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# LIMITING CASES OF ASYMPTOTICALLY POSITIVE LINEAR CONDITIONS AND SOLVABILITY OF STURM–LIOUVILLE BOUNDARY-VALUE PROBLEMS FOR DUFFING EQUATIONS

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Abstract. In this paper we study the solvability of Sturm–Liouville BVPs for Duffing equations by means of homotopy continuation methods. We propose a new kind of solvable conditions on the nonlinear function in the equation. This kind of conditions can be seen as some limiting cases of the well-known asymptotically positive linear conditions. The obtained results generalize and unify some previous results by S. Villegas, T. Ma and L. Sanchez, and Y. Dong, respectively.

## 1. Introduction and main results

Consider the Sturm–Liouville boundary value problem

$$(p(t)x'(t))' + q_0(t)x(t) + h(t, x(t)) + g(t, x(t)) = 0, \tag{1.1}$$

$$x(0)\cos\alpha - p(0)x'(0)\sin\alpha = 0, \tag{1.2}$$

$$x(1)\cos\beta - p(1)x'(1)\sin\beta = 0. \tag{1.3}$$

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where $p\colon[0,1]\to(0,\infty)$ is positive and absolutely continuous, $q_0\in H_0(p,\alpha,\beta)$ (its meaning will be given in the following Definition 1.1); $\alpha,\beta\in\mathbb{R}$ are fixed with $0\le\alpha<\pi, 0<\beta\le\pi$; $h\colon[0,1]\times\mathbb{R}\to\mathbb{R}$ is a $L^1$-Carathéodory function, i.e. $h\ (t,\,\cdot\,)$ is continuous on $\mathbb{R}$ for a.e. $t\in(0,1)$, $h(\,\cdot\,,x)$ is measurable on (0,1) for each $x\in\mathbb{R}$, and for any constant $r>0$ there exits some function $\rho_r\in L^1(0,1)$ such that $|h(t,x)|\le\rho_r(t)$ for a.e. $t\in(0,1)$ and all $x\in\mathbb{R}$ with $|x|\le r$; $g\colon(0,1)\times\mathbb{R}\to\mathbb{R}$ is also a Carathéodory function such that $|g(t,x)|\le\widehat{g}(t)$ for $x\in\mathbb{R}$ a.e. $t\in(0,1)$, where $\widehat{g}\in L^1(0,1)$. In the following we always denote by $f(t,x):=h(t,x)+g(t,x)$.

For readers' convenience we list a definition by the second author bellow.

DEFINITION 1.1 (cf. [1, Definition 2.2]). For any $q\in L^1(0,1)$ we say $q\in H_n(p,\alpha,\beta)$ for some nonnegative integer $n$ if and only if the linear boundary value problem (1.2)–(1.3) and

$$(p(t)x'(t))'+q(t)x(t)=0 \tag{1.4}$$

has a nontrivial solution with exactly $n$ zeros on $(0,1)$.

Because we assumed $q_0\in H_0(p,\alpha,\beta)$, by Definition 1.1, (1.2)–(1.4) has a nontrivial solution $x_0(t)$. Without loss of generality we assume $x_0(t)>0$ for $t\in(0,1)$. For any $a,b\in L^1(0,1)$, let $a\le b$ denote $a(t)\le b(t)$ for a.e. $t\in(0,1)$; and let $a<b$ denote $a\le b$ and $a(t)<b(t)$ for $t$ in a subset of $(0,1)$ with positive measure. Let $C^i(0,1)=\{x\colon[0,1]\to\mathbb{R}\mid x^{(i)}(t)$ is continuous for $t\in[0,1]\}$ with the usual norm and denote by $C^0(0,1)=C(0,1)$. Let $H^1(0,1)=\{x\in C(0,1)\mid x'\in L^2(0,1)\}$ and $W_0^{2,1}(0,1)=\{x\in C^1(0,1)\mid x''\in L^1(0,1)$ and $x$ satisfies (1.2)–(1.3)$\}$. The following theorems are the main results of this paper.

THEOREM 1.2. *Assume that $\alpha\ne0$, $\beta\ne\pi$ and*

(a) *there exists some $r>0$ such that for $|x|\ge r$, and a.e. $t\in(0,1)$ one has*

$$h(t,x)/x\ge0; \tag{1.5}$$

(b) *there exist $q\in L^1(0,1)$ and $q_{1,\alpha}\in H_0(p,\alpha,\pi)$, $q_{2,\beta}\in H_0(p,0,\beta)$ with $0<q\le q_{1,\alpha}-q_0$, $q\le q_{2,\beta}-q_0$ such that for $x\ge r$, and a.e. $t\in(0,1)$ one has*

$$h(t,x)/x\le q(t); \tag{1.6}$$

(c) *there exists $\rho>0$ such that for any $x_+,x_-\in W_0^{2,1}(0,1)$ with $x_+(t)\ge\rho x_0(t)>0$, $x_-(t)\le-\rho x_0(t)<0$, we have*

$$\int_0^1 f(t,x_-(t))x_0(t)\,dt\le0\le\int_0^1 f(t,x_+(t))x_0(t)\,dt, \tag{1.7}$$

*where we recall that $f(t,x)=h(t,x)+g(t,x)$.*

*Then* (1.1)–(1.3) *has at least one solution.*

THEOREM 1.3. *Assume that* $\alpha = 0$ *or* $\beta = \pi$, *assumption* (a) *of Theorem* 1.2 *is satisfied and*

(a) *there exists some function* $F \in L^1(0,1)$ *such that*

$$h(t,x) \le F(t), \tag{1.8}$$

*for* $x \ge r$, *a.e.* $t \in (0,1)$;

(b) *there exists* $\rho > 0$ *such that for any* $x_+, x_- \in W_0^{2,1}(0,1)$ *with* $x_+(t) \ge \rho x_0(t) > 0$, $x_-(t) \le -\rho x_0(t) < 0$, *we have*

$$\int_0^1 f(t, x_-(t))x_0(t)\,dt < 0 < \int_0^1 f(t, x_+(t))x_0(t)\,dt. \tag{1.9}$$

