CONTINUITY OF ATTRACTORS
FOR NET-SHAPED THIN DOMAINS

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Abstract. Consider a reaction-diffusion equation \( u_t = \Delta u + f(u) \) on a family of net-shaped thin domains \( \Omega_\varepsilon \) converging to a one dimensional set as \( \varepsilon \downarrow 0 \). With suitable growth and dissipativeness conditions on \( f \) these equations define global semiflows which have attractors \( A_\varepsilon \). In [4] it has been shown that there is a limit problem which also defines a semiflow having an attractor \( A_0 \), and the family of attractors is upper-semi-continuous at \( \varepsilon = 0 \). Here we show that under a stronger dissipativeness condition the family of attractors \( A_\varepsilon, \varepsilon \geq 0 \), is actually continuous at \( \varepsilon = 0 \).

1. Introduction

Consider domains \( \Omega_\varepsilon \) depending on a parameter \( \varepsilon > 0 \). On \( \Omega_\varepsilon \) we have a reaction-diffusion equation with Neumann boundary condition

\[
(1.1) \quad u_t = \Delta u + f(u) \quad \text{in} \ \Omega_\varepsilon, \quad \partial_\nu u = 0 \quad \text{on} \ \partial\Omega_\varepsilon.
\]

This equation generates a dynamical system if we impose suitable growth and dissipativeness conditions on the nonlinearity \( f \). Then equation (1.1) induces a semiflow \( \pi_\varepsilon \) on some functional space, and this semiflow has an attractor \( A_\varepsilon \). Many authors have asked and answered questions regarding the existence of a limiting dynamical system, as \( \varepsilon \to 0 \). E.g. if there is a equation which induces a semiflow \( \pi_0 \) with attractor \( A_0 \), such that the semiflows \( \pi_\varepsilon \) and attractors \( A_\varepsilon \)

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converge in some sense. We are interested principally in the case that the domains $\Omega_\varepsilon$ are thin domains, that is they are squeezed in some sense as $\varepsilon \to 0$, collapsing to a lower dimensional set. Among others, Hale and Raugel in [8] and [9], Prizzi and Rybakowski in [14], Prizzi, Rinaldi and Rybakowski in [13] and Elsken in [4] have shown for this type of singular perturbations that a limiting semiflow $\pi_0$ exists, it has an attractor $A_0$, $\pi_\varepsilon$ converges to $\pi_0$ in some sense, and the family of attractors $A_\varepsilon$, $\varepsilon \geq 0$, is upper-semi-continuous at $\varepsilon = 0$. Under the assumption that eigenvalues and eigenvectors converge, and some mild geometrical condition, Arrieta and Carvalho show in [2] that the attractors are even continuous.

We want to extend the results of [2] to the case of squeezed domains. That is we will prove the continuity of the attractors $A_\varepsilon$ at $\varepsilon = 0$ for the case of thin net-shaped domains. The results in [2] do not include the case where $|\Omega_\varepsilon| \to 0$, in particular they do not apply to the case of squeezed domains.

The fundamental idea we use is the same as in [2], but due to the singular perturbation of collapsing domains there are additional difficulties which have to be overcome. Roughly the argument is as follows. One knows that the attractors are upper-semi-continuous, and shows that the same is true for all points of equilibrium of the semiflows $\pi_\varepsilon$. We assume that there are only finitely many of these points and that 0 is not in the spectrum of the linearization around each point of equilibrium for the limit flow. Then the same holds for $\pi_\varepsilon$ for $\varepsilon > 0$ small, and the points of equilibrium are continuous at $\varepsilon = 0$. Any point in $A_0$ which is not a point of equilibrium has to lie on a trajectory which is in the unstable manifold of some point of equilibrium of $\pi_0$. Unlike in [2] we use fixed points on spaces of functions with exponential growth to construct the unstable manifolds (see e.g. Schneider [16], Fischer [5] and Rybakowski [15]). We show that given a trajectory $\pi_0(\cdot, u_0)$ converging exponentially as $t \to -\infty$ to a point of equilibrium of $\pi_0$, for $\varepsilon > 0$ small there are trajectories $\pi_\varepsilon(\cdot, u_\varepsilon)$ converging exponentially (as $t \to -\infty$) to some point of equilibrium of $\pi_\varepsilon$, and the $\pi_\varepsilon(\cdot, u_\varepsilon)$ themselves converge (as $\varepsilon \to 0$) in some sense to $\pi_0(\cdot, u_0)$. This then gives the continuity of the attractors.

Our technique works also in the other cases of thin domains mentioned above. We consider here only the case of net-shaped ones because this is the most general case. Also it presents some features which give rise to technical difficulties which are not present in the remaining cases.

The most important one is related to the weaker convergence we have for this case: in [14] and other papers the semiflows converge with respect to the family of norms $\|A_{1/2}^{1/2} \cdot\|_{L^2}$, that is the natural norms of fractional power spaces induced by the abstract linear operator of equation (1.1). For net-shaped thin
domains this is not true in general, and one has to introduce a second family of norms (defined in (1.3)) for the convergence of the semiflows and attractors.

We will now state our main result. Unfortunately the exact definition of net-shaped domains is rather lengthy, so we shall postpone it to the next section and give only the essential features here.

We assume $\Omega_\varepsilon \subset \mathbb{R}^{M+1}$, $\varepsilon > 0$, $M \in \mathbb{N}$ fixed, to be $C^2$, bounded, and to consist of $K_E$ edges and $K_N$ nodes:

$$\Omega_\varepsilon = \bigcup_{j=1}^{K_E} \Omega_{\varepsilon,j} \cup \bigcup_{j=K_E+1}^{K_E+K_N} \Omega_{\varepsilon,j},$$

$K_E, K_N \in \mathbb{N}$. All edges and nodes may have holes or multiple branches. Roughly speaking the edges converge to curves and the nodes to points, as $\varepsilon \to 0$. Each edge $\Omega_{\varepsilon,j}$, $j = 1, \ldots, K_E$, is the transformation of a fixed bounded, Lipschitz domain $G_j$ via a map $\Psi_{\varepsilon,j}$. These maps $\Psi_{\varepsilon,j}$ have a special structure and satisfy $|\det D\Psi_{\varepsilon,j}| \leq C_\varepsilon^M$. For each node $\Omega_{\varepsilon,j}$, $j = K_E + 1, \ldots, K_E + K_N$, there are bounded $G_{\varepsilon,j}$ and a bijection $\Psi_{\varepsilon,j}: G_{\varepsilon,j} \to \Omega_{\varepsilon,j}$ such that $D\Psi_{\varepsilon,j} = \varepsilon E_{M+1}$, where the latter is the unit matrix. Also, each $G_{\varepsilon,j}$ is transformed by bounded diffeomorphisms onto a finite number of fixed domains.

We identify $H^1(\Omega_\varepsilon)$ and $L^2(\Omega_\varepsilon)$ with spaces

$$H_\varepsilon \subset \prod_{j=1}^{K_E} H^1(G_j) \times \prod_{j=K_E+1}^{K_E+K_N} H^1(G_{\varepsilon,j}),$$

$$L_\varepsilon \subset \prod_{j=1}^{K_E} L^2(G_j) \times \prod_{j=K_E+1}^{K_E+K_N} L^2(G_{\varepsilon,j}),$$

respectively (see (2.2), (2.3) below).

The nonlinearity $f: \mathbb{R} \to \mathbb{R}$ is $C^2$. We impose two conditions on it:

(H1) $|f'(s)| \leq C(|s|^{\beta_1} + 1)$ for all $s \in \mathbb{R}$, where $C, \beta_1 \geq 0$ are constants; if $M > 1$, then additionally $\beta_1 \leq p^*/2 - 1$, where $p^* = 2(M+1)/(M-1) > 2$.

(H2) $\limsup_{|s| \to \infty} f(s)/s|s|^{\beta_2} \leq -\xi$, for some $\xi > 0$ and $\beta_2 > 0$.

In this paper we will always impose condition (H1) on $f$. Condition (H2) will be needed for our central result and in part of section three (see Proposition 3.3). Throughout this paper we shall assume at least the following weaker version of (H2) on $f$:

(H2') $\limsup_{|s| \to \infty} f(s)/s \leq -\xi$, for some $\xi > 0$.

It is well known that under these assumptions equation (1.1) can be written as an abstract equation

$$u_t = -A_\varepsilon u + f_\varepsilon(u) \quad t > 0,$$
where $A_\varepsilon : D(A_\varepsilon) \subset L_\varepsilon \rightarrow L_\varepsilon$ is a sectorial operator and $f_\varepsilon : H_\varepsilon \rightarrow L_\varepsilon$ is the Nemitsky operator of $f$. (1.2) induces a semiflow $\pi_\varepsilon$ on $H_\varepsilon$, and this semiflow has a global attractor $A_\varepsilon \subset H_\varepsilon$ (see e.g. [4]).

We need a few notations regarding the limit semiflow. Write $(x, y), x, y \in \mathbb{R}^M$, for a generic point of $\mathbb{R}^{M + 1}$, and set $H^1(U) := \{ u \in H^1(U) : D_y u = 0 \}, \ L^2(U)$ the closure of $H^1(U)$ in $L^2(U)$, for a domain $U \subset \mathbb{R}^{M + 1}$. Denote by $f_0$ and $Df_0$ the Nemitsky operators of $f$ and $f'$ on $\prod_{j=1}^{K_E} H^1_j(G_j)$, respectively. Define norms $| \cdot |_{\varepsilon, d}, \varepsilon > 0, 0 \leq d \leq 1$, on $H_\varepsilon$ by

\begin{equation}
|u_0, \ldots, u_{K_E + K_N}|^2_{\varepsilon, d} := \sum_{j=1}^{K_E} \|u_j\|^2_{L^2(G_j)} + \|D_x u_j\|^2_{L^2(G_j)} + \frac{1}{\varepsilon d^2} \|D_y u_j\|^2_{L^2(G_j)} + \varepsilon \sum_{j=K_E+1}^{K_E+K_N} \|u_j\|^2_{L^2(G_{d,j})} + \frac{1}{\varepsilon^2} \|D u_j\|^2_{L^2(G_{d,j})}.
\end{equation}

For $\varepsilon = 0$ set

\begin{equation}
|u_0, \ldots, u_{K_E}|^2_{0, d} := \sum_{j=1}^{K_E} \|u_j\|^2_{H^1(G_j)}.
\end{equation}

In [4] it is shown that there are linear spaces

\begin{equation}
H_0 = \prod_{j=1}^{K_E} H^1_j(G_j), \quad L_0 = \prod_{j=1}^{K_E} L^2_j(G_j),
\end{equation}

a linear embedding $\Phi_\varepsilon^H : H_0 \rightarrow H_\varepsilon$ and a sectorial operator $A_0 : D(A_0) \subset L_0 \rightarrow L_0$ such that

\begin{equation}
u_t = -A_0 u + f_0(u) \quad \text{for } t > 0
\end{equation}

induces a semiflow $\pi_0$ on $H_0$ which has an attractor $A_0 \subset H_0$. As $\varepsilon \rightarrow 0$, $\pi_\varepsilon$ converges to $\pi_0$ with respect to the family of norms $| \cdot |_{\varepsilon, d}$ for $d < 1$, and the family of attractors $A_\varepsilon, \varepsilon \geq 0$, is upper-semicontinuous at $\varepsilon = 0$ (see Theorem 2.2).

In this article we prove the continuity of these attractors. That is we show

**Theorem 1.1.** Assume $\Omega_\varepsilon$ satisfy the conditions of Section 2 and $f$ (H1), (H2) above. Assume also that the limit semiflow $\pi_0$ has only finitely many points of equilibrium, say $\{u^0_1, \ldots, u^0_{M_0}\} \subset H_0$, and $0$ is not in the spectrum of the linear operators

\begin{equation}
A_0 = Df_0(u^0_j)\text{id} : D(A_0) \rightarrow L_0 \quad \text{for all } j = 1, \ldots, M_0.
\end{equation}

Then the family of attractors $A_\varepsilon, \varepsilon \geq 0$, is continuous at $\varepsilon = 0$, i.e. for $0 \leq d < 1$

\begin{equation}
\lim_{\varepsilon \rightarrow 0} \text{dist}_{\varepsilon, d}(A_\varepsilon, A_0) = 0,
\end{equation}

\begin{equation}
\text{dist}_{\varepsilon, d}(A_\varepsilon, A_0) := \inf \{ \varepsilon \cdot d(u, v) : u \in A_\varepsilon, v \in A_0 \}
\end{equation}
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where for \( U \subset H_{\varepsilon}, V \subset H_0 \) we define

\[
\text{dist}_{\varepsilon,d}(U, V) := \sup_{u \in U} \inf_{v \in V} |u - \Phi_H^\varepsilon v|_{\varepsilon,d} + \sup_{v \in V} \inf_{u \in U} |u - \Phi_H^\varepsilon v|_{\varepsilon,d}.
\]

In section three we prove the attractors \( A_{\varepsilon}, \varepsilon \geq 0 \), to be bounded uniformly in \( L^\infty \) (Proposition 3.3). Thus for \( u \in A_0 \) we have \( Df_0(u): L_0 \to L_0 \) and (1.5) in the theorem above makes sense.

This paper is organized as follows. In section two we present our notations, define net-shaped domains and state some results of [4]. In section three we prove the boundedness of the attractors in \( L^\infty \) and some auxiliary results we shall need in the next section. There we prove Theorem 1.1.

2. Notations and assumptions on \( \Omega_\varepsilon \)

In this section we will present our notations and state the exact requirements on the domains \( \Omega_\varepsilon \). We will also bring some results of [4] we shall need.

In the rest of this paper \( \varepsilon \) will always — unless stated otherwise — denote a number in \([0, 1]\).

\( M \in \mathbb{N} \) is a fixed positive natural number. We will write \((x, y)\) for a generic point in \( \mathbb{R} \times \mathbb{R}^M = \mathbb{R}^{M+1} \). Let \( U \subset \mathbb{R}^{M+1} \) then \( \text{proj}_x(U) \) is the projection onto the first coordinate.

As in [14], [4] and other papers here also the set of functions on an open set \( \Omega \subset \mathbb{R}^{M+1} \) which have derivative 0 in \( y \)-direction plays an important role. We define

\[
H_1^s(\Omega) := \{ u \in H^1(\Omega) : D_y u = 0 \}, \quad L_2^s(\Omega) := H_1^s(\Omega) \subset L_2(\Omega).
\]

\( L_2^s(\Omega) \) is a closed subset of \( L_2(\Omega) \), hence the orthogonal complement exists. Denote it by \( L_2^s(\Omega) \).

For \( n \in \mathbb{N} \) we denote by \( E_n \in \mathbb{R}^{n \times n} \) the unit-matrix and for a vector \( x \in \mathbb{R}^n \) \( \| x \| \) denotes the Euclidian norm.

Let \( V \) be a normed space, \( z \in V \) and \( \delta > 0 \). Then \( B_\delta(z) \subset V \) denotes the open ball around \( z \) with radius \( \delta \).

If \( U \subset \mathbb{R}^n \) then \( |U| \) is the Lebesgue-measure of \( U \). The closure will be denoted by \( \overline{U} \).

We will use the notation \( u_0 \pi t \) for semiflows \( \pi(t, u_0) = u(t), u \) solution of some (abstract) differential equation with initial value \( u_0 \).

In proofs we shall often substitute an index \( \varepsilon_n \) by the simpler \( n \). For example \( A_{\varepsilon_n}, H^1_{\varepsilon_n} \) and \( \| \cdot \|_{\varepsilon_n,d} \) will be \( A_n, H^1_n \) and \( \| \cdot \|_{n,d} \). Also we shall assume constants \( C_1, C_2, \ldots \) to be independent of \( \varepsilon \). If they depend on \( \varepsilon \) we shall always indicate this, writing \( C(\varepsilon) \), or \( C(n) \) if \( \varepsilon = \varepsilon_n \).

We will start defining the domain \( \Omega_\varepsilon \) which, as already mentioned, will be net like and consists of \( K_E \in \mathbb{N} \) edges and \( K_N \in \mathbb{N} \) nodes. More in detail we assume
\(\Omega_\varepsilon \subset \mathbb{R}^{M+1}\) to be bounded, connected and \(C^2\). \(\Omega_\varepsilon = \bigcup_{j=1}^{K_E} \Omega_{\varepsilon,j} \cup \bigcup_{j=K_E+1}^{K_E+K_N} \Omega_{\varepsilon,j}\), where the \(\Omega_{\varepsilon,j}\) are mutually disjoint and satisfy the following.

The edges \(\Omega_{\varepsilon,j}, j = 1, \ldots, K_E\), have a description
\[
\Omega_{\varepsilon,j} = \Psi_{\varepsilon,j}(G_j),
\]
where \(G_j \subset \mathbb{R} \times \mathbb{R}^M\) is open, bounded, connected and Lipschitz. To facilitate notation we assume \(\text{proj}_x(G_j) = [0,1]\).

The transformation \(\Psi_{\varepsilon,j}: G_j \to \Psi_{\varepsilon,j}(G_j) \supset \Omega_{\varepsilon,j}\) is a \(C^1\)-diffeomorphism \(T_{\varepsilon,j}\) which is near to the identity, followed by a contraction \(S_\varepsilon\) in \(y\)-direction and a \(C^1\)-diffeomorphism \(T_j\) which is independent of \(\varepsilon\):
\[
\Psi_{\varepsilon,j} = T_j \circ S_\varepsilon \circ T_{\varepsilon,j}.
\]
Here \(T_{\varepsilon,j}; Q_{1,j} \supset \overline{G_j} \to T_{\varepsilon,j}(\overline{G_j}) \subset Q_{2,j}\) is a \(C^1\)-diffeomorphism, \(Q_{1,j}, Q_{2,j} \subset \mathbb{R}^{M+1}\) fixed, open, bounded sets. \(S_\varepsilon(x,y) := (x, \varepsilon y)\) and \(T_j; \tilde{Q}_j \to T_j(\tilde{Q}_j) \subset \mathbb{R}^{M+1}\) is again a \(C^1\)-diffeomorphism, \(\tilde{Q}_j \supset \bigcup_{0 \leq z \leq 1} S_\varepsilon(T_{\varepsilon,j}(G_j))\) open. Roughly speaking \(T_{\varepsilon,j}\) is there to give some liberty choosing the nodes, \(S_\varepsilon\) is the normal squeezing, and \(T_j\) moves an edge into the right position (i.e. to \([0,1] \times \mathbb{R}^M\), eventually scaling and deforming it in a way independent of \(\varepsilon\).

We want an edge to touch a node only at the sides corresponding to \((\{0\} \times \mathbb{R}^M) \cap \overline{G_j}\) or \((\{1\} \times \mathbb{R}^M) \cap \overline{G_j}\), so we assume
\[
\emptyset \neq \Psi_{\varepsilon,j}^{-1}(\Omega_{\varepsilon,j} \cap \overline{\Omega_{\varepsilon,j}}) \subset \{0\} \times \mathbb{R}^M,\]
if the edge \(\Omega_{\varepsilon,j}\) begins at the node \(\Omega_{\varepsilon,i}\), or
\[
\emptyset \neq \Psi_{\varepsilon,j}^{-1}(\Omega_{\varepsilon,j} \cap \overline{\Omega_{\varepsilon,j}}) \subset \{1\} \times \mathbb{R}^M,
\]
if the edge \(\Omega_{\varepsilon,j}\) ends at the node \(\Omega_{\varepsilon,i}\), for all possible \(i, j\).

We assume also that any edge may only begin or end at a given node, but not both.

Each of the nodes \(\Omega_{\varepsilon,j}, j = K_E + 1, \ldots, K_E + K_N\), converges to a one point set, say \(\Omega_{0,j} = \{z_{0,j}\} \subset T_\varepsilon(\tilde{Q}_j) \subset \mathbb{R}^{M+1}\), for all edges \(\Omega_{\varepsilon,j}\) which either start or end at the node \(\Omega_{\varepsilon,j}\).

We assume the node \(\Omega_{\varepsilon,j}, j = K_E + 1, \ldots, K_E + K_N\), has a description
\[
\Omega_{\varepsilon,j} = \Psi_{\varepsilon,j}(G_{\varepsilon,j}),\]
where \(\Psi_{\varepsilon,j}(z) = \varepsilon z + z_{\varepsilon,j}, z_{\varepsilon,j} \to z_{0,j}\) as \(\varepsilon \to 0\).

