = L\varphi$. The reciprocals of eigenvalues are called *characteristic values* of L. The radius of the spectrum of L, denoted $r(L)$, is given by the well-known spectral radius formula $r(L) = \lim_{n\to\infty} \|L^n\|^{1/n}$.

THEOREM 2.4 ([22]). *Let K be a total cone in a real Banach space X and let $\widehat{L}\colon X \to X$ be a compact linear operator with $\widehat{L}(K) \subseteq K$. If $r(\widehat{L}) > 0$ then there is $\varphi_1 \in K \setminus \{0\}$ such that $\widehat{L}\varphi_1 = r(\widehat{L})\varphi_1$.*

Thus $\lambda_1 := r(\widehat{L})$ is an eigenvalue of $\widehat{L}$, the largest possible real eigenvalue, and $\mu_1 = 1/\lambda_1$ is the smallest positive characteristic value.

LEMMA 2.5. *Assume that* (C_1)–(C_3) *hold and let L be as defined in* (2.2). *Then $r(L) > 0$.*

PROOF. For $u \in K$ and for $t \in [a,b]$ we have

$$Lu(t) \geq \int_a^b c\Phi(s)g(s)u(s)\,ds \geq c\|u\|\bigg(\int_a^b c\Phi(s)g(s)\,ds\bigg).$$

Then

$$\begin{aligned} L^2u(t) &\geq \int_a^b k(t,s)g(s)\bigg[c\|u\|\bigg(\int_a^b c\Phi(s)g(s)\,ds\bigg)\bigg]\,ds \\ &\geq c\|u\|\bigg(\int_a^b c\Phi(s)g(s)\,ds\bigg)^2, \end{aligned}$$

and so

$$\|L^n\|\,\|u\| \geq \|L^n u\| \geq L^n u(t) \geq c\|u\|\bigg(\int_a^b c\Phi(s)g(s)\,ds\bigg)^n.$$

Hence

$$r(L) = \lim_{n\to\infty} \|L^n\|^{1/n} \geq \int_a^b c\Phi(s)g(s)\,ds > 0. \qquad \square$$

Hence we have the following result.

Theorem 2.6. *When* (C_1)–(C_3) *hold,* $r(L)$ *is an eigenvalue of* L *with eigenfunction* φ_1 *in* K.

Proof. $r(L)$ is an eigenvalue of L with eigenfunction in P, by Theorem 2.4. As L maps P into K, the eigenfunction belongs to K. □

Remark 2.7. Although K depends on a, b, the eigenvalue $r(L)$ is defined by the spectral radius formula and is independent of a, b.

Some other constants have previously been used for the type of BVPs we study. The following estimates shows that we obtain better results in the present paper.

Theorem 2.8. *Let* $\mu_1 = 1/r(L)$ *and* $\varphi_1(t)$ *be a corresponding eigenfunction in* P *of norm* 1. *Then* $m \le \mu_1 \le M$, *where*

$$m = \bigg(\sup_{t\in[0,1]} \int_0^1 k(t,s)g(s)\,ds\bigg)^{-1}, \quad M = \bigg(\inf_{t\in[a,b]} \int_a^b k(t,s)g(s)\,ds\bigg)^{-1}. \tag{2.7}$$

If $g(t) > 0$ *for* $t \in [0,1]$ *and* $k(t,s) > 0$ *for* $t,s \in [0,1]$, *the first inequality is strict unless* $\varphi_1(t)$ *is constant for* $t \in [0,1]$. *If* $g(t)\Phi(t) > 0$ *for* $t \in [a,b]$, *the second inequality is strict unless* $\varphi_1(t)$ *is constant for* $t \in [a,b]$.

Proof. We have, for $t \in [0,1]$,

$$\varphi_1(t) = \mu_1 \int_0^1 k(t,s)g(s)\varphi_1(s)\,ds \le \mu_1 \int_0^1 k(t,s)g(s)\,ds. \tag{2.8}$$

Taking the supremum over $t \in [0,1]$ gives

$$1 \le \mu_1 \sup_{t\in[0,1]} \int_0^1 k(t,s)g(s)\,ds = \frac{\mu_1}{m},$$

so that $m \le \mu_1$. When $g(t) > 0$ and $k(t,s) > 0$, if φ_1 is not constant on $[0,1]$ then the inequality in (2.8) is strict.

Secondly we have for each $t \in [a,b]$,

$$\varphi_1(t) \ge \mu_1 \int_a^b k(t,s)g(s)\varphi_1(s)\,ds \ge \mu_1 q(\varphi_1) \int_a^b k(t,s)g(s)\,ds,$$

with a strict inequality if φ_1 is not constant on $[a,b]$. Taking the infimum over $[a,b]$ shows that $M \ge \mu_1$. □

The inequality $m \le \mu_1$ also follows from the facts that

$$\|L\| = \int_0^1 k(t,s)g(s)\,ds = \frac{1}{m} \quad \text{and} \quad r(L) \le \|L\|.$$

In many cases, the inequalities in Theorem 2.8 are strict, we illustrate with some examples later in the paper. For some BVPs, for example with periodic or Neumann BCs the eigenfunction is constant and equality holds. For some of our results below it is useful to know that there is precisely one positive eigenvalue of L with an eigenfunction that is positive on $(0,1)$.

DEFINITION 2.9. We say that L satisfies (UPE) if $r(L)$ is the only positive eigenvalue of L with an eigenfunction in the cone P.

If L is *strongly positive*, that is L maps $P \setminus \{0\}$ into the interior of P, then it follows from Theorem 3.2 of [1] that (UPE) holds. In our case, L is strongly positive if $k(t,s) > 0$ on $[0,1]\times[0,1]$, and $g(s) > 0$ for almost all $s \in [0,1]$, but we make weaker assumptions in general.

We now show such a uniqueness result when the kernel k is symmetric. If the term g were positive almost everywhere then this is essentially well known in L^2 theory.

THEOREM 2.10. *Suppose that* (C_1)–(C_3) *hold with* $g \in L^1$ *and that* k *is symmetric, that is,* $k(t,s) = k(s,t)$ *for almost all* s, t. *Then* L *satisfies* (UPE).