*Then* (1.1)–(1.3) *has at least one solution.*

Several special cases of our theorems were discussed in other papers. In 1995, T. Ma and L. Sanchez in [5] discussed Dirichlet BVP

$$x'' + \pi^2 x + h(t,x) + g(t,x) = 0, \tag{1.10}$$

$$x(0) = 0 = x(1), \tag{1.11}$$

and obtained the following

THEOREM 1.4. *Assume that*

(a) $h(t,x)/x \ge 0$ *for* $|x| \ge r$;

(b) *there exists some function* $F \in L^1(0,1)$ *such that* $h(t,x) \le F(t)$ *for* $x \le -r$;

(c)
$$\int_0^1 f(t,x_-)\sin \pi t\,dt < 0 < \int_0^1 f(t,x_+)\sin \pi t\,dt$$
*where* $x_+(t) \ge \rho \sin \pi t$, $x_-(t) \le -\rho \sin \pi t$, $\rho > 0$ *is a constant.*

*Then* (1.10), (1.11) *has at least one solution.*

REMARK 1.5. This theorem is a special case of Theorem 1.3. In fact, when $\alpha = 0, \beta = \pi$, (1.2), (1.3) reduces to (1.11). And $x = \sin \pi t$ is a nontrivial solution of

$$x'' + \pi^2 x = 0,$$
$$x(0) = 0 = x(1).$$

So $\pi^2 \in H_0(1, 0, \pi)$ and the assumptions in Theorem 1.4 are special cases of Theorem 1.3.

The following two corollaries of Theorem 1.3 are new results.

EXAMPLE 1.6. Let $t_1 \in (0,1)$ be fixed,

$$q_0(t) := \begin{cases} \dfrac{\pi^2}{4t_1^2} & \text{for } t \in (0,t_1), \\ \dfrac{\pi^2}{4(1-t_1)^2} & \text{for } t \in (t_1,1), \end{cases}$$

$$h(t,x) := \begin{cases} xe^{-x} + \sin t & \text{for } x < 0, \\ \arctan x + \sin t & \text{for } x > 0. \end{cases}$$

The following problem

$$x'' + q_0(t)x + h(t,x) = 0, \quad x(0) = 0 = x(1), \tag{1.12}$$

has at least one solution. In fact, let

$$\begin{aligned} x_0(t) &= \sin\frac{\pi}{2t_1}t && \text{for } t \in (0,t_1), \\ x_0(t) &= \sin\frac{\pi}{2(1-t_1)}(t-t_1) && \text{for } t \in (t_1,1). \end{aligned}$$

Then $x = x_0(t)$ is a nontrivial solution of

$$\begin{aligned} &x'' + q_0(t)x = 0, \\ &x(0) = 0 = x(1). \end{aligned}$$

And by Definition 1.1, we have $q_0 \in H_0(1,0,\pi)$. All the assumptions of Theorem 1.3 are satisfied. Hence, (1.11), (1.12) has a solution.

Let $\alpha = 0, \beta = \pi/2$, (1.2), (1.3) reduce to

$$x(0) = 0 = x'(1), \tag{1.13}$$

and $x = \sin(\pi t/2)$ is a solution of (1.13) and $x'' + (\pi^2/4)x = 0$. We can consider (1.13) and

$$x'' + \frac{\pi^2}{4}x + h(t,x) + g(t,x) = 0. \tag{1.14}$$

From Theorem 1.3 we have

COROLLARY 1.7. *Assume that*

(a) $h(t,x)/x \geq 0$ *for* $|x| \geq r$;

(b) *there exists some function* $F \in L^1(0,1)$ *such that* $h(t,x) \leq F(t)$ *for* $x \geq r$;

(c)
$$\int_0^1 f(t,x_-)\sin\frac{\pi}{2}t\,dt < 0 < \int_0^1 f(t,x_+)\sin\frac{\pi}{2}t\,dt$$
*where* $x_+(t) \geq \rho\sin(\pi t/2)$, $x_-(t) \leq -\rho\sin(\pi t/2)t$ *for some* $\rho > 0$.

*Then the problem* (1.13), (1.14) *has at least one solution.*

In 1998 and in 2002, S. Villegas in [4] and Dong in [2] discussed the following Neumann BVP respectively

$$x'' + h(t,x) + g(t,x) = 0, \tag{1.15}$$

$$x'(0) = 0 = x'(1). \tag{1.16}$$

The following result was obtained.

Theorem 1.8. *Assume that*

(a) $h(t,x)/x \geq 0$ *for* $|x| \geq r$;
(b) $h(t,x)/x \leq \pi^2/4$ *for* $x \geq r$;
(c)
$$\int_0^1 f(t,x_-)\sin\frac{\pi}{2}t\,dt \leq 0 \leq \int_0^1 f(t,x_+)\sin\frac{\pi}{2}t\,dt$$
*where* $x_+(t) \geq \rho\sin(\pi t/2) > 0$, $x_-(t) \leq -\rho\sin(\pi t/2) < 0$.

*Then* (1.15), (1.16) *has at least one solution.*

Note that when $\alpha = \beta = \pi/2$, (1.2), (1.3) reduce to (1.16), and $0 \in H_0(1,\pi/2,\pi/2)$, $\pi^2/4 \in H_0(1,\pi/2,\pi) \cap H_0(1,0,\pi/2)$. As a result, Theorem 1.8 is a special case of Theorem 1.2. The following example shows our Theorem 1.2 can be applied to some new problems.

Example 1.9. Let $\lambda_0 > 0$ satisfy

$$\cosh\lambda_0 - \lambda_0\sinh\lambda_0 = 0, \quad h(t,x) = \lambda_0^2(\sin xt)^2 + \sin t \quad \text{as } x \geq 0,$$

and

$$h(t,x) = xe^{-x} + \sin t \quad \text{as } x < 0.$$

Consider the following problem

$$x'' - \lambda_0^2 x + h(t,x) = 0, \tag{1.17}$$

$$x'(0) = 0, \quad x(1) - x'(1) = 0. \tag{1.18}$$

As $\alpha = \pi/2, \beta = \pi/4, p(t) \equiv 1$, (1.2), (1.3) reduce to (1.18). It is easy to check that $x = \cosh\lambda_0 t$ is a solution of (1.18) and $x'' - \lambda_0^2 x = 0$, and $x = t$ is a solution of $x'' = 0$, $x(0) = 0$, $x(1) - x'(1) = 0$. Thus, $-\lambda_0^2 \in H_0(1,\pi/2,\pi/4)$, $0 \in H_0(1,0,\pi/4)$. As before we also have $\pi^2/4 \in H_0(1,\pi/2,\pi)$. By Theorem 1.2, (1.17), (1.18) has a solution.