Note that since all edges are open, each node is closed in \(\Omega_\varepsilon\). It may even have empty interior.

Throughout this article we put the following additional conditions (C1)–(C8) on \(G_j, T_{\varepsilon,j}, T_j\) and \(G_{\varepsilon,i}\), where always \(j = 1, \ldots, K_E, i = K_E + 1, \ldots, K_E + K_N\).

(C1) \(\overline{G_j} \cap (\{0\} \times \mathbb{R}^M)\) or \(\overline{G_j} \cap (\{1\} \times \mathbb{R}^M)\) has finitely many connected components with positive \(M\)-dimensional measure, if the edge \(G_j\) begins or ends at some node, respectively.
There are at most countably many open, connected, pairwise disjoint $U^{ij} \subset G_j$, $l \in I_j$, such that each $U^{ij}$ has connected $x$-crosssections and $E := \{ x \in \mathbb{R} : \text{ there exists } y \in \mathbb{R}^M, (x, y) \in G_j \setminus \bigcup_{l \in I_j} U^{ij} \}$ has at most finitely many points of accumulation.

$T_{\varepsilon,j}(x, y) \rightarrow (x, y), \varepsilon \rightarrow 0$, pointwise for all $(x, y) \in \overline{G}_j$, and if $(T_{\varepsilon,j})_x$ denotes the $x$-component of $T_{\varepsilon,j}$, then $(T_{\varepsilon,j})_x \rightarrow \text{proj}_x|_{\overline{G}_j}$ uniformly on $\overline{G}_j$.

There is a $C > 0$ such that, for all $\varepsilon \leq 1, v \in \mathbb{R}^{M+1}, \|v\| = 1$,

$$\sup_{(x,y)\in\overline{G}_j}\|DT_{\varepsilon,j}(x,y)v\|, \sup_{(x,y)\in T_{\varepsilon,j}(\overline{G}_j)}\|DT_{\varepsilon,j}^{-1}(x,y)v\| < C.$$ 

Define $T_{\varepsilon,j}, T_{\varepsilon,j}^*$ by

$$DT_{\varepsilon,j}(x, y) = E_{M+1} - T_{\varepsilon,j}(x, y),$$

$$(DT_{\varepsilon,j}(x, y))^{-1} = E_{M+1} + T_{\varepsilon,j}^*(x, y).$$

Denote the elements of these matrix-functions by $T_{\varepsilon,j,l,k}$ and $T_{\varepsilon,j,l,k}^*$, $l, k = 0, \ldots, M$. We assume

$$\sup_{0 < \varepsilon \leq 1, (x, y) \in G_j} \left( \frac{1}{\varepsilon} |T_{\varepsilon,j,0,l}(x, y)|, \frac{1}{\varepsilon} |T_{\varepsilon,j,0,l}^*(x, y)| \right) < \infty,$$

$$\lim_{\varepsilon \rightarrow 0} T_{\varepsilon,j}(x, y) = \lim_{\varepsilon \rightarrow 0} T_{\varepsilon,j}^*(x, y) = 0,$$

and there are maps $T_j = (T_{j,1}, \ldots, T_{j,M}): \overline{G}_j \rightarrow \mathbb{R}^M$ such that

$$\lim_{\varepsilon \downarrow 0} T_{\varepsilon,j,0,l}(x, y) = \lim_{\varepsilon \downarrow 0} T_{\varepsilon,j,0,l}^*(x, y) = T_{j,l}(x, y)$$

for all $(x, y) \in G_j$, $l = 1, \ldots, M$.

$G_{\varepsilon,i}$ is bounded independent of $\varepsilon$, i.e. there is a positive $R_\Omega$ such that $G_{\varepsilon,i} \subset B_{R_\Omega}(0)$ for all $0 < \varepsilon \leq 1$.

$\Omega_\varepsilon$ is nicely connected, that is $\Omega_\varepsilon$ connects nicely at all edges.

We say $\Omega_\varepsilon$ connects nicely at the node $G_{\varepsilon,i}$ if the following is satisfied. There are $\delta, C > 0$, and for all edges $G_k$ which begin or end at the node $G_{\varepsilon,i}$ there are open, connected, Lipschitz, pairwise disjoint $G_{i, k, l} \subset G_k$, connected $\omega_{i, k, l, x} \subset \mathbb{R}^M$, $|\omega_{i, k, l, x}| \geq \delta$ for all $x \in I_{i, k}$, where $I_{i, k} = ]0, 0[$ if $G_k$ begins, $I_{i, k} = ]1 - \delta, 1[$ if it ends at $G_{\varepsilon,i}$, such that

$$G_{i, k, l} = \bigcup_{x \in I_{i, k}} \{ x \} \times \omega_{i, k, l, x},$$

$$G_k \cap (I_{i, k} \times \mathbb{R}^M) = \bigcup_{l=1}^{L_{i, k}} G_{i, k, l}$$

for all possible $l, k$. Set $S_\Omega := \{(i, k, l) : i = K_E + 1, \ldots, K_E + K_N, l = 1, \ldots, L_{i, k}, k = 1, \ldots, K_E, G_k \text{ begins or ends at } G_{\varepsilon,i}\}$.
If there are an $\varepsilon_1 > 0$, $(i, k_1, l_1) \in S^\Omega$, $(x_m, y_m) \in \partial G_{i,k_1,l_1}$, $m = 1, 2$, such that $\Psi_{\varepsilon_1, k_1}(x_1, y_1)$ and $\Psi_{\varepsilon_1, k_2}(x_2, y_2)$ belong to the same connected component of $\Omega_{\varepsilon_1, i}$, then there are an open, connected, bounded, Lipschitz $U = U_{i,k_1,l_1,k_2,l_2} \subset \Psi_{\varepsilon_1}^{-1}(\Omega_{\varepsilon_1})$, $r > 0$, both independent of $\varepsilon$, and open

$$U = U_{\varepsilon,i,k_m,l_m} = B_r(z_{\varepsilon,i,k_m,l_m}) \subset U \cap \Psi_{\varepsilon,k_m}^{-1}(G_{i,k_m,l_m}),$$

$$\Psi_{\varepsilon,k_m}^{-1} \circ \Psi_{\varepsilon,i}(U_{\varepsilon,i}) \subset [0, R] \times \mathbb{R}^M$$

if $G_{k_m}$ begins, and

$$\Psi_{\varepsilon,k_m}^{-1} \circ \Psi_{\varepsilon,i}(U_{\varepsilon,i}) \subset (1 - \varepsilon C, 1] \times \mathbb{R}^M$$

if $G_{k_m}$ ends at $G_{\varepsilon,i}$, for all $\varepsilon$ and $m = 1, 2$.

(C8) One of the following holds:

(i) $G_{i,\varepsilon}$ has empty interior for all $\varepsilon > 0$.

(ii) There are $G_{i,1}, \ldots, G_{i,N_i} \subset \mathbb{R}^{M+1}$ open, bounded, connected, Lipschitz, $C > 0$, $\overline{G_{i,k}} \subset Q_k \subset \mathbb{R}^{M+1}$ open, $\Psi_{\varepsilon,i,k} : Q_k \rightarrow \Psi_{\varepsilon,i,k}(Q_k) \subset \mathbb{R}^{M+1}$ $C^1$-diffeomorphisms, $\Psi_{\varepsilon,i,k}(G_{i,k}) \subset G_{\varepsilon,i}$,

$$\frac{1}{C} \leq |\det D\Psi_{\varepsilon,i,k}(z)|, \quad \|D\Psi_{\varepsilon,i,k}(z)v\| \leq C,$$

$$\left|G_{\varepsilon,i} \setminus \bigcup_{k=1}^{N_i} \Psi_{\varepsilon,i,k}(G_{i,k})\right| = 0,$$

for all possible $z, k, \varepsilon$ and $v \in \mathbb{R}^{M+1}$, $\|v\| = 1$. For all $k \in \{1, \ldots, N_i\}$ exist $l \in \{1, \ldots, K_E\}$, open, bounded, connected, Lipschitz $U_{i,k} \subset \Psi_{\varepsilon,l}^{-1}(\Omega_{\varepsilon,l} \cup \Omega_{\varepsilon,l})$, $r > 0$, all independent of $\varepsilon$, and open $U_{\varepsilon,i} = U_{\varepsilon,i,k,l} = B_r(z_{\varepsilon,i,k,l}) \subset U_{i,k} \cap \Psi_{\varepsilon,i}^{-1}(\Omega_{\varepsilon,l})$, such that $|\Psi_{\varepsilon,i,k}^{-1}(U_{i,k} \cap \Psi_{\varepsilon,i,k}(G_{i,k}))| \geq 1/C$, $\Psi_{\varepsilon,l}^{-1} \circ \Psi_{\varepsilon,i}(U_{\varepsilon,i}) \subset [0, R] \times \mathbb{R}^M$ if $G_i$ begins, $\Psi_{\varepsilon,l}^{-1} \circ \Psi_{\varepsilon,i}(U_{\varepsilon,i}) \subset [1 - \varepsilon C, 1] \times \mathbb{R}^M$ if it ends at $G_{\varepsilon,i}$, for all $\varepsilon$.

Proposition 3.1 in [4] states, that if (C1)–(C8) hold, then the following two conditions hold too:

(C9) Define $H_0$ as the set of all $[u] = [u_1, \ldots, u_{K_E}] \in H^1_s(G_1) \times \ldots \times H^1_s(G_{K_E})$ such that there are a constant $\beta > 0$, a sequence $\varepsilon_n \downarrow 0$ (both dependent on $[u]$), and $\tilde{u}_n \in H^1(\Omega_{\varepsilon_n})$ such that $\tilde{u}_n \circ \Psi_{\varepsilon_n,k} \rightharpoonup u_k$ weakly in $H^1(G_k)$, $k = 1, \ldots, K_E$,

$$\sum_{k=1}^{K_E} \frac{1}{\varepsilon_n} \|D_{\varepsilon_n}(\tilde{u}_n \circ \Psi_{\varepsilon_n,k})\|_{L^2(G_k)} + \sum_{k=K_E+1}^{K_E+K_N} \varepsilon_n \|\tilde{u}_n \circ \Psi_{\varepsilon_n,k}\|_{L^2(G_{\varepsilon_n,k})}^2 + \frac{1}{\varepsilon_n} \|D(\tilde{u}_n \circ \Psi_{\varepsilon_n,k})\|_{L^2(G_{\varepsilon_n,k})}^2 < \beta.$$
We assume $H_0$ is a closed subspace of $H^1_s(G_1) \times \ldots \times H^1_s(G_{K_E})$, and for every $\varepsilon > 0$ there is a linear map

$$\Phi^H_\varepsilon : H_0 \rightarrow H^1(G_1) \times \ldots \times H^1(G_{K_E}) \times H^1(G_{\varepsilon,K_E+1}) \times \ldots \times H^1(G_{\varepsilon,K_E+K_N})$$

and a constant $C > 0$, independent of $\varepsilon$, such that for all $[u] = [u_1, \ldots, u_{K_E}]$ we have $(\Phi^H_\varepsilon [u])_k = u_k$, $k = 1, \ldots, K_E$. Also

$$(2.1) \quad C \sum_{k=1}^{K_E} \|u_k\|^2_{H^1(G_k)} \geq - \sum_{k=K_E+1}^{K_E+K_N} \varepsilon (\|\Phi^H_\varepsilon [u]\|_{L^2(G_{\varepsilon,k})})^2 + \frac{1}{\varepsilon} \|D(\Phi^H_\varepsilon [u])_k\|^2_{L^2(G_{\varepsilon,k})} \rightarrow 0,$$

as $\varepsilon \rightarrow 0$, and $\hat{u}_\varepsilon$ defined by

$$\hat{u}_\varepsilon := (\Phi^H_\varepsilon [u])_k \circ \Psi_{\varepsilon,k}^{-1} \quad \text{on } \Omega_{\varepsilon,k}, \quad k = 1, \ldots, K_E + K_N,$$

is a function in $H^1(\Omega_{\varepsilon})$ (i.e. $\Phi^H_\varepsilon [u]$ comes from the $H^1$-function $\hat{u}_\varepsilon$ via the transformations $\Psi_{\varepsilon,k}$).

(C10) If $C > 0$, $\varepsilon_n \rightarrow 0$, $[u_n] \in H^1(\Omega_{\varepsilon_n})$, $\|u_n\|_{\varepsilon_n,1} \leq C$ and $\|u_n\|_{L^2(\Omega_{\varepsilon_n})} = 1$ for all $n$, then

$$\varepsilon_n \sum_{k=K_E+1}^{K_E+K_N} \|u_{n,k}\|^2_{L^2(G_{\varepsilon_n,k})} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

To simplify notations we set $\Phi^H_0 := \text{id}$ on $H_0$.

![Figure 1](image_url)

Figure 1. An example for a net-shaped domain. It is an $L$-shaped one as defined in [9] only that it has holes.

As has already been mentioned in the introduction, we want to identify $H^1(\Omega_\varepsilon)$ with a certain space

$$H_\varepsilon \subset \prod_{j=1}^{K_E} H^1(G_j) \times \prod_{j=K_E+1}^{K_E+K_N} H^1(G_{\varepsilon,j}).$$
To do this define matrix functions $A_{\varepsilon,j} : G_j \to \mathbb{R}^{(M+1) \times (M+1)}$, $j = 1, \ldots, K_E$, by

$$A_{\varepsilon,j}(x,y) := \begin{pmatrix} 1 & \varepsilon & \cdots & \varepsilon \\ \varepsilon & & & \\ & & & \\ \varepsilon & & & 1 \\ & 1/\varepsilon & & \\ & & & \ddots \\ & & 1/\varepsilon & & 1/\varepsilon \end{pmatrix} (DT_{\varepsilon,j}(x,y))^{-1}.$$ 

Note that the norms and determinants of all $A_{\varepsilon,j}$ are bounded from 0 and infinity uniformly in $\varepsilon$ and $(x,y)$, and

$$A_{\varepsilon,j}(x,y) \to \begin{pmatrix} 1 & T_j(x,0) \\ 0 & E_M \end{pmatrix} DT_j^{-1}(x,0)$$

pointwise as $\varepsilon \to 0$ (see [4, Lemma 2.3]).

We divide $\Omega_\varepsilon$ into the above mentioned $K_E$ edges and $K_N$ nodes, which in turn get transformed by $\Psi_{\varepsilon,j}$ into $G_j$, $j = 1, \ldots, K_E$, and $G_{\varepsilon,j}$, $j = K_E + 1, \ldots, K_E + K_N$. Thus we can identify $L^2(\Omega_\varepsilon)$, $H^1(\Omega_\varepsilon)$ with

\begin{equation}
L_\varepsilon := \{ [u] = [u_1, \ldots, u_{K_E+K_N}] : u_j \in L^2(G_j), j = 1, \ldots, K_E, \]
\end{equation}
\begin{equation}
\begin{aligned}
    u_i & \in L^2(G_{\varepsilon,i}), i = K_E + 1, \ldots, K_E + K_N, \\
    \exists \tilde{u} & \in H^1(\Omega_\varepsilon) \tilde{u} \circ \Psi_{\varepsilon,j} = u_j, j = 1, \ldots, K_E + K_N \}
\end{aligned}
\end{equation}

and norms $\| \cdot \|_{L_\varepsilon}$, $\| \cdot \|_{H_\varepsilon}$, respectively. Here we used measures on $\mathbb{R}^{M+1}$ defined by

$$\lambda_{\varepsilon,j}(A) := \int_A | \det DT_{\varepsilon,j}(x,y) | | \det DT_j(S_\varepsilon \circ T_{\varepsilon,j}(x,y)) | \, dx \, dy,$$

$$\lambda_{0,j}(A) := \int_A | \det DT_j(x,0) | \, dx \, dy,$$
for all Lebesgue measurable sets $A \subset \overline{G}_j$, $j = 1, \ldots, K_E$.

Note that $|\det DT_{\varepsilon,j}(x,y)||\det DT_j(S_\varepsilon \circ T_{\varepsilon,j}(x,y))|$ is bounded from 0 and infinity uniformly in $(x,y)$ and $\varepsilon$ (see [4, Lemma 2.1]). Also above expression tends pointwise to $|\det DT_j(x,0)|$ as $\varepsilon \to 0$.

Given $u_j$, $j = 1, \ldots, K_E$ or $j = 1, \ldots, K_E + K_N$, we write $[u]$ for $[u_1, \ldots, u_{K_E}]$ and $[u_1, \ldots, u_{K_E}+K_N]$, respectively. It will be clear from the context which case is meant.

The definition of $L_\varepsilon$ and $H_\varepsilon$ with the respective scalar products in (2.2) and (2.3) is just a change of variables on each subset $\Omega_{\varepsilon,j}$, $j = 1, \ldots, K_E + K_N$, the measures $\lambda_{\varepsilon,j}$ being the Jacobian of the respective transformations dropping the common factor $\varepsilon^{-M}$. Thus $\hat{u} \in L^2(\Omega_\varepsilon)$ if and only if $[u] \in L_\varepsilon$ and $||\hat{u}||_{L^2(\Omega_\varepsilon)}^2 = \varepsilon^M ||[u]||_{L^2_\varepsilon}^2$; $\hat{u} \in H^1(\Omega_\varepsilon)$ if and only if $[u] \in H_\varepsilon$ and $||\hat{u}||_{H^1(\Omega_\varepsilon)}^2 = \varepsilon^M ||[u]||_{H^1_\varepsilon}^2$. Also, if $[u_\varepsilon] \in H_\varepsilon$ is such that $(||u_\varepsilon||_{L^2_\varepsilon})_\varepsilon$ is bounded, then $(\varepsilon^{-M}||\hat{u}_\varepsilon||_{H^1(\Omega_\varepsilon)})_\varepsilon$ is bounded too.

Note that by Lemma 2.7 in [4] there is a constant $C > 0$, independent of $\varepsilon$, such that for all $[u] \in H_\varepsilon$ we have

$$\frac{1}{C} ||[u]||_{H_\varepsilon} \leq ||[u]||_{L^2_\varepsilon,1} \leq C ||[u]||_{H_\varepsilon}.$$  

We have already introduced the space $H_0$ in (C9), let $L_0$ be the closure of $H_0$ in $L^2(G_1) \times L^2(G_2) \times L^2(G_3)$. Then $L_0 = \prod_{j=1}^{K_E} L^2_\varepsilon(G_j)$ (see [4, Lemma 2.5]). We introduce inner products on them by

$$([u],[v])_{L_0} := \sum_{j=1}^{K_E} \int_{G_j} u_j v_j d\lambda_{0,j},$$

$$([u],[v])_{H_0} := ([u],[v])_{L_0} + \sum_{j=1}^{K_E} \int_{G_j} D_x u_j D_x v_j d\lambda_{0,j}.$$  

Denote the respective norms by $|| \cdot ||_{L_0}$ and $|| \cdot ||_{H_0}$.

We need to embed $H_0$ in $H_\varepsilon$ and $L_0$ in $L_\varepsilon$ in order to be able to compare semiflows and attractors. We do this by the linear operator $\Phi_\varepsilon^H$ given in condition (C9) in the case of the $H^1$-spaces.