PROOF. Suppose there are two positive eigenvalues of L with eigenfunctions of norm 1 in K, say

$$\lambda_1\varphi_1(t) = \int_0^1 k(t,s)g(s)\varphi_1(s)\,ds, \qquad \lambda_2\varphi_2(t) = \int_0^1 k(t,s)g(s)\varphi_2(s)\,ds.$$

Then, for a.e. t, we have

$$\lambda_1\varphi_1(t)\varphi_2(t)g(t) = \int_0^1 g(t)\varphi_2(t)k(t,s)g(s)\varphi_1(s)\,ds.$$

Integrating this gives

$$\begin{aligned}\lambda_1\int_0^1 \varphi_1(t)\varphi_2(t)g(t)\,dt &= \int_0^1\int_0^1 g(t)\varphi_2(t)k(t,s)g(s)\varphi_1(s)\,ds\,dt\\ &= \int_0^1 g(s)\varphi_1(s)\,ds\int_0^1 k(t,s)\varphi_2(t)g(t)\,dt = \int_0^1 g(s)\varphi_1(s)\lambda_2\varphi_2(s)\,ds\end{aligned}$$

by the symmetry of k. The interchange of the order of integration is justified by Tonelli's theorem since the iterated integral exists by our assumption $g \in L^1$. Hence,

$$(\lambda_1 - \lambda_2)\int_0^1 \varphi_1(t)\varphi_2(t)g(t)\,dt = 0.$$

Since $g(t) > 0$ on a subset of $[a,b]$ of positive measure and $\varphi_j(t) \ge c > 0$ for $t \in [a,b]$ we must have $\lambda_1 = \lambda_2$. $\square$

3. Fixed point index calculations

If Ω is a bounded open subset of K (in the relative topology) we denote by $\overline{\Omega}$ and $\partial\Omega$ the closure and the boundary relative to K. When D is an open bounded subset of X we write $D_K = D \cap K$, an open subset of K. For $\rho > 0$ we shall use the open sets

$$K_\rho = \{u \in K : \|u\| < \rho\}.$$

We use standard properties of the classical fixed point index for compact maps, see for example [1] or [6] for further information.

A consequence of the properties of index is the following result.

Lemmma 3.1. *Under our assumptions, if $x \neq Ax$ for $x \in \partial D_P$, then $i_{K_{a,b}}(A, D \cap K_{a,b})$ is independent of $[a,b]$ for which* (C_2) *holds.*

Proof. For each $[a,b]$, we have $i_{K_{a,b}}(A, D \cap K_{a,b}) = i_P(A, D \cap P)$ by the permanence property since $A(P) \subset K_{a,b}$ by Lemma 2.2. □

3.1. Fixed point index and eigenvalues. We now give our new results on index calculations for A which involve the interplay between the eigenvalues of L and the behaviour of the nonlinearity near 0 and near infinity. We write $\mu_1(L) = 1/r(L)$ or simply μ_1 when L is clear from the context.

Notation. Let f satisfy (C_4) and let E be a fixed subset of $[0,1]$ of measure zero. We make the following definitions.

$$\begin{aligned} \overline{f}(u) &:= \sup_{t\in[0,1]\setminus E} f(t,u) & \underline{f}(u) &:= \inf_{t\in[0,1]\setminus E} f(t,u) \\ f^0 &= \limsup_{u\to 0+} \overline{f}(u)/u, & f_0 &= \liminf_{u\to 0+} \underline{f}(u)/u, \\ f^\infty &= \limsup_{u\to\infty} \overline{f}(u)/u, & f_\infty &= \liminf_{u\to\infty} \underline{f}(u)/u. \end{aligned}$$

We should really indicate the dependence on E but, for simplicity, we omit this. Also we could assume there were two sets E, one for $\overline{f}$ and one for $\underline{f}$, but this is merely complicating the notation.

Theorem 3.2. *If $0 \le f^0 < \mu_1$, then there exists $\rho_0 > 0$ such that*

$$i_K(A, K_\rho) = 1 \quad \text{for each } \rho \in (0, \rho_0].$$

Proof. Let $\varepsilon > 0$ be such that $f^0 \le \mu_1 - \varepsilon$. Then there exists $\rho_0 > 0$ such that

$$f(t,u) \le (\mu_1 - \varepsilon)u \quad \text{for all } u \in [0, \rho_0] \text{ and almost all } t \in [0,1].$$

Let $\rho \in (0, \rho_0]$. We prove that

$$Au \neq \lambda u \quad \text{for } u \in \partial K_\rho \text{ and } \lambda \ge 1, \tag{3.1}$$

which implies the result. In fact, if (3.1) does not hold, then there exist $u \in \partial K_\rho$ and $\lambda \ge 1$ such that $\lambda u = Au$. This implies

$$\lambda u(t) = \int_0^1 k(t,s) g(s) f(s, u(s))\, ds \le (\mu_1 - \varepsilon) \int_0^1 k(t,s) g(s) u(s)\, ds = (\mu_1 - \varepsilon) L u(t).$$

Thus, we have shown $u(t) \le (\mu_1 - \varepsilon)Lu(t)$. This gives

$$u(t) \le (\mu_1 - \varepsilon)L[(\mu_1 - \varepsilon)Lu(t)] = (\mu_1 - \varepsilon)^2 L^2 u(t)$$

and iterating gives $u(t) \le (\mu_1 - \varepsilon)^n L^n u(t)$ for $n \in \mathbb{N}$. Therefore

$$1 \le (\mu_1 - \varepsilon)^n \|L^n\|,$$

and we have

$$1 \le (\mu_1 - \varepsilon) \lim_{n\to\infty} \|L^n\|^{1/n} = (\mu_1 - \varepsilon)\frac{1}{\mu_1} < 1,$$

a contradiction. It follows that $i_K(A, K_\rho) = 1$. □

THEOREM 3.3. *If $0 \le f^\infty < \mu_1$, then there exists R_0 such that*

$$i_K(A, K_R) = 1 \quad \textit{for each } R > R_0.$$

PROOF. Let $\varepsilon > 0$ satisfy $f^\infty < \mu_1 - \varepsilon$. Then there exists $R_1 > 0$ such that

$$f(t,u) \le (\mu_1 - \varepsilon)u \quad \text{for all } u \ge R_1 \text{ and almost all } t \in [0,1].$$

By (C_4) there exists an L^∞ function ϕ_1 such that

$$f(t,u) \le \phi_1(t) \quad \text{for all } u \in [0, R_1] \text{ and almost all } t \in [0,1].$$

Hence, we have

$$f(t,u) \le (\mu_1 - \varepsilon)u + \phi_1(t) \quad \text{for all } u \in \mathbb{R}_+ \text{ and almost all } t \in [0,1].$$

Since $1/\mu_1$ is the radius of the spectrum of L, $(I/(\mu_1 - \varepsilon) - L)^{-1}$ exists. Let

$$C = \int_0^1 \Phi(s)g(s)\phi_1(s)\,ds \quad \text{and} \quad R_0 = \left(\frac{1}{\mu_1 - \varepsilon}I - L\right)^{-1}\left(\frac{C}{\mu_1 - \varepsilon}\right).$$

We prove that for each $R > R_0$,

$$Au \ne \lambda u \quad \text{for all } u \in \partial K_R \text{ and } \lambda \ge 1. \tag{3.3}$$