The assumption in (a) of Theorem 1.2 is sharp. The following example will illustrate its precise meaning. Note that for the special case (b) of Theorem 1.8, an example has been given in [3]. Let $\phi(t,p,q,a,\gamma)$ be the unique solution of

$$\phi' = \frac{1}{p(t)}\cos^2\phi + q(t)\sin^2\phi, \quad t \in (a,b),$$
$$\phi(a) = \gamma.$$

From [1], $\phi(t,p,q,a,\gamma)$ is monotonously increasing with respect to $q$ and $q \in H_0(p,(a,b),\alpha,\beta)$ if and only if $\phi(b,p,q,a,\alpha) = \beta$, i.e. the following problem

$$\begin{aligned} (p(t)x'(t))' + q(t)x(t) &= 0, \quad t \in (a,b), \\ x(a)\cos\alpha - p(a)x'(a)\sin\alpha &= 0, \\ x(b)\cos\beta - p(b)x'(b)\sin\beta &= 0, \end{aligned}$$

has a nontrivial solution with no zeros on $(a,b)$. Note that $H_0(p,\alpha,\beta) = H_0(p,(0,1),\alpha,\beta)$ from Definition 1.1.

EXAMPLE 1.10. Let $q_{1,\alpha} \in H_0(p,\alpha,\pi)$, $q^+ = q_{1,\alpha} + \varepsilon$ such that

$$\phi(t_1,p,q^+,0,\alpha) = \pi \quad \text{for some } t_1 \in (0,1) \quad \text{and} \quad \phi(1,p,q^+,0,\alpha) < \pi + \beta.$$

Let

$$\begin{aligned} q^-(t) &= q^+(t) && \text{for a.e. } t \in (0,t_1), \\ q^-(t) &= q_{1,\alpha}(t) + \mu && \text{for a.e. } t \in (t_1,1) \end{aligned}$$

such that $\phi(1,p,q^-,t_1,0) = \beta$. Let $f(t) = 0$ for $t \in (0,t_1)$ and $f(t) = 1$ for $t \in (t_1,1)$. Then the following problem (1.2), (1.3) and

$$(p(t)x')' + q^+(t)x^+ - q^-(t)x^- = f(t)$$

has no solutions. In fact, as elements in $L^1(0,t_1)$, $q^+ = q^- \in H_0(p,(0,t_1),\alpha,\pi)$, and hence, the following problem (1.2) and

$$(p(t)x')' + q^+(t)x = 0, \quad t \in (0,t_1), \tag{1.20}$$
$$x(t_1) = 0, \tag{1.21}$$

has a nontrivial solution $x^*(t)$. If $x(t)$ is a solution, then $x$ satisfies (1.20), (1.2). So, $x(t) = Cx^*(t)$, $t \in (0,t_1)$ for some constant $C$. From (1.21) we have $x(t_1) = 0$. In the following we will obtain a contradiction in two cases.

*Case 1.* $x'(t_1) \geq 0$. From (1.19), $x(t) > 0$ for $t \in (t_1,1)$, and $x(t)$ satisfies

$$(p(t)x')' + q^+(t)x = 1, \quad t \in (t_1,1), \tag{1.22}$$
$$x(t_1) = 0, \quad x(1)\cos\beta - p(1)x'(1)\sin\beta = 0. \tag{1.23}$$

There exists $\varepsilon_1 > 0$ such that $\phi(1,p,q^+ + \varepsilon_1,t_1,0) = \beta$, i.e. the problem (1.23) and

$$(p(t)x')' + (q^+(t) + \varepsilon_1)x = 0, \quad t \in (t_1,1) \tag{1.24}$$

has a nontrivial solution $u_0(t) > 0$ for $t \in (t_1, 1)$. Multiplying (1.22) with $u_0(t)$ and integrating over $(t_1, 1)$, we have from (2.24) that

$$\begin{aligned}\int_{t_1}^1 u_0(t)\,dt = \int_{t_1}^1 ((p(t)x')' + (q^+(t) + \varepsilon_1)x)u_0(t)\,dt - \varepsilon_1 \int_{t_1}^1 u_0(t)\,dt \\ = -\varepsilon_1 \int_{t_1}^1 u_0(t)\,dt,\end{aligned}$$

a contradiction.

*Case 2.* $x'(t_1) < 0$. From Case 1, $x(t) < 0$ for $t \in (t_1, 1)$. And $x(t)$ satisfies (1.23) and

$$(p(t)x')' + q^-(t)x = 1, \quad t \in (t_1, 1). \tag{1.25}$$

Because $\phi(1, p, q^-, t_1, 0) = \beta$, there exists $v_0(t) > 0$ for $t \in (t_1, 1)$ such that $x = v_0(t)$ is a nontrivial solution of (1.23) and

$$(p(t)x')' + q^-(t)x = 0, \quad t \in (t_1, 1).$$

Multiplying (1.25) with $v_0(t)$ and integrating over $(t_1, 1)$, we have

$$0 = \int_{t_1}^1 ((p(t)x')' + q^-(t)x(t))v_0(t)\,dt = \int_{t_1}^1 v_0(t)\,dt,$$

a contradiction.

The conditions (1.5), (1.6) and (1.5), (1.8) have some relationship with the well-known asymptotically positive linear conditions. We only explain (a), (b) of Theorem 1.8 as an example.