To embed the $L^2$-spaces define $\Phi_\varepsilon^L: L_0 \rightarrow L_\varepsilon$ by

$$\Phi_0^L[u]|_j := u_j, \quad \text{for} \ j = 1, \ldots, K_E,$$

$$\Phi_\varepsilon^L[u]|_j := 0, \quad \text{for} \ j = K_E + 1, \ldots, K_E + K_N.$$  

Then the $\Phi_\varepsilon^L: L_0 \rightarrow L_\varepsilon$, $\Phi_\varepsilon^H: H_0 \rightarrow H_\varepsilon$ are both linear and bounded, the bound being independent of $\varepsilon \geq 0$. 
We want to write equation (1.1) as an abstract equation. To do so define bilinear forms $a_{\varepsilon}: H_{\varepsilon} \times H_{\varepsilon} \to \mathbb{R}$, $\varepsilon \geq 0$, by

$$
(2.7) \quad a_{\varepsilon}([u],[v]) := \sum_{j=1}^{K_{\varepsilon}} \int_{G_{\varepsilon,j}} (D_{z} u_{j}, \frac{1}{\varepsilon} D_{y} u_{j}) A_{\varepsilon,j} A_{\varepsilon,j}^{T} (D_{z} v_{j}, \frac{1}{\varepsilon} D_{y} v_{j})^{T} d\lambda_{\varepsilon,j} + \frac{1}{\varepsilon} \sum_{j=K_{\varepsilon}+1}^{K_{\varepsilon}+K_{\varepsilon,N}} \int_{G_{\varepsilon,j}} D u_{j} D v_{j} d\varepsilon,z,
$$

$$
(2.8) \quad a_{0}([u],[v]) := \sum_{j=1}^{K_{\varepsilon}} \int_{G_{\varepsilon,j}} (D_{z} u_{j} D_{z} v_{j}) (1,0) D T_{j}^{T} (x,0)^{-2} d\lambda_{0,j}.
$$

It is well known (see e.g. [4]) that for $\varepsilon \geq 0$ these bilinear forms $a_{\varepsilon}$ define linear operators $A_{\varepsilon}: D(A_{\varepsilon}) \subset L_{\varepsilon} \to L_{\varepsilon}$. $D(A_{\varepsilon}) \subset L_{\varepsilon}$, $D(A_{\varepsilon}) \subset H_{\varepsilon}$ densely and the operators $A_{\varepsilon}$ have compact resolvent, are selfadjoint and sectorial. There are complete orthonormal systems (ONS) of $L_{\varepsilon}$ consisting of eigenvectors of $A_{\varepsilon}$. Note that the fractional power space $X_{\varepsilon}^{1/2}$ belonging to $A_{\varepsilon}$ is $H_{\varepsilon}$.

If $f_{\varepsilon}: H_{\varepsilon} \to L_{\varepsilon}$ denote the Nemitsky operators of $f$ for $\varepsilon \geq 0$, i.e. $f_{\varepsilon}$ is defined by $f_{\varepsilon}([u])(z) = f(u_{j}(z)) \text{ for } z \in G_{\varepsilon,j}$ for all possible $j$, then equation (1.1) and — in a certain sense — its limit can be written in an abstract form as

$$
(2.9) \quad [u_{t}] = -A_{\varepsilon}[u] + f_{\varepsilon}([u]), \quad t > 0.
$$

It is clear that it suffices to investigate the behavior of the semiflow generated by equation (2.9) because a simple transformation changes it into the semiflow generated by (1.1) (for $\varepsilon > 0$).

Henceforth we shall only treat equation (2.9).

We cite now some results from [4] regarding the convergence of eigenvectors and eigenvalues of $A_{\varepsilon}$, of the existence of semiflows $\pi_{\varepsilon}$ generated by equation (2.9) and their convergence, and finally of the existence of global attractors $A_{\varepsilon}$ and their upper-semicontinuity.

**Theorem 2.1** (cf. [4]). Denote by $\lambda_{\varepsilon,l}$ the eigenvalues of $A_{\varepsilon}$, $\varepsilon \geq 0$, and assume them to be ordered $0 \leq \lambda_{\varepsilon,1} \leq \lambda_{\varepsilon,2} \leq \ldots$. Denote by $[u_{\varepsilon,l}] \in H_{\varepsilon}$ the corresponding eigenvectors which form a complete ONS of $L_{\varepsilon}$. Let $\varepsilon_{n} \to 0$. Then $\lambda_{\varepsilon_{n},l} \to \lambda_{0,l}$, for all $l \in \mathbb{N}$. There is a subsequence, called $\varepsilon_{n}$ too, and a complete ONS $([u_{l}])_{l}$ of $L_{0}$ consisting of eigenvectors belonging to $\lambda_{0,l}$ such that $||[u_{\varepsilon_{n},l}] - \Phi_{\varepsilon_{n}}^{l}[u_{l}]||_{\varepsilon_{n},d} \to 0$ as $n \to \infty$, for all $0 \leq d < 1$.

**Theorem 2.2** (cf. [4]). Let $\varepsilon_{n} \downarrow 0$, $[u_{n}] \in H_{\varepsilon_{n}}$, $[u_{0}] \in H_{0}$ and $||[u_{n}] - \Phi_{\varepsilon_{n}}^{l}[u_{0}]||_{L_{\varepsilon}} \to 0$, $n \to \infty$. Assume $f$ satisfies (H1) and (H2). Then equation (2.9) generates a global semiflow, called $\pi_{\varepsilon}$, on $H_{\varepsilon}$, for $\varepsilon \geq 0$. If $t_{0}, t_{n} > 0$, $t_{n} \to t_{0}$ as $n \to \infty$, and $\sup_{n \in [0, t_{0}]} ||[u_{n}]\pi_{\varepsilon_{n},l}||_{\varepsilon_{n},d} < \infty$, then for $0 \leq d < 1$

$$
||[u_{n}]\pi_{\varepsilon_{n},l} - \Phi_{\varepsilon_{n}}^{l}([u_{0}]\pi_{0}t_{0})||_{\varepsilon_{n},d} \to 0, \quad \text{as } n \to \infty.
$$
For \( \varepsilon \geq 0 \) the semiflows \( \pi_\varepsilon \) have attractors \( A_\varepsilon \subset H_\varepsilon \) consisting of all full bounded solutions on \( H_\varepsilon \) which attract every bounded set \( B \subset H_\varepsilon \). The family of attractors is upper-semi-continuous at \( \varepsilon = 0 \), i.e. for all \( 0 \leq d < 1 \),

\[
\lim_{\varepsilon \to 0} \sup_{[u] \in A_\varepsilon} \inf_{[v] \in A_0} \|u - \Phi^\varepsilon [v]\|_{\varepsilon,d} = 0.
\]

\[3. \text{Boundedness in } L^\infty \text{ and auxiliary results}\]

We want the Nemitsky operators \( f_\varepsilon \) to be differentiable on the attractors. On way to get this is to show the attractors to be bounded uniformly in \( L^\infty \). Then one can cut \( f \) without changing it on the attractor.

In this section we show the attractors to be bounded uniformly in \( L^\infty \). For this purpose we need the stronger dissipativeness condition (H2) on \( f \), i.e. we suppose \( \beta_2 > 0 \), where in many other papers (e.g. [14], [4]) \( \beta_2 = 0 \) is allowed.

We also provide some results we shall need later, among them a convergence result for eigenvalues and eigenfunctions for the linear problem \([u_t] = -A_\varepsilon [u] + V_\varepsilon [u] \), where \( V_\varepsilon \) are some potentials. Additionally we define a (the usual) Liapunov-function for these semiflows.

For \( \varepsilon > 0 \) we can apply Theorem 2.1 from [3]: with this theorem \([u] \pi_\varepsilon t\) is in \( L^\infty \) for \( t > 0 \), and all \([u] \in H_\varepsilon \). In particular all \( A_\varepsilon, \varepsilon > 0 \), are bounded in \( L^\infty \). But we want a uniform bound on \([u] \pi_\varepsilon t\) independent of \( \varepsilon \) and \([u] \). We cannot apply the results of above paper to this case because \( \Omega_\varepsilon \) collapses to a lower dimensional set, and on the fixed sets \( G_j \) the coefficients in the linear operator tend to infinity. Also we do not have similar results for the limiting case since the abstract theorems do not apply to it.

We shall use functions of the form \( t \mapsto \| [u] \pi_\varepsilon t \|_{L^p} \) to show that after a certain time (independent of \( \varepsilon \)) \([u] \pi_\varepsilon t\) is bounded in \( L^\infty \) by a bound independent of the initial value \([u] \) and \( \varepsilon \). Then the convergence of the semiflows \( \pi_\varepsilon \) to \( \pi_0 \) shows a similar result for \( \pi_0 \). Thus all attractors \( A_\varepsilon, \varepsilon > 0 \), are bounded uniformly in \( L^\infty \).

We need the spaces \( L^p(\Omega_\varepsilon), 1 \leq p \leq \infty \). Dividing \( \Omega_\varepsilon \) as before into edges and nodes, and making the transformations via \( \Psi_{\varepsilon,j} \), each of these spaces corresponds to an \( L^p_\varepsilon \) with norm \( \| \cdot \|_{L^p_\varepsilon} \) defined by

\[
L^p_\varepsilon := \{ [u] = [u_1, \ldots , u_{K_E + K_N}]: u_j \in L^p(G_j), j = 1, \ldots, K_E, \\
u_l \in L^p(G_{\varepsilon,l}), l = K_E + 1, \ldots, K_E + K_N \}, \quad \text{for } \varepsilon > 0,
\]

\[
\|[u]\|_{L^p_\varepsilon} := \left( \sum_{j=1}^{K_E} \int_{G_j} |u_j|^p d\lambda_{e,j} + \varepsilon \sum_{j=K_E+1}^{K_E+K_N} \int_{G_{e,j}} |u_j|^p dz \right)^{1/p}, \quad p < \infty, \varepsilon > 0,
\]

\[
\|[u]\|_{L^\infty_\varepsilon} := \max(\|[u]\|_{L^\infty(G_j)}, \|[u]\|_{L^\infty(G_{\varepsilon,l})}, j = 1, \ldots, K_E, l = K_E + 1, \ldots, K_E + K_N), \quad \varepsilon > 0,
\]
\[ L_p^\varepsilon := \{ [u] = [u_1, \ldots, u_{K_E}] : u_j \in L^p(G_j), \ j = 1, \ldots, K_E \}, \]
\[ \|u\|_{L_p^\varepsilon} := \left( \sum_{j=1}^{K_E} \int_{G_j} |u_j|^p d\lambda_{0,j} \right)^{1/p}, \quad p < \infty, \]
\[ \|u\|_{L_\infty^\varepsilon} := \max(\|u_j\|_{L_\infty(G_j)}, \ j = 1, \ldots, K_E). \]

We need a few technical lemmas.

**Lemma 3.1.** Let \( y : [a, b] \to \mathbb{R} \geq 0 \) be continuous and differentiable on \([a, b] \) with \( y' \leq C_1 - C_2 y^{C_3}, \) where \( C_1 \geq 0, C_2 > 0, C_3 > 1 \) are some given constants. Then for \( t \in [a, b] \) we have either

\[ y(t) \leq 2 \left( \frac{C_1}{C_2} \right)^{1/C_3} \]

or

\[ y(t) \leq ((t - a) \frac{C_2}{2} (C_3 - 1) + y^{1-C_3}(a))^{-1/(C_3-1)}. \]

**Proof.** If \( y(t) > (C_1/C_2)^{1/C_3} \), then \( y' < 0 \). Hence if \( y(a) \leq 2(C_1/C_2)^{1/C_3} =: C_4 \), then \( y(t) \leq C_4 \) for all \( t \).

Now assume \( y(a) > C_4 \). Then as long as \( y(t) \geq C_4 \) we have \( 1/2C_2 y^{C_3}(t) > C_1 \) and thus

\[ t - a \leq \int_a^t \frac{-y'(s)}{C_2 y^{C_3}(s) - C_1} ds = \int_{y(t)}^{y(a)} \frac{dy}{C_2 y^{C_3} - C_1} \leq \int_{y(t)}^{y(a)} \frac{2 dy}{C_2 y^{C_3}} = \frac{2}{C_2(1 - C_3)} (y^{1-C_3}(a) - y^{1-C_3}(t)). \]

We get

\[ y(t) \leq ((t - a) \frac{C_2}{2} (C_3 - 1) + y^{1-C_3}(a))^{-1/(C_3-1)}. \]

**Lemma 3.2.** Assume \( f \) satisfies (H2). Let \( \varepsilon > 0, 2 \leq p < \infty, [u_0] \in H_\varepsilon, \) and set \( u(t) := [u_0]_\varepsilon t, \ t \geq 0. \) Then \( u(t) \in L^\infty_\varepsilon, \) for \( t > 0, \) and we can define \( G_{\varepsilon,p} : \mathbb{R} \to \mathbb{R} \) by \( G_{\varepsilon,p}(t) := \|u(t)\|_{L^p_\varepsilon}. \)

(a) \( G_{\varepsilon,p} \) is differentiable on \( [0, \infty[. \) The derivative is

\[ G'_{\varepsilon,p}(t) = p(\|u(t)\|^{p-2}u(t), \partial_t u(t))_{L^\varepsilon}. \]

(b) There are constants \( C, T > 0, \) independent of \( p, \varepsilon, [u_0], \) such that \( G_{\varepsilon,p}(t) \leq C^p \) for all \( t \geq T. \)

(c) If \( f \) satisfies only (H2'), \( \tilde{C} > 0 \) is such that \( f(s)/s \leq -1/2\tilde{C}, \) \( |s| \geq \tilde{C}, \) and \( u(t) = [u_0] \) is constant, then \( \|u_0\|_{L^\infty} \leq \tilde{C}, \) for \( \varepsilon > 0. \)

**Proof.** First note that condition (H2) implies there is a constant \( C_1 > 0, \) such that \( f(s)/s^{\beta_2} < -1/2\xi, \) for \( |s| \geq C_1. \) The same holds if \( f \) only satisfies (H2') setting \( \beta_2 = 0. \)

In this proof we will work with \( \Omega_\varepsilon \) rather than the partition into \( G_j, G_{\varepsilon,j}. \)
If \([u] \in L^\varepsilon\) and \(w \in L^2(\Omega_\varepsilon)\) are such that \(w \circ \Phi_{\varepsilon,j} = u_j\) for all \(j\), then by construction \(w \in L^p(\Omega_\varepsilon)\) if and only if \([u] \in L^p_{\varepsilon}\), and in this case \(\|w\|_{L^p(\Omega_\varepsilon)} = \varepsilon^M \|u\|_{L^p_{\varepsilon}}\).

For \(t \geq 0\) set \(w(t) := u_j(t) \circ \Phi_{\varepsilon,j}^{-1}, \Omega_{\varepsilon,j} \to \mathbb{R}, j = 1, \ldots, K_E + K_N\), then \(w(t)\) is solution of equation (1.1) with initial value \(w(0) \in H^1(\Omega_\varepsilon)\). Define \(\tilde{G}_{\varepsilon,p}(t) := \|w(t)\|_{L^p(\Omega_\varepsilon)}^p\). If we show \(\tilde{G}_{\varepsilon,p}^t\) is differentiable with derivative \(\tilde{G}_{\varepsilon,p}^t(t) = p(\|w(t)\|^{p-2}w(t), w_t(t))_{L^2(\Omega_\varepsilon)}\), then the first part of the lemma follows immediately.

Let \(0 < t_0\). Then \(w(t_0) \in L^\infty(\Omega_\varepsilon)\), as has already been mentioned. For \(t_0 \leq t\) we can view \(w(t)\) as the solution of the abstract equation

\[
\dot{w}_t = -\tilde{A}_\varepsilon w + f_\varepsilon(w),
\]

where the linear operator \(\tilde{A}_\varepsilon : D(\tilde{A}_\varepsilon) \subset L^2(\Omega_\varepsilon) \to L^2(\Omega_\varepsilon)\) is sectorial. It is well known that the restriction \(\tilde{A}_\varepsilon,p : D(\tilde{A}_\varepsilon,p) \subset L^p(\Omega_\varepsilon) \to L^p(\Omega_\varepsilon)\) is sectorial (see e.g. [12, Theorem 3.1.3]). Hence there is \(T_1 = T_1(w(t_0), \varepsilon, p) > 0\) such that \(t \mapsto w(t) \in L^p(\Omega_\varepsilon)\) is continuous on \([t_0, T_1]\) and differentiable on \([t_0, T_1]\). If \(T_1\) is maximal, then either \(T_1 = \infty\) or \(\|w(t)\|_{L^p(\Omega_\varepsilon)} \to \infty\), as \(t \uparrow T_1\).

The differentiability of \(t \mapsto w(t) \in L^p(\Omega_\varepsilon)\) implies

\[
\tilde{G}_{\varepsilon,p}^t(t) = p(\|w(t)\|^{p-2}w(t), w_t(t))_{L^2(\Omega_\varepsilon)}.
\]

Hence if we knew \(T_1 = \infty\), then the first part of the lemma would have been shown.

To prove \(T_1 = \infty\) and the second part, note that \(w(t) \in L^\infty(\Omega_\varepsilon)\) for \(0 < t < T_1\). Thus \(\|w(t)\|^{p-2}w(t) \in H^1(\Omega_\varepsilon)\) and \(\partial(\|w(t)\|^{p-2}w(t)) = (p-1)|w(t)|^{p-2}\partial w(t)\).

If \(\tilde{a}_\varepsilon\) denotes the bilinear form which generates \(\tilde{A}_\varepsilon\), we get for \(t_0 < t < T_1\)

\[
\begin{align*}
G_{\varepsilon,p}^t(t) &= p_{\varepsilon}^{-M} (-\tilde{a}_\varepsilon(w(t), \|w(t)\|^{p-2}w(t)) + (f_\varepsilon(w(t), \|w(t)\|^{p-2}w(t)))_{L^2(\Omega_\varepsilon)}) \\
&= p_{\varepsilon}^{-M} \int_{\Omega_{\varepsilon}} (-(p-1)|w(t)|^{p-2}\nabla w(t)\nabla w(t) + f(w(t))w(t)|w(t)|^{p-2}) \, dz \\
&\leq p_{\varepsilon}^{-M} \left(\frac{|\Omega_{\varepsilon}|C_1^{p-1}}{\max_{|\Omega_{\varepsilon}|\leq C_1} |f(s)|} - \frac{1}{2} \varepsilon^M \|w(t)\|^{p+\beta_2}_{L^{p+\beta_2}(\Omega_\varepsilon)}\right) \\
&\leq p_{\varepsilon}^{-M} \left(\frac{|\Omega_{\varepsilon}|C_2C_1^{p-1}}{\max_{|\Omega_{\varepsilon}|\leq C_1} |f(s)|} - \frac{1}{2} \varepsilon^M \|w(t)\|^{p+\beta_2}_{L^{p+\beta_2}(\Omega_\varepsilon)}\right),
\end{align*}
\]

where the constant \(C_2\) is independent of \(\varepsilon, p, [u_0], t\). The conditions on the transformations \(\Phi_{\varepsilon,j}\) imply the existence of a constant \(C_3\) such that \(|\Omega_{\varepsilon}| \leq C_3 \varepsilon^M\). Thus there are constants \(C_4, C_5 > 0\), also independent of \(\varepsilon, p \geq \beta_2, [u_0], t\), such that

\[
(3.1) \quad G_{\varepsilon,p}^t(t) \leq p_{\varepsilon}C_4C_1^p - p_{\varepsilon}C_5G_{\varepsilon,p}^{(p+\beta_2)/p}(t), \quad t_0 < t < T_1.
\]
By Lemma 3.1 we have either
\[
G_{\varepsilon,t}(t) \leq 2\frac{C_4}{C_5} \varepsilon^{p/(p+\beta_2)} \leq C_6^p,
\]
where \( C_6 \) is independent of \( \varepsilon, p, [u_0], t, \) or
\[
G_{\varepsilon,p}(t) \leq \left( \frac{t - t_0}{2} p C_5 \frac{\beta_2}{p} + G_{\varepsilon,p}(t_0) \right)^{-p/\beta_2}.
\]
Thus for \( t \uparrow T_1 \) \( G_{\varepsilon,p}(t) \) is bounded, and \( T_1 = \infty \) follows.

Now if \( t \geq T := (1/2) C_5 C_6^2 (\beta_2)^{-1} + t_0, \) then in any case \( G_{\varepsilon,p}(t) \leq C_6^p \) and the first two parts of the lemma have been proven.

To prove part (c) use again inequality (3.1), only that now \( \beta_2 = 0. \) Since \( G'_{\varepsilon,p}(t) \equiv 0 \) in this case, we have
\[
\| [u_0] \|_{L^p} \leq \frac{C_4}{C_5} C_1.
\]

It is well known that if \( u: U \to \mathbb{R}, U \subset \mathbb{R}^{M+1} \) open, then \( \| u \|_\infty \leq C \) if and only if \( \| u \|_{L^p(U)} \leq |U|^{1/p} C \) for all \( p \) big enough (see e.g. [6, Problem 7.1]).