In fact, if not, there exist $u \in \partial K_R$ and $\lambda \ge 1$ such that $\lambda u = Au$. This, together with (3.2), implies

$$u(t) \le (\mu_1 - \varepsilon)Lu(t) + C.$$

This implies

$$\left(\frac{1}{\mu_1 - \varepsilon}I - L\right)u(t) \le \frac{C}{\mu_1 - \varepsilon} \quad \text{and} \quad u(t) \le \left(\frac{1}{\mu_1 - \varepsilon}I - L\right)^{-1}\left(\frac{C}{\mu_1 - \varepsilon}\right) = R_0.$$

Therefore, we have $\|u\| \le R_0 < R$, a contradiction. It follows from (3.3) and properties of index that $i_K(A, K_R) = 1$ for every $R > R_0$. □

Theorem 3.4. *If $\mu_1 < f_0 \leq \infty$, then there exists $\rho_0 > 0$ such that for each $\rho \in (0, \rho_0]$, if $u \neq Au$ for $u \in \partial K_\rho$, then $i_K(A, K_\rho) = 0$.*

Proof. Let $\varepsilon > 0$ satisfy $f_0 > \mu_1 + \varepsilon$. Then there exists $\rho_0 > 0$ such that

$$f(t,u) \geq (\mu_1 + \varepsilon)u \quad \text{for all } u \in [0, \rho_0] \text{ and almost all } t \in [0,1]. \tag{3.4}$$

Let $\rho \in (0, \rho_0]$. We prove that

$$u \neq Au + \beta\varphi_1 \quad \text{for all } u \in \partial K_\rho \text{ and } \beta > 0,$$

where $\varphi_1 \in K$ is the eigenfunction of L with $\|\varphi_1\| = 1$ corresponding to the eigenvalue $1/\mu_1$, which implies the result. In fact, if not, there exist $u \in \partial K_\rho$ and $\beta > 0$ such that $u = Au + \beta\varphi_1$. This implies $u \geq \beta\varphi_1$ and $Lu \geq \beta L\varphi_1 \geq (\beta/\mu_1)\varphi_1$. Using this, together with (3.4), gives

$$u \geq (\mu_1 + \varepsilon)Lu + \beta\varphi_1 \geq (\mu_1 + \varepsilon)\frac{\beta}{\mu_1}\varphi_1 + \beta\varphi_1 > 2\beta\varphi_1.$$

Repeating the process gives $u \geq n\beta\phi_1$ for $n \in \mathbb{N}$, a contradiction. □

Our next index result is a little different, we do not use the eigenvalue $r(L)$ but the eigenvalue of a related linear operator $\widetilde{L} = \widetilde{L}(a,b)$. We shall see later that there are many cases when we can use $r(L)$ but we do not know whether this is always the case. Under the hypotheses (C_1)–(C_3) let $\widetilde{L}$ be defined by

$$\widetilde{L}u(t) = \int_a^b k(t,s)g(s)u(s)\,ds.$$

Then $\widetilde{L}$ is a compact linear operator and $\widetilde{L}(P) \subseteq K$. Hence $r(\widetilde{L})$ is an eigenvalue of $\widetilde{L}$ with an eigenfunction $\widetilde{\varphi}_1$ in K. Let $\widetilde{\mu_1} := 1/r(\widetilde{L})$. Note that $\widetilde{\mu_1} \geq \mu_1$, hence the condition in the following theorem is more stringent than if we could use $r(L)$.

Theorem 3.5. *If $\widetilde{\mu_1} < f_\infty \leq \infty$. Then there exists R_1 such that for each $R \geq R_1$, if $u \neq Au$ for $u \in \partial K_R$, then $i_K(A, K_R) = 0$.*

Proof. Let $R_1 > 0$ be chosen so that $f(t,u)/u > \widetilde{\mu_1}$ for all $u \geq cR_1$, c as in (C_2) and almost all $t \in [0,1]$. We claim that $u \neq Au + \beta\widetilde{\varphi}_1$ for all $\beta > 0$ and $u \in \partial K_R$ when $R \geq R_1$. Note that for $u \in K$ with $\|u\| = R \geq R_1$ we have $u(t) \geq cR_1$ for all $t \in [a,b]$. Now, if our claim is false, then we have

$$u(t) = \int_0^1 k(t,s)g(s)f(s,u(s))\,ds + \beta\widetilde{\varphi}_1(t).$$

Therefore,

$$u(t) \geq \int_a^b k(t,s)g(s)\widetilde{\mu_1}u(s)\,ds + \beta\widetilde{\varphi}_1(t) = \widetilde{\mu_1}\widetilde{L}u(t) + \beta\widetilde{\varphi}_1(t). \tag{3.5}$$

From (3.5) we firstly deduce that $u(t) \geq \beta\widetilde{\varphi}_1(t)$ on $[a,b]$. Then we have

$$\widetilde{\mu_1}\widetilde{L}u(t) \geq \widetilde{\mu_1}\widetilde{L}(\beta\widetilde{\varphi}_1(t)) = \beta\widetilde{\varphi}_1(t).$$

Inserting this into (3.5) we obtain $u(t) \geq 2\beta\widetilde{\varphi}_1(t)$ for $t \in [a,b]$. Repeating this process gives

$$u(t) \geq n\beta\widetilde{\varphi}_1(t) \quad \text{for } t \in [a,b],\ n \in \mathbb{N}.$$

Since $\widetilde{\varphi}_1(t)$ is strictly positive on $[a,b]$ this is a contradiction. □

3.2. A stronger index result. We can give an optimal result when (C_2^*) holds, that is a can be arbitrarily near 0 and b arbitrarily near 1, which does occur for many BVPs, see Remark 3.9below. So suppose that for each $a > 0$, $b < 1$ there are functions $\Phi_{a,b}$, constants $c_{a,b} > 0$ such that (C_2) and (C_3) hold. Take sequences $\{a_n\}$, $\{b_n\}$ such that $a_{n+1} \leq a_n$, $b_n \leq b_{n+1}$ and $a_n \to 0$, $b_n \to 1$. Let $c_n := c_{a_n,b_n}$ and

$$K_n = K_{a_n,b_n} = \{u \geq 0 : \min_{[a_n,b_n]} u(t) \geq c_n\|u\|\}.$$

Define $L_n : C[0,1] \to C[0,1]$ by

$$L_n u(t) = \int_{a_n}^{b_n} k(t,s)g(s)u(s)\,ds.$$

Then L_n is compact and maps P into K_n and we obtain $r(L_n) > 0$ by Lemma 2.5.