As we know (see [8] for references) any $(\mu, \nu) \in \mathbb{R}^2$ satisfying

$$\frac{2}{\pi} = \frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}}, \tag{1.26}$$

is a second resonant point of the Fučik spectrum associated with

$$\begin{gathered} x'' + \mu x^+ - \nu x^- = 0, \\ x'(0) = 0 = x'(1). \end{gathered}$$

So in order to discuss the solvability of (1.15), (1.16), we can assume

$$0 \le h(t,x)/x \le \mu, \quad x \ge r > 0, \quad \text{a.e. } t \in (0,1), \tag{1.27}$$

$$0 \le h(t,x)/x \le \nu, \quad x \le -r, \quad \text{a.e. } t \in (0,1). \tag{1.28}$$

As $\nu \to \infty$, from (1.26) we have $\mu \to \pi^2/4$. And hence (1.27), (1.28) become

$$0 \le h(t,x)/x \le \pi^2/4, \quad x \ge r > 0, \quad \text{a.e. } t \in (0,1), \tag{1.29}$$

$$0 \le h(t,x)/x, \quad x < -r, \quad \text{a.e. } t \in (0,1). \tag{1.30}$$

Because (1.27), (1.28) are the well-known asymptotically positive linear conditions, the conditions (1.29), (1.30) can be seen as some limiting cases of the conditions. One can also refer to [3], [6], [7], [11] and the references therein for other these two kinds of conditions. See also the closely relate paper [12] by Mawhin and Ruiz for references.

In the following sections we give the proofs of Theorems 1.2 and 1.3, respectively.

## 2. Proof of Theorem 1.2

In this section we give

PROOF OF THEOREM 1.2. Let $(Lx)(t) := (p(t)x'(t))' + q_0(t)x(t) + q(t)x(t)$ for any $x \in W_0^{2,1}(0,1)$, $(Nx)(t) := q(t)x(t) + f(t.x(t))$. By assumption (b), $L$ is invertible and $L^{-1}\colon L^1(0,1) \to W_0^{2,1}(0,1)$ is continuous. Because $N\colon H^1(0,1) \to L^1(0,1)$ is continuous and the embedding from $W_0^{2,1}(0,1)$ to $H^1(0,1)$ is compact, we have $L^{-1}N\colon H^1(0,1) \to H^1(0,1)$ is compact. Obviously, (1.1)–(1.3) is equivalent to $x + L^{-1}Nx = 0$, $x \in H^1(0,1)$. In view of Leray–Schauder Principle, in order to prove the solvability of (1.1)–(1.3), we only need discuss $x + (1-\lambda)L^{-1}Nx = 0$, $\lambda \in (0,1)$, $x \in H^1(0,1)$, equivalently we only need to prove that solutions of the following auxiliary problem are *à priori* bounded with respect to the norm $||\cdot||_{H^1}$ of $H^1(0,1)$:

$$\begin{aligned} (p(t)x'(t))' + q_0(t)x(t) + \lambda q(t)x(t) + (1-\lambda)f(t,x(t)) &= 0, \\ x(0)\cos\alpha - p(0)x'(0)\sin\alpha &= 0, \\ x(1)\cos\beta - p(1)x'(1)\sin\beta &= 0. \end{aligned}$$

We argue by contradiction. We assume that $\{x_n\} \subset H^1(0,1)$ with $||x_n||_{H^1} \to \infty$ and $\{\lambda_n\} \subset (0,1)$ such that

$$(p(t)x_n'(t))' + q_0(t)x_n(t) + \lambda_n q(t)x_n(t) + (1-\lambda_n)f(t,x_n(t)) = 0, \tag{2.1}$$

$$x_n(0)\cos\alpha - p(0)x_n'(0)\sin\alpha = 0, \tag{2.2}$$

$$x_n(1)\cos\beta - p(1)x_n'(1)\sin\beta = 0. \tag{2.3}$$

Set $y_n = x_n/||x_n||_{H^1}$, then $||y_n||_{H^1} = 1$. So it is possible to extract a subsequence (denoted also by $\{y_n\}$) converging weakly to some function $y_0 \in H^1(0,1)$ and strongly in $C(0,1)$. In the following we will take three steps to reach a contradiction.

*Step 1.* For every $\varepsilon > 0$ there holds

$$||x_n||_{H^1}^{-1} \int_0^1 |f_n(t,x_n(t))x_0(t)|\,dt \le 2\int_{x_n(t)\ge r} q(t)y_n^+(t)x_0(t)\,dt + \varepsilon, \tag{2.4}$$

for $n$ large enough, where $f_n(t,x_n(t)) := \lambda_n q(t)x_n(t) + (1-\lambda_n)f(t,x_n(t))$.

In fact, using the definition of $f_n$ (2.1) can be abbreviated

$$(p(t)x_n'(t))' + q_0(t)x_n(t) + f_n(t, x_n(t)) = 0, \tag{2.5}$$

and

$$\int_0^1 [(p(t)x_n'(t))' + q_0(t)x_n(t)]x_0(t)\,dt + \int_0^1 f_n(t, x_n(t))x_0(t)\,dt = 0. \tag{2.6}$$

By the definitions of $q_0$ and $x_0$ we have

$$\begin{aligned}
0 = &-\int_0^1 [(p(t)x_n'(t))' + q_0(t)x_n(t)]x_0(t)\,dt = \int_0^1 f_n(t, x_n(t))x_0(t)\,dt \\
= &\int_{x_n(t)\geq r} f_n(t, x_n(t))x_0(t)\,dt + \int_{|x_n(t)|\leq r} f_n(t, x_n(t))x_0(t)\,dt \\
&+ \int_{x_n(t)\leq -r} f_n(t, x_n(t))x_0(t)\,dt,
\end{aligned}$$

and

$$-\int_{x_n(t)\leq -r} f_n(t, x_n(t))x_0(t)\,dt \leq \int_{x_n(t)\geq r} f_n(t, x_n(t))x_0(t)\,dt + C_1.$$

where $C_1$ is a positive constant. From assumptions (a) and (b) we have

$$-\int_{x_n(t)\geq r} f_n(t, x_n(t))x_0(t)\,dt \leq \int_{x_n(t)\geq r} q(t)x_n^+(t)x_0(t)\,dt + \int_0^1 \widehat{g}(t)x_0(t)\,dt,$$