Thus \( \| [u_0] \|_{L^\infty} \leq C_1 \) and the third part is true too. \( \square \)

Now we can prove that the attractors \( A_\varepsilon \) are bounded uniformly in \( L^\infty. \)

**Proposition 3.3.** Assume \( f \) satisfies condition (H2). There is a constant \( C > 0 \) such that \( A_\varepsilon \subset B_C(0) \subset L^\infty_\varepsilon \) for all \( \varepsilon \geq 0. \)

**Proof.** First let \( \varepsilon > 0. \) Let \( T \) and \( C_1 \) be as in Lemma 3.2(b). If \( [u] \in A_\varepsilon, \) then there is a \( [u_0] \in H_\varepsilon \) such that \( [u_0] \pi_2 2T = [u]. \) Thus by Lemma 3.2
\[
\| [u] \|_{L^p_T} \leq C_1 \text{ for all } 2 \leq p < \infty.
\]

Using the same characterization of \( L^\infty \) we used in the proof of Lemma 3.2, there is a \( C_2 > 0 \) and \( \| [u] \|_{L^\infty_\varepsilon} \leq C_2. \) This proves the uniform bound on \( A_\varepsilon \) for \( \varepsilon > 0. \)

Now we bound \( A_0. \) Let \( [u] \in A_0. \) Again there is a \( [u_0] \in H_0 \) such that \( [u_0] \pi_0 2T = [u]. \) Let \( \varepsilon_n \to 0. \) By Theorem 2.2 we have
\[
(\Phi^{\varepsilon_n}_t [u_0]) \pi_2 2T_j(x, y) \to u_j(x, y)
\]
for a.a. \( (x, y) \in G_j, \) and arguing as before we get \( \| u_j \|_{L^\infty(G_j)} \leq C_2, \) for all \( j = 1, \ldots, K_E. \) \( \| [u] \|_{L^\infty_\varepsilon} \leq C_2 \) follows immediately. \( \square \)

Now we bring a few lemmas we shall need in later sections. We start by proving a sufficient condition for convergence in \( | \cdot |_{\varepsilon,d}. \)
**Lemma 3.4.** Let \( \varepsilon_n \to 0 \), \( 0 \leq d < 1 \), \( [u_n] \in D(A) \), \( [u_0] \in H \) and \( \|u_n\|_{H^d}, \|A_{\varepsilon_n}[u_n]\|_{L^\infty} \leq C \), \( C > 0 \) independent of \( n \). Suppose
\[
\|u_n - \Phi_{\varepsilon_n}[u_0]\|_{L^\infty} \to 0, \quad \text{as } n \to \infty,
\]
then
\[
\lim_{n \to \infty} a_{\varepsilon_n}([u_n], \Phi_{\varepsilon_n}[v]) = a_0([u_0], [v]) \quad \text{for all } [v] \in H.
\]
If additionally \( a_{\varepsilon_n}([u_n], [u_n]) \to a_0([u_0], [u_0]) \) as \( n \to \infty \), then
\[
\|u_n - \Phi_{\varepsilon_n}[u_0]\|_{\varepsilon_0, d} \to 0, \quad \text{as } n \to \infty.
\]

**Proof.** Assume the situation of the lemma. By Lemma 2.12 in [4] there is a subsequence, called \( \varepsilon_n \) again, \( [v_0] \in H \), \( \tilde{u}_j \in (L^2(G_j))^M \), \( j = 1, \ldots, K_E \), such that \( u_{n,j} \to v_{0,j} \) weakly in \( H^1(G_j) \) and strongly in \( L^2(G_j) \), \( (1/\varepsilon_n)D_j u_{n,j} \to \tilde{u}_{0,j} \) weakly in \( L^2(G_j) \), \( j = 1, \ldots, K_E \), and \( a_n([u_n], \Phi_n^H[v]) \to a_0([v_0], [v]) \), for all \([v] \in H\), as \( n \to \infty \), \( [v_0] = [u_0] \) and thus (3.2) follows.

Now assume additionally \( a_n([u_n], [u_n]) \to a_0([u_0], [u_0]) \), \( n \to \infty \), then
\[
0 = a_n([u_n], [u_n]) - a_0([u_0], [u_0])
\]
\[
= \sum_{j=1}^{K_E} \int_{G_j} \left( (D_x u_{0,j}, \tilde{u}_{0,j}) \left( \begin{array}{c} 1 \\ T_j(x, y) \\ E_M 
\end{array} \right) D^j \right) \left( 0 \right) d\lambda_0, j
\]
\[
- \int_{G_j} (D_x u_{0,j})^T (1, 0) D^T_j (x, 0) \right|^2 d\lambda_0, j
\]
\[
+ \int_{G_j} \left( |D_x u_{n,j} - (1/\varepsilon_n) D_y u_{n,j}| A_{n,j}(x, y) \sqrt{\det DT_{\varepsilon,j} || \det DT_j (S_{\varepsilon} \circ T_{\varepsilon,j}) ||} \right)^2
\]
\[
- \left( |D_x u_{0,j} - \tilde{u}_{0,j}| \left( \begin{array}{c} 1 \\ T_j(x, y) \\ E_M 
\end{array} \right) D^j \left( 0 \right) \right) \left( (x, 0) \right) \right|^2 \det DT_j (x, 0) dx dy
\]
\[
+ \frac{1}{\varepsilon_n} \sum_{j=K_E+1}^{K_E+K_N} \int_{G_n,j} |D(u_{n,j})|^2 dz.
\]
By Lemmas 2.3 and 2.6 in [4]
\[
E_{2,n,j} \to (D_x u_{0,j}, \tilde{u}_{0,j}) \left( \begin{array}{c} 1 \\ T_j(x, y) \\ E_M 
\end{array} \right) D^j \left( 0 \right) \sqrt{\det DT_j (x, 0)}
\]
weakly in \( L^2 \), thus for all \( j \)
\[
\lim_{n \to \infty} \int_{G_j} (|E_{2,n,j}|^2 - E_{3,j}) dx dy \geq 0.
\]
Recall that the orthogonal complement of \( L^2_s(G_j) \) is \( L^2_s(G_j) \). Decompose \( T_{j,t} \), \( \tilde{u}_{0,j,t} \), by setting \( T = T_{s,j} + T_{l,j} \), \( \tilde{u}_{0,j} = \tilde{u}_{s,0,j} + \tilde{u}_{l,0,j} \), where \( T_{s,j,t}, \tilde{u}_{s,0,j,t} \in L^2_s(G_j) \), \( T_{l,j,t}, \tilde{u}_{l,0,j,t} \in L^2_s(G_j) \), \( j = 1, \ldots, K_E, l = 1, \ldots, M \).

Note that by Proposition 2.1 in [4] for \( u \in H^1_s(G_j) \) we have \( D_\epsilon u \in L^2_s(G_j) \) for all \( j \), and by Lemma 2.12 of the same article

\[
\tilde{u}_{s,0,j} = D_x u_{0,j} ((1,0) D T_j^T (x,0)^{-2}(1,0) D T_j^T (x,0) D T_j(x,0)(0,E_M)^T - T_{s,j}).
\]

Proceeding as in the proof of Lemma 2.13 in [4], we get

\[
E_{1,n,j} = \int_{G_j} (D_x u_{0,j})^2 |(1,0) D T_j^T (x,0)^{-2}(1,0) D T_j^T (x,0) D T_j(x,0)(0,E_M)^T - T_{s,j})| d\lambda_{0,j}
\]

\[
= \int_{G_j} |(0, D_x u_{0,j} T_{l,j} + \tilde{u}_{l,0,j}) D T_j^{-1}(x,0)|^2 d\lambda_{0,j}.
\]

We find

\[
0 \leftarrow \sum_{j=1}^{K_E} E_{5,j} + \int_{G_j} (|E_{2,n,j}|^2 - E_{3,j}) \, dx \, dy + E_{4,n}
\]

and (3.3) implies

\[
E_{5,j} = \lim_{n \to \infty} E_{4,n} = \lim_{n \to \infty} \int_{G_j} (|E_{2,n,j}|^2 - E_{3,j}) \, dx \, dy = 0
\]

for all \( j \). Thus \( D_x u_{s,j} \to D_x u_{0,j} \), \((1/\epsilon_n) D_y u_{n,j} \to \tilde{u}_{0,j} \) strongly in \( L^2 \), which in turn implies \( ||u_n|| - \Phi'_{\epsilon_n}[u_n] ||_{n,d} \to 0, n \to \infty \), for all \( d < 1 \). \( \square \)

We need the uniform boundedness of the attractors \( A_\epsilon \) in \( ||\cdot||_{H_\epsilon} \). Since this is not included in [4], we prove it here. For this we use a Liapunov-function often used for such equations.

Let \( \epsilon \geq 0 \). Define \( F(x) := \int_0^x f(s) \, ds \). Denote by \( F_\epsilon: H_\epsilon \to L^1_\epsilon \) the Nenitsky operator of \( F \). It is well known that \( F_\epsilon \) is well defined, maps bounded sets of \( H_\epsilon \) into bounded sets of \( L^1_\epsilon \), and is Fréchet-differentiable with derivative \( DF_\epsilon([u]) [v] = F_\epsilon([u]) [v] \).

Define \( G_{\epsilon,\epsilon}: H_\epsilon \to \mathbb{R} \) by

\[
G_{\epsilon,\epsilon}([u]) := \frac{1}{2} \alpha_\epsilon([u], [u]) - (F_\epsilon([u]), 1)_{L_\epsilon}
\]

(here \( \epsilon \geq 0 \)). It is well known that \( G_{\epsilon,\epsilon} \) is Fréchet-differentiable, and if \( \sigma_\epsilon(t) \) is a solution of equation (2.9), then \((t \mapsto G_{\epsilon,\epsilon}(\sigma_\epsilon(t)))' = -\partial_t \sigma_\epsilon(t) ||_{L_\epsilon}^2 \). \( G_{\epsilon,\epsilon} \) maps bounded sets of \( H_\epsilon \) into bounded sets of \( \mathbb{R} \). Since \( f \) satisfies condition (H2'), there is a \( C_1 > 0 \) such that \( F(s) \leq -(1/4)\xi s^2 + C_1 \) for all \( s \in \mathbb{R} \). Thus there are \( C_2, C_3 > 0 \) such that

\[
||u||_{H_\epsilon}^2 \leq C_2 (G_{\epsilon,\epsilon}([u]) + C_3) \text{ for all } [u] \in H_\epsilon, \epsilon \geq 0.
\]
By Lemma 3.5 to come the sets of equilibrium points of \( \pi_\varepsilon \) is bounded in \( H_\varepsilon \), and \( \pi_\varepsilon \) is gradient like with respect to \( \mathcal{G}_{\varepsilon,H}, \varepsilon \geq 0 \).

**Lemma 3.5.** For every \( \delta > 0 \) there is a \( C = C(\delta) > 0 \), independent of \( \varepsilon \geq 0 \), such that if \( [u] \in D(A_\varepsilon) \), \( \| - A_\varepsilon [u] + f_\varepsilon ([u]) \|_{L_\varepsilon} \leq \delta \) implies \( \| [u] \|_{H_\varepsilon} < C \).

**Proof.** Let \( [u_0] \in D(A_\varepsilon), \varepsilon \geq 0, \delta > 0 \) and assume

\[
\| - A_\varepsilon [u_0] + f_\varepsilon ([u_0]) \|_{L_\varepsilon} \leq \delta.
\]

\( f \) satisfies (H2') means there are constants \( C_1, C_2 > 0 \) such that

\[
([u], f_\varepsilon ([u]))_{L_\varepsilon} \leq (C_1 - C_2 \| [u] \|_{L_\varepsilon}) \| [u] \|_{L_\varepsilon} \quad \text{for all } [u] \in H_\varepsilon, \varepsilon \geq 0.
\]

Now if \( \| [u_0] \|_{L_\varepsilon} \geq \max((2 + 4/C_2)\delta, C_1/C_2 + 1/2) \), then for \( \varepsilon > 0 \)

\[
-\delta \| [u_0] \|_{L_\varepsilon} \leq (-A_\varepsilon [u_0] + f_\varepsilon ([u_0]), [u_0])_{L_\varepsilon} \\
\leq - \min \left( 1, \frac{1}{2} C_2 \right) (a_\varepsilon([u_0], [u_0]) + \| [u_0] \|_{L_\varepsilon}^2) \\
+ \left( C_1 - \frac{1}{2} C_2 \| [u_0] \|_{L_\varepsilon} \right) \| [u_0] \|_{L_\varepsilon} \\
\leq - \min \left( 1, \frac{1}{2} C_2 \right) \| [u_0] \|_{L_\varepsilon}^2 \leq -2\delta \| [u_0] \|_{L_\varepsilon}^2.
\]

If \( \| [u_0] \|_{L_\varepsilon} < \max((2 + 4/C_2)\delta, C_1/C_2 + 1/2) \), then

\[
-\delta \| [u_0] \|_{L_\varepsilon} \leq (-A_\varepsilon [u_0] + f_\varepsilon ([u_0]), [u_0])_{L_\varepsilon} \\
\leq -(a_\varepsilon([u_0], [u_0]) + \| [u_0] \|_{L_\varepsilon}^2) + C_3 \leq -\| [u_0] \|_{H_\varepsilon}^2 + C_3.
\]

Hence \( - A_\varepsilon [u_0] + f_\varepsilon ([u_0]) \|_{L_\varepsilon} \leq \delta \) implies \( \| [u_0] \|_{H_\varepsilon}^2 \leq C_4 = C_4(\delta) \).

For \( \varepsilon = 0 \) we do no longer have \( a_\varepsilon([u], [u]) + \| [u] \|_{L_\varepsilon}^2 = \| [u] \|_{H_\varepsilon}^2 \) but only \( a_0([u], [u]) + \| [u] \|_{L_\varepsilon}^2 \leq C_5 \| [u] \|_{H_\varepsilon}^2 \). In this case we can adapt the argument above using \( \delta = \delta/C_3 \).

**Lemma 3.6.** Let \( C_1 > 0, \Omega \subset ]0, \infty[ \times \mathbb{R}^M \) be open, bounded, Lipschitz. Then there is a constant \( C_2 = C_2(C_1) > 0 \) such that for all \( u \in H^1(\Omega) \) and \( \varepsilon > 0 \)

\[
\frac{1}{\varepsilon} \| u \|_{L^2((x,y) \in \Omega : 0 < x \leq \varepsilon C_1)}^2 \leq C_2 \| u \|_{H^1(\Omega)}^2.
\]

An analogous statement holds if \( \Omega \subset ]\infty, 1[ \times \mathbb{R}^M \).
Proof. Extend \( u \in H^1(\Omega) \) to \( \tilde{u} \in H^1(\mathbb{R}^{M+1}) \). We get
\[
\frac{1}{\varepsilon} \int_{\{(x,y) \in \Omega \, \mid \, 0 < x \leq \varepsilon C_1\}} u^2 \, dx \, dy \\
\leq \frac{2}{\varepsilon} \int_{\{(x,y) \in \Omega \, \mid \, 0 < x \leq \varepsilon C_1\}} |u(x,y) - \tilde{u}(0,y)|^2 \, dx \, dy \\
+ \frac{2}{\varepsilon} \int_{\{(x,y) \in \Omega \, \mid \, 0 < x \leq \varepsilon C_1\}} \tilde{u}^2(0,y) \, dx \, dy \\
\leq \frac{2}{\varepsilon} \int_0^\varepsilon \|\tilde{u}(x,y) - \tilde{u}(0,y)\|_{L^2(\mathbb{R}^M)}^2 \, dx + 2C_1 \|\tilde{u}(0,\cdot)\|_{L^2(\mathbb{R}^M)}^2 \\
\leq C_2 \varepsilon \int_0^\varepsilon \|\tilde{u}\|_{H^1(\mathbb{R}^{M+1})}^2 \, dx + C_3 \|\tilde{u}\|_{H^1(\mathbb{R}^{M+1})}^2 \leq C_4 \|u\|_{H^1(\Omega)}^2,
\]
where we used Theorem 6.2.29 in [7].

Lemma 3.7. There is a constant \( C > 0 \) such that for all \([u] \in H_\varepsilon, \varepsilon > 0, \)
and \( d > 1/2 \)
\begin{equation}
(3.5) \quad \sum_{j=K_E+1}^{K_E+K_N} \|u_j\|_{L^2(G_{\varepsilon,j})}^2 \leq C \|u\|_{E_{\varepsilon,d}}^2.
\end{equation}

Proof. Fix \( j_0 \in \{K_E+1, \ldots, K_E+K_N\} \) and let \([u] \in H_\varepsilon, 0 < \varepsilon \). If \( G_{\varepsilon,j_0} \)
has empty interior, nothing has to be shown. If this is not the case, by (C8)
there are open, bounded, connected, Lipschitz \( G_{j_0,k} \subset \mathbb{R}^{M+1}, j_0 \in \{1, \ldots, K_E\}, \)
\( U_{j_0,k} \subset \Psi_{\varepsilon,j_0}^{-1}(\Omega_{\varepsilon,j_0} \cap \Omega_{\varepsilon,j_0}), k = 1, \ldots, N_{j_0}, r, C_1 > 0, \) all independent of \( \varepsilon, \)
\( C^1 \)-diffeomorphisms \( \Psi_{z,j_0,k} \), \( z_{j_0,k} \in \mathbb{R}^{M+1} \) such that
\[
\left| G_{\varepsilon,j_0} \setminus \bigcup_{k=1}^{N_{j_0}} \Psi_{z_{j_0},k}(G_{j_0,k}) \right| = 0,
\]
\( B_r(z_{\varepsilon,j_0,k}) \subset U_{j_0,k} \cap \Psi_{\varepsilon,j_0}^{-1}(\Omega_{\varepsilon,j_0}), \)
\( |\Psi_{z_{j_0},k}^{-1}(U_{j_0,k}) \cap \Psi_{z_{j_0},k}(G_{j_0,k})| \geq \frac{1}{C_1}, \)
\( \Psi_{\varepsilon,j_0}^{-1} \circ \Psi_{z_{j_0},k}(B_r(z_{\varepsilon,j_0,k})) \subset [0,\varepsilon C_1] \times \mathbb{R}^M \) or
\( \Psi_{\varepsilon,j_0}^{-1} \circ \Psi_{z_{j_0},k}(B_r(z_{\varepsilon,j_0,k})) \subset [1-\varepsilon C_1, 1] \times \mathbb{R}^M, \)
if the edge \( \Omega_{\varepsilon,j_0} \) begins at the node \( \Omega_{\varepsilon,j_0} \), or if \( \Omega_{\varepsilon,j_0} \) ends at \( \Omega_{\varepsilon,j_0} \), respectively.
Without loss of generality we assume \( \Omega_{z_{j_0}} \) begins at \( \Omega_{z_{j_0}} \). Define
\[
u_{j_0,k,U} := u_{i} \circ \Psi_{z_{j_0},k}^{-1} \circ \Psi_{z_{j_0},k} \text{ on } G_{\varepsilon}, \text{ resp. } G_{\varepsilon,i},
\]
u_{j_0,k} := u_{j_0} \circ \Psi_{z_{j_0},k} \text{ on } G_{j_0,k},
\]
for all possible \( j, k \). Then \( u_{j_0,k,U} \in H^1(U_{j_0,k}), u_{j_0,k} \in H^1(G_{j_0,k}) \). Note that these functions may depend on \( \varepsilon \).
Now, if \( V \subset U \subset \mathbb{R}^{M+1} \) are given, \( U \) is open, bounded, connected, Lipschitz, \( |V| \geq C_2 > 0 \) and \( v \in H^1(U) \), we set \( c_v := |U|^{-1}(v,1)_{L^2(U)} \), \( w := v - c_v \). Then \((w,1)_{L^2(U)} = 0\) and by the generalized Poincaré-inequality (see e.g. [1, 5.15]) there is a \( C_3 = C_3(U) \) such that

\[
\|w\|_{L^2(U)} \leq C_3 \|Dw\|_{L^2(U)} = C_3 \|Dv\|_{L^2(U)}.
\]

Also

\[
|c_v| \leq \frac{1}{C_2} \int_V |c_v| \, dz \leq \frac{1}{C_2} \left( \int_V |v| \, dx + \int_U |w| \, dz \right) \leq \frac{|U|^{1/2}}{C_2} (\|v\|_{L^2(V)} + C_3 \|Dv\|_{L^2(U)}),
\]

(3.6) \( \|v\|_{L^2(U)} \leq |U|^{1/2} \|v| + C_3 \|Dv\|_{L^2(U)} \leq C_4 (\|v\|_{L^2(V)} + \|Dv\|_{L^2(U)}) \)

with some constant \( C_4 = C_4(|U|, C_2) \). Apply this first for

\[
U = G_{j_0,k}, \quad V = \Psi_{\varepsilon,j_0,k}^{-1}(U_{j_0,k} \cap \Psi_{\varepsilon,j_0,k}(G_{j_0,k})),
\]

then

\[
\|u_{j_0,k}\|_{L^2(G_{j_0,k})} \leq C_6 (\|u_{j_0,k}\|_{L^2(\Psi_{\varepsilon,j_0,k}^{-1}(U_{j_0,k} \cap \Psi_{\varepsilon,j_0,k}(G_{j_0,k})))} + \|Du_{j_0,k}\|_{L^2(G_{j_0,k})}).
\]

Apply to this inequality the transformation \( \Psi_{\varepsilon,j_0,k}^{-1} \), then by (C8) there is a constant \( C_6 \) such that

(3.7) \( \|u_{j_0}\|_{L^2(G_{\varepsilon,j_0,k} \cap \Psi_{\varepsilon,j_0,k}(G_{j_0,k}))} \leq C_6 (\|u_{j_0,k}\|_{L^2(U_{j_0,k})} + \|Du_{j_0}\|_{L^2(G_{\varepsilon,j_0,k}))} \)

Now apply inequality (3.6) a second time, with \( U = U_{j_0,k}, \quad V = V(\varepsilon) = B_{\varepsilon}(z_{\varepsilon,j_0,k}) \). There is a \( C_7 \) such that

\[
\|u_{j_0,k,u}\|_{L^2(U_{j_0,k})} \leq C_7 (\|u_{j_0,k,u}\|_{L^2(B_{\varepsilon}(z_{\varepsilon,j_0,k}))} + \|Du_{j_0,k,u}\|_{L^2(U_{j_0,k})})
\]

\[
\leq C_8 (\varepsilon^{-1/2} \|u_{j_0}\|_{L^2((x,y) \in G_{j_0,k} : 0 < x \leq c_1)})
\]

\[
+ \sum_{i=1}^{K_E} \varepsilon^{1/2} \|D_{x_i} u_i\|_{L^2(\Psi_{\varepsilon,i}^{-1}(\Psi_{\varepsilon,j_0}(U_{j_0,k}) \cap \Omega_{\varepsilon,i}))}
\]

\[
+ \varepsilon^{-1/2} \|D_y u_i\|_{L^2(\Psi_{\varepsilon,i}^{-1}(\Psi_{\varepsilon,j_0}(U_{j_0,k}) \cap \Omega_{\varepsilon,i}))}
\]

\[
+ \sum_{i=K_E+1}^{K_N} \|Du_i\|_{L^2(\Psi_{\varepsilon,i}^{-1}(\Psi_{\varepsilon,j_0}(U_{j_0,k}) \cap \Omega_{\varepsilon,i}))}
\]

making the transformations onto \( G_i \) and \( G_{\varepsilon,i} \), respectively, and using the boundedness of \( A_{\varepsilon,i} \).