THEOREM 3.6. *Suppose* (C_1), (C_2^*) *and* (C_3) *hold. Then* $\{r(L_n)\}$ *is increasing and bounded above by* $r(L)$.

PROOF. It suffices to prove $r(L_1) \leq r(L_2)$ by merely changing notation. For $u \in C[0,1]$ we have

$$|L_1 u(t)| \leq \int_{a_1}^{b_1} k(t,s)g(s)|u(s)|\,ds \leq \int_{a_2}^{b_2} k(t,s)g(s)|u(s)|\,ds = L_2|u(t)|.$$

Also

$$|L_1^2 u(t)| \leq L_2|L_1 u(t)| \leq L_2(L_2|u(t)|) = L_2^2|u(t)|.$$

We obtain a similar expression for each integer $m > 1$. Hence

$$|L_1^m u(t)| \leq L_2^m|u(t)| \leq \|L_2^m\|\|u\|,$$

so $\|L_1^m\| \leq \|L_2^m\|$. By the spectral radius formula the result is shown. □

THEOREM 3.7. *Under the same assumptions as in Theorem* 3.6, *if L satisfies* (UPE), *then, with L_n as defined above,* $r(L_n) \to r(L)$.

PROOF. We can write $Lu(t) = L_n u(t) + E_n u(t)$ where

$$E_n u(t) = \int_0^{a_n} k(t,s)g(s)u(s)\,ds + \int_{b_n}^1 k(t,s)g(s)u(s)\,ds,$$

so that $\|E_n\| \to 0$. Let φ_n be an eigenfunction of L_n of norm 1 in K_n corresponding to $r(L_n)$. Then

$$r(L_n)\varphi_n = L_n\varphi_n = L\varphi_n - E_n\varphi_n.$$

As $\|\varphi_n\| = 1$ and L is compact, $\{r(L_n)\varphi_n\}$ has a convergent subsequence. Since $\{r(L_n)\}$ is increasing and bounded it follows that $r(L_n) \to \lambda_0 > 0$. Hence φ_n has a convergent subsequence, say $\varphi_{n_k} \to \varphi_0$, where $\|\varphi_0\| = 1$ and $\varphi_0 \geq 0$ on $[0,1]$. Then $\lambda_0\varphi_0 = L\varphi_0$, so by the uniqueness assumption (UPE), $\lambda_0 = r(L)$. □

For clarity of notation we now let $[a_0, b_0]$ denote a fixed interval for which (C_2) holds and let K_0 denote the corresponding cone. We write $K_{0,R} = K_0 \cap B_R$, where B_R is the open ball of radius R.

Theorem 3.8. *Under the same assumptions as in Theorem* 3.6*, suppose also that L satisfies* (UPE)*. If $\mu_1(L) < f_\infty \leq \infty$ then there exists R_1 such that for $R \geq R_1$, if $u \neq Au$ for $u \in \partial K_{0,R}$, then $i_{K_0}(A, K_{0,R}) = 0$.*

Proof. Let $\varepsilon > 0$ be such that $f_\infty > \mu_1(L) + \varepsilon$ and choose a_1, b_1 so that $\mu_1(L) - \mu_1(L(a_1, b_1)) < \varepsilon$. Then $f_\infty > \mu_1(L(a_1, b_1))$ and Theorem 3.5 implies that $i_{K(a_1,b_1)}(A, K(a_1, b_1) \cap B_R) = 0$. By Lemma 3.1 the result is shown. □

Remark 3.9. By Theorem 2.10 and the remarks preceding that theorem, this result holds whenever L is strongly positive and also whenever k is symmetric. Hence our result includes that of Liu and Li [18] and of Erbe [4] who do not have the term $g(t)$ and consider only separated BCs. However Erbe does consider a Sturm–Liouville differential operator rather than $-u''$. This case also follows from our methods but we give only the simpler case when it is possible to explicitly calculate the constants that occur. We shall see below that our result also holds for some well studied nonlocal boundary conditions.

4. Existence results for integral equations

We first give a new result on existence of at least one nonzero positive solution for the equation

$$u(t) = Au(t) := \int_0^1 k(t,s)g(s)f(s,u(s))\,ds, \quad t \in [0,1]. \tag{4.1}$$

We now choose a fixed $[a, b]$ and corresponding cone K.

Theorem 4.1. *Assume that* (C_1)–(C_4) *hold and that one of the following conditions holds:*

(H_1) $0 \leq f^0 < \mu_1(L)$ *and* $\widetilde{\mu}_1(\widetilde{L}) < f_\infty \leq \infty$.
(H_2) $0 \leq f^\infty < \mu_1(L)$ *and* $\mu_1(L) < f_0 \leq \infty$.

Then (2.1) *has a solution $u \in K$ with $\rho \leq \|u\| \leq R$ for some $0 < \rho < R$. When L satisfies* (UPE) *and* (C_2^*) *holds, we may replace $\widetilde{\mu}_1(\widetilde{L})$ by $\mu_1(L)$ in* (H_1).

Proof. Assume that (H_1) holds. By the first part of (H_1) and Theorem 3.2, there exists $\rho > 0$ such that $i_K(A, K_\rho) = 1$. By the second part of (H_1) and Theorem 3.5, there exists $R > \rho$ such that either A has a fixed point on ∂K_R or

$i_K(A, K_R) = 0$. In the second case, A has a fixed point $u \in K$ with $\rho < \|u\| < R$ by properties of index. The proof is similar when (H_2) holds. The last part follows from Theorem 3.8. □

REMARK 4.2. Note that $i_K(A, K_\rho) = 1$ implies there is a fixed point in K_ρ but this may be 0, we are interested in nontrivial solutions. By the estimates of Theorem 2.8, Theorem 4.1 improves Corollary 2.3 in [13]. It also includes a result in [4].

To obtain the existence of multiple positive solutions we shall use some known results from [13]. Define $\Omega_\rho = \{x \in K : q(x) < c\rho\}$, where q is defined in (2.5) and c is as in (C_2). Better results are obtained by considering the index on this open set than by using sets of the form K_ρ.

We do not give the most general result but state just a more easily checked version of the results of [13]; the more general result can be easily constructed by the reader.

When f depends only on u, we define

$$f^{0,\rho} = \sup\{f(u)/\rho : 0 \le u \le \rho\}, \quad f_{c\rho,\rho} = \inf\{f(u)/\rho : c\rho \le u \le \rho\}.$$

In terms of these numbers the following index result holds.

LEMMA 4.3.

(a) *Suppose there is $\rho > 0$ such that $f^{0,\rho} < m$, m as in* (2.7). *Then $x \ne Ax$ on ∂K_ρ and $i_K(A, K_\rho) = 1$.*
(b) *If $f_{c\rho,\rho} > cM$, where c is as in* (C_2) *and M is as in* (2.7), *then $x \ne Ax$ for $x \in \partial\Omega_\rho$ and $i_K(A, \Omega_\rho) = 0$.*

We now give new results on the existence of at least two positive solutions of (4.1).

