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GLOBAL REGULAR NONSTATIONARY FLOW FOR THE NAVIER–STOKES EQUATIONS IN A CYLINDRICAL PIPE

Wojciech M. Zajączkowski

The paper is devoted to the memory
of prof. Olga Aleksandrovna Ladyzhenskaya
whose ideas and methods are used in this paper.

Abstract. Global existence of regular solutions to the Navier–Stokes equations describing the motion of a fluid in a cylindrical pipe with large inflow and outflow in shown. The global existence is proved under the following conditions:

(1) small variations of velocity and pressure with respect to the variable along the pipe,
(2) inflow and outflow are very close to homogeneous and decay exponentially with time,
(3) the external force decays exponentially with time.

Global existence is proved in two steps. First by the Leray–Schauder fixed point theorem we prove local existence with large existence time which is inversely proportional to the above smallness restrictions. Next the local solution is prolonged step by step.

The existence is proved for a solution without any restrictions on the magnitudes of inflow, outflow, external force and the initial velocity.

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1. Introduction

We consider viscous incompressible fluid motions in a finite cylinder with large inflow and outflow and under boundary slip conditions. Therefore the following initial-boundary value problem is examined

$$
\begin{aligned}
& v_{,t} + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) = f && \text{in } \Omega^T = \Omega\times(0,T),\\
& \operatorname{div} v = 0 && \text{in } \Omega^T,\\
& v\cdot\overline{n} = 0 && \text{on } S_1^T = S_1\times(0,T),\\
& \nu\overline{n}\cdot\mathbb{D}(v)\cdot\overline{\tau}_\alpha + \gamma v\cdot\overline{\tau}_\alpha = 0, \quad \alpha=1,2, && \text{on } S_1^T,\\
& v\cdot\overline{n} = d && \text{on } S_2^T = S_2\times(0,T),\\
& \overline{n}\cdot\mathbb{D}(v)\cdot\overline{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S_2^T,\\
& v|_{t=0} = v(0) && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

where $\Omega\subset\mathbb{R}^3$, $S = S_1\cup S_2 = \partial\Omega$, $v = v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t)) \in \mathbb{R}^3$ is the velocity vector of the fluid motion, $p = p(x,t)\in\mathbb{R}^1$ the pressure, $f = f(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t)) \in \mathbb{R}^3$ the external force field, $\overline{n}$ the unit outward vector normal to the boundary S, $\overline{\tau}_\alpha$, $\alpha = 1,2$, are tangent vectors to S. By dot we denote the scalar product in $\mathbb{R}^3$. $\mathbb{T}(v,p)$ is the stress tensor of the form

$$\mathbb{T}(v,p) = \nu\mathbb{D}(v) - pI,$$

where ν is the constant viscosity coefficient, I the unit matrix and $\mathbb{D}(v)$ is the dilatation tensor

$$\mathbb{D}(v) = \{v_{i,x_j} + v_{j,x_i}\}_{i,j=1,2,3}.$$

Finally $\gamma > 0$ is the slip coefficient.

By $\Omega\subset\mathbb{R}^3$ we denote a cylindrical type domain parallel to the axis x_3 with arbitrary cross section. We assume that S_1 is the part of the boundary which is parallel to the axis x_3 and S_2 is perpendicular to x_3. Hence

$$
\begin{aligned}
S_1 &= \{x\in\mathbb{R}^3 : \varphi(x_1,x_2) = c_0,\ -a < x_3 < a\},\\
S_2(-a) &= \{x\in\mathbb{R}^3 : \varphi(x_1,x_2) < c_0,\ x_3 = -a\},\\
S_2(a) &= \{x\in\mathbb{R}^3 : \varphi(x_1,x_2) < c_0,\ x_3 = a\},
\end{aligned}
$$

where a, c_0 are positive given numbers and $\varphi(x_1,x_2) = c_0$ describes a sufficiently smooth closed curve in the plane $x_3 = \text{const}$.

To describe inflow and outflow we define

$$d_1 = -v\cdot\overline{n}|_{S_2(-a)}, \qquad d_2 = v\cdot\overline{n}|_{S_2(a)}, \tag{1.2}$$

so $d_i \geq 0$, $i = 1,2$, and by $(1.1)_{2,3}$ and (1.2) we have the compatibility condition

$$\Phi \equiv \int_{S_2(-a)} d_1\, dS_2 = \int_{S_2(a)} d_2\, dS_2, \tag{1.3}$$

where Φ is flux.

The aim of this paper is to prove the existence of global regular solutions to problem (1.1) without restrictions on magnitudes of the external force f, initial data $v(0)$, inflow d_1 and outflow d_2. We show existence of solutions by regularizing weak solutions. In general we follow the ideas from [6].

Let us introduce an extension $\alpha = \alpha(x,t) \in \mathbb{R}$ such that

$$\alpha|_{S_2(-a)} = d_1, \qquad \alpha|_{S_2(a)} = d_2. \tag{1.4}$$

Then we introduce the following vector $b = \alpha\overline{e}_3$, where $\overline{e}_3 = (0,0,1)$ is directed along the axis x_3. Let us define the function $u = v - b$. Therefore,

$$\begin{aligned} \operatorname{div} u &= -\operatorname{div} b = -\alpha_{,x_3} && \text{in } \Omega,\\ u\cdot\overline{n} &= 0 && \text{on } S. \end{aligned} \tag{1.5}$$

Equations $(1.1)_{2,3,6}$ and (1.3) imply the compatibility condition

$$\int_\Omega \alpha_{,x_3}\, dx = -\int_{S_2(-a)} \alpha|_{x_3=-a}\, dS_2 + \int_{S_2(a)} \alpha|_{x_3=a}\, dS_2 = 0.$$

Hence we can define a function φ as a solution to the Neumann problem

$$\begin{aligned} \Delta\varphi &= -\operatorname{div} b && \text{in } \Omega,\\ \overline{n}\cdot\nabla\varphi &= 0 && \text{on } S,\\ \int_\Omega \varphi\, dx &= 0. \end{aligned} \tag{1.6}$$

Therefore, we introduce the new function

$$w = u - \nabla\varphi = v - (b + \nabla\varphi) \equiv v - \delta, \tag{1.7}$$

which is a solution to the problem

$$\begin{aligned} &w_{,t} + w\cdot\nabla w + w\cdot\nabla\delta + \delta\cdot\nabla w - \operatorname{div}\mathbb{T}(w,p)\\ &\qquad = f - \delta_{,t} - \delta\cdot\nabla\delta + \nu\operatorname{div}\mathbb{D}(\delta) \equiv F(\delta,f) && \text{in } \Omega^T,\\ &\operatorname{div} w = 0 && \text{in } \Omega^T,\\ &w\cdot\overline{n} = 0 && \text{on } S^T,\\ &\nu\overline{n}\cdot\mathbb{D}(w)\cdot\tau_\alpha + \gamma w\cdot\overline{\tau}_\alpha\\ &\qquad = -\nu\overline{n}\cdot\mathbb{D}(\delta)\cdot\overline{\tau}_\alpha - \gamma\delta\cdot\overline{\tau}_\alpha \equiv B_{1\alpha}(\delta), \quad \alpha = 1,2, && \text{on } S_1^T,\\ &\nu\overline{n}\cdot\mathbb{D}(w)\cdot\overline{\tau}_\alpha = -\nu\overline{n}\cdot\mathbb{D}(\delta)\cdot\overline{\tau}_\alpha \equiv B_{2\alpha}(\delta), \quad \alpha = 1,2, && \text{on } S_2^T,\\ &w|_{t=0} = v(0) - \delta(0) && \text{in } \Omega, \end{aligned} \tag{1.8}$$

where

$$\overline{n}|_{S_1} = \frac{(\varphi_{,x_1}, \varphi_{,x_2}, 0)}{\sqrt{\varphi_{,x_1}^2 + \varphi_{,x_2}^2}}, \quad \overline{\tau}_1|_{S_1} = \frac{(-\varphi_{,x_2}, \varphi_{,x_1}, 0)}{\sqrt{\varphi_{,x_1}^2 + \varphi_{,x_2}^2}}, \quad \overline{\tau}_2|_{S_1} = (0,0,1) \equiv \overline{e}_3$$

and

$$\overline{n}|_{S_2} = \overline{e}_3, \quad \overline{\tau}_1|_{S_2} = \overline{e}_1, \quad \overline{\tau}_2|_{S_2} = \overline{e}_2, \tag{1.9}$$

where $\overline{e}_1 = (1,0,0)$, $\overline{e}_2 = (0,1,0)$.

The aim of this paper is to prove the existence of global solutions to problem (1.1) without any restrictions on the magnitudes of initial velocity, the external force field, the inflow and the outflow. This will be done by increasing regularity of weak solutions. For this purpose we follow the ideas and methods from [6]. To show existence of such solutions we need, however, some small parameters. By such parameters we have L_2-norms of derivatives of v and f with respect to x_3 and derivatives of d_1, d_2 with respect to x', $x' = (x_1, x_2)$.

Hence we introduce the quantities

$$\begin{aligned} h^{(1)} &= v_{,x_3}, & q^{(1)} &= p_{,x_3}, & h^{(2)} &= v_{,x_3x_3}, & q^{(2)} &= p_{,x_3x_3}, \\ w &= v_3, & \chi &= v_{2,x_1} - v_{1,x_2}, & g^{(1)} &= f_{,x_3}, & g^{(2)} &= f_{,x_3x_3}. \end{aligned} \tag{1.10}$$

The paper is divided into the following parts. In Section 2 we introduce notation, define a weak solution to problem (1.8) and find energy type estimate (see Lemmas 2.4, 2.5). The existence of weak solutions to problem (1.8) can be proved by the Galerkin method similarly as in [5].

In Lemma 2.4 the Korn inequality for solutions to problem (1.1) is shown. To prove local existence of solutions with large existence time we need energy type estimates for $h^{(i)}$, $i = 1, 2$. For this purpose in Section 3 there are found problems for $h^{(1)}, q^{(1)}$ (see (3.1)), for $h^{(2)}, q^{(2)}$ (see (3.9)), for w (see (3.6)) and for χ (see (3.3)). We derive the problems in Lemmas 3.1–3.4, respectively. To find energy type estimates for solutions of problems (3.1) and (3.9) we have to make the Dirichlet type conditions homogeneous. For this purpose we construct functions $\widetilde{h}^{(1)}$ and $\widetilde{h}^{(2)}$ which are solutions of problems (3.17) and (3.36), respectively. Utilizing these constructions we define new functions $k^{(1)}$ and $k^{(2)}$, determined by (3.23) and (3.38), which are solutions to problems (3.24) and (3.39), respectively. For solutions of problems (3.24) and (3.39) we have Korn inequalities (see Lemmas 3.6 and 3.9) so we are able to find energy type inequalities (see Lemmas 3.7 and 3.10) because the necessary integration by parts can be performed in view of homogeneous Dirichlet boundary conditions. Utilizing energy type estimates (3.29) and (3.45) for $k^{(1)}$ and $k^{(2)}$ we find energy type inequalities for $h^{(1)}$ and $h^{(2)}$ (see (3.52) and (3.54)). In a similar way we obtain an energy type inequality for χ (see (4.6) and Lemma 4.1).

Having the energy type inequalities for $h^{(i)}$, $i = 1, 2$, and χ we obtain inequality (4.17) for $\|v\|_{W^{2,1}_{27/16}}(\Omega^T)$.

In the case without inflow and outflow this regularity for v was enough to prove long time existence (see [6]). However, in this paper in view of boundary conditions for $h^{(2)}_{i,x_3}$ (see $(3.9)_6$) we need more regularity for v (see (4.43), where $v \in W^{2+\sigma,1+\sigma/2}_\rho(\Omega^T)$, $2+\sigma < 5/\rho$).

Finally,the local existence of solutions to problem (1.1) follows from transformation (5.1) by applying the Leray–Schauder fixed point theorem. The magnitude of the local existence time follows from the proof of Lemma 5.2 and is inversely proportional to the quantity $\eta(T) = \eta_1(T) + \eta_2(T)$, where η_1 is defined in (3.53) and η_2 in (3.55), respectively.

To apply the Leray–Schauder fixed point theorem we have to know that transformation Φ described in (5.5) is compact and continuous. These properties follow from Lemmas 5.1 and 5.4, respectively.

In Section 6 global existence of solutions is proved by prolonging the local solution step by step. For this purpose we need two facts:

(1) The existence time T of the local solution is sufficiently large.
(2) The norms of data at time $t = T$ are not greater than the corresponding norms at $t = 0$.

We present two kinds of proofs: explicit (Theorem 6.4) and non-explicit (Theorem 6.5).

To prove Theorem 6.4 we assume some decay estimates on the external force and on derivatives with respect to x' and t of inflow and outflow.

This implies that we are able to obtain the following decays

$$\|h^{(i)}(t)\|_{L_2(\Omega)} \le c_0 e^{-\delta_i t}, \quad \delta_i > 0,\ i = 1, 2;$$

where c_0 depends on some norms of data. Then for sufficiently large time T we obtain that

$$\|h^{(i)}(T)\|_{L_2(\Omega)} \le \|h^{(i)}(0)\|_{L_2(\Omega)} e^{-\delta_i T/2}, \quad i = 1, 2. \tag{1.11}$$

In view of Theorem 1 we have the existence of solutions to problem (1.1) and the estimate

$$\|h^{(i)}\|_{W^{2+\beta^i,1+\beta^i/2}_\delta(\Omega\times(0,T))} \le A, \quad i = 1, 2,\ \beta^1 = \beta,\ \beta^2 = \beta', \tag{1.12}$$

where constant A does not depend on T which is possible by assuming that η_1 and η_2 (see Theorem 1.1) are sufficiently small.

In view of (1.11), (1.12) and by interpolation we obtain that

$$\|h^{(i)}(t)\|_{W^{\sigma_i}_\rho(\Omega)} \le c_1 e^{-\delta_i' T},$$

where $\sigma_i < 2+\beta^i$, $\delta_i' > 0$, $i=1,2$. Then the proof of Theorem 6.4 follows

$$\|h^{(i)}(T)\|_{W_\delta^{2+\beta^i-2/\delta}(\Omega)} \le \|h^{(i)}(0)\|_{W_\delta^{2+\beta^i-2/\delta}(\Omega)}, \quad i=1,2. \tag{1.13}$$

This gives a possibility to prolong the local existence on the interval $[T,2T]$ and step by step to prove the global existence.

To prove Theorem 6.5 we use only estimate (1.12) with sufficiently large T and with A independent of T. We can express (1.12) in the form

$$\bigg(\int_0^T \|h^{(i)}(t)\|_{W_\delta^{2+\beta^i}(\Omega)}\,dt\bigg)^{1/\delta} \le A.$$

Hence for sufficiently large T there exists $T_* \le T$ such that $\|h^{(i)}(T_*)\|_{W_\delta^{2+\beta^i}(\Omega)}$, $i=1,2$, are so small that

$$\|h^{(i)}(T_*)\|_{W_\delta^{2+\beta^i}(\Omega)} \le \|h^{(i)}(0)\|_{W_\delta^{2+\beta^i}(\Omega)}, \quad i=1,2.$$

Hence we have (1.13) with T replaced by T_*. Therefore the local solution can be prolonged on the interval $[T_*, T_*+T]$. Continuing these considerations we prove global existence.

Finally, we recall definitions of Besov spaces which are necessary for understanding main results.

$$\begin{aligned} W_p^{\sigma,\sigma/2}(\Omega^T) = \bigg\{ & u = u(x,t) : \|u\|_{W_p^{\sigma,\sigma/2}(\Omega^T)} \equiv \|u\|_{L_p(\Omega^T)} \\ & + \bigg(\int_0^T\int_\Omega\int_\Omega \frac{|D_x^{[\sigma]}u(x,t) - D_{x'}^{[\sigma]}u(x',t)|^p}{|x-x'|^{3+p(\sigma-[\sigma])}}\,dx\,dx'\,dt\bigg)^{1/p} \\ & + \bigg(\int_\Omega\int_0^T\int_0^T \frac{|\partial_t^{[\sigma/2]}u(x,t) - \partial_{t'}^{[\sigma/2]}u(x,t')|^p}{|t-t'|^{1+p(\sigma/2-[\sigma/2])}}\,dx\,dt\,dt'\bigg)^{1/p} < \infty\bigg\}, \end{aligned}$$

where $\Omega \subset \mathbb{R}^3$, $1 \le p \le \infty$, $\sigma \in \mathbb{R}_+$-noninteger and $[\sigma]$ is the integer part of σ,

$$\begin{aligned} W_p^\sigma(\Omega) = \bigg\{ & u = u(x) : \|u\|_{W_p^\sigma(\Omega)} \equiv \|u\|_{L_p(\Omega)} \\ & + \bigg(\int_\Omega\int_\Omega \frac{|D_x^{[\sigma]}u(x) - D_{x'}^{[\sigma]}u(x')|^p}{|x-x'|^{3+p(\sigma-[\sigma])}}\,dx\,dx'\bigg)^{1/p} < \infty\bigg\}. \end{aligned}$$

Moreover, we denote $W_r^{2s,s}(\Omega^T) = B_{r,r}^{2s,s}(\Omega^T)$, $W_r^s(\Omega) = B_{r,r}^s(\Omega)$, where $s \in \mathbb{R}_+$ is noninteger, $r \in [1,\infty]$, spaces B are introduced in [2, Chapter 3, Section 18]. The above equalities hold in view of [3], where different kinds of differences were introduced.

Now we formulate main results.

THEOREM 1.1 (Local existence). *Assume that*

(a) $h^{(i)}(0) \in W_\delta^{2+\beta^i-2/\delta}(\Omega)$, $i=1,2$, $\chi(0) \in L_2(\Omega)$, $v(0) \in W_\rho^{2+\sigma-2/\rho}(\Omega)$, *where* $\beta^1=\beta$, $\beta^2=\beta'$, $0<\beta-\beta'<1/2$, $5/\delta<3+\beta-\beta'$, $\beta'<\sigma<1$, $2+\sigma<5/\rho$, $5/\rho<2+\sigma+\beta-\beta'$, $\rho>\delta$, $\rho\in(0,1)$, $5/\delta<3+\beta$, $3/\delta<2+\beta$, $\delta\in(1,2)$, $\beta\in(0,1)$, $v(0)\in W_r^{s-2/r}(\Omega)$, $5/r-11/9\le s$, $v(0)\in H^1(\Omega)$.

(b) $g^{(i)} \in W_\delta^{\beta^i,\beta^i/2}(\Omega^T)$, $i=1,2$, $d_{,x'} \in W_\delta^{3+\beta-1/\delta,3/2+\beta/2-1/(2\delta)}(S_2^T) \cap H^{3/2}(S_2^T)$, $d \in H^{5/2}(S_2^T)$, $d_{0,t} \in L_3(S_2^T)$, $d \in L_2(0,T;L_\infty(S_2))$, $F_i = f_{3,x_i} - f_{i,x_3}$, $i=1,2$, $F'=(F_1,F_2)$, $F_3 = f_{1,x_2}-f_{2,x_1}$, $F' \in W_\delta^{1+\beta'-1/\delta,1/2+\beta'/2-1/(2\delta)}(S_2^T)$, $F_3 \in L_{18/13}(\Omega^T)$.

(c) *Let*

$$
\begin{aligned}
\eta_1(T) = {} & \|d_{,x'}\|_{L_\infty(0,T;W_3^1(S_2))} + \|d_{,t}\|_{L_2(0,T;L_{6/5}(S_2))}\\
& + \|f_3\|_{L_2(0,T;L_{4/3}(S_2))} + \|g^{(1)}\|_{L_2(0,T;L_{6/5}(\Omega))} + \|h^{(1)}(0)\|_{L_2(\Omega)},\\
\eta_2(T) = {} & \|d_{,x'}\|_{L_\infty(0,T;W_3^2(S_2))} + \|d_{,t}\|_{L_2(0,T;W_{6/5}^2(S_2))} + \eta_1(T)\\
& + \varepsilon(\|g^{(1)}\|_{W_\delta^{\beta,\beta/2}(\Omega^T)} + \|d_{,x'}\|_{W_\delta^{2+\beta-1/\delta,1+\beta/2-1/(2/\delta)}(S_2^T)}\\
& + \|h^{(1)}(0)\|_{W_\delta^{2+\beta-2/\delta}(\Omega)})\\
& + c\left(\frac{1}{\varepsilon}\right)\eta_1(T) + \sum_{i=1}^2(\|g^{(i)}\|_{L_2(0,T;L_{6/5}(\Omega))} + \|h^{(i)}(0)\|_{L_2(\Omega)})\\
& + \|f_3\|_{L_2(0,T;L_{4/3}(S_2))} + \|F'\|_{L_2(0,T;L_{4/3}(S_2))},
\end{aligned}
$$

$c(1/\varepsilon)$ *is increasing,*

$$
\begin{aligned}
\overline{G}_1(T) = {} & \|f\|_{L_2(\Omega^T)} + \|f\|_{W_\rho^{\sigma,\sigma/2}(\Omega^T)} + \|f_3\|_{L_2(0,T;L_{4/3}(S_2))}\\
& + \|F_3\|_{L_{18/13}(\Omega^T)} + \|d_{,x'}\|_{L_\infty(0,T;W_3^1(S_2))}\\
& + \|d_{,t}\|_{L_2(0,T;W_{6/5}^1(S_2))} + \|d\|_{L_\infty(S_2^T)}\\
& + \|d\|_{W_{27/16}^{38/27,19/27}(S_2^T)} + \|g^{(1)}\|_{L_2(0,T;L_{6/5}(\Omega))},\\
\overline{G}_0(0) = {} & \|v(0)\|_{W_{27/16}^{22/27}(\Omega)} + \|v(0)\|_{W_r^{s-2/r}(\Omega)} + \|v(0)\|_{W_\rho^{2+\sigma-2/\rho}(\Omega)}\\
& + \|v(0)\|_{1,\Omega} + \|h^{(1)}(0)\|_{W_r^{s-2/r}(\Omega)},\\
k(T) = {} & \sum_{i=1}^2(\|g^{(i)}\|_{W_\delta^{\beta^i,\beta^i/2}(\Omega^T)} + \|h^{(i)}(0)\|_{W_\delta^{2+\beta^i-2/\delta}(\Omega)})\\
& + \|d_{,x'}\|_{W_\delta^{3+\beta'-1/\delta,3/2+\beta'/2-1/(2\delta)}(S_2^T)}\\
& + \|F'\|_{W_\delta^{1+\beta'-1/\delta,1/2+\beta'/2-1/(2\delta)}(S_2^T)},\\
k_3(T) = {} & \|d_1\|_{L_6(0,T;L_3(S_2))} + \|d\|_{W_2^{5/2,5/4}(S_2^T)}.
\end{aligned}
$$

(d) *Assume that there exists a constant* A *such that*

$$\varphi(A,\overline{G}_1(T),\overline{G}_0(0),k_3(T))(\eta_1(T)+\eta_2(T))+c_0k(T))\le A, \tag{1.14}$$

where φ *is an increasing positive function* (*see* (5.21)) *and* (1.14) *might be satisfied for sufficiently small* $\eta_1+\eta_2$.

Then there exists a solution to problem (1.1) *such that*

$$\sum_{i=1}^2\|h^{(i)}\|_{W_\delta^{2+\beta^i,1+\beta^i/2}(\Omega^T)}\le A,$$
$$\|v\|_{W_\rho^{2+\sigma,1+\sigma/2}(\Omega^T)}\le\varphi_0(A,\overline{G}_1(T),\overline{G}_0(0),k_3(T),k(T)).$$

Theorem 1.2 (Explicit proof — global existence). *Let the assumptions of Theorem* 1.1 *be satisfied. Let us assume the decays*

$$\|d_1\|^6_{L_6(0,T;L_3(S_2))}\le\frac{\nu}{2}T,$$

$$\begin{aligned}\gamma_1(t)\equiv\|d_{,x'}(t)\|_{W_3^2(S_2)}+\|d_{,t}(t)\|_{W_{6/5}^2(S_2)}+\|f_3(t)\|_{L_{4/3}(S_2)}\\+\|F'(t)\|_{L_{4/3}(S_2)}+\|g^{(1)}(t)\|_{L_{6/5}(\Omega)}+\|g^{(2)}(t)\|_{L_{6/5}(\Omega)}\le\gamma_1(0)e^{-\delta_1t}.\end{aligned}$$

Then there exists a global solution to problem (1.1) *such that*

$$\sum_{i=1}^2\|h^{(i)}\|_{W_\delta^{2+\beta^i,1+\beta^i/2}(\Omega\times(k'T,(k'+1)T))}\le A\quad\text{for all } k'\in\mathbb{N}$$

and

$$\begin{aligned}&\|v\|_{W_\rho^{2+\sigma,1+\sigma/2}(\Omega\times(k'T,(k'+1)T))}\\&\quad\le\varphi_0(A,\overline{G}_1(k'T,(k'+1)T),\overline{G}_0(0),k_3(k'T,(k'+1)T),k(k'T,(k'+1)T)).\end{aligned}$$

Theorem 1.3 (Non-explicit proof — global existence). *Let the assumptions of Theorem* 1.1 *be satisfied. Then there exists a sequence* $\{T_n\}_{n=1}^\infty$ *increasing to infinity such that in each interval* $[T_n,T_{n+1}]$ *with* $|[T_n,T_{n+1}]|\le T$ *there exists a local solution to problem* (1.1) *satisfying the estimates*

$$\sum_{i=1}^2\|h^{(i)}\|_{W_\delta^{2+\beta^i,1+\beta^i/2}(\Omega\times(T_n,T_{n+1}))}\le A\quad\text{for all } n\in\mathbb{N},$$
$$\begin{aligned}&\|v\|_{W_\rho^{2+\sigma,1+\sigma/2}}(\Omega\times(T_n,T_{n+1}))\\&\quad\le\varphi_0(A,\overline{G}_1(T_n,T_{n+1}),\overline{G}_0(0),k_3(T_n,T_{n+1}),k(T_n,T_{n+1})).\end{aligned}$$

2. Notation and auxiliary results

To simplify considerations we introduce the notation

$$
\begin{aligned}
|u|_{p,Q} &= \|u\|_{L_p(Q)}, && Q \in \{\Omega^T, S^T, \Omega, S\},\ p \in [1,\infty],\\
\|u\|_{s,Q} &= \|u\|_{H^s(Q)}, && Q \in \{\Omega, S\},\ s \in \mathbb{R}_+ \cup \{0\},\\
\|u\|_{s,Q^T} &= \|u\|_{W_2^{s,s/2}(Q^T)}, && Q \in \{\Omega, S\},\ s \in \mathbb{R}_+ \cup \{0\},\\
|u|_{p,q,Q^T} &= \|u\|_{L_q(0,T;L_p(Q))}, && Q \in \{\Omega, S\},\ p,q \in [1,\infty],\\
\|u\|_{s,q,Q^T} &= \|u\|_{W_q^{s,s/2}(Q^T)}, && Q \in \{\Omega, S\},\ s \in \mathbb{R}_+ \cup \{0\},\ q \in [1,\infty],\\
\|u\|_{s,q,Q} &= \|u\|_{W_q^{s}(Q)}, && Q \in \{\Omega, S\},\ s \in \mathbb{R}_+ \cup \{0\},\ q \in [1,\infty].
\end{aligned}
$$

By c we denote a generic constant which changes its magnitude from formula to formula. By $\overline{c}(\sigma)$, $c_k(\sigma)$, $k \in \mathbb{N}$, $\varphi(\sigma)$ we understand generic functions which are always positive and increasing. Moreover, we use the abbreviation form r.h.s. (l.h.s.) for right-hand side (left-hand side). Finally, we do not distinguish the scalar and vector-valued functions.