By Lemma 3.6 there is a \( C_9 \) such that, for \( d > 1/2 \),

\[
\|u_{j_0,k,u}\|_{L^2(U_{j_0,k})} \leq C_9 (\|u_{j_0}\|_{H^1(G_{j_0,k})} + \|u\|_{L^2} d) \leq C_{10} \|u\|_{L^2} d.
\]
Using inequality (3.7) and summing over all $k$, there is a $C_{11}$ such that
\[ \| u_{j0} \|_{L^2(G_{\varepsilon,j0})} \leq C_{11} \| u \|_{L^2} \]
which implies (3.5). \hfill \Box

**Lemma 3.8.** There is a constant $C > 0$ such that $A_\varepsilon \subset B_C(0) \subset H_\varepsilon$ for all $\varepsilon \geq 0$, i.e. the attractors are bounded uniformly in $\| \cdot \|_{H_\varepsilon}$ for $\varepsilon \geq 0$.

**Proof.** Let $[u_\varepsilon]: \mathbb{R} \to H_\varepsilon$ be a full bounded solution of equation (2.9). Then $G_{\varepsilon,H}(\{u_\varepsilon(t)\})$ is bounded, by say $C_1(\varepsilon)$. There is a sequence $t_n \to -\infty$ such that $(t \mapsto G_{\varepsilon,H}(\{u_\varepsilon(t)\}))'|_{t=t_n} = -\| \partial_t [u_\varepsilon(t_n)] \|_{L^2}^2 \to 0$.

Let $C_2 = C(1)$ the constant of Lemma 3.5. For $n$ big enough $\| \partial_t [u_\varepsilon(t_n)] \|_{L^2} < 1$ and this lemma implies $\| [u_\varepsilon(t_n)] \|_{H_\varepsilon} \leq C_2$. Since $G_{\varepsilon,H}(\{u_\varepsilon(t)\})$ is non increasing,
\[ G_{\varepsilon,H}(\{u_\varepsilon(t)\}) \leq C_3 \| u_\varepsilon(t_n) \|_{H_\varepsilon}^2 \leq C_2 C_3 \]
for $t \geq t_n$ and thus for all $t \in \mathbb{R}$. (3.4) now proves the lemma. \hfill \Box

**Lemma 3.9.** Let $\varepsilon_n \to 0$ and assume $\sigma_\varepsilon: \mathbb{R} \to H_{\varepsilon_n}$ are solutions of equation (2.9) with $\| \sigma_\varepsilon(t) \|_{H_{\varepsilon_n}} \leq C$ for all $n$ and $t$. Then there are a constant $C_1 = C_1(C) > 0$ and a solution $\sigma_0: \mathbb{R} \to H_0$ of (2.9) with $\| \sigma_0(t) \|_{H_0} \leq C_1$.

$C_1(C) \to 0$ as $C \to 0$, and there is a subsequence, called $\varepsilon_n$ too, such that $|\sigma_\varepsilon(t) - \Phi_{\varepsilon_n}^H \sigma_0(t)|_{\varepsilon_n,d} \to 0$, as $n \to \infty$, for all $0 \leq d < 1$, $t \in \mathbb{R}$.

**Proof.** Fix $0 \leq d < 1$. For each $k \in \mathbb{N}$ fixed, $\| \sigma_\varepsilon(-k) \|_{H_{\varepsilon_n}}$ is bounded, hence taking a subsequence, called $\varepsilon_n$, again, $\sigma_n(-k)|_{H_{\varepsilon_n}}$ converges weakly in $H^1(G_j)$, $j = 1, \ldots, K_E$. By conditions (C9) and (C10) there is a $[u_{0,k}] \in H_0$ with $\| \sigma_n(-k) - \Phi_{\varepsilon_n}^H [u_{0,k}] \|_{L^2} \to 0$, as $n \to \infty$. We can apply Theorem 2.2 to get
\[ |\sigma_n(-k + t) - \Phi_{\varepsilon_n}^H [u_{0,k}]|_{\varepsilon_n,d} \to 0 \quad \text{as} \quad n \to \infty, \]
for each $t > 0$. Using the Cantor diagonal procedure there is a subsequence and $[u_{0,k}] \in H_0$ such that (3.8) holds for all $k \in \mathbb{N}$, $t > 0$.

Since for $k > l$ and $t > 0$
\[ \sigma_n(-k + t) = \sigma_n(-k + t + k - l) \]
we have $[u_{0,k}]_{\varepsilon_n} = [u_{0,k}]_{\varepsilon_n} + [u_{0,k}]_{\varepsilon_n} + [u_{0,k}]_{\varepsilon_n}$ and we can define $\sigma_0: \mathbb{R} \to H_0$, $\sigma_0(t) := [u_{0,k}]_{\varepsilon_n}$ if $t > -k$. $\sigma_0$ is a solution of equation (2.9) (for $\varepsilon = 0$). (3.8) implies $\| \sigma_0(t) \|_{H_0} \leq C_1$ for all $t \in \mathbb{R}$, and $C_1 = C_1(C) \to 0$ as $C \to 0$. \hfill \Box

We want to prove the convergence of eigenvalues and eigenvectors if the linear operator is $A_\varepsilon$ plus a potential $V_\varepsilon$. We assume that the given potentials $V_\varepsilon: L_\varepsilon \to L_\varepsilon$, $\varepsilon \geq 0$, satisfy the following conditions:

(V1) There is a constant $C_\varepsilon > 0$, independent of $\varepsilon \geq 0$, such that $\| V_\varepsilon[u] \|_{L^2} \leq C_\varepsilon \| u \|_{L^2}$ for all $u \in L_\varepsilon$. 

(V2) For all $\varepsilon \geq 0$ is $V_{\varepsilon}$ symmetric.

(V3) If $[u_{\varepsilon}] \in L_{\varepsilon}$, $[u_{0}] \in L_{0}$, $\|[u_{\varepsilon}] - \Phi_{\varepsilon}^{T}[u_{0}]\|_{L_{\varepsilon}} \to 0$, as $\varepsilon \to 0$, then $\|[V_{\varepsilon}[u_{\varepsilon}] - \Phi_{\varepsilon}^{T}V_{0}[u_{0}]\|_{L_{0}} \to 0$, as $\varepsilon \to 0$.

Note that (V3) implies

(V3') If $[u_{\varepsilon}], [w_{\varepsilon}] \in L_{\varepsilon}$, $[u_{0}], [w_{0}] \in L_{0}$ and $\lim_{\varepsilon \to 0} \|[u_{\varepsilon}] - \Phi_{\varepsilon}^{T}[u_{0}]\|_{L_{\varepsilon}} = 0 = \lim_{\varepsilon \to 0} \|[w_{\varepsilon}] - \Phi_{\varepsilon}^{T}[w_{0}]\|_{L_{0}}$ then

\[ (V_{\varepsilon}[u_{\varepsilon}], w_{\varepsilon})_{L_{\varepsilon}} \to (V_{0}[u_{0}], [w_{0}])_{L_{0}}, \quad \text{as } \varepsilon \to 0. \]

For $\varepsilon \geq 0$ define a bilinear form $b_{\varepsilon} : H_{\varepsilon} \times H_{\varepsilon} \to H_{\varepsilon}$ by $b_{\varepsilon}([u], [v]) = a_{\varepsilon}([u], [v]) + (V_{\varepsilon}[u], [v])_{L_{\varepsilon}}$. In the same way as $a_{\varepsilon}$ this bilinear form $b_{\varepsilon}$ defines an operator $B_{\varepsilon} : D(B_{\varepsilon}) \subset H_{\varepsilon} \to H_{\varepsilon}$, $B_{\varepsilon}$ is selfadjoint, sectorial, and has compact resolvent.

There is a complete ONS $([u_{\varepsilon}^{b}, j])_{j}$ of $L_{\varepsilon}$ consisting of eigenvectors $[u_{\varepsilon}^{b}]$ of $B_{\varepsilon}$ with corresponding eigenvalues $\lambda_{\varepsilon}^{b, j}$. Without loss of generality we can assume these eigenvalues to be ordered $\lambda_{\varepsilon, 1}^{b} \leq \lambda_{\varepsilon, 2}^{b} \leq \ldots$

Note that $B_{\varepsilon} = A_{\varepsilon} + V_{\varepsilon}$, $D(B_{\varepsilon}) = D(A_{\varepsilon})$, $H_{\varepsilon}$ is still the fractional power space $\lambda_{\varepsilon}^{1/2}$, and $\text{dist}(\sigma(A_{\varepsilon}), \sigma(B_{\varepsilon})) \leq \|V_{\varepsilon}\|_{L_{\varepsilon}} \leq C_{V_{\varepsilon}}$ as in (V1) (for the inequality see e.g. Theorem 4.10 in [11, Chapter V]).

**Lemma 3.10.** Theorem 2.1 holds for $B_{\varepsilon}$. I.e. if $\varepsilon_{n} \to 0$, then $\lambda_{\varepsilon_{n}, l}^{b} \to \lambda_{0, l}^{b}$, for all $l \in \mathbb{N}$. There is a subsequence, called $\varepsilon_{n}$ too, and a complete ONS $([u_{n}^{b}, j])_{j}$ of $L_{0}$ consisting of eigenvectors belonging to $\lambda_{0, l}^{b}$ such that $\|[u_{\varepsilon_{n}, l}^{b}] - \Phi_{\varepsilon_{n}}^{T}[u_{l}]\|_{\varepsilon_{n}, l} \to 0$, as $n \to \infty$, for all $0 \leq d < 1$.

**Proof.** Let $\varepsilon_{n} \to 0$ and fix $0 \leq d < 1$. Since $\lambda_{\varepsilon, l}^{b} \to \lambda_{0, l}$, as $n \to \infty$, the remark above implies that $(\lambda_{\varepsilon_{n}, l}^{b})_{n}$ is bounded, for all $l \in \mathbb{N}$. Thus for $l$ fixed, we can take a subsequence, called $\varepsilon_{n}$ again, such that $\lambda_{\varepsilon_{n}, l}^{b} \to \mu_{l}$.

We have

\[ \|[u_{n, l}^{b}]\|_{H_{\varepsilon}}^{2} = a_{\varepsilon}([u_{n, l}^{b}], [u_{n, l}^{b}]) + \|[u_{n, l}^{b}]\|_{L_{\varepsilon}}^{2} \leq |\lambda_{n, l}^{b}| + C_{V_{\varepsilon}} + 1. \]

Thus (C10) (recall (2.4) to bound $| \cdot |_{\varepsilon, l}$) shows

\[ \varepsilon_{n} \sum_{j = K_{\varepsilon} + 1}^{K_{\varepsilon} + K_{N}} \|[u_{n, l,j}^{b}]\|_{L_{\varepsilon}^{2}(G_{n,j})}^{2} \to 0. \]

Also $[u_{n, l,j}^{b}]$ is bounded in $\| \cdot \|_{H^{1}(G_{n,j})}$, for all $j = 1, \ldots, K_{E}$. This in turn implies that taking again a subsequence — there are $u_{n,j} \in H^{1}(G_{j})$ and $u_{n,l,j} \to u_{n,j}$ weakly in $H^{1}$ and strongly in $L^{2}$. Thus $[u_{l}] = [u_{l, 1}, \ldots, u_{l, K_{E}}] \in H_{0}$ (see condition (C9)), $\|[u_{n,l}^{b}] - \Phi_{n}^{T}[u_{l}]\|_{L_{n}} \to 0$, and $1 = \|[u_{n,l}^{b}]\|_{L_{n}} \to \|[u_{l}]\|_{L_{n}}$.

Using Lemma 3.14 (and (V3')) we find for all $[u] \in H_{0}$, as $n \to \infty$

\[ \mu_{l}([u], [u])_{L_{0}} \to \lambda_{n, l}^{b}([u_{n,l}^{b}], [u_{n,l}^{b}])_{L_{n}} = a_{n}([u_{n,l}^{b}], [u_{n,l}^{b}]) + (V_{n}[u_{n,l}], [u_{n,l}])_{L_{n}} \to a_{0}([u], [u]) + (V_{0}[u], [u])_{L_{0}} = b_{0}([u], [u]) \]
For every \( \lambda \in C \) and by (2.1) there is a constant eigenvalue \( \lambda \) assume \( L \) of \( \{ u \} \) and (\( u \)) have above results not only for one \( l \) but for all \( l \in N \). That is we can assume

\[
\lim_{n \to \infty} \lambda^b_{n,l} = \mu, \quad \lim_{n \to \infty} \| u^b_{n,l} - \Phi_n^H[u] \|_{n,d} = 0, \quad \| u \|_{L_0} = 1
\]

and \( (\mu, [u]) \) is an eigenvector, value pair for \( B_0 \), for all \( l \in N \).

If \( l \neq k \), then as \( n \to \infty \),

\[
0 = ([u^b_{n,l}, [u^b_{n,k}])_{L_n} \to ([u], [u])_{L_0}
\]

and \( ([u]) \) is an ONS of \( L_0 \). Assume it is not complete, then there is a \( 0 \neq [u] \in L_0 \) such that \( ([u], [u])_{L_0} = 0 \) for all \( l \). Since there is a complete ONS of \( L_0 \) consisting entirely of eigenvectors of \( B_0 \), we can without loss of generality assume \([u]\) to be such an eigenvector, and in particular \([u] \in H_0 \).

Write \( \Phi^H_n[u] = \sum_{l \geq 1} \alpha_{n,l} [u^b_{n,l}] \) (recall that \( [u^b_{n,l}] \) is an eigenvector for the eigenvalue \( \lambda_{n,l} \) of \( A_n \)). Then for all \( l \)

\[
\alpha_{n,l} = \Phi^H_n[u], [u^b_{n,l}]_{L_n} \to ([u], [u])_{L_0} = 0, \quad \text{as} \quad n \to \infty,
\]

and by (2.1) there is a constant \( C_1 \) such that

\[
\sum_{l \geq 1} \lambda_{n,l} \alpha^2_{n,l} = a_n(\Phi^H_n[u], \Phi^H_n[u]) \leq C_1.
\]

For every \( \delta > 0 \) there are \( n_1, l_1 \in N \) such that for \( n \geq n_1, l \geq l_1 \) we have \( \lambda_{n,l} \geq 1/\delta \). For \( n \geq n_1 \) we get

\[
\sum_{l \geq l_1} \alpha^2_{n,l} \leq \frac{C_1}{\lambda_{n,l_1}} \leq \delta C_1
\]

and thus, as \( n \to \infty \)

\[
\| [u]\|_{L_0}^2 = \| \Phi^H_n[u] \|_{L_0}^2 \leq \sum_{l=1}^{l_1} \alpha^2_{n,l} + C_1 \delta.
\]

Hence \([u] = 0\), which cannot be, and \(([u])_l \) has to be complete.

The only thing we still have to show is \( \lambda^b_{n,l} \to \lambda^b_{0,l} \) for all \( l \) for the original sequence \( \varepsilon_n \). For this it is sufficient to show \( \mu_l = \lambda^b_{0,l} \) for all \( l \).

Assume this to be false. Then there is a \( l_2 \in N \) such that \( \mu_l = \lambda^b_{0,l}, l = 1, \ldots, l_2 - 1, \mu_{l_2} \neq \lambda^b_{0,l_2} \). Now if \( \mu_{l_2} < \lambda^b_{0,l_2} \), then \( \mu_{l_2} = \lambda^b_{0,l} \) for some \( l \in \{1, \ldots, l_2 - 1\} \) and \([u]_l \) is a linear combination of the first \( l_2 - 1 \) eigenvectors.
of $b_0$, i.e. of $[u_1, \ldots, u_{l_2-1}]$, which contradicts the orthogonality of $([u_l])$. If $\mu_l > \lambda_{0,l_2}$, then there is an eigenvector of $B_0$ for the eigenvalue $\lambda_{0,l_2}$ which is orthogonal to all $[u_l]$, which contradicts the completeness of $([u_l])$.

Hence the assumption is false and $\mu_l = \lambda_{0,l}$ for all $l$.

**Lemma 3.11.** Assume $0 \not\in \sigma(B_0)$. Then there are $\epsilon_0, C > 0$ such that for all $0 \leq \epsilon \leq \epsilon_0$ we have $0 \not\in \sigma(B_\epsilon)$ and $\|B_\epsilon^{-1}[u]\|_{H_\sigma} \leq C\|[u]\|_{L_\sigma}$, for all $[u] \in L_\sigma$. Also, if $[u_{\epsilon}] \in L_{\epsilon}$, $[w_{\epsilon}] \in L_0$, $\lim_{\epsilon \to 0} \|[u_{\epsilon}] - \Phi_{\epsilon}^L[w_{\epsilon}]\|_{L_\sigma} = 0$, then as $\epsilon \to 0$, for all $0 \leq d < 1$,

$$\|B_\epsilon^{-1}[u_{\epsilon}] - \Phi_{\epsilon}^H B_0^{-1}[w_{\epsilon}]\|_{\epsilon,d} \to 0.$$  

**Proof.** Fix $0 \leq d < 1$. $0 \not\in \sigma(B_\epsilon)$ for $0 \leq \epsilon \leq \epsilon_0$, for some $\epsilon_0 > 0$ follows directly from the convergence of the eigenvalues of $B_\epsilon$ to those of $B_0$ (Lemma 3.10).