THEOREM 4.4. *Assume that* (C_1)–(C_4) *hold together with one of the following conditions:*

(S_1) $0 \le f^0 < \mu_1$, *$f_{c\rho,\rho} > cM$ for some $\rho > 0$, and $0 \le f^\infty < \mu_1$.*
(S_2) $\mu_1 < f_0 \le \infty$, *$f^{0,\rho} < m$ for some $\rho > 0$, and $\widetilde{\mu}_1 < f_\infty \le \infty$.*

Then (4.1) *has two nonzero solutions in K. When L satisfies* (UPE), *we may replace $\widetilde{\mu}_1$ by μ_1 in* (S_2).

PROOF. Assume that (S_1) holds. By Theorems 3.2, 3.3, there exist $\rho_0 \in (0, c\rho)$ and $R \in (\rho, \infty)$ such that $i_K(A, K_{\rho_0}) = 1$ and $i_K(A, K_R) = 1$. By Lemma 4.3, we have $i_K(A, \Omega_\rho) = 0$. Since $\rho_0 < c\rho$, we have $\overline{K}_{\rho_0} \subset K_{c\rho} \subset \Omega_\rho$. By the additivity property of index, A has a fixed point x_1 in $\Omega_\rho \setminus \overline{K}_{\rho_1}$. Similarly, A has a fixed point x_2 in $K_R \setminus \overline{\Omega}_\rho$. When (S_2) holds, either there are fixed points on ∂K_{ρ_0} for $\rho_0 < \rho$ sufficiently small and on ∂K_R for $R > \rho$ sufficiently large, or a similar fixed point index argument applies. The last part follows from Theorem 3.8. □

Remark 4.5. Theorem 4.4 improves Corollary 2.2 in [13], where m, M are used in (S_1) in place of the better constant μ_1. It is possible to state similar types of results for the existence of $3, 4, \ldots$ solutions by adding to the lists in (S_1) or (S_2) above. We write a result for at least 3 solutions for some BVPs in Theorems 5.3, 6.1 as illustration, but we leave the general case to the reader who may refer to [12], [14] to see the kind of statements that can be made.

5. Separated boundary conditions

We consider the existence of one or several positive solutions for a second order differential equation of the form

$$u''(t) + g(t)f(t, u(t)) = 0 \quad \text{a.e. on } [0, 1], \tag{5.1}$$

subject to the following general separated boundary conditions.

$$\begin{cases} \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0, \end{cases} \tag{5.2}$$

where $\alpha, \beta, \gamma, \delta \geq 0$ and $\Gamma := \gamma\beta + \alpha\gamma + \alpha\delta > 0$.

(5.2) contains the following well known BCs.

(B_1) $u(0) = u(1) = 0$.
(B_2) $u(0) = u'(1) = 0$.
(B_2') $u'(0) = u(1) = 0$.
(B_3) $u(0) = 0$ and $\gamma u(1) = -\delta u'(1)$ with $\gamma, \delta > 0$.
(B_3') $u(1) = 0$ and $\alpha u(0) = \beta u'(0)$ with $\alpha, \beta > 0$.
(B_4) $u'(0) = 0$ and $\gamma u(1) = -\delta u'(1)$ with $\gamma, \delta > 0$.
(B_4') $u'(1) = 0$ and $\alpha u(0) = \beta u'(0)$ with $\alpha, \beta > 0$.

Let k be the Green's function for the equation $-u'' = 0$ subject to the BC (5.2). It is well-known that $k\colon [0, 1] \times [0, 1] \to \mathbb{R}_+$ is given by

$$k(t, s) = \frac{1}{\Gamma}\begin{cases} (\gamma + \delta - \gamma t)(\beta + \alpha s), & \text{if } 0 \leq s \leq t \leq 1, \\ (\beta + \alpha t)(\gamma + \delta - \gamma s), & \text{if } 0 \leq t < s \leq 1. \end{cases} \tag{5.3}$$

(5.1)–(5.2) can be studied via the Hammerstein integral equation

$$u(t) = \int_0^1 k(t, s)g(s)f(s, u(s))\, ds \equiv Au(t), \quad t \in [0, 1]. \tag{5.4}$$

We verify that k satisfies (C_2^*). In fact, we may always choose a, b as follows.

(I) Choose $a, b \in [0, 1]$ such that $-\beta/\alpha < a < b < 1 + \delta/\gamma$, where $\beta/\alpha = \infty$ if $\alpha = 0$ and $\delta/\gamma = \infty$ if $\gamma = 0$.

We may take $\Phi(s) = k(s,s)$, and c to be given by

(II) $c = \min\{(\gamma + \delta - \gamma b)/(\gamma + \delta), (\beta + \alpha a)/(\alpha + \beta)\}$.

In particular, $[a,b]$ can be an arbitrary subinterval of $(0,1)$ in every case. Since also k is symmetric, by theorems 2.10, 3.8 we may always use $r(L)$ in our results for these BCs rather than sometimes having to use $r(\widetilde{L})$.

For a general function g, it may not be simple to determine the principal eigenvalue of the linear integral operator corresponding to (5.4), but *we now assume* that $g(t) \equiv 1$ and obtain the eigenvalue by a direct calculation from the differential equations.

THEOREM 5.1.

(a) *If the* BC *is* (B_1), *then* $\mu_1 = \pi^2$ *and* $\varphi(t) = \sin(\pi t)$.
(b) *If the* BC *is* (B_2) *or* (B_2'), *then* $\mu_1 = (\pi/2)^2$.
(c) *If the* BC *is* (B_3) *or* (B_3'), *then* $(\pi/2)^2 < \mu_1 < \pi^2$.
(d) *If the* BC *is* (B_4) *or* (B_4'), *then* $0 < \mu_1 < (\pi/2)^2$.

PROOF. Finding an eigenvalue of the integral operator L corresponds to finding a nonzero solution of the equation

$$u'' + \mu_1 u = 0 \tag{5.5}$$

subject to the boundary condition (5.2).

Nonzero solutions only exist if $\mu_1 > 0$. Equation (5.5) has general solution:

$$u(t) = A\cos\omega_1 t + B\sin\omega_1 t, \quad t \in [0,1]. \tag{5.6}$$

where $\omega_1^2 = \mu_1$.

(a) (B_1) gives $A = 0$, and $B = 1$ with $\sin\omega_1 = 0$. It follows that $\omega_1 = \pi$ and $\mu_1 = \pi^2$. The corresponding eigenfunction is $\varphi(t) = \sin(\pi t)$

(b) (B_2) gives $A = 0$, $B = 1$ and $\cos\omega_1 = 0$. Thus $\omega_1 = \pi/2$ and $\mu_1 = \pi^2/4$. The eigenfunction is $\varphi(t) = \sin(\pi t/2)$.