We introduce the space

$$
V_2^k(\Omega^T) = \bigg\{ u : \|u\|_{V_2^k(\Omega^T)} = \operatorname*{ess\,sup}_{t\in(0,T)} \|u\|_{H^k(\Omega)} + \bigg(\int_0^T \|\nabla u(t)\|^2_{H^k(\Omega)}\,dt \bigg)^{1/2} < \infty \bigg\}, \quad k \in \mathbb{N}.
$$

Now we recall a certain imbedding for anisotropic Sobolev spaces. Let $\Omega \subset \mathbb{R}^3$. Then we define

$$
\|u\|_{W_2^{1,k}(\Omega)} = \bigg[\int_\Omega (|u|^2 + |\nabla' u|^2 + |\nabla^k_{x_3} u|^2)\,dx \bigg]^{1/2}, \quad k \in \mathbb{N},
$$

where $\nabla' = (\partial_{x_1}, \partial_{x_2})$. From [6] we have

$$
|u|_{q,r,\Omega^T} \le c\|u\|^{2/r}_{L_2(0,T;W_2^{1,k}(\Omega))} \operatorname*{ess\,sup}_t |u|^{1-2/r}_{2,\Omega}, \tag{2.1}
$$

where

$$
\frac{2}{r} + \frac{2k+1}{qk} = \frac{2k+1}{2k}. \tag{2.2}
$$

For $r = q$ we have

$$
q = \frac{2(4k+1)}{2k+1} \equiv q(k) \tag{2.3}
$$

and the inequality

$$
|u|_{q,\Omega^T} \le c\|u\|^{2/q}_{L_2(0,T;W_2^{1,k}(\Omega))} \operatorname*{ess\,sup}_t |u|^{1-2/q}_{2,\Omega}, \tag{2.4}
$$

where $2/q < 1$. By the Young inequality (2.4) gives

$$|u|_{q,\Omega^T} \le \varepsilon^{q/2}\|u\|_{L_2(0,T;W_2^{1,k}(\Omega))} + c\varepsilon^{-q/(q-2)}\operatorname*{ess\,sup}_t |u|_{2,\Omega}, \tag{2.5}$$

where $\varepsilon \in \mathbb{R}_+$.

From (2.3) we see that $q(1) = 10/3$, $q(2) = 18/5$, $q(3) = 26/7$, $q(4) = 34/9$. Moreover,

$$\lim_{k\to\infty} q(k) = 4.$$

In this paper we use frequently the imbedding

$$|\nabla^i u|_{p,q,\Omega^T} \le c\|u\|_{2+\beta,\delta,\Omega^T}, \quad i \le 2,$$

which holds for

$$\frac{5}{\delta} - \frac{3}{p} - \frac{2}{q} + i \le 2 + \beta. \tag{2.6}$$

In the case where either p or q is equal to ∞ then we have the sharp inequality in (2.6). Moreover, from [2, Chapter 3, Section 9] we have

$$|u|_{2(2k+1),\Omega} \le c\|u\|_{W_2^{1,k}(\Omega)}, \quad \Omega \subset \mathbb{R}^3. \tag{2.7}$$

DEFINITION 2.1. By a weak solution to problem (1.8) we mean a function w satisfying the following integral identity

$$\begin{aligned}\int_{\Omega^T} w_{,t}\psi\,dx\,dt &+ \int_{\Omega^T} H(w)\cdot\psi\,dx\,dt \\ &+ \nu\int_{\Omega^T}\mathbb{D}(w)\cdot\mathbb{D}(\psi)\,dx\,dt + \gamma\sum_{\alpha=1}^2\int_{S_1^T} w\cdot\overline{\tau}_\alpha\psi\cdot\overline{\tau}_\alpha\,dS_1\,dt \\ &- \sum_{\sigma,\alpha=1}^2\int_{S_\sigma^T} B_{\sigma\alpha}\psi\cdot\overline{\tau}_\alpha\,dS_\sigma\,dt = \int_{\Omega^T} F\cdot\psi\,dx\,dt,\end{aligned} \tag{2.8}$$

where $H(w) = w\cdot\nabla w + w\cdot\nabla\delta + \delta\cdot\nabla w$, which holds for any sufficiently smooth function ψ such that

$$\operatorname{div}\psi = 0, \quad \psi\cdot\overline{n}|_S = 0,$$

and the first integral of (2.8) is understood in the distributional sense.

Now we shall obtain an estimate for the weak solutions to problem (1.8).

LEMMA 2.2. *Assume that* $d_1 \in L_{3,6}(S_2^T)$, $\nabla\alpha \in L_2(0,T;L_3(\Omega))$, $w(0) \in L_2(\Omega)$,

$$\Gamma^2(t) = |\alpha|_{2,S_1}^2 + |\alpha_{,t}|_{6/5,\Omega}^2 + |\alpha_{,x_3t}|_{6/5,\Omega}^2 + (1 + \|\alpha\|_{1,3,\Omega}^2)|\nabla\alpha|_{2,\Omega}^2 + |f|_{6/5,\Omega}^2$$

and

$$\int_0^T \Gamma^2(t)\,dt < \infty.$$

Then, for $t \le T$, the following estimate holds

$$\|w\|^2_{V_2^1(\Omega^t)} \le ce^{c(|d_1|^6_{3,6,S_2^t}+|\nabla\alpha|^2_{3,2,\Omega^t})}\bigg(\int_0^t \Gamma^2(t')\,dt' + |w(0)|^2_{2,\Omega}\bigg). \tag{2.9}$$

Proof. Omitting the integral with respect to time in (2.8), inserting $\psi = w$ and using $(1.8)_3$ yield

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|w|^2_{2,\Omega} + \int_\Omega [w\cdot\nabla\delta\cdot w + \delta\cdot\nabla ww]\,dx + \nu|\mathbb{D}(w)|^2_{2,\Omega} + \gamma|w\cdot\overline{\tau}_\alpha|^2_{2,S_1} \\ = \sum_{\sigma,\alpha=1}^2 \int_{S_\sigma} B_{\sigma\alpha} w\cdot\overline{\tau}_\alpha\,dS_\sigma + \int_\Omega F\cdot w\,dx,\end{aligned} \tag{2.10}$$

where such procedure can be justified by considerations from [5, Chapter 3].

Now we examine the second term on the l.h.s. of (2.10). First we consider

$$\begin{aligned}\int_\Omega \delta\cdot\nabla w\cdot w\,dx &= \int_\Omega (b+\nabla\varphi)\cdot\nabla w\cdot w\,dx = \int_\Omega (\alpha\overline{e}_3+\nabla\varphi)\cdot\nabla w\cdot w\,dx \\ &= \int_\Omega \alpha w_{,x_3}\cdot w\,dx + \int_\Omega \nabla\varphi\cdot\nabla w\cdot w\,dx \equiv I_1 + I_2,\end{aligned}$$

where

$$I_1 = \frac{1}{2}\int_\Omega \alpha(w^2)_{,x_3}\,dx = \frac{1}{2}\int_\Omega (\alpha w^2)_{,x_3}\,dx - \frac{1}{2}\int_\Omega \alpha_{,x_3} w^2\,dx \equiv I_1^1 + I_1^2. \tag{2.11}$$

Integrating in I_1^1 yields

$$I_1^1 = -\frac{1}{2}\int_{S_2(-a)} d_1 w^2\,dx' + \frac{1}{2}\int_{S_2(a)} d_2 w^2\,dx'.$$

Hence

$$|I_1^1| \le \varepsilon_1'|w_{,x}|^2_{2,\Omega} + c(1/\varepsilon_1')|d_1|^6_{3,S_2}|w|^2_{2,\Omega}.$$

Moreover,

$$|I_1^2| \le \varepsilon_1''|w|^2_{6,\Omega} + c(1/\varepsilon_1'')|\alpha_{,x_3}|^2_{3,\Omega}|w|^2_{2,\Omega}.$$

Summmarizing, we have

$$|I_1| \le \varepsilon_1(|w|^2_{6,\Omega} + |w_{,x}|^2_{2,\Omega}) + c(1/\varepsilon_1)(|d_1|^6_{3,S_2} + |\alpha_{,x_3}|^2_{3,\Omega})|w|^2_{2,\Omega}.$$

Next we examine

$$I_2 = \frac{1}{2}\int_\Omega \nabla\varphi\cdot\nabla w^2\,dx = -\frac{1}{2}\int_\Omega \Delta\varphi w^2\,dx = \frac{1}{2}\int_\Omega \alpha_{,x_3} w^2\,dx,$$

where (1.5), (1.6) were used. Hence

$$|I_2| \le \varepsilon_2|w|^2_{6,\Omega} + c(1/\varepsilon_2)|\alpha_{,x_3}|^2_{3,\Omega}|w|^2_{2,\Omega}.$$

Next we consider

$$\int_\Omega w\cdot\nabla\delta\cdot w\,dx = \int_\Omega w\cdot\nabla\alpha w_3\,dx + \int_\Omega w_i\nabla_i\nabla_j\varphi w_j\,dx \equiv I_3 + I_4,$$

where

$$|I_3| \le \varepsilon_3 |w|_{6,\Omega}^2 + c(1/\varepsilon_3)|\nabla\alpha|_{3,\Omega}^2 |w|_{2,\Omega}^2,$$
$$|I_4| \le \varepsilon_4 |w|_{6,\Omega}^2 + c(1/\varepsilon_4)|\nabla\nabla\varphi|_{3,\Omega}^2 |w|_{2,\Omega}^2 \le \varepsilon_4 |w|_{6,\Omega}^2 + c(1/\varepsilon_4)|\alpha_{,x_3}|_{3,\Omega}^2 |w|_{2,\Omega}^2.$$

Utilizing the above estimates and Lemma 2.4 in (2.10) yields

$$\frac{d}{dt}|w|_{2,\Omega}^2 + \nu\|w\|_{1,\Omega}^2 + \gamma|w\cdot\overline{\tau}_\alpha|_{2,S_1}^2 \le c(|d_1|_{3,S_2}^6 + |\nabla\alpha|_{3,\Omega}^2)|w|_{2,\Omega}^2 + \sum_{\sigma,\alpha=1}^2 \int_{S_\sigma} B_{\sigma\alpha} w\cdot\overline{\tau}_\alpha\, dS_\sigma + \int_\Omega F\cdot w\, dx. \tag{2.12}$$

The last two terms on the r.h.s. of (2.12) take the form

$$-\int_{S_1} (\nu\overline{n}\cdot\mathbb{D}(\delta)\cdot\overline{\tau}_\alpha w\cdot\overline{\tau}_\alpha + \gamma\delta\cdot\overline{\tau}_\alpha w\cdot\overline{\tau}_\alpha)\, dS_1 - \int_{S_2} \nu\overline{n}\cdot\mathbb{D}(\delta)\cdot\overline{\tau}_\alpha w\cdot\overline{\tau}_\alpha\, dS_2 + \int_\Omega (f - \delta_{,t} - \delta\cdot\nabla\delta) w\, dx + \nu\int_\Omega \operatorname{div}\mathbb{D}(\delta)\cdot w\, dx \equiv J_1.$$

Integrating by parts in the last integral of J_1 we obtain

$$J_1 = -\gamma\int_{S_1} \delta\cdot\overline{\tau}_\alpha w\cdot\overline{\tau}_\alpha\, dS_1 + \int_\Omega (f - \delta_{,t} - \delta\cdot\nabla\delta)\cdot w\, dx - \nu\int_\Omega \mathbb{D}(\delta)\cdot\mathbb{D}(w)\, dx \equiv J_2 + J_3 + J_4.$$

Utilizing the form of δ we have

$$|J_2| \le \varepsilon_1\|w\|_{1,\Omega}^2 + c(1/\varepsilon_1)(|\alpha|_{2,S_1}^2 + |\overline{\tau}_\alpha\cdot\nabla\varphi|_{2,S_1}^2) \le \varepsilon_1\|w\|_{1,\Omega}^2 + c(1/\varepsilon_1)(|\alpha|_{2,S_1}^2 + |\nabla\alpha|_{2,\Omega}^2)$$
$$|J_3| \le \varepsilon_2|w|_{6,\Omega}^2 + c(1/\varepsilon_2)(|f|_{6/5,\Omega}^2 + |\delta_{,t}|_{6/5,\Omega}^2 + |\delta\cdot\nabla\delta|_{6/5,\Omega}^2),$$

where

$$|\delta_{,t}|_{6/5,\Omega} \le |\alpha_{,t}|_{6/5,\Omega} + |\nabla\varphi_{,t}|_{6/5,\Omega} \le |\alpha_{,t}|_{6/5,\Omega} + \Big|\int_\Omega \nabla G\alpha_{,y_3t}\, dy_3\Big|_{6/5,\Omega} \le c(|\alpha_{,t}|_{6/5,\Omega} + |\alpha_{,x_3t}|_{6/5,\Omega}),$$

where G is the Green function to problem (1.6) and

$$|\delta\cdot\nabla\delta|_{6/5,\Omega} \le |\delta|_{3,\Omega}|\nabla\delta|_{2,\Omega} \le c\|\alpha\|_{1,3,\Omega}|\nabla\alpha|_{2,\Omega}.$$

Finally,

$$|J_4| \le \varepsilon_3\|w\|_{1,\Omega}^2 + c(1/\varepsilon_3)|\mathbb{D}(\delta)|_{2,\Omega}^2,$$

where

$$|\mathbb{D}(\delta)|_{2,\Omega} \le (|\nabla\alpha|_{2,\Omega} + |\nabla\nabla\varphi|_{2,\Omega}) \le c|\nabla\alpha|_{2,\Omega}.$$

Summarizing

$$\begin{aligned}|J_1| \le \varepsilon\|w\|_{1,\Omega}^2 + c(1/\varepsilon)(|\alpha|_{2,S_1}^2 + |f|_{6/5,\Omega}^2 \\ + |\alpha_{,t}|_{6/5,\Omega}^2 + |\alpha_{,x_3t}|_{6/5,\Omega}^2 + \|\alpha\|_{1,3,\Omega}^2|\nabla\alpha|_{2,\Omega}^2 + |\nabla\alpha|_{2,\Omega}^2).\end{aligned}$$

In view of the above estimates, (2.12) takes the form

$$\begin{aligned}(2.13)\quad & \frac{d}{dt}|w|_{2,\Omega}^2 + \nu\|w\|_{1,\Omega}^2 + \gamma|w\cdot\bar\tau_\alpha|_{2,S_1}^2 \\ & \le c(|d_1|_{3,S_2}^6 + |\nabla\alpha|_{3,\Omega}^2)|w|_{2,\Omega}^2 + c(|\alpha|_{2,S_1}^2 + |f|_{6/5,\Omega}^2 \\ & \quad + |\alpha_{,t}|_{6/5,\Omega}^2 + |\alpha_{,x_3t}|_{6/5,\Omega}^2 + \|\alpha\|_{1,3,\Omega}^2|\nabla\alpha|_{2,\Omega}^2 + |\nabla\alpha|_{2,\Omega}^2) \\ & = c(|d_1|_{3,S_2}^6 + |\nabla\alpha|_{3,\Omega}^2)|w|_{2,\Omega}^2 + c\Gamma^2(t).\end{aligned}$$

From (2.13) we have

$$(2.14)\quad \frac{d}{dt}(|w|_{2,\Omega}^2 e^{-c(|d_1|_{3,6,S_2^t}^6 + |\nabla\alpha|_{3,2,\Omega^t}^2)}) \le c\Gamma^2(t)e^{-c(|d_1|_{3,6,S_2^t}^6 + |\nabla\alpha|_{3,2,\Omega^t}^2)}$$

Integrating (2.14) with respect to time yields (2.9) for $|w|_{2,\Omega}$ only.

Next we obtain from (2.13) the inequality

$$\begin{aligned}(2.15)\quad & \frac{d}{dt}(|w|_{2,\Omega}^2\, e^{-c(|d_1|_{3,6,S_2^t}^6 + |\nabla\alpha|_{3,2,\Omega^t}^2)}) \\ & + (\|w\|_{1,\Omega}^2 + \gamma|w\cdot\bar\tau_\alpha|_{2,S_1}^2)\, e^{-c(|d_1|_{3,6,S_2^t}^6 + |\nabla\alpha|_{3,2,\Omega^t}^2)} \\ & \le c\Gamma^2(t)e^{-c(|d_1|_{3,6,S_2^t}^6 + |\nabla\alpha|_{3,2,\Omega^t}^2)}.\end{aligned}$$

Integrating (2.15) with respect to time yields

$$\begin{aligned}(2.16)\quad & |w(t)|_{2,\Omega}^2 + e^{c(|d_1|_{3,6,S_2^t}^6 + |\nabla\alpha|_{3,2,\Omega^t}^2)} \\ & \cdot\int_0^t (\nu\|w(t')\|_{1,\Omega}^2 + \gamma|w(t')\cdot\bar\tau_\alpha|_{2,S_1}^2)\cdot e^{-c(|d_1|_{3,6,S_2^{t'}}^6 + |\nabla\alpha|_{3,2,\Omega^{t'}}^2)}\,dt', \\ & \le ce^{c(|d_1|_{3,6,S_2^t}^6 + |\nabla\alpha|_{3,2,\Omega^t}^2)} \\ & \cdot\bigg(\int_0^t \Gamma^2(t')e^{-c(|d_1|_{3,6,S_2^{t'}}^6 + |\nabla\alpha|_{3,2,\Omega^{t'}}^2)}\,dt' + |w(0)|_{2,\Omega}^2\bigg).\end{aligned}$$

Simplifying (2.16) implies

$$\begin{aligned}(2.17)\quad & |w(t)|_{2,\Omega}^2 + \int_0^t (\nu\|w(t')\|_{1,\Omega}^2 + \gamma|w(t')\cdot\bar\tau_\alpha|_{2,S_1}^2)\,dt' \\ & \le ce^{c(|d_1|_{3,6,S_2^t}^6 + |\nabla\alpha|_{3,2,\Omega^t}^2)}\bigg(\int_0^t \Gamma^2(t')\,dt' + |w(0)|_{2,\Omega}^2\bigg).\end{aligned}$$

Omitting the first term on the l.h.s. of (2.17) we obtain

$$\begin{aligned}(2.18)\quad &\int_0^t (\nu\|w(t')\|_{1,\Omega}^2 + \gamma|w(t')\cdot\overline{\tau}_\alpha|_{2,S_1}^2)\,dt' \\ &\quad \le c e^{c(|\nabla\alpha|_{3,2,\Omega^t}^2 + |d_1|_{3,6,S_2^t}^6)}\bigg(\int_0^t \Gamma^2(t')\,dt' + |w(0)|_{2,\Omega}^2\bigg).\end{aligned}$$

From (2.18) and (2.9) for $|w|_{2,\Omega}$ we get (2.9). This ends the proof. □

Using that

$$\begin{aligned}(2.19)\quad &|\delta|_{2,\infty,\Omega^T}^2 + \nu\int_0^T \|\delta(t)\|_{1,\Omega}^2\,dt + \gamma\int_0^T |\delta(t)\cdot\overline{\tau}_\alpha|_{2,S_1}^2\,dt \\ &\quad \le c(|\alpha|_{2,\infty,\Omega^T}^2 + |\alpha_{,x_3}|_{2,\infty,\Omega^T}^2) + \int_0^T \|\alpha(t)\|_{1,\Omega}^2\,dt < \infty,\end{aligned}$$

we obtain from Lemma 2.2 the result

Lemma 2.3. *Let the assumptions of Lemma* 2.2 *and* (2.19) *hold. Then the following estimate is valid*

$$\begin{aligned}(2.20)\quad \|v\|_{V_2^0(\Omega^T)}^2 &\le c e^{c(|d_1|_{3,6,S_2^T}^6 + |\nabla\alpha|_{3,2,\Omega^T}^2)}\bigg(\int_0^T \Gamma^2(t)\,dt + |v(0)|_{2,\Omega}^2 \\ &\quad + \|\alpha\|_{2,\infty,\Omega^T}^2 + \|\alpha_{,x_3}\|_{2,\infty,\Omega^T}^2 + \int_0^T \|\alpha(t)\|_{1,\Omega}^2\,dt\bigg) \equiv l_1^2(T),\end{aligned}$$

where $\Gamma(t)$ *is defined by assumptions of Lemma* 2.2.

Finally, we prove the Korn inequality.

Lemma 2.4. *Assume that*

$$(2.21)\qquad E_\Omega(w) = \int_\Omega (w_{i,x_j} + w_{j,x_i})^2\,dx < \infty,$$

where the summation convention over the repeated indices is assumed. Assume also that

$$(2.22)\qquad \sum_{\alpha=1}^2 |w\cdot\overline{\tau}_\alpha|_{2,S_1}^2 < \infty, \quad w\cdot\overline{n}|_S = 0,\ \operatorname{div} w = 0.$$

Then there exists a constant c independent of w such that

$$(2.23)\qquad \|w\|_{1,\Omega}^2 \le c\Big(E_\Omega(w) + \sum_{\alpha=1}^2 |w\cdot\overline{\tau}_\alpha|_{2,S_1}^2\Big) \equiv cI.$$

Proof. Directly from (2.21) it follows

$$E_\Omega(w) = 2\int_\Omega w_{i,x_j}^2\,dx + 2\int_\Omega w_{i,x_j}\cdot w_{j,x_i}\,dx,$$

where in view of $(2.22)_2$ the second integral equals

$$-\int_{S_1} n_{i,x_j} w_i w_j \, dS_1,$$

where we used also that $n_i|_{S_2}$ does not depend on x', $x' = (x_1, x_2)$. Hence, we have

$$|\nabla w|_{2,\Omega}^2 \le cI. \tag{2.24}$$

In the non-axially symmetric case by the contradiction argument (see [7]) we obtain that for any $\delta > 0$ there exists $M(\delta)$ such that

$$|w|_{2,\Omega}^2 \le \delta |\nabla w|_{2,\Omega}^2 + M(\delta) E_\Omega(w). \tag{2.25}$$

From (2.24) and (2.25) we get (2.23).

In the axially symmetric case we have also (2.24). Let

$$w = w' + \frac{\alpha}{|\eta|_{2,\Omega}^2}\eta,$$

where $\eta = (-x_2, x_1, 0)$, $\int_\Omega w' \cdot \eta \, dx = 0$, $\alpha = \int_\Omega w_\eta \, dx$, $w_\eta = w \cdot \eta$.

Since for w' (2.25) holds, we have

$$\begin{aligned} |w'|_{2,\Omega}^2 &\le \delta |\nabla w'|_{2,\Omega}^2 + M(\delta) E_\Omega(w') \\ &= \delta (|\nabla w|_{2,\Omega}^2 + c|\alpha|^2) + cI \le \delta |w_\eta|_{2,\Omega}^2 + cI, \end{aligned} \tag{2.26}$$

where $E_\Omega(\eta) = 0$ was utilized. Using that

$$|w_\varphi|_{2,\Omega}^2 \le c(|w_\varphi|_{2,S_1}^2 + |\partial_r w_\varphi|_{2,\Omega}^2) \le cI, \tag{2.27}$$

where $w_\varphi = w \cdot \bar{e}_\varphi$, $\bar{e}_\varphi = (-\sin\varphi, \cos\varphi, 0)$ and φ is one of the cylindrical coordinates such that $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$.

Assuming that the considered domain is a cylinder of the radius R we have

$$|w_\eta|_{2,\Omega} \le R |w_\varphi|_{2,\Omega}.$$

Hence (2.27) implies

$$|w_\eta|_{2,\Omega} \le cI. \tag{2.28}$$

From (2.24), (2.26) and (2.28) we obtain (2.23) in the axially symmetric case. This ends the proof. □

Let us assume that the extension α defined by (1.4) can be expressed in the form

$$\alpha = \eta_1 d_1 + \eta_2 d_2,$$

where $\eta_i = \eta_i(x_3)$, $i = 1, 2$, is a partition of unity such that $\eta_1 = 1$, $\eta_2 = 0$ near $S_2(-a)$ and $\eta_1 = 0$, $\eta_2 = 1$ near $S_2(a)$.