For given $[u] \in L_\epsilon$ we can use the ONS $(u_{\epsilon}^b, l)$ to find $\|B_\epsilon^{-1}[u]\|_{L_\epsilon} \leq C_1\|[u]\|_{L_\epsilon}$, where $C_1 = C_1(\epsilon_0)$ is independent of $\epsilon$. Thus

$$\|B_\epsilon^{-1}[u]\|_{H_\sigma}^2 = (A_\epsilon B_\epsilon^{-1}[u], B_\epsilon^{-1}[u])_{L_\epsilon} + \|B_\epsilon^{-1}[u]\|_{L_\epsilon}^2 = ([u], B_\epsilon^{-1}[u])_{L_\epsilon} - (V_\epsilon B_\epsilon^{-1}[u], B_\epsilon^{-1}[u])_{L_\epsilon} + \|B_\epsilon^{-1}[u]\|_{L_\epsilon}^2 \leq C_2\|[u]\|_{L_\sigma}^2,$$

where $C_2$ is independent of $\epsilon$ (see (V1)).

Assume the convergence of the resolvents is not true. Then there is a sequence $\epsilon_n \to 0$, $\delta_1 > 0$, $[u_n] \in L_{\epsilon_n}$, $[w_0] \in L_0$ such that $\|B_{\epsilon_n}^{-1}[u_n] - \Phi_{\epsilon_n}^H B_0^{-1}[w_0]\|_{n,d} \geq \delta_1$ for all $n$ and $\|[u_n] - \Phi_{\epsilon_n}^L[w_{\epsilon_n}]\|_{L_\sigma} \to 0$.

Taking a subsequence, called $\epsilon_n$ too, by Lemma 3.10 we can assume $\|[u_{\epsilon_n}^b, l] - \Phi_{\epsilon_n}^H[u_{\epsilon_n}^b, l]\|_{n,d} \to 0$, $n \to \infty$, for all $l \in \mathbb{N}$.

Setting $[w_n] := B_{\epsilon_n}^{-1}[u_n]$, $[w_0] := B_0^{-1}[w_0]$, we see $\|[w_n]\|_{L_\sigma}$, $\|[w_0]\|_{L_\sigma}$ and $\|A_n[w_n]\|_{L_\sigma}$ are bounded. If $[u_n] = \sum_{l \geq 1} \alpha_{n,l}[u_{n,l}^b, l]$, $[w_0] = \sum_{l \geq 1} \alpha_{0,l}[u_{0,l}^b, l]$, then $\alpha_{n,l} \to \alpha_{0,l}$, for all $l$, as $n \to \infty$. Since for all $C > 0$ there is a $l_1 = l_1(C)$ such that $\lambda_{n,l} \geq \lambda_{0,l_1}$ for all $l \geq l_1$, $n \in \mathbb{N}$, we see as $n \to \infty$

$$\|[w_n] - \Phi_{\epsilon_n}^L[w_0]\|_{L_\sigma} \leq \left|\sum_{l \geq l_1} \frac{\alpha_{n,l} l_{\epsilon_n}^b}{\lambda_{n,l}^b} \right| \leq \left|\sum_{l \geq l_1} \frac{\alpha_{0,l} l_{\epsilon_n}^b}{\lambda_{0,l}^b} \right| \leq C\|[u_n]\|_{L_\sigma}/\lambda_{n,l_1}$$
\[ b_n([w_n], [w_n]) = ([u_n], [w_n])_{L_n} = \sum_{l=1}^{n_1} \alpha_{n,l}^2 \lambda_{n,l}^2 + \sum_{l \geq 1}^{\infty} \alpha_{n,l}^2 \lambda_{n,l}^2 \leq \|w_n\|_{L_n}/\lambda_{n,l}, \]
\[ \to \sum_{l \geq 1}^{\infty} \frac{\alpha_{n,l}^2}{\lambda_{n,l}^2} = ([u_0], [w_0])_{L_0} = b_0([w_0], [w_0]). \]

By (V3')
\[ (V_n[w_n], [w_n])_{L_n} \to (V_0[w_0], [w_0])_{L_0}, \quad a_n([w_n], [w_n]) \to a_0([w_n], [w_n]), \]
as \(n \to \infty\). We can apply Lemma 3.4 and get \([w_n] - \Phi_n^H[w_0]_{n,d} \to 0\). This is a contradiction, and the proof is complete. \(\square\)

4. Continuity

In this section we shall prove Theorem 1.1. That is we shall show that the family of attractors \(A_\varepsilon\) is lower-semi-continuous at \(\varepsilon = 0\). Theorem 2.2 then implies the continuity, i.e. Theorem 1.1.

Assume \(f\) satisfies (H2). We know already that there is a \(C_A > 0\) such that \(\|[u]\|_{L^\infty} \leq C_A\) for all \([u] \in A_\varepsilon\) and \(\varepsilon \geq 0\) (see Proposition 3.3). Eventually increasing \(C_A\) we can assume
\[ \frac{f(s)}{s} \leq -\frac{3}{4} \varepsilon, \quad |s| \geq C_A, \]
where \(\varepsilon\) is as in condition (H2).

There is a \(C^2\)-function \(g: \mathbb{R} \to \mathbb{R}\) which coincides with \(f\) on \(|s| \leq 2C_A\), satisfies
\[ g(s) \leq -\frac{1}{2} \varepsilon \quad \text{for } |s| \geq C_A, \quad g''(s) = 0 \quad \text{for } |s| \geq 3C_A, \]
and \(g\) still satisfies conditions (H1) and (H2'). Denote by \(g_\varepsilon\) the Nemitsky operator of \(g\) on \(H_\varepsilon, \varepsilon \geq 0\). The differential equations
\[ (4.1) \quad [u_t] = -A_\varepsilon[u] + g_\varepsilon([u]), \quad t > 0 \]
define semiflows \(\pi_\varepsilon\) on \(H_\varepsilon, \varepsilon \geq 0\). Theorem 2.2 still holds, thus all \(\pi_\varepsilon\) are global semiflows and \(\pi_\varepsilon\) converges to \(\pi_0\) in the sense of this theorem.

On the attractor \(A_\varepsilon\) the semiflows \(\pi_\varepsilon\) and \(\pi_0\) coincide. Also, for all \([u] \in A_\varepsilon, [v] \in L_\varepsilon\), we have \(D\pi_\varepsilon([u]) [v] = D\pi_0([u]) [v], \varepsilon \geq 0\). By Lemma 3.2(c) any \([u]\) which is a point of equilibrium of \(\pi_\varepsilon, \varepsilon > 0\), satisfies \(\|[u]\|_{L^\infty} \leq C_A\), that is \([u] \in A_\varepsilon\). In Theorem 1.1 the condition on the spectrum of \(Df_\varepsilon([u])\), \([u]\) a point of equilibrium for \(\pi_\varepsilon\), becomes simply the following:

The semiflow \(\pi_0\) has only finitely many points of equilibrium \([\tilde{u}_0]^0, \ldots, [\tilde{u}_0]^M_0]\) and 0 is not in the spectrum of the linear operators \(A_0 - Dg_0([\tilde{u}_0]^j)\text{id}: D(A_0) \to L_0\) for all \(j = 1, \ldots, M_0\).
In this section we will consider equation (4.1) and assume the condition above holds.

To simplify notation we will drop the tilde “~” in the notation of the semiflows and their points of equilibrium. That is we shall write \( \pi_\varepsilon \) and \([u^0_m]\) for \( \tilde{\pi}_\varepsilon \) and \([\tilde{u}^0_m]\).

As a first step we show that each \([u^0_m]\) is the limit of a point of equilibrium of \( \pi_\varepsilon \). Before we do this, we need some technical lemmas.

**Lemma 4.1.** Let \( p > 2 \), \( G \subset \mathbb{R}^{M+1} \) be open, bounded, and \( g_c \) the Nemitsky operator of \( g \) on \( G \). Let \( C_1 \geq |G| \). Then \( g_c : L^p(G) \rightarrow L^2(G) \) is \( C^1 \), \( D g_c(u)(v(z)) = g'(u(z))v(z), z \in G \), and there are \( \beta > 1 > \gamma > 0 \), \( C_2 > 0 \), \( \beta, \gamma \), \( C_2 \) independent of \( G \), \( C_2 = C_2(C_1) \), such that for all \( u, v, w \in L^p(G) \)

\[
\begin{align*}
\|g_c(u + v) - g_c(u) - D g_c(u)v\|_{L^2(G)} & \leq C_2 \|v\|^2_{L^2(G)}, \\
\|g_c(u + v) - g_c(u + w) - D g_c(u)(v - w)\|_{L^2(G)} & \leq C_2 \|v - w\|_{L^p(G)}\|v\|^2_{L^2(G)} + \|w\|^2_{L^2(G)} \gamma, \\
\|g_c(u + v) - g_c(u)\|_{L^2(G)} & \leq C_2 \|v\|_{L^2(G)} \quad \text{for all } u, v \in L^2(G).
\end{align*}
\]

**Proof.** Note that \( g''(s) = 0 \) for \( |s| \) big enough shows that \( g' \) and \( g'' \) are bounded. Thus indeed \( g_c : L^2(G) \rightarrow L^2(G) \) and (4.4) holds.

Let \( u, v, w \in L^p(G) \). Then

\[
\begin{align*}
\left( \int_G (g_c(u + v) - g_c(u + w) - D g_c(u)(v - w))^2 \right)^{(p-2)/p} & \leq \|v - w\|^2_{L^p(G)} \left( \int_G (g'(u + w + \xi) - g'(u))^2 \right)^{(p-2)/p} \\
& \leq C_2 \|v - w\|^2_{L^p(G)} \left( \int_G |g''(\xi)|^2 \xi^2 \right)^{(p-2)/p} \\
& \leq C_3 \|v - w\|^2_{L^p(G)} \|w + \xi\|^2_{L^2(G)},
\end{align*}
\]

where \( C_2 \), \( C_3 \) depend only on \( g \) and \( p \), \( \varepsilon = \xi(z) \) is between \( 0 \) and \( v(z) - w(z) \), \( z \in \pi_\varepsilon \).

Choose \( w = 0 \), then \( \|\xi\|^2_{L^2(G)} \leq \|v\|^2_{L^2(G)}. \) Thus \( g_c : L^p(G) \rightarrow L^2(G) \) is Fréchet-differentiable and (4.2) holds with \( \beta = 2 - 2/p \).

If \( w \) is arbitrary, then \( \|w + \xi\|^2_{L^2(G)} \leq \|v\|^2_{L^2(G)} + \|w\|^2_{L^2(G)} \) and (4.3) holds.

Now

\[
\begin{align*}
\int_G ((D g_c(u + v) - D g_c(u))w)^2 \, dz & \leq \|w\|^2_{L^p(G)} \left( \int_G |g'(u + v) - g'(u)|^{2p/(p-2)} \right)^{(p-2)/p}.
\end{align*}
\]
As \( \|v\|_{L^p(G)} \to 0 \), \( g'(u(z)+v(z))-g'(u(z)) \to 0 \) for a.a. \( z \in G \). With the Lebesgue dominated convergence the integral on the right-hand-side above tends to 0 and \( Dg_G: L^p(G) \times L^p(G) \to L^2(G) \) is continuous.

As a direct consequence of condition (C10) of the second section we have

**Lemma 4.2.** If

\[
C(\varepsilon) := \sup \left( \frac{\varepsilon \sum_{j=K_\varepsilon+1}^{K_\varepsilon+K_N} \|u_j\|_{L^2(G, \varepsilon, j)}^2}{\|u\|_{H_\varepsilon}} : 0 \neq [u] \in H_\varepsilon \right),
\]

then \( C(\varepsilon) \to 0 \), as \( \varepsilon \to 0 \).

**Lemma 4.3.** \( g_\varepsilon: H_\varepsilon \to L_\varepsilon \) is \( C^1 \), \( Dg_\varepsilon([u])\langle v \rangle = g'(u_j(z))v_j(z) \) for \( z \in G_j \) or \( z \in G_{\varepsilon, j} \), resp. and all possible \( \varepsilon \geq 0 \). Also the following hold:

(a) Let \( \varepsilon \geq 0 \). \( g_\varepsilon: L_\varepsilon \to L_\varepsilon \) and \( Dg_\varepsilon: L_\varepsilon \times L_\varepsilon \to L_\varepsilon \) are well defined. For each \([u] \in L_\varepsilon\) is \( Dg_\varepsilon([u]): L_\varepsilon \to L_\varepsilon \) a symmetric operator. There is a \( C_1 \) > 0, independent of \( \varepsilon \), such that

\[
\|Dg_\varepsilon([u])\langle v \rangle\|_{L_\varepsilon} \leq C_1\|\langle u \rangle\|_{L_\varepsilon} \quad \text{for all } [u], [v] \in L_\varepsilon,
\]

\[
\|g_\varepsilon([u] + [v]) - g_\varepsilon([u])\|_{L_\varepsilon} \leq C_1\|\langle u \rangle\|_{L_\varepsilon} \quad \text{for all } [u], [v] \in L_\varepsilon.
\]

(b) Let \( 0 \leq d < 1 \). There are \( \beta > 1, \gamma, C_2 = C_2(d) > 0 \), all independent of \( \varepsilon \geq 0 \), and \( C_3(\varepsilon) > 0 \), \( C_3(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \), \( C_3(0) = 0 \), such that, for all \([u], [v], [w] \in H_\varepsilon\),

\[
\|g_\varepsilon([u] + [v]) - g_\varepsilon([u]) - Dg_\varepsilon([u])\langle [v] - [w] \rangle\|_{L_\varepsilon} \leq C_2\|\langle v \rangle\|_{L_\varepsilon}^\beta + C_3(\varepsilon)\|\langle v \rangle\|_{H_\varepsilon},
\]

\[
\|g_\varepsilon([u] + [v]) - g_\varepsilon([u] + [w]) - Dg_\varepsilon([u])\langle [v] - [w] \rangle\|_{L_\varepsilon} \leq C_2\|\langle v \rangle - \langle w \rangle\|_{H_\varepsilon}.
\]

(c) Let \( \varepsilon \geq 0 \). For all \( \tilde{C} > 0 \) there is a \( C_4(\varepsilon) = C_4(\varepsilon, \tilde{C}) > 0 \) such that

\[
\|g_\varepsilon([u] + [v]) - g_\varepsilon([u] + [w]) - Dg_\varepsilon([u])\langle [v] - [w] \rangle\|_{L_\varepsilon} \leq \tilde{C}\|\langle v \rangle - [w]\|_{H_\varepsilon},
\]

for all \([u], [v], [w] \in H_\varepsilon\), \( \|\langle v \rangle\|_{H_\varepsilon}, \|\langle w \rangle\|_{H_\varepsilon} \leq C_4(\varepsilon) \).

(d) If \([u_\varepsilon], [v_\varepsilon] \in L_\varepsilon\), \([u_0], [v_0] \in L_0\) and

\[
\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L_\varepsilon} = \lim_{\varepsilon \to 0} \|v_\varepsilon\|_{L_\varepsilon} = \lim_{\varepsilon \to 0} \|\Phi^L_\varepsilon[u_0]\|_{L_\varepsilon} = 0,
\]

then

\[
\lim_{\varepsilon \to 0} \|g_\varepsilon([u_\varepsilon]) - \Phi^L_\varepsilon g_0([u_0])\|_{L_\varepsilon} = \lim_{\varepsilon \to 0} \|Dg_\varepsilon([u_\varepsilon])\langle v_\varepsilon \rangle - \Phi^L_\varepsilon Dg_0([u_0])\langle v_0 \rangle\|_{L_\varepsilon} = 0.
\]
Proof. Let \( 2 < p \leq p^* = (M + 1)/(M - 1) \). All \( G_j \) and \( \Omega_\varepsilon \) are Lipschitz, hence \( H^1(G_j) \subset L^p(G_j) \), \( H^1(\Omega_\varepsilon) \subset L^p(\Omega_\varepsilon) \).

Thus for \( \varepsilon > 0 \) Lemma 4.1 implies directly \( g_\varepsilon : H_\varepsilon \to L_\varepsilon \) is \( C^1 \). For \( \varepsilon = 0 \) we use the same argument for each \( G_j \) separately to get the same conclusion.

The formula for \( Dg_\varepsilon \) obviously holds. The boundedness of \( g' \), \( \Omega_\varepsilon \) and all \( G_j \) imply that \( g_\varepsilon : L_\varepsilon \to L_\varepsilon, Dg_\varepsilon : L_\varepsilon \times L_\varepsilon \to L_\varepsilon \) are well defined for all \( \varepsilon \geq 0 \) and (a) is true.

Now assume the situation in (d). For \( j \in \{1, \ldots, K_E \} \) we have \( u_{\varepsilon,j} \to u_{0,j}, v_{\varepsilon,j} \to v_{0,j} \) in \( L^2(G_j) \). Hence with Lemma 4.1 follow

\[
\sum_{j=1}^{K_E} \int_{G_j} (g(u_{\varepsilon,j}) - g(u_{0,j}))^2 d\lambda_{\varepsilon,j} \to 0,
\]

\[
\sum_{j=1}^{K_E} \int_{G_j} (g'(u_{\varepsilon,j})v_{\varepsilon,j} - g'(u_{0,j})v_{0,j})^2 d\lambda_{\varepsilon,j} \leq C_6 \sum_{j=1}^{K_E} \left\| g'(u_{\varepsilon,j})(v_{\varepsilon,j} - v_{0,j}) \right\|_{L^2(G_j)}^2 + \left\| (g'(u_{\varepsilon,j}) - g'(u_{0,j}))v_{0,j} \right\|_{L^2(G_j)}^2 \to 0.
\]

For \( j > K_E \) and \( \varepsilon > 0 \) we have \( (\Phi^j_{\varepsilon}[u])_j = 0 \) for all \( [u] \in L_\varepsilon \). Hence

\[
\lim_{\varepsilon \to 0} \sum_{j=1}^{K_{E+K_N}} \|u_{\varepsilon,j}\|_{L^2(G_{\varepsilon,j})}^2 = \lim_{\varepsilon \to 0} \sum_{j=1}^{K_{E+K_N}} \|v_{\varepsilon,j}\|_{L^2(G_{\varepsilon,j})}^2 = 0
\]

and by inequality (4.4) there is a \( C_6 \) such that

\[
\varepsilon \sum_{j=1}^{K_{E+K_N}} \left\| g(u_{\varepsilon,j}) \right\|_{L^2(G_{\varepsilon,j})}^2 \leq 2\varepsilon \sum_{j=1}^{K_{E+K_N}} (C_6 \left\| u_{\varepsilon,j} \right\|_{L^2(G_{\varepsilon,j})}^2 + \left\| v_{\varepsilon,j} \right\|_{L^2(G_{\varepsilon,j})}^2) \to 0,
\]

\[
\varepsilon \sum_{j=1}^{K_{E+K_N}} \left\| Dg(u_{\varepsilon,j})v_{\varepsilon,j} \right\|_{L^2(G_{\varepsilon,j})}^2 \leq C_6 \varepsilon \sum_{j=1}^{K_{E+K_N}} \left\| v_{\varepsilon,j} \right\|_{L^2(G_{\varepsilon,j})}^2 \to 0,
\]

as \( \varepsilon \to 0 \). This proves (d).

To prove (b) assume for a moment \( j \in \{1, \ldots, K_E \} \). By Lemma 4.1 there are \( \beta > 1, \gamma, C_7 > 0, \) all independent of \( \varepsilon \geq 0 \), such that for all \( [u], [v], [w] \in H_\varepsilon \), \( \varepsilon \geq 0 \),

\[
\left\| g(u_j + v_j) - g(u_j) - Dg(u_j)v_j \right\|_{L^2(G_j)}^2 \leq C_7 \left\| v_j \right\|_{L^p(G_j)}^{2\beta} \leq C_8 \left\| v_j \right\|_{H^1(G_j)}^{2\beta} \leq C_9 \left\| v \right\|_{H_\varepsilon}^{2\beta},
\]
Thus the second inequality in (b) holds, and choosing 

$$\varepsilon_0 > 0$$

If \( j > K_E, \varepsilon > 0 \), let \( C_1(\varepsilon) \to 0 \) be the constant from Lemma 4.2. Then

$$\varepsilon\|g(u_j + v_j) - g(u_j + w_j) - Dg(u_j)(v_j - w_j)\|_{L^2(G_{\varepsilon})} \leq C_{10}\|v_j - w_j\|^2_{\varepsilon,d}(\|v_j\|^2_{L^2\varepsilon} + \|u_j\|^2_{L^2\varepsilon})^2\gamma.$$ 

If \( j > K_E, \varepsilon > 0 \), let \( C_1(\varepsilon) \to 0 \) be the constant from Lemma 4.2. Then

$$\varepsilon\|g(u_j + v_j) - g(u_j + w_j) - Dg(u_j)(v_j - w_j)\|_{L^2(G_{\varepsilon,j})} \leq C_{11}(\varepsilon)\|v_j - w_j\|^2_{L^2\varepsilon,j} \leq C_{11}(\varepsilon)\|v_j - w_j\|^2_{L^2\varepsilon,j}.$$ 

Thus the second inequality in (b) holds, and choosing \( |w| = 0 \) is the first one too.