(c) (B_3) gives $\mu_1 = \omega_1^2$ where ω_1 is the root of the equation $\tan(\omega) = -\delta\omega/\gamma$. The value of ω_1 may be found numerically in any given case. We note that we always have $\pi/2 < \omega_1 < \pi$ and $\pi^2/4 < \mu_1 < \pi^2$ and $\varphi(t) = \sin(\omega_1 t)$.

(d) If $u \neq 0$ satisfies (B_4), then $\mu_1 = \omega_1^2$ where ω_1 is the root of the equation $\omega\tan(\omega) = \gamma/\delta$, again to be found numerically. Hence, $0 < \omega_1 < \pi/2$ and $0 < \mu_1 < \pi^2/4$ and $\varphi(t) = \cos(\omega_1 t)$.

The other BCs are equivalent to these ones via the change of variable from t to $1-t$. □

Let m, M be as defined in 2.8 which showed that $m \leq \mu_1 \leq M$. The following examples illustrate their values.

Example 5.2. (a) Under (B_1), we have $m = 8$ and if $a = 1/4$ and $b = 3/4$, then $M = 16$. Hence, we have

$$m = 8 < \mu_1 = \pi^2 < M = 16.$$

Note that, for an interval $[a, 1-a]$ the choice of $[1/4, 3/4]$ is optimal in giving the smallest M, see [15].

(b) Under (B_2), we have $m = 2$ and if $a = 1/2$ and $b = 1$, then $M = 4$. Again this choice of $[a, b]$ is optimal. Hence, $m = 2 < \mu_1 = \pi^2/4 < M = 4$.

We can now easily state new existence results for multiple positive solutions for these BCs using Theorems 4.1, 4.4 and their extensions, but we only give a more easily verified version of the result for the existence of at least 3 solutions as an illustration and leave other cases to the reader.

Theorem 5.3. *The* BVP (5.1), (5.2) *has at least three positive solutions if either of the following list of conditions hold, where* $\mu_1 = 1/r(L)$ *for the corresponding linear integral operator.*

(T_1) *There exist* $0 < \rho_1 < c\rho_2 < \infty$, *such that*

$$\mu_1 < f_0 \le \infty, \quad f^{0,\rho_1} < m, \quad f_{c\rho_2,\rho_2} > cM, \quad 0 \le f^\infty < \mu_1.$$

(T_2) *There exist* $0 < \rho_1 < \rho_2 < \infty$, *such that*

$$0 \le f^0 < \mu_1, \quad f_{c\rho_1,\rho_1} > cM, \quad f^{0,\rho_2} < m, \quad \mu_1 < f_\infty \le \infty.$$

Proof. Suppose that (T_1) holds. By Theorem 3.4 there exists $\rho_0 > 0$ sufficiently small so that either $u = Au$ for $u \in \partial K_{\rho_0}$, or $i_K(A, K_{\rho_0}) = 0$. In the second case we have

$$i_K(A, K_{\rho_1} \setminus \overline{K}_{\rho_0}) = 1, \quad \text{and} \quad i_K(A, \Omega_{\rho_2} \setminus \overline{K}_{\rho_1}) = -1.$$

This gives two nonzero solutions. Also, by Theorem 3.8, there is R sufficiently large such that either $u = Au$ for $u \in \partial K_R$, or $i_K(A, K_R) = 0$. This gives a third nonzero solution, either on ∂K_R or in $K_R \setminus \overline{\Omega}_{\rho_2}$. In the first case we have a first nonzero solution on ∂K_{ρ_0} rather than in $K_{\rho_1} \setminus \overline{K}_{\rho_0}$ and then the proof proceeds as before. When (T_2) holds the proof is similar. □

We note that our results are improvements of previous work, for example we improve on Theorem 3.4.5 in [6, p. 214], but their result does allow f to take negative values. Our results also generalize Corollary 3.1 in [13], Theorems 1 and 2 in [18], Theorem 2.4 and Corollary 2.5 of [4], and Theorem 4 in [5]. Li and Han [17] give a result for two positive solutions for the BCs $u(0) = 0, u(1) = 0$ by using a generalization of the Leggett–Williams theorem. They use the value π^2 by a direct calculation but do not mention that this is μ_1, the eigenvalue.

6. Nonlocal multi-point BVPs

Some nonlocal BVPs known as multi-point BVPs have been studied extensively by Gupta and co-authors, see for example [7], [8]. We consider the case when the nonlinearity does not depend on the first derivative, and the BVPs are of the form

$$-u''(t) = g(t)f(t,u(t)), \quad t \in (0,1), \tag{6.1}$$

with one of the boundary conditions

$$u'(0) = 0, \quad u(1) = \sum_{i=1}^{m} \alpha_i u(\eta_i), \quad 0 < \eta_i < 1 \text{ and } \alpha_i \geq 0, \quad \sum_{i=1}^{m} \alpha_i < 1, \tag{6.2}$$

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m} \alpha_i u(\eta_i), \quad 0 < \eta_i < 1 \text{ and } \alpha_i \geq 0, \quad \sum_{i=1}^{m} \alpha_i \eta_i < 1. \tag{6.3}$$

Here we suppose $0 < \eta_1 < \cdots < \eta_m < 1$, where $m \geq 1$. These problems are called $(m+2)$-point BVPs. The study of existence of a positive solution of such problems has been done by Ma [19] but only when f is either sub- or super-linear. More general BCs have been studied in [20], again for f either sub- or super-linear. Multiple positive solutions for both 3-point cases has been done by Webb [23]. Multiple positive solutions for the 1-dimensional p-Laplacian with the BC (6.3) and with a more complicated version of BC (6.2) have been given by Bai and Fang in [2], [3] by following the methodology of [14] and utilising some results of Ma [19], to obtain generalisations of [14]. However, they do not employ eigenvalues. Our results improve the work of [19] by allowing more general behaviour of the function f and establishing existence of multiple solutions. In the special case $p = 2$ our results also improve on those of [2], [3]. We illustrate this with some specific numbers below.

To apply our results to these m-point problems it is necessary to show that (C_2) holds for the corresponding kernel. Explicit forms of the kernel have been given in [7], [8].

However, it seems to be a tricky calculation to verify (C_2) from these explicit forms, so we make an observation which simplifies the calculation.