Lemma 2.5. *Assume that* $d \in L_\infty(0,T;W_3^1(S_2)) \cap L_2(0,T;H^1(S_2))$, $d_{,t} \in L_2(0,T;L_{6/5}(S_2))$, $f \in L_2(0,T;L_{6/5}(\Omega))$, $v(0) \in L_2(\Omega)$. *Then*

$$\|v\|_{V_2^0(\Omega^t)} \le l_1(t),$$

$$(2.29)\qquad l_1^2(t) \le \varphi(\|d\|_{L_\infty(0,t;W_3^1(S_2))}, t)[\|d\|^2_{L_2(0,t;H^1(S_2))} + |d_{,t}|^2_{6/5,2,S_2^t} + |f|^2_{6/5,2,\Omega^t} + |v(0)|^2_{2,\Omega}],\quad t \le T,$$

where φ *is an increasing positive function.*

3. Basic formulations

To prove the existence of global solutions to problem (1.1) we follow [6]. Therefore we need problems for quantities (1.10) First we have

Lemma 3.1. *The quantities* $h^{(1)}$, $q^{(1)}$ *are solutions to the problem*

$$(3.1)\qquad \begin{aligned} & h^{(1)}_{,t} - \operatorname{div} \mathbb{T}(h^{(1)}, q^{(1)}) = -v \cdot \nabla h^{(1)} - h^{(1)} \cdot \nabla v + g^{(1)} && \text{in } \Omega^T,\\ & \operatorname{div} h^{(1)} = 0 && \text{in } \Omega^T,\\ & \overline{n} \cdot h^{(1)} = 0 && \text{on } S_1^T,\\ & \nu \overline{n} \cdot \mathbb{D}(h^{(1)}) \cdot \overline{\tau}_\alpha + \gamma h^{(1)} \cdot \overline{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S_1^T,\\ & h^{(1)}_i = -d_{,x_i}, \quad i = 1,2, && \text{on } S_2^T,\\ & h^{(1)}_{3,x_3} = \Delta' d && \text{on } S_2^T,\\ & h^{(1)}|_{t=0} = h^{(1)}(0) && \text{in } \Omega, \end{aligned}$$

where $\Delta' = \partial_{x_1}^2 + \partial_{x_2}^2$, *d replaces* d_1 *and* d_2, *because* $d|_{S_2(-a)} = d_1$, $d|_{S_2(+a)} = d_2$.

Proof. Equations $(3.1)_{1,2,3,4,7}$ follow directly from corresponding equations in (1.1) by differentiation with respect to x_3, because S_1 is parallel to the axis x_3.

To show $(3.1)_{5,6}$ we recall that

$$(3.2)\qquad v_3|_{S_2} = d, \quad (v_{i,x_3} + v_{3,x_i})|_{S_2} = 0, \quad i = 1,2.$$

Hence $v_{i,x_3}|_{S_2} = -d_{,x_i}$, $i = 1,2$, and $(3.1)_5$ holds.

From $(1.1)_2$ we have $v_{3,x_3x_3}|_{S_2} = -(v_{1,x_3x_1} + v_{2,x_3x_2})|_{S_2} = d_{,x_1x_1} + d_{,x_2x_2} = \Delta' d$. Hence $(3.1)_6$ follows. This ends the proof. □

Lemma 3.2. *The function* $\chi = v_{2,x_1} - v_{1,x_2}$ *is a solution to the problem*

$$(3.3)\qquad \begin{aligned} & \chi_{,t} + v \cdot \nabla \chi - h_3^{(1)} \chi + h_2^{(1)} w_{,x_1} - h_1^{(1)} w_{,x_2} - \nu \Delta \chi = F_3 && \text{in } \Omega^T,\\ & \chi|_{S_1} = -v_i(n_{i,x_j}\tau_{1j} + \tau_{1i,x_j} n_j) + \frac{\gamma}{\nu} v_j \tau_{1j} \\ & \qquad + v \cdot \overline{\tau}_1(\tau_{12,x_1} - \tau_{11,x_2}) \equiv \chi_* && \text{on } S_1^T,\\ & \chi_{,x_3} = 0 && \text{on } S_2^T,\\ & \chi|_{t=0} = \chi(0) && \text{in } \Omega, \end{aligned}$$

where $F_3 = f_{2,x_1} - f_{1,x_2}$, $\overline{n}, \overline{\tau}_1, \overline{\tau}_2$ *are defined by* (1.9) *and* $w = v_3$.

Proof. Differentiating the first equation of $(1.1)_1$ with respect to x_2, the second equation of $(1.1)_1$ with respect to x_1, and subtracting the results yield $(3.3)_1$.

To show $(3.3)_2$ we extend vectors $\overline{\tau}_1$, $\overline{n}$ into neighbourhood of S_1. In this neighbourhood $v' = (v_1, v_2)$ can be expressed in the form

$$v' = v \cdot \overline{\tau}_1 \overline{\tau}_1 + v \cdot \overline{n}\overline{n}.$$

Then

$$\begin{aligned} \chi|_{S_1} &= [(v \cdot \overline{\tau}_1 \tau_{12} + v \cdot \overline{n} n_2)_{,x_1} - (v \cdot \overline{\tau}_1 \tau_{11} + v \cdot \overline{n} n_1)_{,x_2}]|_{S_1} \\ &= [-\overline{n} \cdot \nabla(v \cdot \overline{\tau}_1) + v \cdot \overline{\tau}_1 (\tau_{12,x_1} - \tau_{11,x_2}]|_{S_1}, \end{aligned} \tag{3.4}$$

where $(1.1)_3$ was employed and τ_{1i}, n_i are the i-th Cartesian coordinates.

Utilizing $(1.1)_3$ in $(1.1)_4$ for $\alpha = 1$ yields

$$\nu \overline{n} \cdot \nabla(v \cdot \overline{\tau}_1) - \nu v_i (n_{i,x_j} \tau_{1j} + \tau_{1i,x_j} n_j) + \gamma v \cdot \overline{\tau}_1 = 0. \tag{3.5}$$

Exploiting (3.5) in (3.4) yields $(3.3)_2$. By the definition of χ and $(3.1)_5$ we have

$$\chi_{,x_3}|_{S_2} = (v_{2,x_1x_3} - v_{1,x_2x_3})|_{S_2} = -(d_{,x_1x_2} - d_{,x_2x_1})|_{S_2} = 0.$$

This ends the proof. □

Subsequently we formulate a problem for w.

Lemma 3.3. *Function w is a solution to the problem*

$$\begin{aligned} & w_{,t} + v \cdot \nabla w - \nu \Delta w = q^{(1)} + f_3 && \text{in } \Omega^T, \\ & w_{,n} + \gamma w = 0 && \text{on } S_1^T, \\ & w = d && \text{on } S_2^T, \\ & w|_{t=0} = w(0) && \text{in } \Omega. \end{aligned} \tag{3.6}$$

Proof. We have to show $(3.6)_2$ only. Equation $(1.1)_4$ for $\alpha = 2$ takes the form

$$n_i (v_{i,x_3} + v_{3,x_i}) + \gamma v_3 = 0 \quad \text{on } S_1^T. \tag{3.7}$$

Since $\overline{n}|_{S_1}$ does not depend on x_3, (3.7) implies $(3.6)_2$. □

Let Ω' be a cross-section of Ω by the plane P perpendicular to the axis x_3. Then $\partial\Omega' = S_1 \cap P = S_1'$. Therefore, we can consider the problem

$$\begin{aligned} v_{2,x_1} - v_{1,x_2} &= \chi && \text{in } \Omega', \\ v_{1,x_1} + v_{2,x_2} &= -h_3^{(1)} && \text{in } \Omega', \\ v' \cdot \overline{n}' &= 0 && \text{on } S_1', \end{aligned} \tag{3.8}$$

where $v' = (v_1, v_2)$, $\overline{n}' = (1/|\nabla\varphi|)(\varphi_{,x_1}, \varphi_{,x_2})$, $\Omega' = \Omega \cap \{\text{plane}\, x_3 = \text{const} \in (-a,a)\}$, $S_1' = S_1 \cap \{\text{plane}\, x_3 = \text{const} \in (-a,a)\}$, and x_3, t are treated as parameters.

LEMMA 3.4. *Quantities $h^{(2)}$, $q^{(2)}$ are solutions of the problem*

$$
(3.9)\quad
\begin{aligned}
&h^{(2)}_{,t} - \operatorname{div}\mathbb{T}(h^{(2)}, q^{(2)}) = -h^{(2)}\cdot\nabla v - 2h^{(1)}\cdot\nabla h^{(1)} && \\
&\qquad - v\cdot\nabla h^{(2)} + g^{(2)} \equiv \gamma_0 && \text{in } \Omega^T,\\
&\operatorname{div} h^{(2)} = 0 && \text{in } \Omega^T,\\
&\overline{n}\cdot h^{(2)} = 0 && \text{on } S_1^T,\\
&\nu\overline{n}\cdot\mathbb{D}(h^{(2)})\cdot\overline{\tau}_\alpha + \gamma h^{(2)}\cdot\overline{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S_1^T,\\
&h^{(2)}_3 = \Delta' d \equiv \gamma_3 && \text{on } S_2^T,\\
&h^{(2)}_{i,x_3} = -\frac{2}{\nu} d_{,x_i t} + 3\Delta' d_{,x_i} + \frac{1}{\nu}F_i'|_{S_2} + \frac{1}{\nu}[v_j(h^{(1)}_i - d_{,x_i})_{,x_j} && \\
&\qquad + d(h^{(1)}_{i,x_3} - h^{(1)}_{3,x_i}) + h^{(1)}_3(h^{(1)}_i - d_{,x_i}) && \\
&\qquad + h^{(1)}_j v_{i,x_j} - d_{,x_j} v_{j,x_i}]|_{S_2} \equiv \gamma_i, \quad i = 1,2, && \text{on } S_2^T,\\
&h^{(2)}|_{t=0} = h^{(2)}(0) && \text{in } \Omega,
\end{aligned}
$$

where $d_1 = d|_{S_2(-a)}$, $d_2 = d|_{S_2(a)}$, $\Delta' = \partial_{x_1}^2 + \partial_{x_2}^2$, $F_i' = f_{3,x_i} - f_{i,x_3}$ and $\nabla' d = (d_{,x_1}, d_{,x_2})$. Moreover, the summation convention over j from 1 to 2 is used in $(3.9)_6$.

PROOF. We have to prove $(3.9)_{5,6}$ only. Since

$$
(3.10)\qquad v_3|_{S_2} = d, \quad \frac{\partial v_i}{\partial x_3}\bigg|_{S_2} = -\frac{\partial v_3}{\partial x_i}\bigg|_{S_2} = -d_{,x_i}, \quad i = 1,2,
$$

we get

$$
(3.11)\qquad h^{(2)}_3 = \frac{\partial^2 v_3}{\partial x_3^2} = -\sum_{i=1}^2 \frac{\partial^2 v_i}{\partial x_i \partial x_3} = d_{,x_1x_1} + d_{,x_2x_2} \quad \text{on } S_2.
$$

Hence $(3.9)_5$ holds.

To show $(3.9)_6$ we consider first two components of $(1.1)_1$,

$$
(3.12)\qquad v_{i,t} - \nu\Delta' v_i - \nu\partial_{x_3}^2 v_i + \nabla_i p = -v\cdot\nabla v_i + f_i, \quad i = 1,2.
$$

Differentiating (3.12) with respect to x_3 and projecting the result on S_2 yield

$$
(3.13)\qquad
\begin{aligned}
h^{(2)}_{i,x_3}|_{S_2} = {}& -\frac{1}{\nu} d_{,x_i t} + \Delta' d_{,x_i} + \frac{1}{\nu}\nabla_i q^{(1)}|_{S_2}\\
&+ \frac{1}{\nu}(h^{(1)}\cdot\nabla v_i + v\cdot\nabla h^{(1)}_i)|_{S_2} - \frac{1}{\nu} g_i|_{S_2}, \quad i = 1,2.
\end{aligned}
$$

To calculate the third term on the r.h.s. of (3.13) we calculate $q^{(1)}$ from the third component of (3.12)

$$q^{(1)}|_{S_2} = (-v_{3,t} + \nu\Delta' v_3 + \nu\partial_{x_3}^2 v_3 - v\cdot\nabla v_3 + f_3)|_{S_2} = -d_{,t} + 2\nu\Delta' d - \bigg(\sum_{i=1}^2 v_i v_{3,x_i} + v_3 v_{3,x_3}\bigg)\bigg|_{S_2} + f_3|_{S_2}, \tag{3.14}$$

where we used that $\partial_{x_3}^2 v_3|_{S_2} = \Delta' d$.

Differentiating (3.14) with respect to x_i and utilizing in (3.13) gives

$$h_{i,x_3}^{(2)} = -\frac{2}{\nu} d_{,x_i t} + 3\Delta' d_{,x_i} + \frac{1}{\nu}[(v\cdot\nabla v_i)_{,x_3} - (v\cdot\nabla v_3)_{,x_i}]|_{S_2} + \frac{1}{\nu}(f_{3,x_i} - f_{i,x_3})|_{S_2}, \quad i=1,2. \tag{3.15}$$

Now we examine the third term on the r.h.s. of (3.15). Utilizing (3.10) we get

$$\begin{aligned} &[(v_j\cdot v_{i,x_j} + v_3 v_{i,x_3})_{,x_3} - (v_j v_{3,x_j} + v_3 v_{3,x_3})_{,x_i}]|_{S_2} \\ &= [v_j(v_{i,x_3} - v_{3,x_i})_{,x_j} + v_3(v_{i,x_3} - v_{3,x_i})_{,x_3} \\ &\quad + v_{3,x_3}(v_{i,x_3} - v_{3,x_i}) + v_{j,x_3} v_{i,x_j} - v_{j,x_i} v_{3,x_j}]|_{S_2} \\ &= [v_j(h_i^{(1)} - d_{,x_i})_{,x_j} + d(h_{,x_3}^{(1)} - h_{3,x_i}^{(1)}) \\ &\quad + h_3^{(1)}(h_i^{(1)} - d_{,x_i}) + h_j^{(1)} v_{i,x_j} - d_{,x_j} v_{j,x_i}]|_{S_2}, \quad i=1,2, \end{aligned} \tag{3.16}$$

where the summation over j from 1 to 2 is used. Exploiting (3.16) in (3.15) yields $(3.9)_6$. □

To find the energy estimate for solutions of problem (3.1) we are looking for a function $\widetilde{h}^{(1)}$ such that

$$\begin{aligned} \operatorname{div}\widetilde{h}^{(1)} &= 0 \qquad \text{in } \Omega, \\ \widetilde{h}^{(1)}|_{S_1} &= 0, \\ \widetilde{h}_i^{(1)}|_{S_2} &= -d_{,x_i}, \quad i=1,2, \\ \widetilde{h}_{3,x_3}^{(1)}|_{S_2} &= 0. \end{aligned} \tag{3.17}$$

First we construct explicitly the function

$$\begin{aligned} \overline{h}_i^{(1)}|_{S_2} &= -d_{,x_i}, \quad i=1,2, \\ \overline{h}_3^{(1)}|_{S_2} &= 0. \end{aligned}$$

The construction with be done in such a way that $\overline{h}^{(1)}\cdot\overline{n}|_{S_1} = 0$. This is possible because the following compatibility condition

$$\sum_{i=1}^2 \overline{n}_i|_{S_1}\cdot d_{,x_i}|_{\overline{S}_1\cap\overline{S}_2} = 0$$

holds.

Let η_i, $i=1,2$, be the functions introduced before (see the end of Section 2). Then we construct $\overline{h}^{(1)}$ in the form

$$\begin{aligned} \overline{h}_i^{(1)} &= -(\eta_1 d_{1,x_i} + \eta_2 d_{2,x_i}), \quad i=1,2,\\ \overline{h}_3^{(1)} &= 0. \end{aligned} \tag{3.18}$$

Hence, $\overline{h}^{(1)}$ is a solution to the problem

$$\begin{aligned} \operatorname{div} \overline{h}^{(1)} &= -(\eta_1 \Delta' d_1 + \eta_2 \Delta' d_2),\\ \overline{h}^{(1)} \cdot \overline{\tau}_\alpha|_{S_1} &= \overline{h}^{(1)} \cdot \overline{\tau}_\alpha|_{S_1}, && \alpha=1,2,\\ \overline{h}^{(1)} \cdot \overline{n}|_{S_1} &= 0,\\ \overline{h}_i^{(1)}|_{S_2} &= -d_{,x_i}, && i=1,2,\\ \overline{h}_3^{(1)}|_{S_2} &= 0. \end{aligned} \tag{3.19}$$

Now we define a function φ such that

$$\Delta\varphi = -(\eta_1\Delta' d_1 + \eta_2 \Delta' d_2), \qquad \overline{n}\cdot\nabla\varphi|_S = 0, \tag{3.20}$$

and we are looking for a function α such that

$$\begin{aligned} -\Delta\alpha + \nabla\sigma &= 0,\\ \operatorname{div}\alpha &= 0,\\ \alpha\cdot\overline{\tau}_\alpha|_{S_1} &= -\overline{\tau}_\alpha\cdot\nabla\varphi|_{S_1} + \overline{h}^{(1)}\cdot\overline{\tau}_\alpha|_{S_1}, && \alpha=1,2,\\ \alpha\cdot\overline{n}|_{S_1} &= 0,\\ \alpha_i|_{S_2} &= -\nabla_i\varphi|_{S_2}, && i=1,2,\\ \alpha_3|_{S_2} &= 0. \end{aligned} \tag{3.21}$$

Then

$$\widetilde{h}^{(1)} = \overline{h}^{(1)} - (\alpha + \nabla\varphi) \tag{3.22}$$

is a solution to problem (3.17).

The above construction of solutions to (3.19) can be found in [4].

Lemma 3.5. *Function* (3.22) *satisfies*

$$\begin{aligned} \|\widetilde{h}^{(1)}\|_{1,\sigma,\Omega} &\le c\|d_{,x'}\|_{1,\sigma,S_2},\\ |\widetilde{h}_{,t}^{(1)}|_{\sigma,\Omega} &\le c|d_{,x't}|_{\sigma,S_2}, \end{aligned} \tag{3.22'}$$

where $\sigma\in(1,\infty)$.

Proof. From (3.18) we have

$$\|\overline{h}^{(1)}\|_{1,\sigma,\Omega} \le c\|d_{,x'}\|_{1,\sigma,S_2}, \qquad |\overline{h}_{,t}^{(1)}|_{\sigma,\Omega} \le c|d_{,x't}|_{\sigma,S_2}.$$

There exists a Green function for (3.20) such that

$$\varphi(x) = \int_\Omega G(x,y)\partial_{y_i}(\eta_1 d_{1,y_i} + \eta_2 d_{2,y_i})\,dy = -\int_\Omega \nabla_y G(\eta_1 d_{1,y_i} + \eta_2 d_{2,y_i})\,dy,$$

where $n_i d_{,x_i}|_{S_1} = 0$ and $h_i^{(1)} = -d_{,x_i}$, $i = 1,2$, are used. Then

$$\nabla_x\varphi = \int \nabla_x\nabla_{y_i} G(x,y)(\eta_1 d_{1,y_i} + \eta_2 d_{2,y_i})\,dy$$

and

$$\|\nabla\varphi\|_{1,\sigma,\Omega} \le c\|d_{,x'}\|_{1,\sigma,S_2}, \qquad |\nabla\varphi_t|_{\sigma,\Omega} \le c|d_{,x't}|_{\sigma,S_2}.$$

Utilizing the existence of the Green function to problem (3.21) we have

$$\alpha_i(x) = \int_{S_1} \frac{\partial G_{i\alpha}}{\partial n_{S_1}}(-\overline{\tau}_\alpha\cdot\nabla\varphi + \overline{h}^{(1)}\cdot\overline{\tau}_\alpha)\,dS_1 + \int_{S_2}\frac{\partial G_{ij}}{\partial n_{S_2}}(-\nabla_j\varphi)\,dS_2.$$

Therefore the estimates hold

$$\begin{aligned}\|\alpha\|_{1,\sigma,\Omega} &\le c(\|\nabla\varphi\|_{1,\sigma,\Omega} + \|\overline{h}^{(1)}\|_{1,\sigma,\Omega}) \le c\|d_{,x'}\|_{1,\sigma,S_2},\\ |\alpha_{,t}|_{\sigma,\Omega} &\le c(|\nabla\varphi_{,t}|_{\sigma,\Omega} + |\overline{h}^{(1)}_{,t}|_{\sigma,\Omega}) \le c|d_{,x't}|_{\sigma,S_2}.\end{aligned}$$

From the above estimates we obtain (3.22′). □

Let us introduce the new function

$$k^{(1)} = h^{(1)} - \widetilde{h}^{(1)}. \tag{3.23}$$

Then $k^{(1)}$ is a solution to the problem

$$\begin{aligned}
&k^{(1)}_{,t} - \operatorname{div}\mathbb{T}(h^{(1)}, q^{(1)}) = -v\cdot\nabla h^{(1)} \\
&\qquad - h^{(1)}\cdot\nabla v - \widetilde{h}^{(1)}_{,t} + g^{(1)} \equiv b_1 && \text{in } \Omega^T,\\
&\operatorname{div} k^{(1)} = 0 && \text{in } \Omega^T,\\
&\overline{n}\cdot k^{(1)} = 0 && \text{on } S_1^T\\
&\nu\overline{n}\cdot\mathbb{D}(h^{(1)})\cdot\overline{\tau}_\alpha + \gamma h^{(1)}\cdot\overline{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S_1^T,\\
&k_i^{(1)} = 0, \quad i = 1,2, && \text{on } S_2^T,\\
&h^{(1)}_{3,x_3} = \Delta' d && \text{on } S_2^T,\\
&k^{(1)}|_{t=0} = h^{(1)}(0) - \widetilde{h}^{(1)}(0) && \text{in } \Omega,
\end{aligned} \tag{3.24}$$

where $h^{(1)}$ on the l.h.s. depends on $k^{(1)}$ by (3.23).

Now we find a Korn inequality for solutions of problem (3.24).

LEMMA 3.6. *Assume that* Ω *is not axially symmetric,*

$$E_\Omega(k^{(1)}) = \sum_{i,j=1}^{3} \int_\Omega (k^{(1)}_{i,x_j} + k^{(1)}_{j,x_i})^2\,dx,$$

$$E_\Omega(k^{(1)}) + \sum_{\alpha=1}^{2} |k^{(1)}\cdot\overline{\tau}_\alpha|^2_{2,S_1} < \infty,$$

$$|\Delta' d|_{2,S_2} + |\widetilde{h}^{(1)}_{3,x_3}|_{2,\Omega} + |\widetilde{h}^{(1)}_3|_{2,S_2} < \infty.$$

Then for solutions of problem (3.24) *we have*

$$\|k^{(1)}\|^2_{1,\Omega} \le c(E_\Omega(k^{(1)}) + \sum_{\alpha=1}^{2} |k^{(1)}\cdot\overline{\tau}_\alpha|^2_{2,S_1}) + c(|\Delta' d|^2_{2,S_2} + |\widetilde{h}^{(1)}_{3,x_3}|^2_{2,S_2} + |\widetilde{h}^{(1)}_{3,x_3}|^2_{2,\Omega}. \tag{3.25}$$

PROOF. First we calculate

$$E_\Omega(k^{(1)}) = 2|\nabla k^{(1)}|^2_{2,\Omega} + 2\int_\Omega k^{(1)}_{i,x_j} k^{(1)}_{j,x_i}\,dx,$$

where utilizing $(3.24)_2$ the second intergral takes the form

$$\int (k^{(1)}_{i,x_j} k^{(1)}_j)_{,x_i}\,dx = \int_{S_1} k^{(1)}_{i,x_j} k^{(1)}_j n_i\,dS_1 + \int_{S_2} k^{(1)}_{i,x_j} k^{(1)}_j n_i\,dS_2 \equiv I_1,$$

where the summation convention is assumed.

In view of $(3.24)_{5,6}$ the second integral in I_1 equals

$$\int_{S_2} (\Delta' d - \widetilde{h}^{(1)}_{3,x_3}) k^{(1)}_3\,dS_2.$$

By $(3.24)_3$ the first integral in I_1 assumes the form

$$-\int_{S_1} n_{i,x_j} k^{(1)}_i k^{(1)}_j\,dS_1.$$

Hence

$$|\nabla k^{(1)}|^2_{2,\Omega} \le cE_\Omega(k^{(1)}) + c\sum_{\alpha=1}^{2} |k^{(1)}\cdot\overline{\tau}_\alpha|^2_{2,S_1} + c\left|\int_{S_2} (\Delta' d - \widetilde{h}^{(1)}_{3,x_3}) k^{(1)}_3\,dS_2\right|. \tag{3.26}$$

By the Poincaré inequality and $(3.24)_5$ we obtain

$$|k^{(1)}_i|_{2,\Omega} \le c|\nabla k^{(1)}_i|_{2,\Omega}, \quad i = 1, 2. \tag{3.27}$$

Since

$$\int_\Omega h^{(1)}\,dx = \int_{S_2(a)} d_2\,dS_2 - \int_{S_2(-a)} d_1\,dS_1 = 0,$$

relation (3.23) implies

$$\int_\Omega k^{(1)}_3\,dx = -\int_\Omega \widetilde{h}^{(1)}_3\,dx.$$

Hence

$$\int_\Omega \bigg|k_3^{(1)} - \int_\Omega k_3^{(1)}\,dx\bigg|^2\,dx \le c\int_\Omega |\nabla k_3^{(1)}|_{2,\Omega}^2$$

so

$$|k_3^{(1)}|_{2,\Omega}^2 \le c\bigg(|\nabla k_3^{(1)}|_{2,\Omega}^2 + \bigg|\int_\Omega \widetilde{h}_3^{(1)}\,dx\bigg|^2\bigg). \tag{3.28}$$

From (3.26)–(3.28) we obtain (3.25). □

Now we shall obtain an energy type estimate for solutions of problem (3.24).