and hence,

$$\begin{aligned}
\int_0^1 |f_n(t,x_n(t))x_0(t)|\,dt = &\int_{x_n(t)\geq r} |f_n(t, x_n(t))x_0(t)|\,dt \\
&+ \int_{|x_n(t)|\leq r} |f_n(t, x_n(t))x_0(t)|\,dt + \int_{x_n(t)\leq -r} |f_n(t, x_n(t))x_0(t)|\,dt \\
= &\int_{x_n(t)\geq r} f_n(t, x_n(t))x_0(t)\,dt - \int_{x_n(t)\leq -r} f_n(t, x_n(t))x_0(t)\,dt \\
&+ \int_{|x_n(t)|\leq r} |f_n(t, x_n(t))x_0(t)|\,dt \leq 2\int_{x_n(t)\geq r} q(t)x_n^+(t)x_0(t)\,dt + C_2
\end{aligned}$$

where $C_2$ is a positive constant. This yields (2.4).

*Step 2.* Denote by $y_0^+(t) = \max\{y_0(t), 0\}$ for $t \in [0,1]$. We have

$$y_0^+ \neq 0. \tag{2.7}$$

In fact, if $y_0^+ = 0$, from (2.4) we have

$$||x_n||_{H^1}^{-1} \int_0^1 |f_n(t, x_n(t))x_0(t)|\,dt \to 0. \tag{2.8}$$

Let $y \in C^2(0,1)$ denote an arbitrarily function satisfying the boundary value conditions (1.2), (1.3). By (2.5),

$$\int_0^1 [(p(t)x_n'(t))' + q_0(t)x_n(t)]y(t)\,dt + \int_0^1 f_n(t,x_n(t))y(t)\,dt = 0.$$

And hence,

$$\begin{aligned}
&\left|\int_0^1 [(p(t)y_n'(t))' + q_0(t)y_n(t)]y(t)\,dt\right| \\
&\qquad = ||x_n||_{H^1}^{-1}\left|\int_0^1 f_n(t,x_n(t))y(t)\,dt\right| \\
&\qquad \le ||x_n||_{H^1}^{-1}\int_0^1 |f_n(t,x_n(t))y(t)|\,dt \\
&\qquad = ||x_n||_{H^1}^{-1}\int_0^1 |f_n(t,x_n(t))x_0(t)||y(t)/x_0(t)|\,dt \\
&\qquad \le C_3||y||_{C^1}||x_n||_{H^1}^{-1}\int_0^1 |f_n(t,x_n(t))x_0(t)|\,dt.
\end{aligned} \tag{2.9}$$

Here in the last inequality we used an inequality in [13, Lemma 3] as following

$$|y(t)| \le C_3||y||_{C^1}x_0(t), \quad t \in [0,1]. \tag{2.10}$$

From (2.8) and (2.10) we have

$$\int_0^1 [(p(t)y_n'(t))' + q_0(t)y_n(t)]y(t)\,dt \to 0.$$

Because $y_n \to y_0$ in $C(0,1)$, integrating by parts we have

$$\int_0^1 [(p(t)y'(t))' + q_0(t)y(t)]y_0(t)\,dt = 0. \tag{2.11}$$

Let $y(t)$ satisfy $y(0) = y(1) = y'(0) = y'(1)$. It is obvious that such a $y$ satisfies (1.2), (1.3). From (2.11), we have

$$\int_0^1 y'(t)[p(t)y_0'(t) + \varphi(t)]\,dt = 0 \tag{2.12}$$

where

$$\varphi(t) = \int_0^t q_0(\tau)y_0(\tau)\,d\tau. \tag{2.13}$$

Let $C_4 = \int_0^1 [p(t)y_0'(t) + \varphi(t)]\,dt$ and $\psi(t) := p(t)y_0'(t) + \varphi(t) - C_4$, from (2.12), (2.13) we have

$$\int_0^1 y'(t)\psi(t)\,dt = 0. \tag{2.14}$$

In the following we prove

$$\psi(t) = 0 \quad \text{for a.e. } t \in (0,1). \tag{2.15}$$

In fact, because $\int_0^1 \psi(t)\,dt = 0$, if (2.15) is not satisfied, then there exist two subsets $E_1, E_2 \subset (0,1)$ with non zero measures such that $\psi(t) > 0$ for $t \in E_1$, $\psi(t) < 0$ for $t \in E_2$. Let

$$e(t) := \begin{cases} a \ > 0 & \text{for } t \in E_1, \\ b < 0 & \text{for } t \in E_2, \\ 0 & \text{for } t \in (0,1) \setminus (E_1 \cup E_2). \end{cases} \tag{2.16}$$

satisfy $\int_0^1 e(t)\,dt = a \cdot \text{meas}(E_1) + b \cdot \text{meas}(E_2) = 0$. By the knowledge of smooth approximations, there exists $z_\varepsilon \in C^2(0,1)$ with $z_\varepsilon(0) = z_\varepsilon(1) = 0$ and $\int_0^1 z_\varepsilon(t)\,dt = 0$ such that

$$z_\varepsilon \to e \quad \text{in } L^2(0,1). \tag{2.17}$$

Denote by $y(t) = \int_0^t z_\varepsilon(\tau)\,d\tau$, then $y \in W_0^{2,1}(0,1)$ and $y(0) = y(1) = y'(0) = y'(1) = 0$. From (2.14) we have

$$\int_0^1 z_\varepsilon(t)\psi(t)\,dt = 0. \tag{2.18}$$

Let $\varepsilon \to 0^+$, from (2.17) we have

$$0 = \int_0^1 e(t)\psi(t)\,dt = a\int_{E_1} \psi(t)\,dt + b\int_{E_2} \psi(t)\,dt > 0.$$

This is a contradiction. Now we have proved (2.15). From (2.15) it follows that

$$(p(t)y_0'(t))' + \varphi'(t) = 0, \quad \text{a.e. } t \in (0,1);$$

i.e.

$$(p(t)y_0'(t))' + q_0(t)y_0(t) = 0, \quad \text{a.e. } t \in (0,1). \tag{2.19}$$