For the BC (6.2) let $\Lambda_1 = \sum_{i=1}^{m} \alpha_i < 1$. We seek a solution of $-u''(t) = f(t,u(t))$ with the BC (6.2) via the integral operator A_1 defined by (see [8])

$$\begin{aligned} A_1 u(t) = \frac{1}{1-\Lambda_1}\Big[\int_0^1 (1-s) f(s,u(s))\, ds \\ - \sum_{i=1}^{m} \alpha_i \int_0^{\eta_i} (\eta_i - s) f(s,u(s))\, ds\Big] - \int_0^t (t-s) f(s,u(s))\, ds. \end{aligned}$$

Our observation is that A_1 can be written as follows.

$$(6.4)\quad A_1u(t) = \frac{1}{1-\Lambda_1}\int_0^1 \sum_{i=1}^m \alpha_i h_1(\eta_i,s) f(s,u(s))\,ds + \int_0^1 h_1(t,s) f(s,u(s))\,ds,$$

where

$$h_1(t,s) = \begin{cases} 1-t & \text{for } s \le t, \\ 1-s & \text{for } s > t, \end{cases}$$

is the kernel for the BCs $u'(0)=0, u(1)=0$. This is readily checked. Therefore the kernel may be written

$$(6.5)\qquad k_1(t,s) = \frac{1}{1-\Lambda_1}\sum_{i=1}^m \alpha_i h_1(\eta_i,s) + h_1(t,s).$$

However, for the kernel h_1 it is easy to check that there are Φ, $[a,b]$, c such that $h_1(t,s) \le \Phi(s)$ and $h_1(t,s) \ge c\Phi(s)$ for all $t \in [a,b]$ where $a \ge 0$ and $b < 1$ may be arbitrary. In fact we may take $a = 0, b < 1$ and $\Phi(s) = 1-s$ with $c = 1-b$. This also follows from the general separated BC case.

We now can see that, for the $m+2$-point problem, we may take an arbitrary $[a,b] \subset [0,1]$. In fact, from (6.5), we have

$$k_1(t,s) \le \left(\frac{1}{1-\Lambda_1}\right)\Phi(s) \quad \text{for } s,t \in [0,1],$$

and $k_1(t,s) \ge h_1(t,s) \ge c\Phi(s)$. Also if we choose $a \in [0,\eta_1], b \in [\eta_m, 1)$, then we have

$$k_1(t,s) \ge c\left(\frac{1}{1-\Lambda_1}\right)\Phi(s) \quad \text{for } s \in [0,1],\ t \in [a,b].$$

In other words, we have flexibility to choose $[a,b]$ to best suit our needs but we do not search for an optimal choice here.

Similarly, for the BC (6.3) we have, [7],

$$A_2u(t) = \frac{1}{1-\Lambda_2}\left[\int_0^1 t(1-s) f(s,u(s))\,ds - \sum_{i=1}^m \alpha_i \int_0^{\eta_i} t(\eta_i - s) f(s,u(s))\,ds\right] - \int_0^t (t-s) f(s,u(s))\,ds.$$

where $\Lambda_2 = \sum_{i=1}^m \alpha_i\eta_i < 1$. We observe that the corresponding kernel can be written

$$(6.6)\qquad k_2(t,s) = \frac{t}{1-\Lambda_2}\sum_{i=1}^m \alpha_i h_2(\eta_i,s) + h_2(t,s)$$

where

$$h_2(t,s) = \begin{cases} s(1-t) & \text{for } s \le t, \\ t(1-s) & \text{for } s > t, \end{cases}$$

is the kernel corresponding to the BCs $u(0) = 0$, $u(1) = 0$. Again for the kernel h_2 it is easy to check that we may choose $a > 0$, $b < 1$, $\Phi(s) = s(1-s)$ and $c = \min\{a, 1-b\}$. For the $m+2$-point problem we may take arbitrary $[a,b] \subset (0,1)$ and, as above, we verify that (C_2) holds.

Now let $L_i = \int_0^1 k_i(t,s)g(s)u(s)\,ds$ for $i = 1,2$ be the corresponding integral operators.

From the above we see that L_1 is strongly positive, when g is positive almost everywhere on $[0,1]$, hence $r(L_1) > 0$ and (UPE) is satisfied. L_2 is not strongly positive even for $g \equiv 1$ since $L_2u(0) = 0$. But we shall show below that, when $g \equiv 1$, L_2 does satisfy (UPE), see Subsection 6.2. Hence we can apply our results above to obtain multiple positive solutions. We state just one such result leaving the obvious statements of other results to the reader.

Theorem 6.1. *The the* BVP (6.1), *with the* BC*s* (6.2) *has at least three positive solutions if either of the following list of conditions hold, where* μ_1 *denotes* $1/r(L_1)$ *for the corresponding linear operator with kernel defined in* (6.5).

(T_1) *There exist* $0 < \rho_1 < c\rho_2 < \infty$, *such that*

$$\mu_1 < f_0 \le \infty, \quad f^{0,\rho_1} < m, \quad f_{c\rho_2,\rho_2} > cM, \quad 0 \le f^\infty < \mu_1.$$

(T_2) *There exist* $0 < \rho_1 < \rho_2 < \infty$, *such that*

$$0 \le f^0 < \mu_1, \quad f_{c\rho_1,\rho_1} > cM, \quad f^{0,\rho_2} < m, \quad \widetilde{\mu}_1 < f_\infty \le \infty.$$

When g *is positive almost everywhere in* $[0,1]$ *the result holds with* $\widetilde{\mu}_1(\widetilde{L}_1)$ *replaced by* $\mu_1(L_1)$ *in* (T_2). *The same result holds for the* BVP (6.1), *with the* BC*s* (6.3) *with* μ_1 *standing for* $1/r(L_2)$. *When* $g \equiv 1$ *we may replace* $\widetilde{\mu}_1(\widetilde{L}_2)$ *in* (T_2) *by* $\mu_1(L_2)$.

The proof is the same as that of Theorem 5.3. This gives improved versions of the results of [2], [3] specialised to $p = 2$. However they also have a more complicated BC at 0. We could also give results for that BC using our general approach, but we omit the tedious calculations.

It is useful to know the values of the principal eigenvalue that occurs in each problem but it is not clear how to calculate this from the integral equation with a nonsymmetric kernel. We *assume* that $g(t) \equiv 1$ and obtain the eigenvalue from the differential equations.