Lemma 3.7. *Assume that v is the weak solution to problem* (1.1). *Assume that*

$$\begin{aligned} l_3^2 = {} & l_1^2\sup_t \|d_{,x'}\|_{1,S_2}^2 + (1+|d|_{3,\infty,S_2^t}^2)\int_0^t \|d_{,x'}\|_{1,S_2}^2\,dt' \\ & + \int_0^t (\|d_{,t}\|_{1,6/5,S_2}^2 + \|d_{,x'}\|_{2,3/2,S_2}^2)\,dt' \\ & + |f_3|_{4/3,2,S_2^t}^2 + |g^{(1)}|_{6/5,2,\Omega^t}^2 + |k^{(1)}(0)|_{2,\Omega}^2, \end{aligned}$$

where l_1 is defined by (2.20). *Then solutions of the problem* (3.24) *satisfy*

$$\begin{aligned} |k^{(1)}|_{2,\Omega}^2 + \nu\int_0^t \|k^{(1)}(t')\|_{1,\Omega}^2\,dt' + \gamma|k^{(1)}\cdot\overline{\tau}_\alpha|_{2,S_1^t}^2 \\ \le c\exp(|d_1|_{3,6,S_2^t}^6 + |\nabla v|_{3,2,\Omega^t}^2)l_3^2. \end{aligned} \tag{3.29}$$

Proof. Multiplying $(3.24)_1$ by $k^{(1)}$ and integrating over Ω yield

$$\frac{1}{2}\frac{d}{dt}|k^{(1)}|_{2,\Omega}^2 - \int_\Omega \operatorname{div}\mathbb{T}(h^{(1)},q^{(1)})\cdot k^{(1)}\,dx = \int_\Omega b_1\cdot k^{(1)}\,dx. \tag{3.30}$$

Integrating by parts in the second term on the l.h.s. of (3.30) yields

$$\begin{aligned} -\int_{S_1}\overline{n}\cdot\mathbb{T}(h^{(1)},q^{(1)})\cdot\overline{\tau}_\alpha k^{(1)}\cdot\overline{\tau}_\alpha\,dS_1 - \int_{S_2}\overline{n}\cdot\mathbb{T}(h^{(1)},q^{(1)})\cdot\overline{n}k^{(1)}\cdot\overline{n}\,dS_2 \\ + \nu\int_\Omega \mathbb{D}(h^{(1)})\cdot\mathbb{D}(k^{(1)})\,dx \equiv I_1 + I_2 + I_3. \end{aligned}$$

In view of $(3.24)_4$ we get

$$I_1 = \gamma\int_{S_1} h^{(1)}\cdot\overline{\tau}_\alpha k^{(1)}\cdot\overline{\tau}_\alpha\,dS_1 = \gamma|k^{(1)}\cdot\overline{\tau}_\alpha|_{2,S_1}^2 + \gamma\int_{S_1}\widetilde{h}^{(1)}\cdot\overline{\tau}_\alpha k^{(1)}\cdot\overline{\tau}_\alpha\,dS_1.$$

From $\overline{n}|_{S_2} = \overline{e}_3$ and $(3.24)_6$ it follows

$$\begin{aligned} I_2 &= -\int_{S_2}(2\nu h_{3,x_3}^{(1)} - q^{(1)})k_3^{(1)}\,dS_2 \\ &= -\nu\int_{S_2} h_{3,x_3}^{(1)}k_3^{(1)}\,dS_2 - \int_{S_2}(\nu h_{3,x_3}^{(1)} - q^{(1)})k_3^{(1)}\,dS_2 \equiv I_2^1 + I_2^2, \end{aligned}$$

where

$$|I_2^1| \le \varepsilon_1 \|k^{(1)}\|_{1,\Omega}^2 + c(1/\varepsilon_1)|\Delta' d|_{2,S_2}^2,$$

and to examine I_2^2 we take the third component of $(1.1)_1$ and project it on S_2. Then we get

$$d_{,t} + v' \cdot \nabla' d + dh_3^{(1)} - \nu\Delta' d - f_3 = \nu h_{3,x_3}^{(1)} - q^{(1)} \quad \text{on } S_2, \tag{3.31}$$

where $v' = (v_1, v_2)$, $\nabla' = (\partial_{x_1}, \partial_{x_2})$, $\Delta' = \partial_{x_1}^2 + \partial_{x_2}^2$.

Using (3.31) in I_2 yields

$$\begin{aligned} |I_2^2| &= \bigg| \int_{S_2} (d_{,t} + v' \cdot \nabla' d + dh_3^{(1)} - \nu\Delta' d - f_3) k_3^{(1)}\, dS_2 \bigg| \\ &\le \varepsilon_2 |k_3^{(1)}|_{4,S_2}^2 + c(1/\varepsilon_2)(|d_{,t}|_{4/3,S_2}^2 + |v'|_{4\lambda_1/3,S_2}^2 |\nabla' d|_{4\lambda_2/3,S_2}^2 \\ &\quad + |\Delta' d|_{4/3,S_2}^2 + |f_3|_{4/3,S_2}^2) + \bigg| \int_{S_2} dh_3^{(1)} k_3^{(1)}\, dS_2 \bigg|, \end{aligned}$$

where $1/\lambda_1 + 1/\lambda_2 = 1$ and

$$\int_{S_2} dh_3^{(1)} k_3^{(1)}\, dS_2 = \int_{S_2} d|k_3^{(1)}|^2\, dS_2 + \int_{S_2} d\widetilde{h}_3^{(1)} k_3^{(1)}\, dS_2 \equiv \widetilde{I}_1 + \widetilde{I}_2,$$

where

$$|\widetilde{I}_1| \le |d|_{3,S_2} |k_3^{(1)}|_{3,S_2}^2 \le \varepsilon_3 \|k_3^{(1)}\|_{1,\Omega}^2 + c(1/\varepsilon_3)|d|_{3,S_2}^6 |k_3^{(1)}|_{2,\Omega}^2,$$

and

$$|\widetilde{I}_2| \le \varepsilon_4 \|k_3^{(1)}\|_{1,\Omega}^2 + c(1/\varepsilon_4)|d|_{3,S_2}^2 \|\widetilde{h}^{(1)}\|_{1,\Omega}^2.$$

Finally, to examine I_3 we have to use the Korn inequality (3.25).

Next we examine the r.h.s. of (3.30). Utilizing the form of b_1 we obtain

$$\begin{aligned} &-\int_\Omega v \cdot \nabla k^{(1)} \cdot k^{(1)}\, dx - \int_\Omega v \cdot \nabla \widetilde{h}^{(1)} \cdot k^{(1)}\, dx \\ &\qquad - \int_\Omega h^{(1)} \cdot \nabla v \cdot k^{(1)}\, dx + \int_\Omega (-\widetilde{h}_{,t}^{(1)} + g^{(1)}) \cdot k^{(1)}\, dx \equiv \sum_{i=4}^7 I_i. \end{aligned}$$

In view of the boundary conditions for v we obtain

$$\begin{aligned} I_4 &= -\frac{1}{2} \int_\Omega v \cdot \nabla (k^{(1)})^2\, dx \le \frac{1}{2} \int_{S_2} d_1 (k^{(1)})^2\, dS_2 \\ &\le c|d_1|_{3,S_2} |k^{(1)}|_{3,S_2} \le c|d_1|_{3,S_2} (\varepsilon^{1/3} |\nabla k^{(1)}|_{2,\Omega}^2 + c\varepsilon^{-5/3} |k^{(1)}|_{2,\Omega}^2) \\ &\le \varepsilon_5 |\nabla k^{(1)}|_{2,\Omega}^2 + c(1/\varepsilon_5)|d_1|_{3,S_2}^6 |k^{(1)}|_{2,\Omega}^2. \end{aligned}$$

By the Hölder and Young inequalities we get

$$\begin{aligned} |I_5| &\le \varepsilon_6 |k^{(1)}|_{6,\Omega}^2 + c(1/\varepsilon_6)|v|_{3,\Omega}^2 |\nabla \widetilde{h}^{(1)}|_{2,\Omega}^2, \\ |I_6| &\le \varepsilon_7 |k^{(1)}|_{6,\Omega}^2 + c(1/\varepsilon_7)(|\nabla v|_{3,\Omega}^2 |k^{(1)}|_{2,\Omega}^2 + |\nabla v|_{2,\Omega}^2 |\widetilde{h}^{(1)}|_{3,\Omega}^2), \\ |I_7| &\le \varepsilon_8 |k^{(1)}|_{6,\Omega}^2 + c(1/\varepsilon_8)(|\widetilde{h}_{,t}^{(1)}|_{6/5,\Omega}^2 + |g^{(1)}|_{6/5,\Omega}^2). \end{aligned}$$

Utilizing the above estimates in (3.30) and assuming that $\varepsilon_1-\varepsilon_8$ are sufficiently small yield

$$
\begin{aligned}
(3.32)\quad &\frac{1}{2}\frac{d}{dt}|k^{(1)}|_{2,\Omega}^2+\nu\|k^{(1)}\|_{1,\Omega}^2+\gamma|k^{(1)}\cdot\overline{\tau}_\alpha|_{2,S_1}^2\\
&\le-\int_{S_1}\widetilde{h}^{(1)}\cdot\overline{\tau}_\alpha k^{(1)}\cdot\overline{\tau}_\alpha\,dS_1+c(|\Delta'd|_{2,S_2}^2+|d_{,t}|_{\frac{4}{3},S_2}^2\\
&\quad+|v'|_{\frac{4}{3}\lambda_1S_2}^2|\nabla'd|_{\frac{4}{3}\lambda_2,S_2}^2+|d|_{3,S_2}^2\|\widetilde{h}^{(1)}\|_{1,\Omega}^2+|\Delta'd\|_{\frac{4}{3},S_2}^2+|f_3|_{\frac{4}{3},S_2}^2)\\
&\quad+c(|h_{3,x_3}^{(1)}|_{2,S_2}^2+|\widetilde{h}_3^{(1)}|_{2,\Omega}^2)+c(|d_1|_{3,S_2}^6|k^{(1)}|_{2,\Omega}^2+|v|_{3,\Omega}^2|\nabla\widetilde{h}^{(1)}|_{2,\Omega}^2\\
&\quad+|\nabla v|_{3,\Omega}^2|k^{(1)}|_{2,\Omega}^2+|\widetilde{h}_{,t}^{(1)}|_{6/5,\Omega}^2+|g^{(1)}|_{6/5,\Omega}^2+|\nabla v|_{2,\Omega}^2|\widetilde{h}^{(1)}|_{3,\Omega}^2).
\end{aligned}
$$

The first integral on the r.h.s. of (3.32) is estimated by

$$\varepsilon_9|k^{(1)}\cdot\overline{\tau}_\alpha|_{2,S_1}^2+c(1/\varepsilon_9)|\widetilde{h}^{(1)}\cdot\overline{\tau}_\alpha|_{2,S_1}^2,$$

where the second norm is bounded by $\|\widetilde{h}^{(1)}\|_{1,\Omega}^2$, and we use the imbedding

$$|v'|_{4/3\lambda_1,S_2}\le c\|v\|_{1,\Omega},\quad \text{where } \lambda_1=3.$$

Utilizing the above estimates in (3.32) yields

$$
\begin{aligned}
(3.33)\quad &\frac{d}{dt}|k^{(1)}|_{2,\Omega}^2+\nu\|k^{(1)}\|_{1,\Omega}^2+\gamma|k^{(1)}\cdot\overline{\tau}_\alpha|_{2,S_1}^2\\
&\le c(|d_1|_{3,S_2}^6+|\nabla v|_{3,\Omega}^2)|k^{(1)}|_{2,\Omega}^2\\
&\quad+c(\|v\|_{1,\Omega}^2|d_{,x'}|_{2,S_2}^2+|v|_{3,\Omega}^2|\nabla\widetilde{h}^{(1)}|_{2,\Omega}^2+|\nabla v|_{2,\Omega}^2|\widetilde{h}^{(1)}|_{3,\Omega}^2)\\
&\quad+c(|\Delta'd|_{2,S_2}^2+|d_{,t}|_{4/3,S_2}^2+|d|_{3,S_2}^2\|\widetilde{h}^{(1)}\|_{1,\Omega}^2+\|\widetilde{h}_{,t}^{(1)}\|_{6/5,\Omega}^2\\
&\quad+|\widetilde{h}_3^{(1)}|_{2,\Omega}^2+|\widetilde{h}_{3,x_3}^{(1)}|_{2,S_2}^2+\|\widetilde{h}^{(1)}\|_{1,\Omega}^2+|f_3|_{4/3,S_2}^2+|g^{(1)}|_{6/5,\Omega}^2).
\end{aligned}
$$

Using Lemma 3.5 in (3.33) gives

$$
\begin{aligned}
(3.34)\quad &\frac{d}{dt}|k^{(1)}|_{2,\Omega}^2+\nu\|k^{(1)}\|_{1,\Omega}^2+\gamma|k^{(1)}\cdot\overline{\tau}_\alpha|_{2,S_1}^2\\
&\le c(|d_1|_{3,S_2}^6+|\nabla v|_{3,\Omega}^2)|k^{(1)}|_{2,\Omega}^2+c\|v\|_{1,\Omega}^2\|d_{,x'}\|_{1,S_2}^2\\
&\quad+c[(1+|d|_{3,S_2}^2)\|d_{,x'}\|_{1,S_2}^2+\|d_{,t}\|_{1,6/5,S_2}^2\\
&\quad+\|d_{,x'}\|_{2,3/2,S_2}^2+|f_3|_{4/3,S_2}^2+|g^{(1)}|_{6/5,\Omega}^2],
\end{aligned}
$$

where we used that

$$|\widetilde{h}_{3,x_3}^{(1)}|_{2,S_2}\le c\|\widetilde{h}^{(1)}\|_{2,3/2,\Omega}\le c\|d_{,x'}\|_{2,3/2,S_2}$$

which can be proved in the same way as Lemma 3.5.

Integrating (3.34) with respect to time implies (3.29). □

Now we examine problem (3.9). Assume that $\overline{h}^{(2)}$ is such that

$$\overline{n}\cdot\overline{h}^{(2)}|_{S_1}=0,\quad \overline{n}\cdot\overline{h}^{(2)}|_{S_2}=\Delta'd,$$

so

$$\overline{h}_3^{(2)} = \eta_1\Delta' d_1 + \eta_2\Delta' d_2, \quad \overline{h}_i^{(2)} = 0, \quad i = 1,2$$

Since $\operatorname{div}\overline{h}^{(2)} \neq 0$ we construct a function φ such that

$$\begin{aligned} \Delta\varphi &= -\operatorname{div}\overline{h}^{(2)} && \text{in } \Omega, \\ \overline{n}\cdot\nabla\varphi &= 0 && \text{on } S. \end{aligned} \tag{3.35}$$

Compatibility condition is satisfied because $\int_{S_2}\overline{h}^{(2)}\,dS_2 = \int_{\overline{S}_2\cap\overline{S}_1}\overline{n}|_{S_1}\cdot d_{,x'} = 0$. Then

$$\widetilde{h}^{(2)} = \overline{h}^{(2)} + \nabla\varphi. \tag{3.36}$$

LEMMA 3.8. *Function $\widetilde{h}^{(2)}$ satisfies*

$$|\widetilde{h}^{(2)}|_{\sigma,\Omega} \le c\|d_{,x'}\|_{1,\sigma,S_2}, \qquad |\widetilde{h}^{(2)}_{,t}|_{\sigma,\Omega} \le c\|d_{,x't}\|_{1,\sigma,S_2}. \tag{3.37}$$

PROOF. For function $\overline{h}^{(2)}$ we have

$$|\overline{h}^{(2)}|_{\sigma,\Omega} \le c|d_{,x'x'}|_{\sigma,S_2}, \qquad |\overline{h}^{(2)}_{,t}|_{\sigma,\Omega} \le c|d_{,x'x't}|_{\sigma,S_2}.$$

Let $G(x,y)$ be the Green function for solutions of problem (3.35). Then

$$\varphi(x) = \int_\Omega G(x,y)\operatorname{div}\overline{h}^{(2)}dy = \int_\Omega \nabla_y G(x,y)\overline{h}^2 dy - \int_{S_2} G\overline{n}\cdot\overline{h}^{(2)}\,dS_2,$$

where we used that $\overline{n}\cdot\overline{h}^{(2)}|_{S_1} = 0$. Since

$$\nabla\varphi = \int_\Omega \nabla_x\nabla_y G\overline{h}^{(2)}dy - \int_{S_2}\nabla_x G\overline{n}\cdot\overline{h}^{(2)}\,dS_2,$$

we obtain

$$|\nabla\varphi|_{\sigma,\Omega} \le c\|d_{,x'}\|_{1,\sigma,S_2}, \qquad |\nabla\varphi_{,t}|_{\sigma,\Omega} \le \|d_{,x't}\|_{1,\sigma,S_2}.$$

In view of the above estimates and (3.36) we obtain (3.37). □

Introducing the function

$$k^{(2)} = h^{(2)} - \widetilde{h}^{(2)} \tag{3.38}$$

we see that it is a solution to the problem

$$\begin{aligned} & k^{(2)}_{,t} - \operatorname{div}\mathbb{T}(h^{(2)},q^{(2)}) = \gamma_0 - \widetilde{h}^{(2)}_{,t} \equiv \widetilde{\gamma}_0 && \text{in } \Omega^T, \\ & \operatorname{div} k^{(2)} = 0 && \text{in } \Omega^T, \\ & \overline{n}\cdot k^{(2)} = 0 && \text{on } S_1^T, \\ & \nu\overline{n}\cdot\mathbb{D}(h^{(2)})\cdot\overline{\tau}_\alpha + \gamma h^{(2)}\cdot\overline{\tau}_\alpha = 0 && \text{on } S_1^T, \\ & k_3^{(2)} = 0 && \text{on } S_2^T, \\ & h^{(2)}_{i,x_3} = \gamma_i, \quad i = 1,2, && \text{on } S_2^T, \\ & k^{(2)}|_{t=0} = h^{(2)}(0) - \widetilde{h}^{(2)}(0) \equiv k^{(2)}(0) && \text{in } \Omega, \end{aligned} \tag{3.39}$$

where $\gamma_0, \gamma_1, \gamma_2$ are defined in (3.9).

First we shall obtain the Korn-type inequality for solutions to problem (3.39).

Lemma 3.9. *Assume that*

$$E_\Omega(k^{(2)}) = \sum_{i,j=1}^3 \int_\Omega (k^{(2)}_{i,x_j} + k^{(2)}_{j,x_i})^2\,dx < \infty,$$
$$|k^{(2)} \cdot \overline{\tau}_\alpha|_{2,S_1} < \infty, \quad \alpha = 1,2.$$

Then solutions of (3.39) *satisfy*

$$\|k^{(2)}\|^2_{1,\Omega} \le c\bigg(E_\Omega(k^{(2)}) + \sum_{\alpha=1}^2 |k^{(2)} \cdot \overline{\tau}_\alpha|^2_{2,S_1}\bigg). \tag{3.40}$$

Proof. We have

$$\begin{aligned} E_\Omega(k^{(2)}) &= \sum_{i,j=1}^3 \int_\Omega (k^{(2)}_{i,x_j} + k^{(2)}_{j,x_i})^2\,dx \\ &= 2|\nabla k^{(2)}|^2_{2,\Omega} + 2\sum_{i,j=1}^3 \int_\Omega k^{(2)}_{i,x_j} k^{(2)}_{j,x_i}\,dx. \end{aligned} \tag{3.41}$$

Integrating by parts and utilizing $(3.39)_2$ in the second integral on the r.h.s. yield

$$\begin{aligned} \sum_{i,j=1}^3 \int_\Omega (k^{(2)}_{i,x_j} k^{(2)}_j)_{,x_i}\,dx &= \sum_{i,j=1}^3 \int_S n_i k^{(2)}_{i,x_j} k^{(2)}_j\,dS \\ &= \sum_{i,j=1}^3 \int_{S_1} n_i k^{(2)}_{i,x_j} k^{(2)}_j\,dS_1 + \sum_{i,j=1}^3 \int_{S_2} n_i k^{(2)}_{i,j} k^{(2)}_j\,dS_2 \equiv I_1 + I_2. \end{aligned}$$

Employing $(3.39)_3$ in I_1 implies

$$I_1 = -\sum_{i,j=1}^3 \int_{S_1} n_{i,x_j} k^{(2)}_i k^{(2)}_j\,dS_1$$

and $(3.39)_5$ in I_2 gives

$$I_2 = \sum_{\alpha=1}^2 \int_{S_2} k^{(2)}_{3,x_\alpha} k^{(2)}_\alpha\,dS_2 = 0.$$

Therefore, (3.41) yields

$$|\nabla k^{(2)}|^2_{2,\Omega} \le c\bigg(E_\Omega(k^{(2)}) + \sum_{\alpha=1}^2 |k^{(2)} \cdot \overline{\tau}_\alpha|^2_{2,S_1}\bigg). \tag{3.42}$$

Using $(3.39)_3$, the fact that Ω is a cylindrical domain, so $\overline{n}|_{S_1}\cdot\overline{e}_3=0$, we have by the Poincaré inequality the estimate

$$\sum_{i=1}^2|k_i^{(2)}|_{2,\Omega}\le c\sum_{i=1}^2|\nabla k_i^{(2)}|_{2,\Omega}. \tag{3.43}$$

We outline a proof of this inequality. Let s_1, s_2 be two points of S_1 with the same coordinate x_3. Let $\overline{n}(s_1)$, $\overline{n}(s_2)$ be the normal vectors to S_1 at points s_1, s_2, respectively. Then by the Poincaré inequality we have $|k^{(2)}\cdot\overline{n}(s_i)|_{2,\Omega}\le c|\nabla(k^{(2)}\cdot\overline{n}(s_i))|_{2,\Omega}$. Since $\overline{n}(s_1)$, $\overline{n}(s_2)$ are finearly independent, (3.43) holds.

Similarly, $(3.39)_5$ implies

$$|k_3^{(2)}|_{2,\Omega}\le c|\nabla k_3^{(2)}|_{2,\Omega}. \tag{3.44}$$

From (3.42)–(3.44) we obtain the Korn inequality (3.40). □

Next we shall obtain an energy type estimate for solutions to problem (3.39).

LEMMA 3.10. *Assume that v is a weak solution to problem* (1.1). *Assume that*

$$\begin{aligned}
l_4&=\|d_{,x'}\|_{L_\infty(0,t;H^2(S_2))}+\|d_{,x'}\|_{L_2(0,t;H^2(S_2)}+\|d_{,t}\|_{L_2(0,t;H^1(S_2)}<\infty,\\
l_5&=\|v\|_{L_2(0,t;W_3^1(\Omega))}+\|v\|_{L_\infty(0,t;H^1(S_2))}\\
&\quad+\|h^{(1)}\|_{L_2(0,t;W_3^1(\Omega))}+\|h^{(1)}\|_{L_\infty(0,t;H^1(S_2))}+\|h^{(1)}\|_{L_2(0,t;H^1(S_2))}<\infty,\\
l_6&=\sum_{i=1}^2(|g^{(i)}|_{6/5,2,\Omega^t}+|k^{(i)}(0)|_{2,\Omega})+|f_3|_{4/3,2,S_2^t}+|F'|_{4/3,2,S_2^t}<\infty.
\end{aligned}$$

Assume that $|d|_{\infty,S_2^t}<\infty$. Then solutions to problem (3.39) *satisfy*

$$\begin{aligned}
&\sum_{i=1}^2\biggl(|k^{(i)}(t)|_{2,\Omega}^2+\nu\int_0^t\|k^{(i)}(t')\|_{1,\Omega}^2\,dt'+\gamma\int_0^t|k^{(i)}(t')\cdot\overline{\tau}_\alpha|_{2,S_1}^2\,dt'\biggr)\\
&\quad\le c\exp c[|d_1|_{3,6,S_2^t}^6+\|v\|_{L_2(0,t;W_3^1(\Omega))}^2+\|h^{(1)}\|_{L_2(0,t;W_3^1(\Omega))}^2]\\
&\qquad\cdot[(1+|d|_{\infty,S_2^t}^2+l_5^2)(l_4^2+l_5^2)+l_6^2].
\end{aligned} \tag{3.45}$$

PROOF. Multiplying $(3.39)_1$ by $k^{(2)}$ and integrating the result over Ω yield

$$\begin{aligned}
&\frac12\frac{d}{dt}|k^{(2)}|_{2,\Omega}^2-\int_\Omega\operatorname{div}\mathbb{T}(h^{(2)},q^{(2)})\cdot k^{(2)}\,dx\\
&\quad=-\int_\Omega(h^{(2)}\cdot\nabla v+2h^{(1)}\cdot\nabla h^{(1)}+v\cdot\nabla h^{(2)})\cdot k^{(2)}\,dx\\
&\qquad+\int_\Omega(g^{(2)}-\widetilde{h}_{,t}^{(2)})\cdot k^{(2)}\,dx.
\end{aligned} \tag{3.46}$$

Integrating by parts in the second term on the l.h.s. of (3.46) implies

$$-\int_{S_1}\overline{n}\cdot\mathbb{T}(h^{(2)},q^{(2)})\cdot\overline{\tau}_\alpha k^{(2)}\cdot\overline{\tau}_\alpha\,dS_1-\int_{S_2}\overline{n}\cdot\mathbb{T}(h^{(2)},q^{(2)})\cdot\overline{\tau}_\alpha k^{(2)}\cdot\overline{\tau}_\alpha\,dS_2$$
$$+\nu\int_\Omega\mathbb{D}(h^{(2)})\cdot\mathbb{D}(k^{(2)})\,dx\equiv I_1+I_2+I_3.$$

In view of $(3.39)_4$ we obtain

$$\begin{aligned}I_1&=\gamma\int_{S_1}h^{(2)}\cdot\overline{\tau}_\alpha k^{(2)}\cdot\overline{\tau}_\alpha\,dS_1\\&=\gamma|k^{(2)}\cdot\overline{\tau}_\alpha|^2_{2,S_1}+\gamma\int_{S_1}\widetilde{h}^{(2)}\cdot\overline{\tau}_\alpha k^{(2)}\cdot\overline{\tau}_\alpha\,dS_1\equiv I_1^1+I_1^2.\end{aligned}$$