Analogously, either using Lemma 4.1 directly for \( \Omega^\varepsilon \), \( \varepsilon > 0 \), or for each \( G_j \) separately \( \varepsilon = 0 \), one proves (c).

Now we prove the continuity — in a certain sense — of the equilibrium points of \( \pi_\varepsilon \). Recall that \( \{u_0^\varepsilon, \ldots, u_{M_0}^\varepsilon\} \) are the points of equilibrium of \( \pi_\varepsilon \).

**Lemma 4.4.** Fix \( m \in \{1, \ldots, M_0\} \) and \( 0 \leq d < 1 \). There are \( \varepsilon_0, C > 0 \) and for all \( 0 \leq \varepsilon \leq \varepsilon_0 \) there are \( [u_0^\varepsilon, \ldots, u_{M_0}^\varepsilon] \in D(A_\varepsilon) \) such that \( \|\|u_0^\varepsilon\|\|_{L^2\varepsilon} \leq C \), \( A_\varepsilon[u_0^\varepsilon] = g_\varepsilon([u_0^\varepsilon]) \) and \( \|\|u_0^\varepsilon - \Phi_\varepsilon[u_0^\varepsilon]\|\|_{\varepsilon,d} \to 0 \), as \( \varepsilon \to 0 \).

**Proof.** Recall that \( \Phi_0^\varepsilon = \text{id} \) on \( H_\varepsilon \). For \( \varepsilon \geq 0 \) set \( V_\varepsilon = V_\varepsilon(m): L_\varepsilon \to L_\varepsilon \) by

$$V_\varepsilon[u] := Dg_\varepsilon(\Phi_\varepsilon^H[u_0^\varepsilon])(u).$$

The potentials \( V_\varepsilon \) satisfy conditions (V1)–(V3) of section three. In particular there is a linear operator \( B_\varepsilon : = A_\varepsilon - V_\varepsilon : D(A_\varepsilon) \to L_\varepsilon \) which is selfadjoint, sectorial, has compact resolvent, and there are complete ONS \( \{[u_0^\varepsilon, \ldots, u_{M_0}^\varepsilon]\} \) of \( L_\varepsilon \) consisting of eigenvectors with corresponding eigenvalues \( \lambda_\varepsilon^b \leq \lambda_\varepsilon^{d} \leq \ldots \).

By Lemma 3.10 the \( \lambda_\varepsilon^b \to \lambda_0^b \), \( \varepsilon \to 0 \), for all \( l \), and by Lemma 3.11 the assumption \( 0 \notin \sigma(B_0) \) shows \( 0 \notin \sigma(B_\varepsilon) \), \( 0 \leq \varepsilon \leq \varepsilon_0 \) for some \( \varepsilon_0 > 0 \).

For \( \varepsilon \geq 0 \) define \( T_\varepsilon = T_\varepsilon\varepsilon,m: H_\varepsilon \to H_\varepsilon \) by

$$T_\varepsilon[u] := B_\varepsilon^{-1}(g_\varepsilon([u]) - V_\varepsilon[u]).$$

We shall show that \( T_\varepsilon \) has a fixed point, which will be \( [u_0^\varepsilon, \ldots, u_{M_0}^\varepsilon] \).

Using Lemmas 3.11 and 4.3(b) there are constants \( C_1, \gamma > 0 \), independent of \( \varepsilon \), and \( C_2(\varepsilon) \to 0 = C_2(0) \) such that for \( [v], [w] \in L_\varepsilon \)

$$\|T_\varepsilon[v] - T_\varepsilon[w]\|_{H_\varepsilon} \leq C_1\|[v] - [w]\|_{\varepsilon,d}(\||v| - \Phi_\varepsilon[u_0^\varepsilon]|^2_{H_\varepsilon} + \|\|w - \Phi_\varepsilon[u_0^\varepsilon]\|_{H_\varepsilon})^\gamma + C_2(\varepsilon)\|[v] - [w]\|_{H_\varepsilon}.$$ 

Since there is a \( C_3 \) such that \( \|\|L_\varepsilon \leq C_3\|\|_{\varepsilon,d} \leq (C_3)^2\|\|_{H_\varepsilon} \), there is a \( C_4 \) such that for \( [v], [w] \in \{[u] \in H_\varepsilon : [u] - \Phi_\varepsilon[u_0^\varepsilon]|_{\varepsilon,d} \leq C_4\} \)

$$|T_\varepsilon[v] - T_\varepsilon[w]|_{\varepsilon,d} \leq \frac{1}{2}\|[v] - [w]\|_{\varepsilon,d} + C_2(\varepsilon)C_3\|[v] - [w]\|_{H_\varepsilon},$$

(4.5)

$$\|T_\varepsilon[v] - T_\varepsilon[w]\|_{H_\varepsilon} \leq \frac{1}{2}\|[v] - [w]\|_{\varepsilon,d} + C_2(\varepsilon)\|[v] - [w]\|_{H_\varepsilon}.$$
Recall that $\Phi^H_\varepsilon: H_0 \to H_\varepsilon$ is a bounded operator, the bound being independent of $\varepsilon$. Thus by Lemma 4.3(a) $\|g_\varepsilon(\Phi^H_\varepsilon[u_m^0])\|_{L_\varepsilon}$ is bounded and by Lemma 3.11 there is a $C_5 > 4C_4$ such that $\|\Phi^H_\varepsilon[u_m^0]\|_{H_\varepsilon}, \|T_\varepsilon \Phi^H_\varepsilon[u_m^0]\|_{H_\varepsilon} \leq 1/2C_5$.

Eventually decreasing $\varepsilon_0 > 0$ so that $C_2(\varepsilon) \leq (1/4)\min(1, C_4/C_5)$, we get for $[v]$ as above

$$\|T_\varepsilon[v]\|_{H_\varepsilon} \leq \frac{3}{4} C_4 + \frac{1}{2} C_5 + \frac{\|v\|_{H_\varepsilon}}{4} < \frac{3}{4} C_5 + \frac{1}{4} \|v\|_{H_\varepsilon}.$$ 

Now $T_0[u_m^0] = [u_m^0]$ and

$$\|g_\varepsilon(\Phi^H_\varepsilon[u_m^0]) - Dg_\varepsilon(\Phi^H_\varepsilon[u_m^0]) \Phi^H_\varepsilon[u_m^0] - \Phi^L_\varepsilon(g_\varepsilon(\Phi^H_\varepsilon)[u_m^0]) - Dg_\varepsilon(\Phi^H_\varepsilon)[u_m^0]\|_{L_\varepsilon} \to 0$$

by Lemma 4.3(d), thus $|T_\varepsilon \Phi^H_\varepsilon[u_m^0] - \Phi^H_\varepsilon T_0[u_m^0]|_{\varepsilon,d} \to 0$ by Lemma 3.11, as $\varepsilon \to 0$.

Putting all together, and eventually decreasing $\varepsilon_0 > 0$ further, for $0 < \varepsilon \leq \varepsilon_0$

$$T_\varepsilon:\{[u] \in H_\varepsilon : \|u\|_{H_\varepsilon} \leq C_5, \quad \|u - \Phi^H_\varepsilon[u_m^0]\|_{\varepsilon,d} \leq C_4\}$$

$$\rightarrow \{[v] \in H_\varepsilon : \|v\|_{H_\varepsilon} \leq C_5, \quad \|v - \Phi^H_\varepsilon[u_m^0]\|_{\varepsilon,d} \leq C_4\}.$$ 

$A_\varepsilon$ has compact resolvent, hence $D(A_\varepsilon) \subset H_\varepsilon$ compactly and $T_\varepsilon: H_\varepsilon \to H_\varepsilon$ is completely continuous. By the Schauder fixed-point theorem $T_\varepsilon$ has a fixed-point, say $[u_m^\varepsilon]$ and $\|u_m^\varepsilon - \Phi^H_\varepsilon[u_m^0]\|_{\varepsilon,d} \leq C_4$. Since we can choose $C_4$ arbitrarily small (and decrease $\varepsilon_0$ with $C_4$), we can assume $\|u_m^\varepsilon - \Phi^H_\varepsilon[u_m^0]\|_{\varepsilon,d} \to 0$, as $\varepsilon \to 0$.

Obviously $[u_m^\varepsilon] \in D(A_\varepsilon)$ and $g([u_m^\varepsilon]) = A_\varepsilon[u_m^\varepsilon]$. By Lemma 3.2(c)

$$\|[u_m^\varepsilon]\|_{L^\infty} \leq C_6.$$ 

Now we can show that the points of equilibrium depend continuously on $\varepsilon$ at $\varepsilon = 0$.

**Lemma 4.5.** The family of points of equilibrium

$$E_\varepsilon := \{[u] \in D(A_\varepsilon) : A_\varepsilon[u] = g_\varepsilon([u])\}$$

is continuous at $\varepsilon = 0$, i.e. for $0 \leq d < 1$

$$\lim_{\varepsilon \to 0} \text{dist}_{\varepsilon,d}(E_\varepsilon, E_0) = 0,$$

where $\text{dist}_{\varepsilon,d}$ is defined in Theorem 1.1.

**Proof.** Fix $0 \leq d < 1$. We have to show

$$\lim_{\varepsilon \to 0} \sup_{[u] \in E_\varepsilon} \inf_{[v] \in E_0} \|[u] - \Phi^H_\varepsilon[v]\|_{\varepsilon,d} = \lim_{\varepsilon \to 0} \sup_{[v] \in E_0} \inf_{[u] \in E_\varepsilon} \|[u] - \Phi^H_\varepsilon[v]\|_{\varepsilon,d} = 0.$$ 

Assume the first limit is not 0. Then there is a sequence $\varepsilon_n \to 0$, $\delta > 0$, $[u_n] \in E_n$ such that

$$\inf_{[v] \in E_0} \|[u_n] - \Phi^H_\varepsilon[v]\|_{n,d} \geq \delta$$

for all $n$.
By Lemma 3.8 $\mathcal{A}_\varepsilon$ are bounded in $|| \cdot ||_{H_\varepsilon}$ uniformly in $\varepsilon$, hence, taking a subsequence, by (C9) there is a $[u_0] \in H_0$ and by (C10) $||[u_n] - \Phi_n[u_0]||_{L_n} \to 0$ as $n \to \infty$.

By Theorem 2.2, for all $t > 0$,

$$||u_n - \Phi_n[u_0]|_{n,d} = ||u_n|_{n,d} - \Phi_n[u_0]|_{n,d} \to 0$$

as $n \to \infty$. Hence $[u_0]$ is a point of equilibrium for $\pi_0$, that is $[u_0] \in \mathcal{E}_0$, and we have a contradiction.

Assume now the second limit in (4.6) is not 0. Then there is a sequence $\varepsilon_n \to 0$, $[v_n] \in \mathcal{E}_0$, and $\delta > 0$ such that

$$\inf_{[u] \in \mathcal{E}_n} ||[u] - \Phi_n[u_n]|_{n,d} \geq \delta$$

for all $n$. By assumption $\mathcal{E}_0$ is finite, hence taking a subsequence we can without loss of generality assume $[v_n] = [u_0^n]$ for all $n$.

By Lemma 4.4 for $\varepsilon$ sufficiently small there are $[u_0^n] \in \mathcal{E}_\varepsilon$ and $||u_0^n - \Phi_{\varepsilon}[u_0^n]|_{\varepsilon,d} \to 0$, as $\varepsilon \to 0$. This is a contradiction, and the lemma has been proven.

Now we prove that the family of attractors is lower semicontinuous at $\varepsilon = 0$. We do this by essentially proving the continuity of the unstable manifolds for the points of equilibrium $[u_0^n]$ of $\pi_\varepsilon$. Since all the semiflows are gradient-like, any $[u_0] \in \mathcal{A}_0 \setminus \mathcal{E}_0$ has to be in the unstable manifold of some $[u_0^n] \in [E]_0$. Thus the convergence of the unstable manifolds allows to get $[u_\varepsilon] \in [A]_\varepsilon$ which converge to the given $[u_0]$.

We look first at what happens around a given point of equilibrium $[u_0^n] \in \mathcal{E}_0$, $m \in \{1, \ldots, M_0\}$. To simplify notations, we drop in what follows the index “$m$”.

Set $[u^n] := [u_0^n]$ and let $0 \leq d < 1$, $\varepsilon_n \to 0$. By Lemma 4.5 there are $[u^n] \in \mathcal{E}_{\varepsilon_n}$ and $||u^n - \Phi_{\varepsilon}[u_0^n]|_{\varepsilon,d} \to 0$, as $n \to \infty$. Set $B_n: D(A_{\varepsilon_n}) \to L_{\varepsilon_n}$, $B_n[u] := A_{\varepsilon_n}[u] - Dg_{\varepsilon_n}([u^n])[u]$. Define $B_0: D(A_0) \to L_0$ by $B_0[u] := A_0[u] - Dg_0([u^n])[u]$. (Note that $B_n$ is not the operator $B_\varepsilon$ of the proof of Lemma 4.4.)

We shall in what follows abuse notation and include the limit case $\varepsilon = 0$ by writing something is defined (or holds) for all $n \geq 0$. E.g. we would say $B_n[u] := A_n[u] - Dg_n([u^n])[u]$, $n \geq 0$, to define the operators $B_n$, $B_0$ above.

Set $V_n[u] := -Dg_n([u^n])[u]$ for $n \geq 0$. (Again note that $V_n$ is not the $V_\varepsilon$ of the proof of Lemma 4.4.) Then by Lemma 4.3(d) the potentials $V_n$ satisfy conditions (V1)–(V3) of Section 3, and Lemmas 3.10 and 3.11 hold. In particular the operators $B_n$ have all the properties stated in Section 3.
Thus, eventually taking a subsequence, we have eigenvalue, eigenvector pairs $(\lambda^b_{n,l}, [u^b_{n,l}])$ of $B_n$, $\lambda^b_{n,1} \leq \lambda^b_{n,2} \leq \ldots$, $(u^b_{n,l})_l$ is a complete ONS of $L_n$, and
\[
\lim_{n \to \infty} \lambda^b_{n,l} - \lambda^b_{n,1} = \lim_{n \to \infty} \|u^b_{n,l} - \Phi^H_{\varepsilon_n} [u^b_{n,1}]\| = 0,
\]
for all $0 \leq d < 1$, $l \in \mathbb{N}$.

Assume $\pi_0$ has an unstable manifold at $[u^0]$. Then by Lemma 3.11 there is a $t_1 \geq 1$ such that (eventually taking again a subsequence) $\lambda^b_{n,t_1} < 0 < \lambda^b_{n,t_1+1}$, for all $n \geq 0$. Fix $C_B > 0$ such that
\[
\frac{1}{2} \lambda^b_{n,t_1} < -C_B \quad \text{for all } n \geq 0.
\]
Define for $n \geq 0$
\[
W_n := \left\{ [u] \in H_n : [u] = \sum_{l=1}^{t_1} \alpha_l [u^b_{n,l}], \alpha_l \in \mathbb{R} \right\},
\]
\[
W_n^\perp := \left\{ [u] \in L_n : ([u], [w])_{L_n} = 0 \text{ for all } [w] \in W_n \right\},
\]
\[
P_n : L_n \to W_n \text{ orthogonal projection }, \quad Q_n := \text{id} - P_n ; L_n \to W_n^\perp,
\]
\[
h_n : H_n \to L_n, \quad h_n([u]) := g_n([u] + [u^u]) - g_n([u^u]) - Dg_n([u^u])[u],
\]
\[
B_{1,n} := B_n|_{W_n} : W_n \to W_n, \quad B_{2,n} := B_n|_{D(A_n) \cap W_n^\perp} : D(A_n) \cap W_n^\perp \to W_n^\perp.
\]

We shall need the space of functions on $[-\infty, 0]$ which decrease at least with $e^{C_B t}$ as $t \to -\infty$. For $\sigma : [-\infty, 0] \to H_n$, $n \geq 0$, define
\[
\|\sigma\|_{H_n} := \sup_{t \leq 0} (e^{C_B t} ||\sigma(t)||_{H_n}),
\]
\[
BH_n := \{ \sigma : [-\infty, 0] \to H_n : ||\sigma||_{H_n} < \infty \}.
\]
$BH_n$ with $\| \cdot \|_{H_n}$ is a Banach space for all $n \geq 0$.

Note that $[u(t)]$ satisfies equation (4.1) (with $\varepsilon = \varepsilon_n$ or $\varepsilon = 0$) if and only if $[v] = [v(t)] := [u(t)] - [u^u]$ satisfies
\[
(4.7) \quad [v]_1 = -B_n[v] + h_n([v]).
\]

We construct the unstable manifold via a contraction map on the space of functions with exponential growth (as $t \to -\infty$; see e.g. [16], [5], [15]). For this we need the following well known result we state without proof.

**Lemma 4.6.** Let $n \geq 0$ and $\sigma : [-\infty, 0] \to H_n$. $\sigma$ is a solution of equation (4.7) and $\|\sigma(t)\|_{H_n} \to 0$ as $t \to -\infty$ if and only if $\sigma \in BH_n$ and
\[
\sigma(t) = e^{-B_{1,n} t} P_n \sigma(0) + \int_0^t e^{-B_{1,n}(t-s)} P_n h_n(\sigma(s)) \, ds
\]
\[
+ \int_{-\infty}^t e^{-B_{2,n}(t-s)} Q_n h_n(\sigma(s)) \, ds.
\]
A list of some properties of $h_n$ follows, the proof is a simple application of Lemma 4.3.

**Lemma 4.7.**

(a) $h_n(0) = 0 = Dh_n(0)$ and $h_n$ is $C^1$.

(b) There is a constant $C_1 > 0$, independent of $n$, such that

$$
\|h_n([u] + [v]) - h_n([u])\|_{\mathcal{L}_{\varepsilon_n}} \leq C_1\|v\|_{\mathcal{L}_{\varepsilon_n}} \text{ for all } [u], [v] \in L_{\varepsilon_n}, \ n \geq 0.
$$

(c) For all $\tilde{C} > 0$ there are $C_2 = C_2(\tilde{C}) > 0$, independent of $n$, and $C_3(n) > 0$, independent of $\tilde{C}$, $C_3(n) \to 0$ as $n \to \infty$, $C_3(0) = 0$, such that for all $n \geq 0$, $[u], [v] \in H_{\varepsilon_n}$, $\|u\|_{\varepsilon_n,d}, \|v\|_{\varepsilon_n,d} \leq C_2$

$$
\|h_n([u] + [v]) - h_n([u])\|_{\mathcal{L}_{\varepsilon_n}} \leq \tilde{C}\|v\|_{\varepsilon_n,d} + C_3(n)\|v\|_{\mathcal{L}_{\varepsilon_n}}.
$$

(d) If $[u_n] \in L_{\varepsilon_n}$, $[u_0] \in L_0$, $\lim_{n \to \infty} \|u_n\| - \Phi_{\varepsilon_n}^n[u_0]\|_{\mathcal{L}_{\varepsilon_n}} = 0$, then

$$
\|h_n([u_n]) - \Phi_{\varepsilon_n}^n h_0([u_0])\|_{\mathcal{L}_{\varepsilon_n}} \to 0 \quad n \to \infty.
$$

The fixed points of the maps $\Psi_n$ we define in the following lemma define the unstable manifold near to a point of equilibrium $[u^0]$.