In the following, we prove $y_0$ satisfies the boundary value conditions (2.2) and (2.3). From (2.11), (2.19), we get

$$[p(t)y_0(t)y'(t) - p(t)y_0'(t)y(t)]_0^1 - \int_0^1 [(p(t)y_0'(t))' + q_0(t)y_0(t)]y(t)\,dt = 0$$

and

$$p(1)y_0(1)y'(1) - p(1)y_0'(1)y(1) = p(0)y_0(0)y'(0) - p(0)y_0'(0)y(0). \tag{2.20}$$

Because $y$ is arbitrary, by changing $y$ satisfying $y(1) = y'(1) = 0$ and $y(0) \neq 0$ from (2.20) and that $y$ satisfies (2.2) we have $y_0$ satisfies (1.2) and in a similar way $y_0$ satisfies (1.3). Now we have proved that $y_0$ is a nontrivial solution of (1.2)–(1.4) with $q$ replaced by $q_0$. Because $x_0$ is also a nontrivial solution, so

$y_0 = c \cdot x_0$ for some constant $c$. Since we assumed $y_0^+ = 0$, then $y_0(t) < 0$ for $t \in [0,1]$ and $c < 0$. From (2.6) we also have

$$\lambda_n \int_0^1 q(t)x_n(t)x_0(t)\,dt + (1-\lambda_n)\int_0^1 f(t,x_n(t))x_0(t)\,dt = 0.$$

and

$$\int_0^1 f(t,x_n(t))x_0(t)\,dt > 0. \tag{2.21}$$

Because $x_n(t) = ||x_n||_{H^1} y_n(t)$, and $x_n$ satisfies (2.1)–(2.3) and (2.8), we can prove $y_n \to y_0$ in $C^1(0,1)$. Let $y_n = y_0 + \widetilde{y_n}$, then $\widetilde{y_n} \to 0$ in $C^1(0,1)$. Making use of (2.10) again, we have

$$\widetilde{y_n}(t) \le C_3||\widetilde{y_n}||_{C^1}|y_0(t)| = -C_3||\widetilde{y_n}||_{C^1} y_0(t).$$

As a result, $x_n(t) \le -\rho x_0(t)$ for $n$ large enough. And hence, (2.21) contradicts (1.7), and (2.7) is proved.

From the proof of (2.7) we can also find a point $t_1 \in [0,1]$ such that $y_0(t_1)=0$. In fact if it is not the case we have $y_0(t) > 0$ for $t \in [0,1]$. This will also lead to a contradiction. Now for $y_0$ we have three cases

(1) $y_0(a) = 0 = y_0(b)$, $y_0(t) > 0$ for $t \in (a,b) \subset [0,1]$;
(2) $y_0(t_1) = 0$, $y_0(t_1) > 0$ for $t \in (t_1, 1]$;
(3) $y_0(t_1) = 0$, $y_0(t_1) > 0$ for $t \in [0, t_1)$.

*Step 3.* Case (1) leads to a contradiction. In fact let

$$q_n(t) = \begin{cases} h_n(t,x_n)/x_n & \text{if } x_n(t) \ge r, \\ q(t) & \text{if } x_n(t) \le r, \end{cases} \tag{2.22}$$

$$\xi_n(t) = q_0(t) + \lambda_n q(t) + (1-\lambda_n)q_n(t), \tag{2.23}$$

$$g_n(t) = (1-\lambda_n)(f(t,x_n(t)) - q_n(t)x_n(t)), \tag{2.24}$$

then (2.1) becomes

$$(p(t)y_n'(t))' + \xi_n(t)y_n(t) + ||x_n||_{H^1}^{-1} g_n(t) = 0. \tag{2.25}$$

From (2.22), (2.23), and assumptions (a), (b), we have

$$q_0(t) \le \xi_n(t) \le q_0(t) + q(t) = \min\{q_{1,\alpha}(t), q_{2,\beta}(t)\}$$

for a.e. $t \in (a,b)$, and $\xi_n \rightharpoonup \xi_0$ in $L^2(a,b)$ by going to subsequences if necessary and $q_0 \le \xi_0 \le q_0 + q$. Furthermore, by (2.24) there exists some $\overline{g} \in L^1(a,b)$ such that

$$|h(t,x_n(t)) - q_n(t)x_n(t)| \le \overline{g}(t) \quad \text{for a.e. } t \in (a,b).$$

From (2.25) and the Arzela–Ascoli theorem, we may assume $y_n \to y_0$ in $C^1(0,1)$.

Taking limits in (2.25) as $n \to \infty$ we have

$$(p(t)y_0'(t))' + \xi_0(t)y_0(t) = 0, \quad \text{a.e. } t \in (a,b).$$

By the Sturm comparison theorem, this is impossible since $y_0(a) = 0 = y_0(b)$ and $q_0 \le \xi_0 \le q_0 + q$. The proof is complete. $\square$

## 3. Proof of Theorem 1.3

In this section we prove Theorem 1.3. To this end we need a continuation theorem, which can be found in [9] and [10].

Let $X, Z$ be Banach spaces, $L\colon \operatorname{dom} L \cap X \to Z$ be a linear operator. Recall that if $\dim \operatorname{Ker} L = \dim(Z/\operatorname{Im} L) < \infty$ and $\operatorname{Im} L$ is closed in $Z$, then $L$ will be called a Fredholm mapping of index zero. In this case there exist continuous projectors $P\colon X \to X, Q\colon Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q$ and $L|\operatorname{dom} L \cap \operatorname{Ker} P\colon \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is invertible. As usual its inverse is denoted by $K_P$. If $\Omega$ is an open bounded subset of $X$, a map $N\colon X \to Z$ will be called $L$-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N\colon \overline{\Omega} \to X$ is compact.