By $(3.39)_6$ we get

$$\begin{aligned}I_2&=-\int_{S_2}(h^{(2)}_{\alpha,x_3}+h^{(2)}_{3,x_\alpha})k^{(2)}\cdot\overline{\tau}_\alpha\,dS_2\\&=-\int_{S_2}\bigg[-\frac{2}{\nu}d_{,x_\alpha t}+3\Delta' d_{,x_\alpha}+\frac{1}{\nu}F'_\alpha|_{S_2}\\&\quad+[v_j(h^{(1)}_\alpha-d_{,x_\alpha})_{,x_j}+d(h^{(1)}_{\alpha,x_3}-h^{(1)}_{3,x_\alpha})\\&\quad+h^{(1)}_3(h^{(1)}_\alpha-d_{,x_\alpha})+h^{(1)}_jv_{\alpha,x_j}-d_{,x_j}v_{j,x_\alpha}]|_{S_2}\bigg]k^{(2)}\cdot\overline{\tau}_\alpha\,dS_2,\end{aligned}$$

where the summation convention over the repeated indices, $\alpha,\ j=1,2$, is assumed. Continuing, we have

$$|I_1^2|\le\varepsilon_1|k^{(2)}|^2_{4,S_1}+c(1/\varepsilon_1)|\widetilde{h}^{(2)}|^2_{4/3,S_1},$$

and

$$\begin{aligned}|I_2|\le&\,\varepsilon_2|k^{(2)}|^2_{4,S_2}+c(1/\varepsilon_2)(|d_{,x't}|^2_{4/3,S_2}+|\Delta' d_{,x'}|^2_{4/3,S_2}+|F'|^2_{4/3,S_2})\\&+c(1/\varepsilon_2)(|vh^{(1)}_{,x'}|^2_{4/3,S_2}+|vd_{,x'x'}|^2_{4/3,S_2}+|dh^{(1)}_{,x}|^2_{4/3,S_2}+|(h^{(1)})^2|^2_{4/3,S_2}\\&+|h^{(1)}d_{,x'}|^2_{4/3,S_2}+|h^{(1)}v_{,x'}|^2_{4/3,S_2}+|d_{,x'}v_{,x}|^2_{4/3,S_2})\\\equiv&\,\varepsilon_2|k^{(2)}|^2_{4,S_2}+c(1/\varepsilon_2)(I_2^1+I_2^2).\end{aligned}$$

By the Hölder inequality we have

$$\begin{aligned}I_2^2\le&\,c(|v|^2_{4,S_2}|h^{(1)}_{,x'}|^2_{2,S_2}+|v|^2_{4,S_2}\|d_{,x'}\|^2_{1,S_2}+|d|^2_{\infty,S_2}|h^{(1)}_{,x}|^2_{4/3,S_2}\\&+|h^{(1)}|^2_{4,S_2}|h^{(1)}|^2_{2,S_2}+|h^{(1)}|^2_{2,S_2}|d_{,x'}|^2_{4,S_2}\\&+|h^{(1)}|^2_{4,S_2}|v_{,x'}|^2_{2,S_2}+\|d_{,x'}\|^2_{1,S_2}|v_{,x}|^2_{2,S_2}).\end{aligned}$$

Finally,

$$I_3=\nu|\mathbb{D}(k^{(2)}|^2_{2,\Omega}+\nu\int_\Omega\mathbb{D}(\widetilde{h}^{(2)})\cdot\mathbb{D}(k^{(2)})\,dx\equiv I_3^1+I_3^2,$$

where

$$|I_3^2|\le\varepsilon_3\|k^{(2)}\|^2_{1,\Omega}+c(1/\varepsilon_3)\|\widetilde{h}^{(2)}\|^2_{1,\Omega}.$$

We express the first integral on the r.h.s. of (3.46) in the form

$$-\int_\Omega h^{(2)}\cdot\nabla v\cdot k^{(2)}\,dx-\int_\Omega 2h^{(1)}\cdot\nabla h^{(1)}k^{(2)}\,dx - \int_\Omega v\cdot\nabla k^{(2)}k^{(2)}\,dx-\int_\Omega v\cdot\nabla\widetilde{h}^{(2)}\cdot k^{(2)}\,dx\equiv\sum_{i=4}^7 I_i.$$

First we examine I_4 expressed in the form

$$I_4=-\int_\Omega k^{(2)}\cdot\nabla v\cdot k^{(2)}\,dx-\int_\Omega\widetilde{h}^{(2)}\cdot\nabla v\cdot k^{(2)}\,dx\equiv I_4^1+I_4^2,$$

where

$$\begin{aligned}|I_4^1|&\le\varepsilon_3|k^{(2)}|_{6,\Omega}^2+c(1/\varepsilon_3)|\nabla v|_{3,\Omega}^2|k^{(2)}|_{2,\Omega}^2,\\ |I_4^2|&\le\varepsilon_4|k^{(2)}|_{6,\Omega}^2+c(1/\varepsilon_4)|\nabla v|_{2,\Omega}^2|\widetilde{h}^{(2)}|_{3,\Omega}^2.\end{aligned}$$

Next, we have

$$\begin{aligned}|I_5|&\le\varepsilon_5|k^{(2)}|_{6,\Omega}^2+c(1/\varepsilon_5)(|\nabla h^{(1)}|_{3,\Omega}^2|k^{(1)}|_{2,\Omega}^2+|\nabla h^{(1)}|_{2,\Omega}^2|\widetilde{h}^{(1)}|_{3,\Omega}^2),\\ |I_6|&\le\varepsilon_6|\nabla k^{(2)}|_{2,\Omega}^2+c(1/\varepsilon_6)|v|_{3,\Omega}^2|k^{(2)}|_{2,\Omega}^2\\ |I_7|&\le\varepsilon_7|k^{(2)}|_{6,\Omega}^2+c(1/\varepsilon_7)|v|_{3,\Omega}^2|\nabla\widetilde{h}^{(2)}|_{2,\Omega}^2.\end{aligned}$$

Utilizing the above estimates with $\varepsilon_1-\varepsilon_7$ sufficiently small and the Korn inequality (3.40) in (3.46) we obtain

$$\begin{aligned}(3.47)\qquad&\frac{d}{dt}|k^{(2)}|_{2,\Omega}^2+\nu\|k^{(2)}\|_{1,\Omega}^2+\gamma|k^{(2)}\cdot\overline{\tau}_\alpha|_{2,S_1}^2\\ &\le c(\|v\|_{1,3,\Omega}^2|k^{(2)}|_{2,\Omega}^2+|\nabla h^{(1)}|_{3,\Omega}^2|k^{(1)}|_{2,\Omega}^2+\|v\|_{1,\Omega}^2\|\widetilde{h}^{(2)}\|_{1,\Omega}^2\\ &\quad+|\nabla h^{(1)}|_{2,\Omega}^2|\widetilde{h}^{(1)}|_{3,\Omega}^2)+c(\|\widetilde{h}^{(2)}\|_{1,\Omega}^2+|g^{(2)}|_{6/5,\Omega}^2+|\widetilde{h}_{,t}^{(2)}|_{6/5,\Omega}^2\\ &\quad+\|d_{,x'}\|_{2,4/3,S_2}^2+\|d_{,t}\|_{1,4/3,S_2}^2+|F'|_{4/3,S_2}^2)\\ &\quad+c(|v|_{4,S_2}^2|h_{,x'}^{(1)}|_{2,S_2}^2+|v|_{4,S_2}^2\|d_{,x'}\|_{1,S_2}^2+|d|_{\infty,S_2}^2|h_{,x}^{(1)}|_{4/3,S_2}^2\\ &\quad+|h^{(1)}|_{4,S_2}^2|h^{(1)}|_{2,S_2}^2+|h^{(1)}|_{2,S_2}^2|d_{,x'}|_{4,S_2}^2\\ &\quad+|h^{(1)}|_{4,S_2}^2|v_{,x'}|_{2,S_2}^2+\|d_{,x'}\|_{1,S_2}^2|v_{,x}|_{2,S_2}^2).\end{aligned}$$

Utilizing Lemmas 3.5 and 3.8 in (3.47) yields

$$\begin{aligned}(3.48)\qquad&\frac{d}{dt}|k^{(2)}|_{2,\Omega}^2+\nu\|k^{(2)}\|_{1,\Omega}^2+\gamma|k^{(2)}\cdot\overline{\tau}_\alpha|_{2,S_1}^2\le c(\|v\|_{1,3,\Omega}^2|k^{(2)}|_{2,\Omega}^2\\ &\quad+|\nabla h^{(1)}|_{3,\Omega}^2|k^{(1)}|_{2,\Omega}^2+\|v\|_{1,\Omega}^2\|d_{,x'}\|_{2,S_2}^2+|\nabla h^{(1)}|_{2,\Omega}^2\|d_{,x'}\|_{2,S_2}^2)\\ &\quad+c(\|d_{,x'}\|_{2,4/3,S_2}^2+\|d_{,t}\|_{1,4/3,S_2}^2+|g^{(2)}|_{6/5,\Omega}^2+|F'|_{4/3,S_2}^2)\\ &\quad+c(|v|_{4,S_2}^2|h_{,x'}^{(1)}|_{2,S_2}^2+|v|_{4,S_2}^2\|d_{,x'}\|_{1,S_2}^2\\ &\quad+|d|_{\infty,S_2}^2|h_{,x}^{(1)}|_{4/3,S_2}^2+|h^{(1)}|_{4,S_2}^2|h^{(1)}|_{2,S_2}^2+|h^{(1)}|_{2,S_2}^2|d_{,x'}|_{4,S_2}^2\\ &\quad+|h^{(1)}|_{4,S_2}^2|v_{,x}|_{2,S_2}^2+\|d_{,x'}\|_{1,S_2}^2|v_{,x}|_{2,S_2}^2).\end{aligned}$$

Adding (3.34) and (3.48) implies

$$
\begin{aligned}
(3.49)\quad & \frac{d}{dt}(|k^{(1)}|_{2,\Omega}^2 + |k^{(2)}|_{2,\Omega}^2) + \nu(\|k^{1)}\|_{1,\Omega}^2 + \|k^{(2)}\|_{1,\Omega}^2) \\
& \quad + \gamma(|k^{1)} \cdot \overline{\tau}_\alpha|_{2,S_1}^2 + |k^{(2)} \cdot \overline{\tau}_\alpha|_{2,S_1}^2) \\
& \leq c(|d_1|_{3,S_2}^6 + |\nabla v|_{3,\Omega}^2 + |\nabla h^{(1)}|_{3,\Omega}^2)|k^{1)}|_{2,\Omega}^2 + c\|v\|_{1,3,\Omega}^2 |k^{(2)}|_{2,\Omega}^2 \\
& \quad + c(\|v\|_{1,\Omega}^2 + |\nabla h^{(1)}|_{2,\Omega}^2)\|d_{,x'}\|_{2,S_2}^2 + c((1+|d|_{3,S_2}^2)\|d_{,x'}\|_{2,S_2}^2 \\
& \quad + \|d_{,t}\|_{1,S_2}^2 + |g^{(1)}|_{6/5,\Omega}^2 + |g^{(2)}|_{6/5,\Omega}^2 + |f_3|_{4/3,S_2}^2 + |F'|_{4/3,S_2}^2) \\
& \quad + c(|v|_{4,S_2}^2 |h_{,x'}^{(1)}|_{2,S_2}^2 + |v|_{4,S_2}^2 \|d_{,x'}\|_{1,S_2}^2 + |d|_{\infty,S_2}^2 |h_{,x}^{(1)}|_{4/3,S_2}^2 \\
& \quad + |h^{(1)}|_{4,S_2}^2 |h^{(1)}|_{2,S_2}^2 + |h^{(1)}|_{2,S_2}^2 |d_{,x'}|_{4,S_2}^2 \\
& \quad + |h^{(1)}|_{4,S_2}^2 |v_{,x'}|_{2,S_2}^2 + \|d_{,x'}\|_{1,S_2}^2 |v_{,x}|_{2,S_2}^2).
\end{aligned}
$$

Integrating (3.49) with respect to time yields

$$
\begin{aligned}
|k^{(1)}(t)|_{2,\Omega}^2 & + |k^{(2)}(t)|_{2,\Omega}^2 + \nu \int_0^t (\|k^{(1)}(t')\|_{1,\Omega}^2 + \|k^{(2)}(t')\|_{1,\Omega}^2)\, dt' \\
& + \gamma \int_0^t (|k^{(1)}(t') \cdot \overline{\tau}_\alpha|_{2,S_1}^2 + |k^{(2)}(t') \cdot \overline{\tau}_\alpha|_{2,S_1}^2)\, dt' \\
& \leq c \exp c \Big[|d_1|_{3,6,S_2^t}^6 + \int_0^t (\|v(t')\|_{1,3,\Omega}^2 + \|h^{(1)}(t')\|_{1,3,\Omega}^2)\, dt' \Big] \\
& \cdot \Big[\sup_t \|d_{,x'}\|_{2,S_2}^2 \int_0^t (\|v(t')\|_{1,\Omega}^2 + \|h^{(1)}(t')\|_{1,\Omega}^2)\, dt' \\
& + (1 + |d|_{3,\infty,S_2^t}^2) \int_0^t (\|d_{,x'}(t')\|_{2,S_2}^2 + \|d_{,t'}(t')\|_{1,S_2}^2)\, dt' \\
& + \sup_t |v(t')|_{4,S_2}^2 \int_0^t |h_{,x'}^{(1)}|_{2,S_2}^2\, dt' + \sup_t \|d_{,x'}\|_{1,S_2}^2 \int_0^t |v(t')|_{4,S_2}^2\, dt' \\
& + |d|_{\infty,S_2^t}^2 \int_0^t |h_{,x}^{(1)}(t')|_{4/3,S_2}^2\, dt' \\
& + \int_0^t |h^{(1)}(t')|_{2,S_2}^2\, dt' (\sup_t |h^{(1)}|_{4,S_2}^2 + \sup_t |d_{,x'}|_{4,S_2}^2) \\
& + \sup_t |v_{,x'}|_{2,S_2}^2 \int_0^t |h^{(1)}|_{4,S_2}^2\, dt' + \sup_t \|d_{,x'}\|_{1,S_2}^2 \int_0^t |v_{,x'}(t')|_{2,S_2}^2\, dt' \\
& + |g^{(1)}|_{6/5,2,\Omega^t}^2 + |g^{(2)}|_{6/5,2,\Omega^t}^2 + |f_3|_{4/3,2,S_2^t}^2 \\
& + |F'|_{4/3,2,S_2^t}^2 + |k^{(1)}(0)|_{2,\Omega}^2 + |k^{(2)}(0)|_{2,\Omega}^2 \Big].
\end{aligned}
$$

Simplifying, we have

$$
(3.50)\quad \sum_{i=1}^2 \Big(|k^{(i)}(t)|_{2,\Omega}^2 + \nu \int_0^t \|k^{(i)}(t')\|_{1,\Omega}^2\, dt' + \gamma \int_0^t |k^{(i)}(t') \cdot \overline{\tau}_\alpha|_{2,S_1}^2\, dt' \Big)
$$

$$
\begin{aligned}
&\le c\exp c\bigg[|d_1|^6_{3,6,S_2^t} + \int_0^t \|v(t')\|^2_{1,3,\Omega} + \|h^{(1)}(t')\|^2_{1,3,\Omega})\,dt'\bigg]\\
&\quad\cdot[(\|v\|^2_{L_2(0,t;W^1_3(\Omega))} + \|h^{(1)}\|^2_{L_2(0,t;W^1_3(\Omega))})\|d_{,x'}\|^2_{L_\infty(0,t;H^2(S_2))}\\
&\quad + (1+|d|^2_{\infty,S_2^t})(\|d_{,x'}\|^2_{L_2(0,t;H^2(S_2))}\\
&\quad + \|d_{,t}\|^2_{L_2(0,t;H^1(S_2))}) + \sup_t |v(t)|^2_{4,S_2}|h^{(1)}_{,x'}|^2_{2,S_2^t}\\
&\quad + \|v\|^2_{L_2(0,t;L_4(S_2))}\|d_{,x'}\|^2_{L_\infty(0,t;H^1(S_2))} + |d|^2_{\infty,S_2^t}|h^{(1)}_{,x'}|^2_{2,S_2^t}\\
&\quad + (\sup_t |h^{(1)}|^2_{4,S_2} + \sup_t |d_{,x'}|^2_{4,S_2})|h^{(1)}|^2_{2,S_2^t}\\
&\quad + \sup_t |v_{,x'}|^2_{2,S_2}\|h^{(1)}\|^2_{L_2(0,t;L_4(S_2))} + |v_{,x'}|^2_{2,S_2^t}\|d_{,x'}\|^2_{L_\infty(0,t;H^1(S_2))}\\
&\quad + |g^{(1)}|^2_{6/5,2,\Omega^t} + |g^{(2)}|^2_{6/5,2,\Omega^t} + |f_3|^2_{4/3,2,S_2^t}\\
&\quad + |F'|^2_{4/3,2,S_2^t} + |k^{(1)}(0)|^2_{2,\Omega} + |k^{(2)}(0)|^2_{2,\Omega}].
\end{aligned}
$$

Utilizing expressions l_4 and l_6 we obtain from (3.50) the inequality

$$
\begin{aligned}
(3.51)\quad &\sum_{i=1}^2\bigg(|k^{(i)}(t)|^2_{2,\Omega} + \nu\int_0^t \|k^{(i)}(t')\|^2_{1,\Omega}\,dt' + \gamma\int_0^t |k^{(i)}(t')\cdot\bar\tau_\alpha|^2_{2,S_1}\,dt'\bigg)\\
&\le c\exp c[|d_1|^6_{3,6,S_2^t} + \|v\|^2_{L_2(0,t;W^1_3(\Omega))} + \|h^{(1)}\|^2_{L_2(0,t;W^1_3(\Omega))}]\\
&\quad\cdot[l_4^2(\|v\|^2_{L_2(0,t;W^1_3(\Omega))} + \|h^{(1)}\|^2_{L_2(0,t;W^1_3(\Omega))})\\
&\quad + (1+|d|^2_{\infty,S_2^t})(l_4^2 + |h^{(1)}_{,x'}|^2_{2,S_2^t})\\
&\quad + l_4^2(\|v\|^2_{L_2(0,t;L_4(S_2))} + |h^{(1)}|^2_{2,S_2^t} + |v_{,x'}|^2_{2,S_2^t})\\
&\quad + \sup_t |v(t)|^2_{4,S_2}|h^{(1)}_{,x'}|^2_{2,S_2^t} + \sup_t |h^{(1)}|^2_{4,S_2}|h^{(1)}|^2_{2,S_2^t}\\
&\quad + \sup_t |v_{,x'}|^2_{2,S_2}\|h^{(1)}\|^2_{L_2(0,t;L_4(S_2))} + l_6^2].
\end{aligned}
$$

Utilizing the form of l_5 inequality (3.51) implies (3.45). □

In view of Lemmas 3.5 and 3.7 we obtain from (3.22′) and (3.29) the following inequality

$$
\begin{aligned}
(3.52)\quad &|h^{(1)}(t)|^2_{2,\Omega} + \nu\int_0^t \|h^{(1)}(t')\|^2_{1,\Omega}\,dt' + \gamma|h^{(1)}\cdot\bar\tau_\alpha|^2_{2,S_1^t}\\
&\qquad\le \varphi(|d_1|_{3,6,S_2^t}, |\nabla v|_{3,2,\Omega^t}, l_1, |d|_{3,\infty,S_2^t})\eta_1^2(t),
\end{aligned}
$$

where φ is an increasing positive function, l_1 is defined by (2.20) and

$$
\begin{aligned}
(3.53)\quad \eta_1(t) &= \sup_{t'\le t}\|d_{,x'}(t')\|_{1,S_2} + \|d_{,x'}\|_{L_2(0,t;H^1(S_2))}\\
&\quad + \|d_{,t}\|_{L_2(0,t;W^1_{6/5}(S_2))} + |f_3|_{4/3,2,S_2^t} + |g^{(1)}|_{6/5,2,\Omega^t} + |h^{(1)}(0)|_{2,\Omega}.
\end{aligned}
$$

Next by Lemmas 3.8 and (3.10) we get from (3.37) and (3.51) the inequality

$$\begin{aligned}(3.54)\quad &\sum_{i=1}^{2}\bigg(|h^{(i)}|_{2,\Omega}^2+\nu\int_0^t\|h^{(i)}(t')\|_{1,\Omega}^2\,dt'+|h^{(i)}\cdot\bar\tau_\alpha|_{2,S_1^t}^2\bigg)\\ &\le\varphi(|d_1|_{3,6,S_2^t},|d|_{\infty,S_2^t},\|v\|_{L_2(0,t;W_3^1(\Omega))},\|v\|_{L_2(0,t;H^1(S_2))},\\ &\quad\|v\|_{L_\infty(0,t;H^1(S_2))},\|h^{(1)}\|_{L_2(0,t;W_3^1(\Omega))},\|h^{(1)}\|_{L_\infty(0,t;L_4(S_2))})\cdot\eta_2(t),\end{aligned}$$

where φ is an increasing positive function and

$$(3.55)\qquad \eta_2(t)=l_4(t)+l_6(t)+\|h^{(1)}\|_{L_2(0,t;H^1(S_2))},$$

where l_4 and l_5 are defined in assumptions of Lemma 3.10.

4. Estimates

First we replace (2.20) by more appropriate energy estimate

$$(4.1)\quad |v(t)|_{2,\Omega}+\sqrt{\nu}\bigg(\int_0^t\|v(t')\|_{W_2^{1,2}(\Omega)}^2\,dt'\bigg)^{1/2}\le l_1(t)+|\partial_{x_3}^2v|_{2,\Omega^t}\equiv l_2(t),$$

for $t\le T$. In view of (2.4), inequality (4.1) implies

$$(4.2)\qquad |v|_{18/5,\Omega^t}\le cl_2(t),\quad t\le T.$$

Now we shall obtain an estimate for solutions of problem (3.3). First we construct a function $\widetilde\chi$ described by the following problem

$$(4.3)\qquad \begin{aligned}\widetilde\chi_{,t}-\nu\Delta\widetilde\chi&=0 &&\text{in }\Omega^T,\\ \widetilde\chi|_{S_1}&=\chi_* &&\text{on }S_1^T,\\ \widetilde\chi_{,x_3}|_{S_2}&=0 &&\text{on }S_2^T,\\ \widetilde\chi|_{t=0}&=\chi_0 &&\text{in }\Omega.\end{aligned}$$

To show existence of such function we need the following compatibility condition

$$\begin{aligned}\chi_{*,x_3}=\sum_{i,j=1}^{2}\bigg[&-d_{,x_i}(n_{i,x_j}\tau_{1j}+\tau_{1i,x_j}n_j)+\frac{\gamma}{\nu}d_{,x_j}\tau_{1j}\\ &-d_{,x_i}\tau_{1i}(\tau_{12,x_1}-\tau_{11,x_2})\bigg]\bigg|_{\bar S_1\cap\bar S_2}=0.\end{aligned}$$

Moreover, we need the compatibility conditions

$$(4.4)\qquad \chi_0|_{S_1}=\chi_*|_{t=0},\qquad \chi_{0,x_3}|_{S_2}=0,$$

which imply restrictions on χ_0. From (4.4) we have that χ_0 depends on v in a similar way as χ_*. This fact is important for below estimations.