**Lemma 4.8.** Recall that $l_1$ is the index of the last negative eigenvalue of $B_n$. For $\xi \in \mathbb{R}^{l_1}$, $\sigma \in BH_n$, $t \leq 0$ define

$$
\Psi_n(\xi, \sigma)(t) := e^{-B_1,n t} \sum_{i=1}^{l_1} \xi_i[u_{n,i}^0] + \int_0^t e^{-B_1,n(t-s)} P_n h_n(\sigma(s)) \, ds
$$

$$
+ \int_{-\infty}^t e^{-B_2,n(t-s)} Q_n h_n(\sigma(s)) \, ds.
$$

Then $\Psi_n : \mathbb{R}^{l_1} \times BH_n \to BH_n$ is continuous and $\Psi_n(\xi, \cdot) : BH_n \to BH_n$ is completely continuous for each $\xi \in \mathbb{R}^{l_1}$, $n \geq 0$.

**Proof.** Set

$$
\|\sigma\|_{L_n} := \sup_{t \leq 0} (e^{-C_n t} \|\sigma(t)\|_{L_n}),
$$

$$
BL_n := \{\sigma : [-\infty, 0] \to L_n : \|\sigma\|_{L_n} < \infty\}.
$$

By Lemma 4.7(b) for $\sigma \in BL_n$ and $s \leq 0$

$$
\|h_n(\sigma(s))\|_{L_n} \leq C_1\|\sigma(s)\|_{L_n} \leq C_1 e^{C_n s}\|\sigma\|_{L_n}
$$
for some constant $C_1 > 0$. There is a $C_2 > 0$ such that $\|u\|^2_{H_n} \leq b_n([u], [u]) + C_2\|u\|^2_{L_n}$. We get for $t \leq 0$

\[
\begin{align*}
(4.8) \quad e^{-C_2 t} \|\Psi_n(\xi, \sigma)(t)\|_{H_n} & \leq C_4(\|\xi\| + \|\sigma\|_{L_n}) \\
& + \int_{t-(\lambda_{n,1} + C_2)^{-1/2}}^{t} \frac{e^{-C_2 s} \|h_n(\sigma(s))\|_{L_n}}{\sqrt{2(t-s)}} ds \\
& + \int_{-\infty}^{t-(\lambda_{n,1} + C_2)^{-1/2}} (\lambda_{n,1} + C_2)^{1/2} e^{-C_2 (t-s)} \|h_n(\sigma(s))\|_{L_n} ds \\
& \leq C_4(\|\xi\| + \|\sigma\|_{L_n}).
\end{align*}
\]

Hence $\Psi_n(\xi, \cdot) : BL_n \to BH_n$ maps bounded sets into bounded sets. In a completely analogous way one shows that if $\|\cdot\|_{\alpha,n}$ is the norm of the fractional power space $X^{\alpha}_n$ of $B_n$, $1/2 < \alpha < 1$, then

\[
(4.9) \quad e^{-C_2 t} \|\Psi_n(\xi, \sigma)(t)\|_{\alpha,n} \leq C_5(\|\xi\| + \|\sigma\|_{L_n}).
\]

$\Psi_n$ is obviously continuous with respect to $\xi$. To prove continuity in $\sigma$ let $\sigma, \sigma_1 \in BH_n$, $t \leq 0$ and assume $\|\sigma_1\|_{H_n} \to 0$. Then

\[
\begin{align*}
\|\Psi_n(\xi, \sigma + \sigma_1)(t) - \Psi_n(\xi, \sigma)(t)\|_{H_n} & \leq C_6 \left( \int_{0}^{t} e^{-\lambda_{n,1} (t-s) + C_B s} \|\sigma_1\|_{H_n} ds \right. \\
& \left. + \int_{t-(\lambda_{n,1} + C_2)^{-1/2}}^{t} \frac{e^{CN_B s}}{\sqrt{t-s}} \|\sigma_1\|_{H_n} ds + \int_{-\infty}^{t-(\lambda_{n,1} + C_2)^{-1/2}} (\lambda_{n,1} + C_2)^{1/2} e^{-C_2 (t-s)} \|\sigma_1\|_{H_n} ds \right) \\
& \leq C_7 e^{C_2 t} \|\sigma_1\|_{H_n} \to 0
\end{align*}
\]

and $\Psi_n$ is indeed continuous.

If we show $\{\Psi_n(\xi, \sigma) : \|\sigma\|_{H_n} \leq C\}$ is compact for all $C > 0$, then $\Psi_n(\xi, \cdot)$ is completely continuous.

So let $\sigma_j \in BH_n$ be a sequence for which $\sup_{t \leq 0} (e^{-C_2 t} \|\sigma_j\|_{H_n}) \leq C$. Let $\bigcup_{t \geq 1} t_i = Q \cap \mathbb{R} [0, \infty, 0]$. For each $i$ fixed, $(\Psi_n(\xi, \sigma_j)(t_i))_j$ is in a compact set by (4.9), hence taking a subsequence, called $\sigma_j$ too,

\[
(4.10) \quad \|\Psi_n(\xi, \sigma_j)(t_i) - \mu(t_i)\|_{H_n} \to 0,
\]

as $j \to \infty$, for some $\mu(t_i) \in H_n$. With the Cantor diagonal procedure there is a subsequence, called $\sigma_j$ again, such that for all $i$ (4.10) holds.

$t \mapsto \Psi_n(\xi, \sigma)(t) \in H_n$ is continuous: by Lemma 4.6 it is a solution of equation (4.7), as such it is continuous (see e.g. [10, Theorem 3.5.2]). But we need more, namely bounds independent of $j$. To get them let $\sigma \in BC(0) \subset BH_n$, $\tau_1 < \tau_2 \leq 0$, for
\[ \tau_2 - \tau_1 < 1. \text{ Arguing as in (4.8)} \]
\[ \| \Psi_n(\xi, \sigma)(\tau_1) - \Psi_n(\xi, \sigma)(\tau_2) \|_{H_n} \]
\[ \leq \| (e^{-B_1,n(\tau_1 - \tau_2)} - \text{id})e^{-B_1,n(s)} \sum_{i=1}^{t_1} \xi_i[u_{n,i}] \|_{H_n} \]
\[ + \int_{\tau_1}^{\tau_2} \| (e^{-B_1,n(\tau_1 - \tau_2)} - \text{id})e^{-B_1,n(s)} P_n h_n(\sigma(s)) \|_{H_n} \, ds \]
\[ + \int_{\tau_1}^{\tau_2} \| e^{-B_2,n(s)} P_n h_n(\sigma(s)) \|_{H_n} \, ds \]
\[ + \int_{\tau_1}^{-\infty} \| (e^{-B_2,n(s)} - \text{id})e^{-B_2,n(s)} Q_n h_n(\sigma(s)) \|_{H_n} \, ds \]
\[ \leq C_8 \left( \| \xi \|_{H_n} + \sqrt{\tau_2 - \tau_1} + (\tau_2 - \tau_1)^{\tilde{\alpha}} \| \sigma \|_{L_n} \right) \]
\[ \leq C_9(\tau_2 - \tau_1)^{\tilde{\alpha}} \| \xi \|_{H_n} + \| \sigma \|_{L_n}, \]

where the constants \( C_8, C_9 \) are independent of \( \tau_1, \tau_2 \), and \( 0 < \tilde{\alpha} < 1/2 \). Thus for \( \delta_1 > 0 \), \( i, \tilde{i}, |t_i - t_{\tilde{i}}| \leq \min(1, (\delta_1/4((\| \xi \| + C)C_9))^{1/\tilde{\alpha}}) \), there is a \( j = \tilde{j}(\delta_1, i, \tilde{i}) \), such that
\[ \| \mu(t_i) - \mu(t_{\tilde{i}}) \|_{H_n} \leq \| \mu(t_i) - \Psi_n(\xi, \sigma_j)(t_i) \|_{H_n} \]
\[ + \| \Psi_n(\xi, \sigma_j)(t_i) - \Psi_n(\xi, \sigma_j(t_{\tilde{i}})) \|_{H_n} + \| \Psi_n(\xi, \sigma_j)(t_{\tilde{i}}) - \mu(t_{\tilde{i}}) \|_{H_n} \leq \delta_1. \]

Hence we can define \( \mu(t) \) for all \( t \leq 0 \) by continuously extending \( \mu(t_i) \). Then for \( t \leq 0 \) and \( t_i \in \mathbb{Q} \) near to \( t \)
\[ e^{-C_8 t^2} \| \Psi_n(\xi, \sigma_j)(t) - \mu(t) \|_{H_n} \leq e^{-C_8 t^2} \| \Psi_n(\xi, \sigma_j)(t) - \Psi_n(\xi, \sigma_j)(t_i) \|_{H_n} \]
\[ + \| \Psi_n(\xi, \sigma_j)(t_i) - \mu(t_i) \|_{H_n} + \| \mu(t_i) - \mu(t) \|_{H_n} \]
shows \( \| \Psi_n(\xi, \sigma_j) - \mu \|_{H_n} \to 0 \) as \( j \to \infty \). I.e. we have found a convergent subsequence, and \( \{ \Psi_n(\xi, \sigma) : \| \sigma \|_{H_n} \leq C \} \) is indeed compact in \( BH_n \). \( \square \)

**Lemma 4.9.** There is a subsequence, called \( \varepsilon_n \) too, constants \( C_1, C_2, C_3 > 0 \), maps \( \sigma^*_n : B_{C_1}(0) \subset \mathbb{R}^{l_i} \to BH_n \), such that
\[ \sigma^*_n(\xi)(t) = e^{-B_1,n(t-s)} \sum_{i=1}^{t_1} \xi_i[u_{n,i}] + \int_0^t e^{-B_1,n(t-s)} P_n h_n(\sigma^*_n(\xi)(s)) \, ds \]
\[ + \int_{-\infty}^t e^{-B_2,n(t-s)} Q_n h_n(\sigma^*_n(\xi)(s)) \, ds, \]

\[ \tau_2 - \tau_1 < 1. \]
for all \( t \leq 0, n \geq 0 \). \( \sigma^*_n(\xi)(\cdot) \) can be extended to a function on \( \mathbb{R} \) in such a way that it is a solution of equation (4.7). It is the only solution \( \sigma \) in \( \{ \sigma \in BH_n : \| \sigma \|_{H_n} \leq C_2, \| \sigma \|_{H_n} \leq C_3 \} \) with \( P_n \sigma(0) = \sum_{i=1}^{i_1} \xi_i u^{0}_{i} \). Moreover,

\[
|\sigma^*_n(\xi)(t) - \Phi^H_{\varepsilon_n} \sigma^0_0(\xi)(t)|_{\varepsilon_n} \to 0
\]
as \( n \to \infty \), for all \( \xi \in B_{C_{\varepsilon}}(0) \subset \mathbb{R}^l \), \( t \in \mathbb{R} \).

**Proof.** Let \( \Psi_n \) be as in Lemma 4.8. We know already that for each fixed \( \xi \in \mathbb{R}^l \) is \( \Psi_n(\xi, \cdot) : BH_n \to BH_n \) a completely continuous map. We claim that, with some restrictions on \( \xi \) and \( \sigma \), this map is a contraction.

Given \( \tilde{C} > 0 \), by Lemma 4.7(c) there are \( C_1 = C_1(\tilde{C}) > 0, C_2(n) > 0, C_2(n) \to 0 \) as \( n \to \infty \), \( C_2(0) = 0 \), such that

\[
|h_n(\sigma(s) + \sigma_1(s)) - h_n(\sigma(s))|_{L_n} \leq \tilde{C}|\sigma_1(s)|_{n,d} + C_2(n)|\sigma_1(s)|_{H_n},
\]
whenever \( |\sigma(s)|_{n,d}, |\sigma_1(s)|_{n,d} \leq C_1 \). Thus, if

\[
|\sigma|_{n,d} := \sup_{t \leq 0} (e^{-Cn t}|\sigma(t)|_{n,d})
\]
we find for \( |\sigma|_{n,d}, |\sigma_1|_{n,d} \leq C_1, t \leq 0 \)

\[
e^{-Cn t}\|\Psi_n(\xi, \sigma + \sigma_1)(t) - \Psi_n(\xi, \sigma)(t)\|_{H_n}
\leq C_3(\tilde{C}|\sigma_1|_{n,d} + C_2(n)|\sigma_1|_{H_n}) \leq C_3(C_2 \tilde{C} + C_2(n))|\sigma_1|_{H_n},
\]
where \( C_3, C_4 \) do not depend on \( \tilde{C} \), and \( C_4 \) is such that \( |\cdot|_{n,d} \leq C_4\|\cdot\|_{H_n} \) for all \( n \geq 0 \).

Let \( C_5 \) denote the constant in (4.8). Choose \( \tilde{C} \leq 1/(4C_3 C_4) \), \( C_0 = C_0(\tilde{C}) \leq C_1/C_4, C_7 = C_7(\tilde{C}) \leq C_0/(2C_3) \) and, taking a subsequence, assume \( C_2(n) \leq 1/(4C_3) \). Then inequalities (4.8) and (4.11) show

\[
\|\Psi_n(\xi, \sigma)\|_{H_n} \leq \|\Psi_n(\xi, \sigma) - \Psi_n(\xi, 0)\|_{H_n} + \|\Psi_n(\xi, 0)\|_{H_n}
\leq (C_3 C_4 \tilde{C} + C_3 C_2(n))|\sigma|_{H_n} + C_5|\xi| \leq C_0,
\]

\[
|\Psi_n(\xi, \sigma)|_{n,d} \leq C_4 C_0 \leq C_1,
\]
for all \( n \geq 0, |\xi| \leq C_7, \|\sigma\|_{H_n} \leq C_0, |\sigma|_{n,d} \leq C_1 \). If additionally \( |\sigma_1|_{n,d} \leq C_1 \) we get

\[
\|\Psi_n(\xi, \sigma + \sigma_1) - \Psi_n(\xi, \sigma)\|_{H_n} \leq \frac{1}{2}|\sigma_1|_{H_n},
\]

Thus, for these \( \xi \),

\[
\Psi_n(\xi, \cdot) : \{ \sigma \in BH_n : \| \sigma \|_{H_n} \leq C_2, \| \sigma \|_{H_n} \leq C_3 \}
\to \{ \sigma \in BH_n : \| \sigma \|_{H_n} \leq C_2, \| \sigma \|_{H_n} \leq C_3 \}
\]
is contracting. Hence for each \( \xi \in B_{C_{\varepsilon}} (0) \subset \mathbb{R}^l \) there is a unique \( \sigma^*_n(\xi) \) in above set with \( \sigma^*_n(\xi) = \Psi_n(\xi, \sigma^*_n(\xi)) \). By Lemma 4.6 the map \( t \mapsto \sigma^*_n(\xi)(t) \) is a solution
of equation (4.7) for all \( n \geq 0 \). Since this equation is just the original (4.1) under the substitution \( [u(t)] \rightarrow [u(t)] + [u^n] \), and the semiflows \( \pi_x \) are global ones, all \( \sigma_n^*(\xi)(t) \) can be extended to full solutions \( \sigma_n^*(\xi) : \mathbb{R} \rightarrow H_n \) of (4.7).

Note that \( \sigma_n^*(\xi)(t) \in \mathcal{A}_n \) for all \( t \in \mathbb{R} \), \( n \geq 0 \).

The only thing we have not shown already is the convergence \( |\sigma_n^*(\xi) - \Phi_n^H \sigma_0^*(\xi)|_{n,d} \rightarrow 0 \), as \( n \rightarrow \infty \).

The solutions \( \sigma_n^*(\xi)(t) + [u^n] \) are in the respective attractors, thus by Lemma 3.8 there is a constant \( C_8 \) such that

\[
|\sigma_n^*(\xi)(t) + [u^n]|_{H_n} \leq C_8,
\]

for all \( n \geq 0 \), \( t \in \mathbb{R} \) and \( \xi \). But then Lemma 3.9 shows the existence of a solution \( \sigma^*(\xi)(t) + [u^0] \) with \( ||\sigma^*(\xi)(t) + [u^0]||_{H_0} \leq C_9 \), some constant \( C_9 > 0 \), and, taking again a subsequence,

\[
|\sigma_n^*(\xi)(t) - \Phi_n^H \sigma_0^*(\xi)(t)|_{n,d} \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad \text{for all } t \in \mathbb{R}.
\]

We have

\[
P_0(\sigma^*(\xi)(0)) \leftarrow P_n(\sigma_n^*(\xi)(0)) = \sum_{l=1}^{l_1} \xi_l[u_{n,d}^l] \rightarrow \sum_{l=1}^{l_1} \xi_l[u_0^l] \quad \text{as } n \rightarrow \infty.
\]

If \( ||\xi|| \) is small, we can choose \( C_7 \) and \( C_8 \) small, and by Lemma 3.9 \( ||\sigma^*(\xi)||_{H_0} \) is small too. By Lemma 4.6 \( \sigma^*(\xi) = \Psi_0(\xi, \sigma^*(\xi)) \) and the uniqueness of \( \sigma_0^* \) on \( \overline{B_{C_6}(0)} \subset B_{H_0} \) yields \( \sigma^*(\xi) = \sigma_0^*(\xi) \). Thus

\[
|\sigma_n^*(\xi)(t) - \Phi_n^H \sigma_0^*(\xi)(t)|_{n,d} \rightarrow 0,
\]

as \( n \rightarrow \infty \), for each \( t \in \mathbb{R} \) fixed. \( \square \)

Now we can prove our main theorem:

**Proof of Theorem 1.1.** Fix \( 0 \leq d < 1 \). We shall show that for given \( \delta > 0 \), \( [u_0] \in \mathcal{A}_0 \) there is a \( 0 < \varepsilon_0 \), and for all \( 0 < \varepsilon \leq \varepsilon_0 \) there are \( [u_\varepsilon] \in \mathcal{A}_\varepsilon \) such that \( ||[u_\varepsilon] - \Phi_\varepsilon^H [u_0]||_{\varepsilon,d} \leq \delta \). Together with Theorem 2.2 this proves Theorem 1.1.

Assume that these \( [u_\varepsilon] \) do not exist, then there are \( [u_0] \in \mathcal{A}_0 \), \( \varepsilon_n \rightarrow 0 \), \( \delta > 0 \) such that for all \( n \in \mathbb{N} \)

\[
\inf_{[u] \in \mathcal{A}_n} ||[u] - \Phi_n^H [u_0]||_{n,d} \geq \delta.
\]

There is a full solution \( \sigma_0 : \mathbb{R} \rightarrow H_0 \) of (4.1) such that \( [u_0] = \sigma_0(0) \). By Lemma 4.5 \( [u_0] \) is no point of equilibrium. We have already shown in Section 3 that \( \pi_0 \) is gradient like, thus \( \sigma_0(t) \rightarrow [u_0^m] \), as \( t \rightarrow -\infty \), for some \( m \in \{1, \ldots, M_0\} \). This implies \( [u_0^m] \) has an unstable manifold and we can use Lemma 4.9.

Let \( C_1 \), \( C_2 \) be as in this lemma. Note that \( \sigma_0(t + t_0) - [u_0^m] \in B_{H_0} \) for all \( t_0 \), and \( ||\sigma_0(\cdot + t_0) - [u_0^m]||_{H_0} \leq C_3 e^{C_4 t_0} \) by Lemma 4.6.
Setting $\sigma^*_0(t) := \sigma_0(t + t_0) - [u^0_m]$ and choosing $t_0 < 0$ small enough, we have $\|\sigma^*_0\|_{H_0} < C_2$ and if $\xi$ is defined by

$$P_0\sigma^*_0(0) = \sum_{i=1}^{l_1} \xi_i [u^0_{0, i}],$$

then also $\|\xi\| < C_1/2$. $\sigma^*_0$ solves (4.7), and Lemma 4.9 shows $\sigma^*_0(t) = \sigma^*_0(\xi)(t)$, $\sigma^*_0(\xi)$ as in this lemma.

Now $\sigma^*_0(\xi)(-t_0) + [u^0_m] \in A_n$, and since $\|u^0_m - \Phi^H_n[u^0_m]\|_{n,d} \to 0$

$$|\sigma^*_0(\xi)(-t_0) + [u^0_m] - \Phi^H_n[u^0_m]|_{n,d} \leq |\sigma^*_0(\xi)(-t_0) - \Phi^H_n\sigma^*_0(\xi)(-t_0)|_{n,d}$$

$$+ |\Phi^H_n(\sigma^*_0(\xi)(-t_0) - [u^0] + [u^0_m])|_{n,d} + \|u^0_m - \Phi^H_n[u^0_m]\|_{n,d} \to 0,$$

as $n \to \infty$. This contradicts our assumption, and the proof is complete. □

References


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