6.1. BVP (6.2). The problem $u'' + \omega^2 u = 0$, with BC (6.2) has nontrivial solutions of the form $\varphi(t) = \cos(\omega t)$. The principal eigenvalue is ω_1^2 where ω_1 is the smallest positive solution of the equation

$$\cos\omega = \sum_{i=1}^{m} \alpha_i \cos(\eta_i \omega). \tag{6.7}$$

Letting $f(x) := \cos x - \sum_{i=1}^{m} \alpha_i \cos(\eta_i x)$ we see that

$$f(0) = 1 - \sum_{i=1}^{m} \alpha_i > 0, \quad f(\pi/2) = -\sum_{i=1}^{m} \alpha_i \cos(\eta_i \pi/2) < 0$$

hence there is a solution between 0 and $\pi/2$. The corresponding eigenfunction $\varphi_1(t) = \cos(\omega_1 t) > 0$ on $[0,1]$ and $0 < \mu_1 < \pi^2/4$, the actual value would have to calculated numerically from (6.7). Since $f'(x) < 0$ on $(0,\pi/2)$ there is precisely one zero of f in $[0,\pi/2]$ so any other positive eigenvalue has an eigenfunction which must change sign on $[0,1]$.

6.2. BVP (6.3). The problem $u'' + \omega^2 u = 0$, with BC (6.3) has nontrivial solutions of the form $\varphi(t) = \sin(\omega t)$. The smallest positive ω is the smallest positive solution of the equation

$$\sin\omega = \sum_{i=1}^{m} \alpha_i \sin(\eta_i \omega). \tag{6.8}$$

Letting $g(x) := \sin x - \sum_{i=1}^{m} \alpha_i \sin(\eta_i x)$ we have

$$g(0) = 0, \quad g'(0) = 1 - \sum_{i=1}^{m} \alpha_i \eta_i > 0, \quad \text{and} \quad g(\pi) = -\sum_{i=1}^{m} \alpha_i \sin(\eta_i \pi) < 0.$$

Thus there is a solution between 0 and π. The corresponding eigenfunction $\varphi_1(t) = \sin(\omega_1 t) > 0$ on $[0,1]$ and $0 < \mu_1 < \pi^2$. The precise value of μ_1 would be found numerically from (6.8) in a given case. We also show explicitly that g has no other zeros on $[0,\pi]$. In fact

$$g'(x) = \cos x - \sum_{i=1}^{m} \alpha_i \eta_i \cos(\eta_i x)$$

has precisely one zero on $[0,\pi/2]$ since $\sum_{i=1}^{m} \alpha_i \eta_i < 1$, using the argument above for (6.2). On $[\pi/2,\pi]$, as cos is decreasing, $\cos x < \cos(\eta_i x)$ and

$$g'(x) = \left(1 - \sum_{1}^{m} \alpha_i \eta_i\right) \cos(x) + \sum_{1}^{m} \alpha_i \eta_i (\cos(x) - \cos(\eta_i x)).$$

Since $0 < \sum_{i=1}^{m} \alpha_i \eta_i < 1$ we obtain $g'(x) < 0$ on $(\pi/2,\pi)$. Thus g' has precisely one zero in $(0,\pi)$ and therefore g has at most two zeros on $[0,\pi]$. One of these is 0 the other is the unique positive solution with a positive eigenfunction.

Remark 6.2. The conditions $\sum_{i=1}^{m} \alpha_i < 1$ for the BCs (6.2) and $\sum_{i=1}^{m} \alpha_i \eta_i < 1$ for BCs (6.3) are optimal. This may be seen from the 3-point cases. For BC (6.2), if $\alpha > 1$ there is no positive eigenvalue with a positive eigenfunction.

Also, in the 3-point case for BC (6.3), if $\alpha\eta > 1$ it may be checked that there is no positive eigenvalue with a positive eigenfunction.

EXAMPLE 6.3. By way of illustration, we now compute a few numbers for the 3-point problem with the BC $u'(0) = 0$, $u(1) = \alpha u(\eta)$. We first take $\eta = 1/2$, $\alpha = 3/4$. Then ω_1 is the smallest positive solution of the equation

$$\cos(\omega) = (3/4)\cos(\omega/2).$$

Hence we find $\omega_1 = 0.8103$, $\mu_1 = 0.6565$ (rounded to 4 decimal places). We also need to compute m, M. In [24], $[a, b]$ is found so as to make $M_{a,b}$ as small as possible. Using these expressions we get

$$m = 8/13 = 0.6154, \quad \mu_1 = 0.6565, \quad M = 0.8163 \quad \text{(to 4 decimal places)}.$$

Taking the formulae from [3] and applying them to the special case that we have here, their corresponding numbers are

$$m_{BF} = 8/13, \quad M_{BF} = 8/9 = 0.8889.$$

Secondly we take $\eta = 3/4$, $\alpha = 1/2$. From [24], we get

$$m = 32/23 = 1.3913, \quad M_{0,b_0} = 64/25 = 2.56 \quad \text{and} \quad \mu_1 = 1.6382.$$

The corresponding number calculated from the formulae in [3] is $M_{BF} = 32/7 = 4.5714$.

Hence our numbers m, M are better than those of [3]. Of course we also use μ_1 which gives even better results whatever $[a, b]$ is chosen.

EXAMPLE 6.4. We also give some numbers for the BC (6.3) and compare them with the numbers found in [2]. In this case Bai and Fang do not get the optimal m because they discard two negative terms in one of their index calculations, but their M is optimal when $\alpha \geq 1$ but not in other cases (see [24]). When $\alpha = 1/2, \eta = 1/2$ we obtain, using a notation and formula from [24], $m = 288/49 = 5.8776$, $m_{BF} = 1.5$, $M(\eta, b_3) = 8$, $M_{BF} = M(\eta, 1) = 12$, and $\mu_1 = 6.9497$.

REMARK 6.5. Our results can be applied to obtain existence of one or several positive radial solutions in an annulus for the equation

$$\triangle u + h(|x|)f(u) = 0, \quad |x| \in [R_1, R_0],\ x \in \mathbb{R}^n,\ n \geq 2. \tag{6.9}$$

with either local or nonlocal boundary conditions. The local BCs are of the form

$$\begin{cases} \alpha u(x) + \beta' \partial u/\partial r(x) = 0 & \text{on } |x| = R_0, \\ \gamma u(x) - \delta' \partial u/\partial r(x) = 0 & \text{on } |x| = R_1, \end{cases} \tag{6.10}$$

where $r = |x|$, $\partial u/\partial r$ denotes differentiation in the radial direction and $\alpha, \beta', \gamma, \delta' > 0$ with $\gamma\beta' + \alpha\gamma + \alpha\delta' > 0$.

For positive radial solutions we can write (6.9)–(6.10) in the form

$$u''(r) + \frac{n-1}{r}u'(r) + h(r)f(u(r)) = 0 \quad \text{a.e. on } [R_1, R_0]. \tag{6.11}$$

It is known that (6.11) and the corresponding BCs can be transformed into (5.1)–(5.2), see for example [15] for explicit formulas to achieve this. Similarly we can deal with nonlocal BCs that transform into (6.2) or (6.3). We do not state the large number of obvious theorems that may be written.

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