Lemma 3.1. *Suppose that $X, Z$ are Banach spaces, $L\colon \operatorname{dom} L \cap X \to Z$ is a Fredholm operator of index zero and $N\colon X \to Z$ is $L$-compact on $\overline{\Omega}$, where $\Omega$ is an open bounded subset of $X$. If the following conditions are satisfied:*

(a) *For each $\lambda \in (0,1)$, every solution $x$ of $Lx = \lambda Nx$ satisfies $x \notin \partial\Omega$.*
(b) *$QNx \neq 0$ for $x \in \partial\Omega \cap \operatorname{Ker} L$ and $\deg(\Lambda QN, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, where $Q\colon Z \to Z$ is a continuous projector with $\operatorname{Im} L = \operatorname{Ker} Q$ and $\Lambda\colon \operatorname{Im} Q \to \operatorname{Ker} L$ is an isomorphism.*

*Then the operator equation $Lx = Nx$ has one solution.*

Proof of Theorem 1.3. Without loss of generality, we assume $\alpha = 0$. Now the boundary conditions (1.2), (1.3) can be rewritten

$$x(0) = 0, \tag{3.1}$$

$$x(1)\cos\beta - p(1)x'(1)\sin\beta = 0. \tag{3.2}$$

Denote by $X = C^1(0,1), Y = L^1(0,1)$, and define

$$\begin{gathered}
\operatorname{dom} L := W_0^{2,1}(0,1), \\
L\colon \operatorname{dom} L \subset X \to Y, \quad x(t) \mapsto (p(t)x'(t))' + q_0(t)x(t), \\
N\colon X \to Y, \quad x(t) \mapsto f(t, x(t)).
\end{gathered}$$

Then $\operatorname{Ker} L = \operatorname{span}\{x_0\}$, and we claim

$$\operatorname{Im} L = \left\{ x \in L^1(0,1) \;\middle|\; \int_0^1 x(t)x_0(t)\,dt = 0 \right\}. \tag{3.3}$$

In fact, let $u_1$ be the unique solution of

$$\begin{aligned}(p(t)u'(t))' + q_0(t)u(t) &= 0,\\ u(0) = 1, \quad p(0)u'(0) &= 0,\end{aligned}$$

and let $u_2$ be the unique solution of

$$\begin{aligned}(p(t)u'(t))' + q_0(t)u(t) &= 0,\\ u(0) = 0, \quad p(0)u'(0) &= 1.\end{aligned}$$

Then $u_2(t)/x_0(t) \equiv$ constant and $u_1$, $u_2$ are linearly independent. For any $f \in \operatorname{Im} L$, assume $x(t)$ is a solution of $Lx = f$, then $x$ satisfies (3.1), (3.2) and

$$(p(t)x'(t))' + q_0(t)x(t) + e(t) = 0. \tag{3.4}$$

The general solution of (3.4) is

$$x(t) = \bigg(C_1 + \int_0^t u_2(\tau)e(\tau)\,d\tau\bigg)u_1(t) + \bigg(C_2 - \int_0^t u_1(\tau)e(\tau)\,d\tau\bigg)u_2(t). \tag{3.5}$$

From (3.1) and (3.2), we have $C_1 = 0$ and $f \in \operatorname{Im} L$ if and only if $\int_0^1 e(t)x_0(t)\,dt = 0$. And hence (3.3) is proved. Thus $\dim(\operatorname{Ker} L) = 1 = \dim(Z/\operatorname{Im} L)$, and $L$ is a Fredholm operator. For any $x \in L^1(0,1)$ we have a decomposition $x(t) = ax_0(t) + u(t)$ with

$$\int_0^1 u(t)x_0(t)\,dt = 0, \tag{3.6}$$

i.e.

$$a = \int_0^1 x(t)x_0(t)\,dt\bigg(\int_0^1 x_0(t)^2\,dt\bigg)^{-1}.$$

Define $Q\colon x(t) \mapsto ax_0(t)$ for any $x \in L^1(0,1)$ and $P = Q|_X$, then $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q$. And for any $e \in \operatorname{Im} L$, from (3.5) we have

$$(K_Pe)(t) = \int_0^t u_2(\tau)e(\tau)\,d\tau u_1(t) + \bigg(C_2 - \int_0^t u_1(\tau)e(\tau)\,d\tau\bigg)u_2(t), \tag{3.7}$$

and because $u = K_Pe$ satisfies (3.6), $C_2$ can be determined by

$$C_2\int_0^1 u_2^2(t)\,dt = \int_0^1\int_0^t e(\tau)(u_1(\tau)u_2(t) - u_2(\tau)u_1(t))\,d\tau\,dt. \tag{3.8}$$

By definition,

$$((I-Q)Nx)(t) = f(t,x(t)) - \int_0^1 f(\tau,x(\tau))x_0(\tau)\,d\tau\bigg(\int_0^1 x_0(t)^2\,d\tau\bigg)^{-1}x_0(t).$$

Combining (3.7) and (3.8) we know that for any $\Omega \subset X$ is open and bounded, $K_P(I-Q)N(\overline{\Omega})$ is bounded and from the Ascoli–Arzela theorem, $K_P(I-Q)N\colon \overline{\Omega} \to X$ is compact. And hence, $N$ is $L$-compact.

In order to finish the proof we only need to verify two conditions.

*Step 1.* Consider the Brouwer degree $d(\Lambda QN|_{\operatorname{Ker} L}, \Omega, 0)$. For every $x \in \operatorname{Ker} L$, we have $x(t) = ax_0(t)$ for a real number $a$.

So we can choose $\Omega = \{ax_0(t) \mid |a| \le r\}$, $\Lambda = I$ and

$$\begin{aligned}(\Lambda QNx)(t) &= \int_0^1 f(t, x(t))x_0(t)\,dt x_0(t)\\ &= \int_0^1 f(t, ax_0(t))x_0(t)\,dt x_0(t) := \psi(a)x_0(t).\end{aligned}$$

Because $a\psi(a) = a\int_0^1 f(t, ax_0(t))x_0(t)\,dt > 0$ as $a \ge r$, it follows that

$$d(\Lambda QN|_{\operatorname{Ker} L}, \Omega, 0) = d(\psi, [-r, r], 0) = d(I, [-r, r], 0) = 1.$$