Introducing the new function

$$(4.4')\qquad \chi'=\chi-\widetilde\chi$$

we see that it is a solution to the following problem

$$
(4.5)\qquad
\begin{aligned}
&\chi'_{,t} + v\cdot\nabla\chi' - h_3^{(1)}\chi' + h_2^{(1)}w_{,x_1} - h_1^{(1)}w_{,x_2} - \nu\Delta\chi' \\
&\qquad = F_3 - v\cdot\nabla\widetilde{\chi} + h_3^{(1)}\widetilde{\chi} && \text{in } \Omega^T,\\
&\chi' = 0 && \text{on } S_1^T,\\
&\chi'_{,x_3} = 0 && \text{on } S_2^T,\\
&\chi'|_{t=0} = \chi(0) - \chi_0 && \text{in } \Omega.
\end{aligned}
$$

LEMMA 4.1. *Assume that v is the weak solution to problem* (1.1), *assume that*

$$
\begin{aligned}
&h^{(1)} \in L_2(0,T;L_3(\Omega)) \cap L_\infty(0,T;L_{5/2}(\Omega)) \cap W_{r'}^{s's'/2}(\Omega^T),\\
&h^{(2)} \in L_2(0,T;H^1(\Omega)),
\end{aligned}
$$

where $5/r' - s' \le 3/2$, $v \in W_r^{s,s/2}(\Omega^T)$ with $5/r - 11/9 \le s$. Assume that $d \in L_2(0,T;L_\infty(S_2))$, $v(0) \in W_r^{s-2/r}(\Omega)$, $\chi(0) \in L_2(\Omega)$, $h^{(1)}(0) \in W_{r'}^{s'-2/r'}(\Omega)$, $F_3 \in L_{18/13}(\Omega^T)$. Then solutions of problem (3.4) *satisfy*

$$
\begin{aligned}
(4.6)\quad &|\chi(t)|_{2,\Omega}^2 + \nu\int_0^t \|\chi(t')\|_{W_2^{1,2}(\Omega)}^2\,dt' \le c\exp[c(|h^{(1)}|_{3,2,\Omega^t}^2 + |d|_{\infty,2,S_2^t}^2)]\\
&\quad\cdot[(l_1^2(t) + \|h^{(1)}\|_{s',r',\Omega^t}^2 + |h^{(1)}|_{5/2,\infty,\Omega^t}^2 + 1)\\
&\quad\cdot(\|v\|_{s,r,\Omega^t}^2 + \|v(0)\|_{s-2/r,r,\Omega}^2)\\
&\quad + l_1^2(t)|h^{(1)}|_{5/2,\infty,\Omega^t}^2 + |F_3|_{18/13,\Omega^t}^2 + |\chi(0)|_{2,\Omega}^2]\\
&\quad + c(\|h^{(1)}\|_{s',r',\Omega^t}^2 + \|h^{(1)}(0)\|_{s'-2/r',r',\Omega}^2 + |\nabla h^{(2)}|_{2,\Omega^t}^2),\quad t\le T.
\end{aligned}
$$

PROOF. Multiplying $(4.5)_1$ by χ', integrating the result over Ω and using $(1.1)_3$ we get

$$
\begin{aligned}
(4.7)\qquad \frac12\frac{d}{dt}|\chi'|_{2,\Omega}^2 + \nu|\nabla\chi'|_{2,\Omega}^2 &= -\int_{S_2} d\chi'^2\,dS_2\\
&\quad + \int_\Omega h_3^{(1)}\chi'^2\,dx - \int_\Omega (h_2^{(1)}w_{,x_1} - h_1^{(1)}w_{,x_2})\chi'\,dx\\
&\quad + \int_\Omega F_3\chi'\,dx - \int_\Omega v\cdot\nabla\widetilde{\chi}\chi'\,dx + \int_\Omega h_3^{(1)}\widetilde{\chi}\chi'\,dx.
\end{aligned}
$$

Estimating the first term on the r.h.s. by

$$
\begin{aligned}
|d|_{\infty,S_2}|\chi'|_{2,S_2}^2 &\le |d|_{\infty,S_2}\Big(\varepsilon_1|\nabla\chi'|_{2,\Omega}^2 + \frac{c}{\varepsilon_1}|\chi'|_{2,\Omega}^2\Big)\\
&\le \varepsilon_2|\nabla\chi'|_{2,\Omega}^2 + \frac{c}{\varepsilon_2}|d|_{\infty,S_2}^2|\chi'|_{2,\Omega}^2,
\end{aligned}
$$

and the second term on the r.h.s. of (4.7) by

$$
\varepsilon_3|\chi'|_{6,\Omega}^2 + c(1/\varepsilon_3)|h^{(1)}|_{3,\Omega}^2|\chi'|_{2,\Omega}^2,
$$

utilizing the Poincaré inequality in (4.7), assuming $\varepsilon_1-\varepsilon_3$ sufficiently small and integrating the result with respect to time yield

$$\begin{aligned}(4.8)\quad |\chi'(t)|_{2,\Omega}^2 &+\nu\int_0^t \|\chi'(t')\|^2_{W_2^{1,2}(\Omega)}\,dt' \le c\exp[c(|h^{(1)}|^2_{3,2,\Omega^t}+|d|^2_{\infty,2,S_2^t})]\\ &\cdot\bigg[\int_{\Omega^t}|h^{(1)}||w_{,x'}||\chi'|\,dx\,dt'+\int_{\Omega^t}|F_3\chi'|\,dx\,dt'\\ &+\int_{\Omega^t}|v\cdot\nabla\widetilde\chi\chi'|\,dx\,dt'+\int_{\Omega^t}|h^{(1)}\widetilde\chi\chi'|\,dx\,dt'+|\chi(0)-\chi_0|^2_{2,\Omega}\bigg]\\ &+c\int_0^t|\partial_{x_3}^2\chi'(t')|^2_{2,\Omega}\,dt'.\end{aligned}$$

By the Hölder and Young inequalities and (2.7) we estimate the first integral on the r.h.s. of (4.8) in the following way

$$\begin{aligned}\int_0^t dt'[\varepsilon_1|\chi'(t')|^2_{10,\Omega}&+c(1/\varepsilon_1)|h^{(1)}(t')|^2_{5/2,\Omega}|w_{,x}(t')|^2_{2,\Omega}]\\ &\le\varepsilon_1\int_0^t\|\chi'(t')\|^2_{W_2^{1,2}(\Omega)}\,dt'+c(1/\varepsilon_1)|h^{(1)}|^2_{5/2,\infty,\Omega^t}l_1^2(t).\end{aligned}$$

We bound the second term on the r.h.s. of (4.8) by

$$|\chi'|_{18/5,\Omega^t}|F_3|_{\frac{18}{13},\Omega^t}\le\varepsilon_2|\chi'|^2_{18/5,\Omega^t}+c(1/\varepsilon_2)|F_3|^2_{18/13,\Omega^t}.$$

By the Hölder inequality the third integral on the r.h.s. of (4.8) is estimated by

$$|v|_{18/5,\Omega^t}|\nabla\widetilde\chi|_{9/4,\Omega^t}|\chi'|_{18/5,\Omega^t}\equiv I_1.$$

In view of (2.5) and (4.2) we have

$$I_1\le\varepsilon_3|\chi'|^2_{18/5,\Omega^t}+c(1/\varepsilon_3)l_2^2(t)|\nabla\widetilde\chi|^2_{9/4,\Omega^t},$$

where to estimate the last norm we apply the imbedding

$$|\nabla\widetilde\chi|_{9/4,\Omega^t}\le c\|\widetilde\chi\|_{s,r,\Omega^t}$$

with

$$(4.9)\qquad 5/r-11/9\le s,\quad 1<s\in\mathbb{R}_+,\quad r\in(1,9/4).$$

Applying the Hölder inequalities the fourth term on the r.h.s. of (4.8) is bounded by

$$|\chi'|_{18/5,\Omega^t}|h^{(1)}\widetilde\chi|_{18/13,\Omega^t}\equiv I_2.$$

Assuming that $\widetilde\chi\in W_r^{s,s/2}(\Omega^T)$ with s,r satisfying (4.9) we obtain that $\widetilde\chi\in L_{45/11}(\Omega^T)$. Hence we get

$$I_2\le\varepsilon_4|\chi'|^2_{18/5,\Omega^t}+c(1/\varepsilon_4)|h^{(1)}|^2_{90/43,\Omega^t}\|\widetilde\chi\|^2_{s,r,\Omega^t}.$$

Utilizing the above estimates in (4.8) and assuming that $\varepsilon_1-\varepsilon_4$ are sufficiently small we obtain

$$
\begin{aligned}
(4.10)\quad |\chi'(t)|_{2,\Omega}^2 &+ \nu\int_0^t \|\chi'(t')\|^2_{W_2^{1,2}(\Omega)}\,dt' \le c\exp[c|h^{(1)}|^2_{3,2,\Omega^t} + |d|^2_{\infty,2,S_2^t})]\\
&\cdot[(l_2^2(t)+|h^{(1)}|^2_{90/43,\Omega^t})\|\widetilde{\chi}\|^2_{s,r,\Omega^t} + l_1^2(t)|h^{(1)}|^2_{5/2,\infty,\Omega^t}\\
&+|F_3|^2_{18/13,\Omega^t}+|\chi(0)-\chi_0|^2_{2,\Omega}] + c\int_0^t |\partial^2_{x_3}\chi'(t')|^2_{2,\Omega}\,dt'.
\end{aligned}
$$

In view of the imbedding

$$
|\widetilde{\chi}|_{2,\infty,\Omega^T} + \|\widetilde{\chi}\|_{L_2(0,T;H^1(\Omega))} \le c\|\widetilde{\chi}\|_{s,r,\Omega^T},
$$

where s and r satisfy (4.9) and in view of transformation (4.4') we obtain from (4.10) the inequality

$$
\begin{aligned}
(4.11)\quad |\chi(t)|_{2,\Omega}^2 &+ \nu\int_0^t \|\chi(t')\|^2_{W_2^{1,2}(\Omega)}\,dt' \le c\exp[c(|h^{(1)}|^2_{3,2,\Omega^t} + |d|^2_{\infty,2,S_2^t})]\\
&\cdot[(l_2^2(t)+|h^{(1)}|^2_{90/43,\Omega^t}+1)\|\widetilde{\chi}\|^2_{s,r,\Omega^t} + l_1^2(t)|h^{(1)}|^2_{5/2,\infty,\Omega^t}\\
&+|F_3|^2_{18/13,\Omega^t}+|\chi(0)-\chi_0|^2_{2,\Omega}]\\
&+c\int_0^t |\partial^2_{x_3}\widetilde{\chi}(t')|^2_{2,\Omega}\,dt' + c\int_0^t |\nabla'\partial^2_{x_3}v|^2_{2,\Omega}\,dt'.
\end{aligned}
$$

Solving problem (4.3) we get

$$
(4.12)\qquad \|\widetilde{\chi}\|_{s,r,\Omega^T} \le c(\|\chi_*\|_{s-1/r,r,S_1^T} + \|\chi_0\|_{s-2/r,r,\Omega}).
$$

In view of $(3.3)_2$ we need that $v\in W_r^{s,s/2}(\Omega^T)$ and the estimate holds

$$
\|\chi_*\|_{s-1/r,r,S_1^T} \le c\|v\|_{s,r,\Omega^T}.
$$

In virtue of compatibility condition $(4.4)_1$ we get

$$
\|\chi_0\|_{s-2/r,r,\Omega} \le c\|\chi_0|_{S_1}\|_{s-3/r,r,S_1} \le c\|v(0)|_{S_1}\|_{s-3/r,S_1} \le c\|v(0)\|_{s-2/r,\Omega}.
$$

Therefore, instead of (4.12) we obtain

$$
\|\widetilde{\chi}\|_{s,r,\Omega^T} \le c(\|v\|_{s,r,\Omega^T} + \|v(0)\|_{s-2/r,r,\Omega}).
$$

Assuming that $s-2/r\ge 0$ which holds for $r\ge 27/11$ we have

$$
|\chi_0|_{2,\Omega} \le c\|v(0)\|_{s-2/r,r,\Omega}.
$$

Otherwise compatibility conditions (4.4) are satisfied.

To estimate the last but one term on the r.h.s. of (4.11) we express $\partial_{x_3}^2\widetilde{\chi}$ in the form $\partial_{x_3}\widetilde{\chi}'$. Then (4.3) implies that $\widetilde{\chi}'$ is a solution to the problem

$$\begin{aligned} \widetilde{\chi}'_{,t} - \nu\Delta\widetilde{\chi}' &= 0 && \text{in } \Omega^T, \\ \widetilde{\chi}' = \chi'_* &\equiv \chi_{*,x_3} && \text{on } S_1^T, \\ \widetilde{\chi}' &= 0 && \text{on } S_2^T, \\ \widetilde{\chi}'|_{t=0} = \chi'_0 &\equiv \chi_{0,x_3} && \text{in } \Omega. \end{aligned}$$

Since $\chi_* = \sum_{i=1}^2 a_i v_i$ we have that $\chi'_* = \sum_{i=1}^2 a_i h_i$. Then

$$\begin{aligned} \|\widetilde{\chi}'\|_{L_2(0,t;H^1(\Omega))} &\le c\|\widetilde{\chi}'\|_{s',r',\Omega^t} \le c(\|\chi'_*\|_{s'-1/r',r',S_1^t} + \|\chi'_0\|_{s'-2/r',r',\Omega}) \\ &\le c(\|h^{(1)}\|_{s',r',\Omega^t} + \|h^{(1)}(0)\|_{s'-2/r',r',\Omega}), \end{aligned}$$

where

$$\frac{5}{r'} - s' \le \frac{3}{2}. \tag{4.13}$$

Finally, we express the last term on the r.h.s. of (4.11) as $\int_0^t |\nabla h^{(2)}(t')|_{2,\Omega}^2\,dt'$ and we have that $l_2(t) = l_1(t) + |\nabla h^{(1)}|_{2,\Omega^t}$.

In view of the above considerations (4.11) takes the form

$$\begin{aligned} |\chi(t)|_{2,\Omega}^2 + \nu\int_0^t \|\chi(t')\|_{W_2^{1,2}(\Omega)}^2\,dt' &\le c\exp[c(\left|h^{(1)}\right|_{3,2,\Omega^t}^2 + \left|d\right|_{\infty,2,S_2^t}^2)] \\ &\quad\cdot [(l_1^2(t) + |\nabla h^{(1)}|_{2,\Omega^t}^2 + |h^{(1)}|_{90/43,\Omega^t}^2 + 1) \\ &\quad\cdot (\|v\|_{s,r,\Omega^t}^2 + \|v(0)\|_{s-2/r,r,\Omega}^2) \\ &\quad + l_1^2(t)\left|h^{(1)}\right|_{5/2,\infty,\Omega^t}^2 + |F_3|_{18/13,\Omega^t}^2 + |\chi(0)|_{2,\Omega}^2] \\ &\quad + c(\|h^{(1)}\|_{s',r',\Omega^t}^2 + \|h^{(1)}(0)\|_{s'-2/r',r',\Omega}^2 + |\nabla h^{(2)}|_{2,\Omega^t}^2), \end{aligned} \tag{4.14}$$

where s, r satisfy (4.9) and s', r' — (4.13).

Using the imbeddings

$$|h^{(1)}|_{90/43,\Omega^t} \le c\left|h^{(1)}\right|_{5/2,\infty,\Omega^t}, \qquad |\nabla h^{(1)}|_{2,\Omega^t} \le c\|h^{(1)}\|_{s',r',\Omega^t}$$

we obtain from (4.14) the inequality (4.6). □

Now we consider problem (3.8).

Lemma 4.2. *Assume that* $h^{(1)}, \chi \in V_2^0(\Omega^T) \cap L_2(0,T;W_2^{1,2}(\Omega))$. *Then solutions of problem* (3.8) *satisfy, for* $v' = (v_1, v_2)$ *and* $t \le T$,

$$\begin{aligned} \sup_{t'\le t}\|v'(t')\|_{1,\Omega} + \|\nabla v'\|_{L_2(0,t;W_2^{1,2}(\Omega))} &\le c(\left|\chi\right|_{2,\infty,\Omega^t} + \|\chi\|_{L_2(0,t;W_2^{1,2}(\Omega))} \\ &\quad + \left|h^{(1)}\right|_{2,\infty,\Omega^t} + \|h^{(1)}\|_{L_2(0,t;W_2^{1,2}(\Omega))}) \equiv cA_1(t). \end{aligned} \tag{4.15}$$

PROOF. For solutions of problem (3.8) we have

$$\sup_{t'\le t}\int_{-a}^{a}\|v'(t',x_3)\|_{1,\Omega'}^2\,dx_3+\int_{-a}^{a}\|\nabla' v'(t',x_3)\|^2_{L_2(0,t;W_2^1(\Omega'))}\,dx_3 \le c(\|\chi\|^2_{V_2^0(\Omega^t)}+\|h^{(1)}\|^2_{V_2^0(\Omega^t)}), \tag{4.16}$$

where index Ω' in the norms on the l.h.s. of (4.16) means that derivatives with respect to $x'=(x_1,x_2)$ appear only. Adding derivatives with respect x_3 in corresponding norms to (4.16) implies (4.15). □

Now we obtain an estimate for velocity.

LEMMA 4.3. *Assume that*

$$\begin{aligned}\gamma_1(t)&=\|h^{(1)}\|_{s',r',\Omega^t}+|\nabla h^{(2)}|_{2,\Omega^t}<\infty, s',r' \quad \text{satisfy}(4.27),\\ G_1(t)&=|f|_{27/16,\Omega^t}+|f|_{6/5,2,\Omega^t}+|F_3|_{18/13,\Omega^t}\\ &\quad+\|d\|_{38/27,27/16,S_2^t}+\|d\|_{L_\infty(0,t;W_3^1(S_2))}+|d_{,t}|_{6/5,2,S_2^t}<\infty,\\ G_0(0)&=\|v(0)\|_{22/27,27/16,\Omega}+|\chi(0)|_{2,\Omega}+\|v(0)\|_{s-2/r,r,\Omega}\\ &\quad+\|h^{(1)}(0)\|_{s'-2/r',r',\Omega}<\infty,\quad \textit{where } s,r \textit{ satisfy } (4.25) \textit{ and } (4.9).\end{aligned}$$

Then the following inequality holds

$$\|v\|_{2,27/16,\Omega^t}\le G(\gamma_1(t),G_1(t),G_0(0)), \tag{4.17}$$

where G is an increasing positive function.

PROOF. In view of (2.1) and (2.2) we get from (4.15) the inequality

$$|v'|_{q,r,\Omega^t}+|v'_{,x}|_{q,r,\Omega^t}\le cA_1(t), \tag{4.18}$$

where

$$\frac{2}{r}+\frac{5}{2q}=\frac{5}{4}.$$

Hence

$$\|v'\|_{L_r(0,t;W_q^1(\Omega))}\le cA_1(t).$$

Let us use the imbedding

$$|v'|_{\sigma,\Omega}\le c\|v'\|_{1,q,\Omega}, \tag{4.19}$$

with

$$\frac{1}{q}=\frac{1}{3}+\frac{1}{\sigma}.$$

Utilizing (4.19) in (4.18) yields

$$|v'|_{\sigma,r,\Omega^t}\le cA_1(t), \tag{4.20}$$

where

$$\frac{2}{r}+\frac{5}{2\sigma}=\frac{5}{12}. \tag{4.21}$$

Inserting $\sigma = r$ in (4.21) gives that $r = 54/5$. Then (4.20) takes the form

$$|v'|_{54/5,\Omega^t} \le cA_1(t). \tag{4.22}$$

To increase regularity of v we consider the problem

$$\begin{aligned} & v_{,t} - \operatorname{div}\mathbb{T}(v,p) = -v'\cdot\nabla v - wh^{(1)} + f && \text{in } \Omega^T,\\ & \operatorname{div} v = 0 && \text{in } \Omega^T,\\ & v\cdot\overline{n}|_{S_1} = 0, \quad \overline{n}\cdot\mathbb{T}(v,p)\cdot\overline{\tau}_\alpha|_{S_1} = -\gamma v\cdot\overline{\tau}_\alpha|_{S_1}, \quad \alpha = 1,2, && \text{on } S_1^T,\\ & v\cdot\overline{n}|_{S_2} = d, \quad \overline{n}\cdot\mathbb{T}(v,p)\cdot\overline{\tau}_\alpha|_{S_2} = 0, \quad \alpha = 1,2, && \text{on } S_2^T,\\ & v|_{t=0} = v(0) && \text{in } \Omega. \end{aligned} \tag{4.23}$$

In view of the proof of Lemma 4.4 from [6] we have

$$\begin{aligned} |v'\cdot\nabla v|_{27/16,\Omega^t} &\le |v'|_{54/5,\Omega^t}|\nabla v|_{2,\Omega^T} \le cl_1(t)A_1,\\ |wh^{(1)}|_{27/16,\Omega^t} &\le |w|_{18/5,\Omega^t}|h^{(1)}|_{54/17,\Omega^t} \le cl_2(t)|h^{(1)}|_{54/17,\Omega^t}. \end{aligned}$$

In view of the above estimates we obtain for solutions of problem (4.23) the inequality

$$\begin{aligned} \|v\|_{2,27/16,\Omega^t} \le{}& c(l_1A_1 + l_2|h^{(1)}|_{54/17,\Omega^t}) + c\|v\|_{11/27,27/16,S_1^t}\\ &+ c(|f|_{27/16,\Omega^t} + \|d\|_{38/27,27/16,S_2^t} + \|v(0)\|_{22/27,27/16,\Omega}). \end{aligned} \tag{4.24}$$

Exploiting the form of A_1 defined by the r.h.s. of (4.15) and (4.6) and using the interpolation inequalities

$$\begin{aligned} \|v\|_{11/27,27/16,S_1^T} &\le \varepsilon_1\|v\|_{2,27/16,\Omega^T} + c(1/\varepsilon_1)|v|_{2,\Omega^T},\\ \|v\|_{s,r,\Omega^T} &\le \varepsilon_2^{1-\varkappa_1}\|v\|_{2,27/16,\Omega^T} + c\varepsilon_2^{-\varkappa_1}|v|_{2,\Omega^T}, \end{aligned}$$

where

$$\varkappa_1 = \frac{1}{2}\left(\frac{80}{27} - \frac{5}{r} + s\right) < 1 \tag{4.25}$$

we obtain from (4.24) the inequality

$$\begin{aligned} \|v\|^2_{2,27/16,\Omega^t} \le{}& cl_1^2\exp[c(\left|h^{(1)}\right|^2_{3,2,\Omega^t} + \left|d\right|^2_{\infty,2,S_2^t})]\\ &\cdot[(l_1^2 + \left|h^{(1)}\right|^2_{s',r',\Omega^t} + \left|h^{(1)}\right|^2_{5/2,\infty,\Omega^t} + 1)\|v(0)\|^2_{s-2/r,r,\Omega}\\ &+ (l_1^2 + \|h^{(1)}\|^2_{s',r',\Omega^t} + \|h^{(1)}\|^2_{5/2,\infty,\Omega^t} + 1)^{1/(1-\varkappa_1)}l_1^2\\ &+ l_1^2\left|h^{(1)}\right|^2_{5/2,\infty,\Omega^t} + |F_3|^2_{18/13,\Omega^t} + |\chi(0)|^2_{2,\Omega}]\\ &+ cl_1^2(\left|h^{(1)}\right|^2_{s',r',\Omega^t} + \|h^{(1)}(0)\|^2_{s'-2/r',r',\Omega} + |\nabla h^{(2)}|^2_{2,\Omega^t})\\ &+ c(l_2^2|h^{(1)}|^2_{54/17,\Omega^t} + |f|^2_{27/16,\Omega^t} + \|d\|^2_{38/27,27/16,S_2^t}\\ &+ \|v(0)\|^2_{22/27,27/16,\Omega} + l_1^2). \end{aligned} \tag{4.26}$$

Assuming that

$$5/r' - 6/5 \le s' \tag{4.27}$$

we have the imbeddings

$$\big|h^{(1)}\big|_{3,2,\Omega^t} + \big|h^{(1)}\big|_{5/2,\infty,\Omega^t} + |h^{(1)}|_{54/17,\Omega^t} \le c\|h^{(1)}\|_{s',r',\Omega^t}.$$

Then (4.26) takes the form

$$\begin{aligned}\|v\|_{2,27/16,\Omega^t} \le{}& cl_1 \exp c(\|h^{(1)}\|^2_{s',r',\Omega^t} + \big|d\big|^2_{\infty,2,S_2^t})\\ &\cdot [(l_1 + \|h^{(1)}\|_{s',r',\Omega^t} + 1)\|v(0)\|_{s-2/r,r,\Omega}\\ &+ (l_1 + \|h^{(1)}\|_{s',r',\Omega^t} + 1)^{1/(1-\varkappa_1)} l_1\\ &+ l_1\|h^{(1)}\|_{s',r',\Omega^t} + |F_3|_{18/13,\Omega^t} + |\chi(0)|_{2,\Omega}]\\ &+ c(l_1 + \|h^{(1)}\|_{s',r',\Omega^t})\|h^{(1)}\|_{s',r',\Omega^t} + cl_1|\nabla h^{(2)}|_{2,\Omega^t}\\ &+ c(l_1\|h^{(1)}(0)\|_{s'-2/r',r',\Omega} + |f|_{27/16,\Omega^t}\\ &+ \|d\|_{38/27,27/16,S_2^t} + \|v(0)\|_{22/27,27/16,\Omega} + l_1),\end{aligned} \tag{4.28}$$

where we used that $l_2 \le l_1 + \|h^{(1)}\|_{s',r',\Omega^t}$. From (2.29) we obtain

$$l_1(t) \le \varphi_1(\|d\|_{L_\infty(0,t;W_3^1(S_2))}, \big|d_{,t}\big|_{6/5,2,S_2^t} + \big|f\big|_{6/5,2,\Omega^t} + |v(0)|_{2,\Omega} + \|h^{(1)}\|_{s',r',\Omega^t}),$$

where φ_1 is a positive function, nonlinear with respect to the first argument and linear with respect to the second one. Utilizinig the above inequalities in (4.28) yields (4.17). □

To prove the existence of local solutions to problem (1.1) we apply the Leray–Schauder fixed point theorem. We show existence of a fixed point of a transformation generated by problems (3.1) and (3.9). First we examine problem (3.1).

LEMMA 4.4. *Let* $v \in W_r^{2,1}(\Omega^T)$, $r > 5/3$. *Let* $h^{(1)} \in L_2(\Omega^T)$, $g^{(1)} \in W_\delta^{\beta,\beta/2}(\Omega^T)$, $h^{(1)}(0) \in W_\delta^{2+\beta-2/\delta}(\Omega)$, $d_{,x'} \in W_\delta^{2+\beta-1/\delta,1+\beta/2-1/(2\delta)}(S_2^T)$, $\beta > 0$, $\delta > 1$. *Then solutions of* (3.1) *satisfy*

$$\begin{aligned}\|h^{(1)}\|_{2+\beta,\delta,\Omega^T} + \|\nabla q^{(1)}\|_{\beta,\delta,\Omega^T} \le{}& \varphi(\|v\|_{2,r,\Omega^T})|h^{(1)}|_{2,\Omega^T}\\ &+ c(\|g^{(1)}\|_{\beta,\delta,\Omega^T} + \|d_{,x'}\|_{2+\beta-1/\delta,\delta,S_2^T} + \|h^{(1)}(0)\|_{2+\beta-2/\delta,\delta,\Omega})\\ &\equiv \varphi(\|v\|_{2,r,\Omega^T})|h^{(1)}|_{2,\Omega^T} + cl_6(T),\end{aligned} \tag{4.29}$$

where φ *is an increasing positive function.*

PROOF. Applying [1] to problem (3.1) yields

$$\begin{aligned}\|h^{(1)}\|_{2+\beta,\delta,\Omega^T} + \|\nabla q^{(1)}\|_{\beta,\delta,\Omega^T} \le{}& c(\|v\cdot\nabla h^{(1)}\|_{\beta,\delta,\Omega^T} + \|h^{(1)}\cdot\nabla v\|_{\beta,\delta,\Omega^T}\\ &+ \|g^{(1)}\|_{\beta,\delta,\Omega^T} + \|d_{,x'}\|_{2+\beta-1/\delta,\delta,S_2^T} + \|h^{(1)}(0)\|_{2+\beta-2/\delta,\delta,\Omega}).\end{aligned} \tag{4.30}$$

Utilizing Lemma 2.2 from [8] to the first term on the r.h.s. of (4.30) implies

$$\|v\cdot\nabla h^{(1)}\|_{\beta,\delta,\Omega^T}\le\|v\|_{\beta+\varepsilon/\delta,\delta_1,\Omega^T}|\nabla h^{(1)}|_{\delta_1',\Omega^T} + \|\nabla h^{(1)}\|_{\beta+\varepsilon/\delta,\delta_2,\Omega^T}|v|_{\delta_2',\Omega^T}\equiv I_1, \tag{4.31}$$

where $\varepsilon>0$ is assumed as arbitrary small, $1/\delta_i+1/\delta_i'=1/\delta$, $i=1,2$. To estimate I_1 we use the imbeddings

$$\|v\|_{\beta+\varepsilon/\delta,\delta_1,\Omega^T}+|v|_{\delta_2',\Omega^T}\le c\|v\|_{2,r,\Omega^T} \tag{4.32}$$

with

$$\frac{5}{r}-\frac{5}{\delta_1}+\beta<2,\quad \frac{5}{r}-\frac{5}{\delta_2'}\le 2 \tag{4.33}$$

and the interpolation inequalities

$$\begin{aligned} |\nabla h^{(1)}|_{\delta_1',\Omega^T} &\le \varepsilon_1^{1-\varkappa_1}\|h^{(1)}\|_{2+\beta,\delta,\Omega^T}+c\varepsilon_1^{-\varkappa_1'}|h^{(1)}|_{2,\Omega^T},\\ \|\nabla h^{(1)}\|_{\beta+\frac{\varepsilon}{\delta},\delta_2,\Omega^T} &\le \varepsilon_2^{1-\varkappa_2}\|h^{(1)}\|_{2+\beta,\delta,\Omega^T}+c\varepsilon_2^{-\varkappa_2'}|h^{(1)}|_{2,\Omega^T}, \end{aligned} \tag{4.34}$$

with

$$\begin{aligned} \varkappa_1 &= \left(\frac{5}{\delta}-\frac{5}{\delta_1'}+1\right)\frac{1}{2+\beta}<1,\\ \varkappa_1' &= \left(\frac{5}{2}-\frac{5}{\delta_1'}+1\right)\frac{1}{2+\beta},\\ \varkappa_2 &= \left(\frac{5}{\delta}-\frac{5}{\delta_2}+\beta+\frac{\varepsilon}{\delta}+1)\right)\frac{1}{2+\beta}<1,\\ \varkappa_2' &= \left(\frac{5}{2}-\frac{5}{\delta_2}+\beta+\frac{\varepsilon}{\delta}+1\right)\frac{1}{2+\beta}. \end{aligned} \tag{4.35}$$

From (4.33) and (4.35) we obtain the following restrictions

$$\frac{5}{r}+\beta<2+\frac{5}{\delta_1}<3+\beta,\quad \frac{5}{r}\le 2+\frac{5}{\delta_2'}<3,\quad \frac{1}{\delta_i}+\frac{1}{\delta_i'}=1,\quad i=1,2, \tag{4.36}$$

which hold for $r>5/3$. Applying Lemma 2.2 from [8] to the second term on the r.h.s. of (4.30) yields

$$\|h^{(1)}\cdot\nabla v\|_{\beta,\delta,\Omega^T}\le\|\nabla v\|_{\beta+\varepsilon/\delta,\delta_3,\Omega^T}|h^{(1)}|_{\delta_3',\Omega^T} + \|h^{(1)}\|_{\beta+\varepsilon/\delta,\delta_4,\Omega^T}|\nabla v|_{\delta_4',\Omega^T}\equiv I_2, \tag{4.37}$$

where $\varepsilon>0$ is arbitrary small and $1/\delta_i+1/\delta_i'=1/\delta$, $i=3,4$.