*Step 2.* We are trying to make sure that the solution $x$ of $Lx + \lambda Nx = 0$, $\lambda \in (0,1)$ is bounded in the space $C^1(0,1)$. Obviously, $Lx+\lambda Nx = 0$ is equivalent to (3.1), (3.2) and

$$(p(t)x'(t))' + q_0(t)x(t) + \lambda f(t, x(t)) = 0. \tag{3.9}$$

From (3.9) and the definition of $x_0$, we have

$$\int_0^1 f(t, x(t))x_0(t)\,dt = 0. \tag{3.10}$$

For any $x \in L^1(0,1)$, let $x(t) = ax_0(t) + u(t)$ with $u(t)$ satisfying (3.6). Then we have

$$(p(t)u'(t))' + q_0(t)u(t) + \lambda f(t, x(t)) = 0, \tag{3.11}$$

$$u(0) = 0, \tag{3.12}$$

$$u(1)\cos\beta - p(1)u'(1)\sin\beta = 0. \tag{3.13}$$

From the former discussion and (3.7), (3.8) we find

$$\begin{aligned}u(t) = \int_0^t \lambda u_2(\tau)f(\tau, x(\tau))\,d\tau u_1(t)&\\ + \left(C_3 - \int_0^t \lambda u_1(\tau)f(\tau, x(\tau))\,d\tau\right)&u_2(t),\end{aligned} \tag{3.14}$$

and $C_3$ satisfies

$$C_3\int_0^1 u_2^2(t)\,dt = \int_0^1\int_0^t \lambda f(\tau, x(\tau))u_2(t)(u_1(\tau)u_2(t) - u_2(\tau)u_1(t))\,d\tau\,dt \tag{3.15}$$

From (3.14) we have

$$u'(t) = \int_0^t \lambda u_2(\tau)f(\tau, x(\tau))d\tau u_1'(t) + \left(C_3 - \int_0^t \lambda u_1(\tau)f(\tau, x(\tau))\,d\tau\right)u_2'(t). \tag{3.16}$$

And hence, there exists a constant $C_4 > 0$ such that

$$||u||_{C^1} \leq C_4 \int_0^1 |f(\tau, x(\tau))| \, d\tau. \tag{3.17}$$

On the other hand, from assumption (a) we have

$$\begin{aligned}\int_0^1 |f(t,x(t))x_0(t)| \, dt &= \int_{x(t)\geq r} |f(t,x(t))x_0(t)| \, dt \\ &\quad + \int_{|x(t)|\leq r} |f(t,x(t))x_0(t)| \, dt + \int_{x(t)\leq -r} |f(t,x(t))x_0(t)| \, dt \\ &\leq \int_{x(t)\geq r} F(t)x_0(t) \, dt + \int_{|x(t)|\leq r} \rho_r(t)x_0(t) \, dt + \int_{x(t)\leq -r} |f(t,x(t))x_0(t)| \, dt.\end{aligned}$$

From (3.10) we have

$$\begin{aligned}\int_{x(t)\leq -r} f(t,x(t))x_0(t) \, dt &= -\int_{x(t)\leq -r} f(t,x(t))x_0(t) \, dt \\ &= \int_{x(t)\geq r} f(t,x(t))x_0(t) \, dt + \int_{|x(t)|\leq r} f(t,x(t))x_0(t) \, dt.\end{aligned}$$

So,

$$\int_0^1 |f(t,x(t))x_0(t)| \, dt \leq 2\int_0^1 (F(t) + \rho_r(t))x_0(t) \, dt := C_5. \tag{3.18}$$

For any given $\varepsilon > 0$, there exists an integrable function $D = D_\varepsilon(t) > 0$ such that

$$|h(t,x(t))| \leq \varepsilon |x \cdot h(t,x(t))| + D(t)$$

for a.e. $t \in (0,1), x \in \mathbb{R}$. In fact, let $D(t) = 1$ for $x \geq 1/\varepsilon$; let $D(t) = 2\rho_r(t)$ for $-1/\varepsilon \leq x \leq 1/\varepsilon$; let $D(t) = 2F(t)$ for $x \leq -1/\varepsilon$. So we obtain

$$\begin{aligned}\int_0^1 |f(t,x(t))| \, dt &\leq \varepsilon \int_0^1 |x(t)h(t,x(t))| \, dt + \int_0^1 (D(t) + \widehat{g}(t)) \, dt \\ &\leq \varepsilon \int_0^1 |x(t)||h(t,x(t))| \, dt + C_6 \\ &= \varepsilon \int_0^1 |ax_0(t) + \ u(t)||h(t,x(t))| \, dt + C_6 \\ &\leq \varepsilon|a| \int_0^1 |h(t,x(t))x_0(t)| \, dt + \varepsilon \int_0^1 |h(t,x(t))||u(t)| \, dt + C_6 \\ &\leq \varepsilon|a|C_4 + \varepsilon \int_0^1 |h(t,x(t))|x_0(t) \frac{|u(t)|}{x_0(t)} \, dt + C_6 \\ &\leq \varepsilon|a|C_5 + \varepsilon \sup_{t\in[0,1]} \frac{|u(t)|}{x_0(t)} C_7 + C_6 \leq \varepsilon C_5|a| + \varepsilon C_5 C_7 ||u||_{C^1} + C_6\end{aligned} \tag{3.19}$$

where $C_6$, $C_7$ are all positive constants and $\sup_{t\in(0,1)}|u(t)|/x_0(t) \le C_7||u||_{C^1}$ from (2.10). From (3.17), (3.19) we have

$$||u||_{C^1} \le \delta|a| + C_8 \tag{3.20}$$

where $\delta > 0$ is sufficient small and $C_8 > 0$.

Now we claim the solutions of (3.1)–(3.3) are bounded with respect to the norm $||\cdot||_{C^1}$. If not, $||x_n||_{C^1} \to \infty$. Writing $x_n = a_n x_0 + u_n$, from (3.20) we have $|a_n| \to \infty$ and $u_n/a_n \to 0$ in $C^1(0,1)$. Assume $a_n \to \infty$, as in the proof of Theorem 1.2, we have $x_n(t) \ge \rho x_0(t)$ for $n$ large enough. But from (3.10) we obtain

$$\int_0^1 (f(t,x_n(t))x_0(t)\,dt = 0.$$

This is a contradiction to assumption (b). The whole proof is complete. □

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