To estimate I_2 we utilize the imbeddings

$$\|\nabla v\|_{\beta+\varepsilon/\delta,\delta_3,\Omega^T}+|\nabla v|_{\delta_4',\Omega^T}\le c\|v\|_{2,r,\Omega^T}, \tag{4.38}$$

with

$$\frac{5}{r}-\frac{5}{\delta_3}+\beta+1<2,\qquad \frac{5}{r}-\frac{5}{\delta_4'}+1\le 2, \tag{4.39}$$

and the interpolation inequalities

$$
(4.40)\qquad \begin{aligned} |h^{(1)}|_{\delta_3',\Omega^T} &\le \varepsilon_3^{1-\varkappa_3}\|h^{(1)}\|_{2+\beta,\delta,\Omega^T} + c\varepsilon_3^{-\varkappa_3'}|h^{(1)}|_{2,\Omega^T},\\ \|h^{(1)}\|_{\beta+\varepsilon/\delta,\delta_4,\Omega^T} &\le \varepsilon_4^{1-\varkappa_4}\|h^{(1)}\|_{2+\beta,\delta,\Omega^T} + c\varepsilon_4^{-\varkappa_4'}|h^{(1)}|_{2,\Omega^T}, \end{aligned}
$$

with

$$
(4.41)\qquad \begin{aligned} \varkappa_3 &= \left(\frac{5}{\delta} - \frac{5}{\delta_3'}\right)\frac{1}{2+\beta} < 1, & \varkappa_3' &= \left(\frac{5}{2} - \frac{5}{\delta_3'}\right)\frac{1}{2+\beta},\\ \varkappa_4 &= \left(\frac{5}{\delta} - \frac{5}{\delta_4} + \beta + \frac{\varepsilon}{\delta}\right)\frac{1}{2+\beta} < 1, & \varkappa_4' &= \left(\frac{5}{2} - \frac{5}{\delta_4} + \beta + \frac{\varepsilon}{\delta}\right)\frac{1}{2+\beta}. \end{aligned}
$$

From (4.39) and (4.41) we have the restrictions

$$
(4.42)\qquad \frac{5}{r} + \beta < 1 + \frac{5}{\delta_3} < 3 + \beta, \quad \frac{5}{r} \le 1 + \frac{5}{\delta_4'} < 3, \quad \frac{1}{\delta_i} + \frac{1}{\delta_i'} = \frac{1}{\delta}, \quad i = 3, 4,
$$

which holds for $r > 5/3$. Exploiting the above estimates in (4.30) and assuming that ε_i, $i = 1, \ldots, 4$, are sufficiently small we obtain (4.29). □

To examine problem (3.9) we need more regularity for v. Hence we have

Lemma 4.5. *Assume that* $v \in W_r^{2,1}(\Omega^T)$, $r = 27/16$. *Assume*

$$
\begin{aligned} G_2(T) = {} & |d|_{\infty,S_2^T} + \|d_{,x'}\|_{L_\infty(0,T;W_3^1(S_2))} + \|d_{,t}\|_{L_2(0,T;W_{6/5}^1(S_2))} + |F_3|_{18/13,\Omega^T}\\ & + |f|_{2,\Omega^T} + \|f\|_{\sigma,\rho,\Omega^T} + |f_3|_{4/3,2,S_2^T} + |g^{(1)}|_{6/5,2,\Omega^T} < \infty,\\ G_0^1(0) = {} & \|v(0)\|_{s-2/r,r,\Omega} + |\chi(0)|_{2,\Omega} + \|v(0)\|_{1,\Omega}\\ & + \|v(0)\|_{2+\sigma-2/\rho,\rho,\Omega} + \|h^{(1)}(0)\|_{s'-2/r',r',\Omega} < \infty. \end{aligned}
$$

Assume that $2+\sigma < 5/\rho$, $5/r - 11/9 \le s$, $5/r' - s' \le 6/5$, $80/27 - 5/r + s < 2$. *Then, for* $t \le T$,

$$
(4.43)\qquad \|v\|_{2+\sigma,\rho,\Omega^t} \le \varphi(\|v\|_{2,27/16,\Omega^t}, \|h^{(1)}\|_{s',r',\Omega^t}, |\nabla h^{(2)}|_{2,\Omega^t}, t, G_2(T), G_0^1(0)),
$$

where φ *is an increasing positive function.*

Proof. To show the lemma we use problem (1.1) in the form (4.23). For solutions of (4.23) we have

$$
(4.44)\qquad \begin{aligned} \|v\|_{2,2,\Omega^t} &\le c(|v'\cdot\nabla v|_{2,\Omega^t} + |w\cdot h^{(1)}|_{2,\Omega^t} + |f|_{2,\Omega^t}\\ &\quad + \|v\|_{1/2,2,S_1^t} + \|v(0)\|_{1,\Omega})\\ &\le c(|v'|_{54/5,\Omega^t}|\nabla v|_{27/11,\Omega^t} + |w|_{5,\Omega^t}|h^{(1)}|_{10/3,\Omega^t} + |f|_{2,\Omega^t}\\ &\quad + \|v\|_{2,27/16,\Omega^t} + \|v(0)\|_{1,\Omega})\\ &\le c(A_1(t)\|v\|_{2,27/16,\Omega^t} + \|v\|_{2,27/16,\Omega^t}\|h^{(1)}\|_{V_2^0(\Omega^t)}\\ &\quad + |f|_{2,\Omega^t} + \|v\|_{2,27/16,\Omega^t} + \|v(0)\|_{1,\Omega}) \equiv I_1, \end{aligned}
$$

where we used (4.22) and imbeddings

$$|v|_{5,\Omega^t}+|\nabla v|_{27/11,\Omega^t}\le c\|v\|_{2,27/16,\Omega^t}.$$

Moreover, we have the imbedding

$$|\nabla v|_{3,2,\Omega^t}\le c\|v\|_{2,27/16,\Omega^t}.$$

From (2.29) we get

$$l_1^2(t)\le\varphi(\|d\|_{L_\infty(0,t;W_3^1(S_2))},t)[\|d\|_{L_2(0,t;H^1(S_2))}+|d_{,t}|_{6/5,2,S_2^t}+|f|_{6/5,2,\Omega^t}+|v(0)|_{2,\Omega}].$$

From the assumptions of Lemma 3.7 we obtain

$$\begin{aligned}l_3(t)\le{}&l_1\|d_{,x'}\|_{L_\infty(0,t;W_3^1(S_2))}+(1+\|d\|_{L_\infty(0,t;L_3(S_2))})\|d_{,x'}\|_{L_\infty(0,t;H^1(S_2))}\\&+\|d_{,t}\|_{L_2(0,t;W_{6/5}^1(S_2))}+|f_3|_{4/3,2,S_2^t}+|g^{(1)}|_{6/5,2,\Omega^t}+|h^{(1)}(0)|_{2,\Omega}\\\le{}&c(l_1+|d|_{3,\infty,S_2^t})\eta_1.\end{aligned}$$

Next, (3.52) implies

$$\|h^{(1)}\|_{V_2^0(\Omega^t)}\le\varphi(t,l_1,\|d\|_{L_\infty(0,t;L_3(S_2))},\|v\|_{2,27/16,\Omega^t})\eta_1,$$

where η_1 is defined by (3.53).

In view of (4.6) we have

$$\begin{aligned}A_1\le{}&c\exp c(|h^{(1)}|^2_{3,2,\Omega^t}+|d|^2_{\infty,2,S_2^t})[(l_1^2+\|h^{(1)}\|^2_{s',r',\Omega^t}+1)\\&\cdot(\|v\|^2_{2,27/16,\Omega^t}+\|v(0)\|^2_{s-2/r,r,\Omega})+l_1^2\|h^{(1)}\|^2_{s',r',\Omega^t}\\&+|F_3|^2_{18/13,\Omega^t}+|\chi(0)|^2_{2,\Omega}]+c(\|h^{(1)}\|^2_{s',r',\Omega^t}\\&+\|h^{(1)}\|^2_{V_2^0(\Omega^t)}+|\nabla h^{(2)}|^2_{2,\Omega^t}+\|h^{(1)}(0)\|^2_{s'-2/r',r',\Omega})\\={}&\varphi(\|v\|_{2,27/16,\Omega^t},\|h^{(1)}\|_{s',r',\Omega^t},t,l_7)+c(\|h^{(1)}\|^2_{V_2^0(\Omega^t)}+|\nabla h^{(2)}|^2_{2,\Omega^t}),\end{aligned}$$

where

$$l_7=|d|_{\infty,S_2^t}+|F_3|_{18/13,\Omega^t}+\|v(0)\|_{s-2/r,r,\Omega}+|\chi(0)|_{2,\Omega}+\|h^{(1)}(0)\|_{s'-2/r',r',\Omega}+l_1.$$

Utilizing the above estimates in (4.44) yields

$$\|v\|_{2,2,\Omega^t}\le\varphi(\|v\|_{2,27/16,\Omega^t},\|h^{(1)}\|_{s',r',\Omega^t},t,l_8)+c|\nabla h^{(2)}|_{2,\Omega^t}\equiv cl_9,\tag{4.45}$$

where

$$l_8=l_7+\|d_{,x'}\|_{L_\infty(0,t;W_3^1(S_2))}+\|d_{,t}\|_{L_2(0,t;W_{6/5}^1(S_2))}+|f_3|_{4/3,2,S_2^t}+|g^{(1)}|_{6/5,2,\Omega^t}.$$

Having that $v\in W_2^{2,1}(\Omega^T)$ we are looking for solutions of (4.23) such that $v\in W_\rho^{2+\sigma,1+\sigma/2}(\Omega^T)$. For this purpose we examine

$$\|v\cdot\nabla v\|_{\sigma,\rho,\Omega^t}\le|v|_{\rho_1^1,\Omega^t}\|\nabla v\|_{\sigma+\varepsilon,\rho_2^1,\Omega^t}+\|v\|_{\sigma+\varepsilon,\rho_1^2,\Omega^t}|\nabla v|_{\rho_2^2,\Omega^t}\le c\|v\|^2_{2,2,\Omega^t},$$

where $1/\rho_1^i + 1/\rho_2^i = 1/\rho$, $i = 1,2$, $\varepsilon > 0$ and the second inequality holds for $2+\sigma < 5/\rho$. Finally, for $3/2+\sigma \le 5/\rho$ we have

$$\|v\|_{1+\sigma-1/\rho,\rho,S_1^t} \le c\|v\|_{2,2,\Omega^t}.$$

Hence for solutions of problem (4.23) we obtain the estimate

$$\begin{aligned}(4.46)\quad \|v\|_{2+\sigma,\rho,\Omega^t} &+ \|\nabla p\|_{\sigma,\rho,\Omega^t}\\ &\le c(\|v\|_{2,2,\Omega^t}^2 + \|v\|_{2,2,\Omega^t} + \|f\|_{\sigma,\rho,\Omega^t} + \|v(0)\|_{2+\sigma-2/\rho,\rho,\Omega}).\end{aligned}$$

From (4.46) and (4.45) the inequality (4.43) follows. □

Now we examine problem (3.9).

LEMMA 4.6. *Assume that*

$v \in W_\rho^{2+\sigma,1+\sigma/2}(\Omega^T) \cap W_2^{2,1}(\Omega^T)$,
$h^{(1)}, h^{(2)} \in L_2(\Omega^T)$, $h^{(1)} \in W_\delta^{2+\beta,1+\beta/2}(\Omega^T)$,
$g^{(2)} \in W_\delta^{\beta',\beta'/2}(\Omega^T)$,
$d_{,t} \in W_\delta^{2+\beta'-1/\delta,1+\beta'/2-1/(2\delta)}(S_2^T)$,
$d_{,x'} \in W_\delta^{3+\beta'-1/\delta,3/2+\beta'/2-1/(2\delta)}(S_2^T)$,
$F' \in W_\delta^{1+\beta'-1/\delta,1/2+\beta'/2-1/(2\delta)}(S_2^T)$,
$h^{(2)}(0) \in W_\delta^{2+\beta'-2/\delta}(\Omega)$, $d \in H^{5/2}(S_2^T)$,

where $0<\beta'<\beta<1$, $5/\delta < 3+\beta-\beta'$, $\beta'<\sigma<1$, $5/\rho < 2+\sigma+\beta-\beta'$, $\rho>\delta$. *Then solutions of problem* (3.9) *satisfy the inequality*

$$\begin{aligned}(4.47)\quad \|h^{(2)}\|_{2+\beta',\delta,\Omega^T} &+ \|\nabla q^{(2)}\|_{\beta',\delta,\Omega^T} \le \varphi(\|v\|_{2,27/16,\Omega^T})|h^{(2)}|_{2,\Omega^T}\\ &+ \varphi\bigg(\frac{1}{\varepsilon}\|h^{(1)}\|_{2+\beta,\delta,\Omega^T}, \|v\|_{2+\sigma,\rho,\Omega^T}, \|d\|_{5/2,2,S_2^T}, \|d_{,x'}\|_{3+\beta'-1/\delta,\delta,S_2^T}\bigg)\\ &\cdot |h^{(1)}|_{2,\Omega^T} + \varepsilon\|h^{(1)}\|_{2+\beta,\delta,\Omega^T} + c\|d_{,x'}\|_{3+\beta'-1/\delta,\delta,S_2^T}\|v\|_{2+\sigma,\rho,\Omega^T}\\ &+ c(\|g^{(2)}\|_{\beta',\delta,\Omega^T} + \|d_{,x'}\|_{3+\beta'-1/\delta,\delta,S_2^T}\\ &+ \|F'\|_{1+\beta'-1/\delta,\delta,S_2^T} + \|h^{(2)}(0)\|_{2+\beta'-2/\delta,\delta,\Omega}).\end{aligned}$$

PROOF. Applying [1] to problem (3.9) yields

$$\begin{aligned}(4.48)\quad \|h^{(2)}\|_{2+\beta',\delta,\Omega^T} &+ |\nabla q^{(2)}|_{\beta',\delta,\Omega^T} \le c(\|v\cdot\nabla h^{(2)}\|_{\beta',\delta,\Omega^T}\\ &+ \|h^{(1)}\cdot\nabla h^{(1)}\|_{\beta',\delta,\Omega^T} + \|h^{(2)}\cdot\nabla v\|_{\beta',\delta,\Omega^T} + \|g^{(2)}\|_{\beta',\delta,\Omega^T}\\ &+ \|d_{,x't}\|_{1+\beta'-1/\delta,\delta,S_2^T} + \|\Delta' d_{,x'}\|_{1+\beta'-1/\delta,\delta,S_2^T} + \|F'\|_{1+\beta'-1/\delta,\delta,S_2^T}\\ &+ \|vh_{,x'}^{(1)}\|_{1+\beta'-1/\delta,\delta,S_2^T} + \|vd_{,x'x'}\|_{1+\beta'-1/\delta,\delta,S_2^T}\\ &+ \|dh_{,x}^{(1)}\|_{1+\beta'-1/\delta,\delta,S_2^T} + \|h^{(1)}h^{(1)}\|_{1+\beta'-1/\delta,\delta,S_2^T}\\ &+ \|h^{(1)}d_{,x'}\|_{1+\beta'-1/\delta,\delta,S_2^T} + \|h^{(1)}v_{,x}\|_{1+\beta'-1/\delta,\delta,S_2^T}\\ &+ \|d_{,x'}v_{,x}\|_{1+\beta'-1/\delta,\delta,S_2^T} + \|h^{(2)}(0)\|_{2+\beta'-2/\delta,\delta,\Omega}).\end{aligned}$$

Employing Lemma 2.2 from [8] to the first and the third terms on the r.h.s. of (4.48) yields the same relations as (4.31)–(4.42) with the difference that $h^{(1)}$ is replaced by $h^{(2)}$ and β by β'.

Applying Lemma 2.2 from [8] to the second term on the r.h.s. of (4.48) we obtain

$$\|h^{(1)}\cdot\nabla h^{(1)}\|_{\beta,\delta,\Omega^T}\le c\|h^{(1)}\|_{\beta+\varepsilon/\delta,\delta_5,\Omega^T}|\nabla h^{(1)}|_{\delta_5',\Omega^T} + c\|\nabla h^{(1)}\|_{\beta+\varepsilon/\delta,\delta_6,\Omega^T}|h^{(1)}|_{\delta_6',\Omega^T}\equiv I_3, \tag{4.49}$$

where $1/\delta_i+1/\delta_i'=1/\delta$, $i=5,6$, $\varepsilon>0$.

Utilizing the imbedding

$$\|h^{(1)}\|_{\beta+\varepsilon/\delta,\delta_5,\Omega^T}+\|\nabla h^{(1)}\|_{\beta,\varepsilon/\delta,\delta_6,\Omega^T}\le c\|h^{(1)}\|_{2+\beta,\delta,\Omega^T}$$

and the interpolation inequality

$$|h^{(1)}|_{\delta_6',\Omega^T}+|\nabla h^{(1)}|_{\delta_5',\Omega^T}\le\bar\varepsilon_1\|h^{(1)}\|_{2+\beta,\delta,\Omega^T}+c(1/\bar\varepsilon_1)|h^{(1)}|_{2,\Omega^T}$$

which hold together for $5/\delta<3+\beta$ we obtain that

$$I_3\le\varepsilon_5\|h^{(1)}\|_{2+\beta,\delta,\Omega^T}+\varphi(1/\varepsilon_5,\|h^{(1)}\|_{2+\beta,\delta,\Omega^T})|h^{(1)}|_{2,\Omega^T}.$$

Finally we examine the terms from eighth to fourteenth on the r.h.s. (4.48). To simplify considerations we introduce $\widetilde d$ which is an extension of d into Ω. The eighth term,

$$\begin{aligned}\|vh^{(1)}_{,x'}\|_{1+\beta'-1/\delta,\delta,S_2^T}&\le c\|vh^{(1)}_{,x'}\|_{1+\beta',\delta,\Omega^T}\\&\le c|v|_{\delta_7',\Omega^T}\|h^{(1)}_{,x'}\|_{1+\beta'+\varepsilon/\delta,\delta_7,\Omega^T}+c|h^{(1)}_{,x'}|_{\delta_8,\Omega^T}\|v\|_{1+\beta'+\varepsilon/\delta,\delta_8',\Omega^T}\equiv I_4,\end{aligned}\tag{4.50}$$

where $1/\delta_i+1/\delta_i'=1/\delta$, $i=7,8$. By the estimate

$$|v|_{\delta_7',\Omega^T}+\|v\|_{1+\beta'+\varepsilon/\delta,\delta_8,\Omega^T}\le c\|v\|_{2+\sigma,\rho,\Omega^T}$$

and the interpolation inequality

$$\|h^{(1)}_{,x'}\|_{1+\beta'+\varepsilon/\delta,\delta_7,\Omega^T}+|h^{(1)}_{,x'}|_{\delta_8',\Omega^T}\le\bar\varepsilon_2\|h^{(1)}\|_{2+\beta,\delta,\Omega^T}+c(1/\bar\varepsilon_2)|h^{(1)}|_{2,\Omega^T}, \tag{4.51}$$

which hold together for $5/\rho<2+\sigma+\beta-\beta'$ we obtain

$$I_4\le\varepsilon_6\|h^{(1)}\|_{2+\beta,\delta,\Omega^T}+\varphi(1/\varepsilon_6,\|v\|_{2+\sigma,\rho,\Omega^T})|h^{(1)}|_{2,\Omega^T}.$$

The nineth term,

$$\begin{aligned}\|vd_{,x'x'}\|_{1+\beta'-1/\delta,\delta,S_2^T}&\le c\|v\widetilde d_{,x'x'}\|_{1+\beta',\delta,\Omega^T}\\&\le c\|v\|_{2,2,\Omega^T}\|\widetilde d_{,x'x'}\|_{2+\beta,\delta,\Omega^T}\le c\|v\|_{2,2,\Omega^T}\|d_{,x'}\|_{3+\beta-1/\delta,\delta,S_2^T}.\end{aligned}$$

The tenth term,

$$\begin{aligned}\|dh^{(1)}_{,x}\|_{1+\beta'-1/\delta,\delta,S^T}&\le\|\widetilde dh^{(1)}_{,x}\|_{1+\beta',\delta,\Omega^T}\le c\|\widetilde d\|_{3,2,\Omega^T}\|h^{(1)}\|_{2+\beta',\delta,\Omega^T}\\&\le\varepsilon_7\|h^{(1)}\|_{2+\beta,\delta,\Omega^T}+\varphi(1/\varepsilon_7,\|d\|_{5/2,2,S_2^T})|h^{(1)}|_{2,\Omega^T},\end{aligned}\tag{4.52}$$

where we used that $\beta' < \beta$.

The eleventh term,

$$\|h^{(1)}h^{(1)}\|_{1+\beta',\delta,\Omega^T} \le c\|h^{(1)}\|_{1+\beta'+\varepsilon/\delta,\delta_9,\Omega^T}|h^{(1)}|_{\delta_9',\Omega^T} \equiv I_4, \tag{4.53}$$

where $1/\delta_9 + 1/\delta_9' = 1/\delta$.

To estimate I_4 we use the interpolation inequalities

$$\|h^{(1)}\|_{1+\beta'+\varepsilon/\delta,\delta_9,\Omega^T} + |h^{(1)}|_{\delta_9',\Omega^T} \le \overline{\varepsilon}_3\|h^{(1)}\|_{2+\beta,\delta,\Omega^T} + c(1/\overline{\varepsilon}_3)|h^{(1)}|_{2,\Omega^T},$$

which hold for $5/\delta - 5/\delta_9 + 1 + \beta' < 2 + \beta$ and $5/\delta - 5/\delta_9' < 2 + \beta$. Therefore, for $5/\delta < 3 + \beta - \beta'$ it follows

$$I_4 \le \varepsilon_7\|h^{(1)}\|_{2+\beta,\delta,\Omega^T} + \varphi(1/\varepsilon_7, \|h^{(1)}\|_{2+\beta,\delta,\Omega^T})|h^{(1)}|_{2,\Omega^T}.$$

The twelveth term,

$$\begin{aligned}\|h^{(1)}d_{,x'}\|_{1+\beta'-1/\delta,\delta,S_2^T} &\le \|h^{(1)}\widetilde{d}_{,x'}\|_{1+\beta',\delta,\Omega^T} \\ &\le c|h^{(1)}|_{\delta_{10}',\Omega^T}\|\widetilde{d}_{,x'}\|_{1+\beta'+\varepsilon/\delta,\delta_{10},\Omega^T} \\ &\quad + c\|h^{(1)}\|_{1+\beta'+\varepsilon/\delta,\delta_{11},\Omega^T}|\widetilde{d}_{,x'}|_{\delta_{11}',\Omega^T} \equiv I_5,\end{aligned} \tag{4.54}$$

where $1/\delta_i + 1/\delta_i' = 1/\delta$, $i = 10, 11$. Hence for $5/\delta < 4 + \beta - \beta'$ and by an interpolation inequality we have

$$I_5 \le \varepsilon_8\|h^{(1)}\|_{2+\beta,\delta,\Omega^T} + \varphi(1/\varepsilon_8, \|d_{,x'}\|_{3+\beta'-1/\delta,\delta,S_2^T})|h^{(1)}|_{2,\Omega^T}.$$

The thirteenth term,

$$\begin{aligned}&\|h^{(1)}v_{,x}\|_{1+\beta'-1/\delta,\delta,S_2^T} \le \|h^{(1)}v_{,x}\|_{1+\beta',\delta,\Omega^T} \\ &\quad \le c|h^{(1)}|_{\delta_{12}',\Omega^T}\|v_{,x}\|_{1+\beta'+\varepsilon/\delta,\delta_{12},\Omega^T} + c\|h^{(1)}\|_{1+\beta'+\varepsilon/\delta,\delta_{13},\Omega^T}|v_{,x}|_{\delta_{13}',\Omega^T} \equiv I_6,\end{aligned}$$

where $1/\delta_i + 1/\delta_i' = 1/\delta$, $i = 12, 13$. Assuming the inequalities

$$\begin{aligned}&\frac{5}{\rho} - \frac{5}{\delta_{12}} + 2 + \beta' < 2 + \sigma, \qquad \frac{5}{\delta} - \frac{5}{\delta_{12}'} < 2 + \beta, \\ &\frac{5}{\delta} - \frac{5}{\delta_{13}} + 1 + \beta' < 2 + \beta, \qquad \frac{5}{\rho} - \frac{5}{\delta_{13}'} < 1 + \sigma,\end{aligned}$$

which hold for $\sigma > \beta'$, $5/\rho < 2 + \sigma + \beta - \beta'$, $\rho > \delta$. Hence, applying interpolation inequalities, we have

$$I_6 \le \varepsilon_9\|h^{(1)}\|_{2+\beta,\delta,\Omega^T} + \varphi(1/\varepsilon_9, \|v\|_{2+\sigma,\rho,\Omega^T})|h^{(1)}|_{2,\Omega^T}.$$

Finally, we estimate the fourteenth term in the way

$$\begin{aligned}&\|d_{,x'}v_{,x}\|_{1+\beta'-1/\delta,\delta,S_2^T} \le \|\widetilde{d}_{,x'}v_{,x}\|_{1+\beta',\delta,\Omega^T} \\ &\quad \le c\|\widetilde{d}_{,x'}\|_{1+\beta'+\varepsilon/\delta,\delta_{14},\Omega^T}|v_{,x}|_{\delta_{14}',\Omega^T} + c|\widetilde{d}_{,x'}|_{\delta_{15},\Omega^T}\|v_{,x}\|_{1+\beta'+\varepsilon/\delta,\delta_{15}',\Omega^T} \\ &\quad \le c\|\widetilde{d}_{,x'}\|_{3+\beta',\delta,\Omega^T}\|v\|_{2+\sigma,\rho,\Omega^T} \le c\|d_{,x'}\|_{3+\beta'-1/\delta,\delta,S_2^T}\|v\|_{2+\sigma,\rho,\Omega^T},\end{aligned}$$

which hold for $1/\delta_i + 1/\delta_i' = 1/\delta$, $i = 14, 15$ and $5/\rho < 3 + \sigma$.

The interpolation inequalities, used in this proof, can be found in [2, Sections 10 and 15] for the anisotropic case.

Summarizing the above considerations we show (4.47). This ends the proof.□

5. Local existence and uniqueness

To prove the existence of local solutions to problem (1.1) we are looking for a fixed point of the transformation

$$(h^{(1)}, h^{(2)}) = \Phi(\widetilde{h}^{(1)}, \widetilde{h}^{(2)}, \lambda), \quad \lambda \in [0,1], \tag{5.1}$$

defined by the following system of problems

$$\begin{aligned}
&h^{(1)}_{,t} - \operatorname{div} \mathbb{T}(h^{(1)}, q^{(1)}) = -\lambda[v(\widetilde{h}^{(1)}, \widetilde{h}^{(2)}) \cdot \nabla \widetilde{h}^{(1)} \\
&\qquad + \widetilde{h}^{(1)} \cdot \nabla v(\widetilde{h}^{(1)}, \widetilde{h}^{(2)})] + g^{(1)} && \text{in } \Omega^T,\\
&\operatorname{div} h^{(1)} = 0 && \text{in } \Omega^T,\\
&\overline{n} \cdot h^{(1)} = 0 && \text{on } S_1^T,\\
&\nu \overline{n} \cdot \mathbb{D}(h^{(1)}) \cdot \overline{\tau}_\alpha + \gamma h^{(1)} \cdot \overline{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S_1^T,\\
&h^{(1)}_i = -d_{,x_i}, \quad i = 1,2, && \text{on } S_2^T,\\
&h^{(1)}_{3,x_3} = \Delta' d && \text{on } S_2^T,\\
&h^{(1)}|_{t=0} = h^{(1)}(0) && \text{in } \Omega,
\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}
&h^{(2)}_{,t} - \operatorname{div} \mathbb{T}(h^{(2)}, q^{(2)}) = -\lambda[v(\widetilde{h}^{(1)}, \widetilde{h}^{(2)}) \cdot \nabla \widetilde{h}^{(2)} \\
&\qquad + \widetilde{h}^{(1)} \cdot \nabla \widetilde{h}^{(1)} + \widetilde{h}^{(2)} \cdot \nabla v(\widetilde{h}^{(1)}, \widetilde{h}^{(2)})] + g^{(2)} && \text{in } \Omega^T,\\
&\operatorname{div} h^{(2)} = 0 && \text{in } \Omega^T,\\
&\overline{n} \cdot h^{(2)} = 0 && \text{on } S_1^T,\\
&\nu \overline{n} \cdot \mathbb{D}(h^{(2)}) \cdot \overline{\tau}_\alpha + \gamma h^{(2)} \cdot \overline{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S_1^T,\\
&h^{(2)}_3 = \Delta' d && \text{on } S_2^T,\\
&h^{(2)}_{i,x_3} = -\frac{2}{\nu} d_{,x_i t} + 3\Delta' d_{,x_i} + \frac{1}{\nu} F_i' + \frac{1}{\nu}[v_j(h^{(1)}_i - d_{,x_i})_{,x_j} \\
&\qquad + d(h^{(1)}_{i,x_3} - h^{(1)}_{3,x_i}) + h^{(1)}_3(h^{(1)}_i - d_{,x_i}) \\
&\qquad + h^{(1)}_j v_{i,x_j} - d_{,x_j} v_{j,x_i}]|_{S_2}, \quad i = 1,2, && \text{on } S_2^T,\\
&h^{(2)}|_{t=0} = h^{(2)}(0) && \text{in } \Omega.
\end{aligned} \tag{5.3}$$

Moreover, the dependence $v = v(\widetilde{h}^{(1)}, \widetilde{h}^{(2)})$ is determined by Lemmas 4.3 and 4.5.

The main problem of this section is to show existence of a fixed point of transformation (5.1) for $\lambda = 1$. The above presentation suggests that the Leray–Schauder fixed point theorem should be applied.

To define the domain of transformation Φ we first examine the mapping $v = v(h^{(1)}, h^{(2)})$ defined by Lemmas 4.3, 4.5. In view of assumptions of Lemma 4.3 we define

$$\begin{aligned}\mathfrak{M}_0(\Omega^T) &= \{h^{(1)} : h^{(1)} \in W_{r'}^{s',s'/2}(\Omega^T),\ 5/r' - s' \le 3/2\},\\ \mathfrak{N}_0(\Omega^T) &= \{h^{(2)} : h^{(2)} \in L_2(0,T;H^1(\Omega))\}.\end{aligned}$$

Then Lemmas 4.3 and 4.5 imply the mapping

$$v\colon \mathfrak{M}_0(\Omega^T) \times \mathfrak{N}_0(\Omega^T) \to W_\rho^{2+\sigma,1+\sigma/2}(\Omega^T). \tag{5.4}$$

Let Φ_1 be a part of transformation Φ defined by problem (5.2). Let v be given. Then to determine the domain of Φ_1 we look for inequalities (4.31) and (4.37). They imply the space

$$\begin{aligned}\mathfrak{M}_1(\Omega^T) = \{h^{(1)} : h^{(1)} \in L_{\delta_3'}(\Omega^T) \cap W_{\delta_4}^{\beta+\varepsilon/\delta,\beta/2+\varepsilon/(2\delta)}(\Omega^T),&\\ \nabla h^{(1)} \in L_{\delta_1'}(\Omega^T) \cap W_{\delta_2}^{\beta+\varepsilon/\delta,\beta/2+\varepsilon/2\delta}(\Omega^T)\}&\end{aligned}$$

where

$$\begin{aligned}\frac{5}{r} + \beta + \frac{5}{\delta_1'} &< 2 + \frac{5}{\delta} < 3 + \beta + \frac{5}{\delta_1'},\\ \frac{5}{r} + \frac{5}{\delta_2} &< 2 + \frac{5}{\delta} < 3 + \frac{5}{\delta_2},\\ \frac{5}{r} + \beta + \frac{5}{\delta_3'} &< 1 + \frac{5}{\delta} < 3 + \beta + \frac{5}{\delta_3'},\\ \frac{5}{r} + \frac{5}{\delta_4} &< 1 + \frac{5}{\delta} < 3 + \frac{5}{\delta_4}.\end{aligned}$$

Hence Lemma 4.4 implies the mapping

$$\Phi_1\colon \mathfrak{M}_0(\Omega^T) \cap \mathfrak{M}_1(\Omega^T) \times \mathfrak{N}_0(\Omega^T) \to W_\delta^{2+\beta,1+\beta/2}(\Omega^T).$$

Now we examine problem (5.3). Let Φ_2 be a mapping determined by (5.3). To define the domain of Φ_2 we examine the proof of Lemma 4.6. Let $\mathfrak{N}_1(\Omega^T)$ be defined in the same way as space $\mathfrak{M}_1(\Omega^T)$, where $h^{(1)}$ is replaced by $h^{(2)}$ and β by $\beta' < \beta$ (in reality we assume that β' is very close to β).

Examining inequalities in the proof of Lemma 4.6 we introduce the space

$$\begin{aligned}\mathfrak{M}_2(\Omega^T) = \Big\{h^{(1)} : h^{(1)} \in \bigcap_{j=1}^{5} L_{\sigma_j}(\Omega^T) \cap W_{\overline{\sigma}_j}^{\beta+\varepsilon/\delta,\beta/2+\varepsilon/2\delta}(\Omega^T),&\\ \nabla h^{(1)} \in L_{\delta_5'}(\Omega^T) \cap W_{\delta_6}^{\beta+\varepsilon/\delta,\beta/2+\varepsilon/2\delta}(\Omega^T)\Big\},&\end{aligned}$$

where $1/\delta_i + 1/\delta_i' = 1/\delta$, $i = 5, 6$, $\sigma_j \in \{\delta_5', \delta_6', \delta_9', \delta_{10}', \delta_{12}'\}$, $\overline{\sigma}_j \in \{\delta_5, \delta_7, \delta_9, \delta_{11}, \delta_{12}\}$, $\varepsilon > 0$ arbitrary small and $5/\delta < 3 + \beta$.

Finally the last but one term on the r.h.s. of (4.52) implies the space

$$\mathfrak{M}_3(\Omega^T) = \{h^{(1)} : h^{(1)} \in W_\delta^{2+\beta',1+\beta'/2}(\Omega^T)\}.$$

Hence Lemma 4.6 implies the mapping

$$\Phi_2: \mathfrak{M}_0(\Omega^T) \cap \mathfrak{M}_2(\Omega^T) \cap \mathfrak{M}_3(\Omega^T) \times \mathfrak{N}_0(\Omega^T) \cap \mathfrak{N}_1(\Omega^T) \to W_\delta^{2+\beta',1+\beta'/2}(\Omega^T).$$

Hence $\Phi = \{\Phi_1, \Phi_2\}$. Let

$$\mathfrak{M}(\Omega^T) = \bigcap_{i=0}^{3} \mathfrak{M}_i(\Omega^T), \qquad \mathfrak{N}(\Omega^T) = \bigcap_{i=0}^{1} \mathfrak{N}_i(\Omega^T).$$

Then

$$\Phi: \mathfrak{M}(\Omega^T) \times \mathfrak{N}(\Omega^T) \to W_\delta^{2+\beta,1+\beta/2}(\Omega^T) \times W_\delta^{2+\beta',1+\beta'/2}(\Omega^T). \tag{5.5}$$

Lemma 5.1. *Assume that*

$g^{(1)} \in W_\delta^{\beta,\beta/2}(\Omega^T)$, $g^{(2)} \in W_\delta^{\beta',\beta'/2}(\Omega^T)$,
$h^{(1)}(0) \in W_\delta^{2+\beta-2/\delta}(\Omega)$, $h^{(2)}(0) \in W_\delta^{2+\beta'-2/\delta}(\Omega)$,
$d_{,t} \in W_\delta^{2+\beta'-1/\delta,1+\beta'/2-1/(2\delta)}(S_2^T)$,
$d_{,x'} \in W_\delta^{3+\beta-1/\delta,3/2+\beta/2-1/(2\delta)}(S_2^T)$,
$F' \in W_\delta^{1+\beta'-1/\delta,1/2+\beta'/2-1/(2\delta)}(S_2^T)$,
$d_{,t} \in L_3(S_2^T)$,
$\beta' < \beta$, $5/\delta < 3+\beta-\beta'$, $\beta' < \sigma < 1$, $2+\sigma < 5/\rho < 2+\sigma+\beta-\beta'$,
$\rho > \delta$,
$f \in W_\rho^{\sigma,\sigma/2}(\Omega^T)$, $F_3 \in L_{18/13}(\Omega^T)$, $\chi(0) \in L_2(\Omega)$, $v(0) \in W_\rho^{2+\sigma-2/\rho}(\Omega)$.

Assume that $\delta \in (1,2)$, $\beta \in (0,1)$ *are such that* $5/\delta < 3+\beta$, $3/\delta < 2+\beta$. *Then the imbeddings*

$$W_\delta^{2+\beta,1+\beta/2}(\Omega^T) \subset \mathfrak{M}(\Omega^T)), \qquad W_\delta^{2+\beta',1+\beta'/2}(\Omega^T) \subset \mathfrak{N}(\Omega^T), \tag{5.6}$$

are compact.

Proof. In view of interpolation inequalities (4.31) (4.37) (4.49)–(4.54) and definition of space $\mathfrak{M}_3(\Omega^T)$ we have to show only that the imbeddings

$$W_\delta^{2+\beta,1+\beta/2}(\Omega^T) \subset \mathfrak{M}_0(\Omega^T), \tag{5.7}$$

$$W_\delta^{2+\beta',1+\beta'/2}(\Omega^T) \subset \mathfrak{N}_0(\Omega^T) \tag{5.8}$$

are compact.

To show (5.7) we check that the following imbeddings are compact

$$W_\delta^{2+\beta,1+\beta/2}(\Omega^T) \subset W_r^{s',s'/2}(\Omega^T) \quad \text{if} \quad \frac{5}{\delta} - \left(\frac{5}{r'} - s'\right) < 2+\beta. \tag{5.9}$$

To show (5.8) we look for the imbedding

$$W_\delta^{2+\beta',1+\beta'/2}(\Omega^T) \subset L_2(0,T;H^1(\Omega)) \tag{5.10}$$

which is compact for $5/\delta < 7/2 + \beta'$.

Since the condition $5/\delta < 3 + \beta$ must be satisfied we have the following restriction on β': $0 < \beta - \beta' < 1/2$. Hence (5.6) are compact. □

Now we find an estimate for a fixed point of mapping Φ. Let

$$\gamma(T) = \sum_{i=1}^{2}(\|h^{(i)}\|_{2+\beta^i,\delta,\Omega^T} + \|\nabla q^{(i)}\|_{\beta^i,\delta,\Omega^T}) \equiv \sum_{i=1}^{2}\overline{\gamma}_i(T),$$

where $\beta^1 = \beta$, $\beta^2 = \beta'$. Let

$$\begin{aligned} k_1(T) &= \|g^{(1)}\|_{\beta,\delta,\Omega^T} + \|d_{,x'}\|_{2+\beta-1/\delta,\delta,S_2^T} + \|h^{(1)}(0)\|_{2+\beta-2/\delta,\delta,\Omega}, \\ k_2(T) &= \|g^{(2)}\|_{\beta',\delta,\Omega^T} + \|d_{,t}\|_{2+\beta'-1/\delta,\delta,S_2^T} + \|d_{,x'}\|_{3+\beta'-1/\delta,\delta,S_2^T} \\ &\quad + \|F'\|_{1+\beta'-1/\delta,\delta,S_2^T} + \|h^{(2)}(0)\|_{2+\beta'-2/\delta,\delta,\Omega}, \\ k_3(T) &= |d_1|_{3,6,S_2^T} + \|d\|_{5/2,2,S_2^T}. \end{aligned} \tag{5.11}$$

Let

$$k(T) = \sum_{i=1}^{2} k_i(T). \tag{5.12}$$

Lemma 5.2. *Let the assumptions of Lemma* 5.1 *hold. Then there exists a constant A sufficiently large* (*see* (5.22)) *satisfying inequality* (5.21) *such that for sufficiently small $\eta_1(T) + \eta_2(T)$* (*see* (3.53), (3.55)) *we have*

$$\gamma(T) \le A. \tag{5.13}$$

Proof. In view of imbeddings (5.9) and (5.10) we have $\gamma_1(T) \le c\gamma(T)$. Therefore (4.17) and (4.43) imply

$$\begin{aligned} \|v\|_{2,27/16,\Omega^T} &\le G(\gamma(T), G_1(T), G_0(0)), \\ \|v\|_{2+\sigma,\rho,\Omega^T} &\le G(\gamma(T), \overline{G}_1(T), \overline{G}_0(0)), \end{aligned} \tag{5.14}$$

where $\overline{G}_1 = G_1 + G_2$, $\overline{G}_0 = G_0 + G_0^1$.

From (4.29) we obtain

$$\overline{\gamma}_1(T) \le \varphi(\|v\|_{2,\frac{27}{16},\Omega^T})|h^{(1)}|_{2,\Omega^T} + ck_1(T). \tag{5.15}$$

From (4.47) we have

$$\begin{aligned} \overline{\gamma}_2(T) \le{}& \varphi(\|v\|_{2,27/16,\Omega^T})|h^{(2)}|_{2,\Omega^T} + \varphi(\overline{\gamma}_1(T), \|v\|_{2+\sigma,\rho,\Omega^T})|h^{(1)}|_{2,\Omega^T} \\ &+ \varepsilon\overline{\gamma}_1(T) + \varphi\left(\frac{1}{\varepsilon}, k_3(T)\right)|h^{(1)}|_{2,\Omega^T} + ck_2(T). \end{aligned} \tag{5.16}$$

From (5.14)–(5.16) we obtain for sufficiently small ε the inequality

$$\begin{aligned} \gamma(T) \le{}& \overline{\varphi}_0(\gamma(T), \overline{G}_1(T), \overline{G}_0(0), k_3(T), \overline{\gamma}_1(T)) \\ &\cdot (|h^{(1)}|_{2,\Omega^T} + |h^{(2)}|_{2,\Omega^T}) + ck(T). \end{aligned} \tag{5.17}$$

From (3.53) we have

$$|h^{(1)}|_{2,\Omega^T} \le \varphi(k_3(T), \|v\|_{2,27/16,\Omega^T})\eta_1(T) \tag{5.18}$$

and (3.54) yields

$$|h^{(2)}|_{2,\Omega^T} \le \varphi(k_3(T), \|v\|_{2+\sigma,\rho,\Omega^T})\eta_2(T). \tag{5.19}$$

Utilizing (5.14), (5.15), (5.18) and (5.19) in (5.17) we obtain

$$\gamma(T) \le \varphi_0(\gamma(T), \overline{G}_1(T), \overline{G}_0(0), k(T), k_3(T), l_3(T)) \cdot (\eta_1 + \eta_2) + c_0 k(T), \tag{5.20}$$

where φ_0 is an increasing positive function.

From (5.20) it follows that for sufficiently small $\eta_1 + \eta_2$ there exists a positive constant A such that

$$\varphi_0(A, \overline{G}_1, \overline{G}_0, k, k_3, l_3)(\eta_1 + \eta_2) + c_0 k \le A, \tag{5.21}$$

$$c_0 k < A. \tag{5.22}$$

Hence, (5.21) implies (5.13). □

Finally, we show the uniform continuity of mapping Φ.

Lemma 5.3. *Let the assumptions of Lemma* 5.2 *hold. Then mapping* Φ *is uniformly continuous in the product* $\mathfrak{M}(\Omega^T) \times \mathfrak{N}(\Omega^T) \times [0,1]$.

Proof. The uniform continuity with respect to $\lambda \in [0,1]$ is evident. Therefore we examine a uniform continuity with respect to elements of $\mathfrak{M}(\Omega^T) \times \mathfrak{N}(\Omega^T)$ for any $\lambda \in [0,1]$. Since the dependence on λ is very elementary we omit λ in the below considerations because it does not change the proof.

Let $(\widetilde{h}_s^{(1)}, \widetilde{h}_s^{(2)}) \in \mathfrak{M}(\Omega^T) \times \mathfrak{N}(\Omega^T)$, $s = 1,2$, be two elements. Therefore we consider the following problem

$$\begin{aligned}
&h_{s,t}^{(1)} - \operatorname{div} \mathbb{T}(h_s^{(1)}, q_s^{(1)}) = -v_s \cdot \nabla \widetilde{h}_s^{(1)} \\
&\qquad - \widetilde{h}_s^{(1)} \cdot \nabla v_s + g^{(1)} && \text{in } \Omega^T, \\
&\operatorname{div} h_s^{(1)} = 0 && \text{in } \Omega^T, \\
&h_s^{(1)} \cdot \overline{n} = 0 && \text{on } S_1^T, \\
&\nu \overline{n} \cdot \mathbb{D}(h^{(1)}) \cdot \overline{\tau}_\alpha + \gamma h^{(1)} \cdot \overline{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S_1^T, \\
&h_{si}^{(1)} = -d_{,x_i}, \quad i = 1,2, && \text{on } S_2^T, \\
&h_{s3,x_3}^{(1)} = \Delta' d && \text{on } S_2^T, \\
&h_s^{(1)}|_{t=0} = h^{(1)}(0) && \text{in } \Omega,
\end{aligned} \tag{5.23}$$

where $s=1,2$, and $v_s=v(h_s^{(1)},h_s^{(2)})$ for $s=1,2$. Moreover, we have

$$
(5.24)\qquad
\begin{aligned}
&\chi_{s,t}+v_s\cdot\nabla\chi_s+\widetilde{h}_{2s}^{(1)}w_{s,x_1}-\widetilde{h}_{s1}^{(1)}w_{s,x_2}\\
&\qquad-\nu\Delta\chi_3=F_3 && \text{in } \Omega^T,\\
&\chi_s=\sum_{i=1}^2 v_{si}a_i\equiv\chi_{s*} && \text{on } S_1^T,\\
&\chi_{s,x_3}=0 && \text{on } S_2^T,\\
&\chi_s|_{t=0}=\chi(0) && \text{in } \Omega,
\end{aligned}
$$

where $s=1,2$, a_i, $i=1,2$, depend on S_1 and are defined by $(3.3)_2$.

Next we have the elliptic problem

$$
\begin{aligned}
v_{s2,x_1}-v_{s1,x_2}&=\chi_s && \text{in } \Omega',\\
v_{s1,x_1}+v_{s2,x_2}&=-h_{s3}^{(1)} && \text{in } \Omega',\\
v_s'\cdot\overline{n}'&=0 && \text{on } S_1',
\end{aligned}
$$

where $s=1,2$, Ω' and S_1' are cross-sections of Ω and S_1 with a plane perpendicular to axis x_3, $\overline{n}'=(n_1,n_2,0)$.

Finally, we consider the problem

$$
(5.25)\qquad
\begin{aligned}
&h_{s,t}^{(2)}-\operatorname{div}\mathbb{T}(h_s^{(2)},q_s^{(2)})\\
&\qquad=-[v_s\cdot\nabla\widetilde{h}_s^{(2)}+\widetilde{h}_s^{(1)}\cdot\nabla\widetilde{h}_s^{(1)}+\widetilde{h}_s^{(2)}\cdot\nabla v_s]+g^{(2)} && \text{in } \Omega^T,\\
&\operatorname{div}h_s^{(2)}=0 && \text{in } \Omega^T,\\
&\overline{n}\cdot h_s^{(2)}=0 && \text{on } S_1^T,\\
&\nu\overline{n}\cdot\mathbb{D}(h_s^{(2)})\cdot\overline{\tau}_\alpha+\gamma h_s^{(2)}\cdot\overline{\tau}_\alpha=0,\quad \alpha=1,2, && \text{on } S_1^T,\\
&h_{s3}^{(2)}=\Delta' d && \text{on } S_2^T,\\
&h_{si,x_3}^{(2)}=-\frac{2}{\nu}d_{,x_it}+3\Delta' d_{,x_i}+\frac{1}{\nu}F_i'\\
&\qquad+\frac{1}{\nu}[v_{sj}(\widetilde{h}_{si}^{(1)}-d_{,x_i})_{,x_j}\\
&\qquad+d(\widetilde{h}_{si,x_3}^{(1)}-\widetilde{h}_{s3,x_i}^{(1)})+\widetilde{h}_{s3}^{(1)}(\widetilde{h}_{si}^{(1)}-d_{,x_i})\\
&\qquad+\widetilde{h}_{sj}^{(1)}v_{si,x_j}-d_{,x_j}v_{sj,x_i}]|_{S_2}\equiv\gamma_{si},\quad i=1,2, && \text{on } S_2^T,\\
&h_s^{(2)}|_{t=0}=h^{(2)}(0) && \text{in } \Omega,
\end{aligned}
$$

where $s=1,2$.

First we examine problem (5.24). Let us introduce a function $\widetilde{\chi}_s$ as a solution to the problem

$$
\begin{aligned}
\widetilde{\chi}_{s,t}-\nu\Delta\widetilde{\chi}_s&=0 && \text{in } \Omega^T,\\
\widetilde{\chi}_s&=\widetilde{\chi}_{s*} && \text{on } S_1^T,\\
\widetilde{\chi}_{s,x_3}&=0 && \text{on } S_2^T,\\
\widetilde{\chi}_s|_{t=0}&=0 && \text{in } \Omega,
\end{aligned}
$$

where $s = 1, 2$. Introducing the new function $\chi'_s = \chi_s - \widetilde{\chi}_s$, for $s = 1, 2$, we see that it is a solution to the problem

$$\begin{aligned} &\chi'_{s,t} + v_s \cdot \nabla\chi'_s - \widetilde{h}^{(1)}_{s3}\chi'_s + \widetilde{h}^{(1)}_{s2} w_{s,x_1} - \widetilde{h}^{(1)}_{s1} w_{s,x_2} \\ &\qquad - \nu\Delta\chi'_s = F_3 - v_s \cdot \nabla\widetilde{\chi}_s + \widetilde{h}^{(1)}_{s3}\widetilde{\chi}_s && \text{in } \Omega^T, \\ &\chi'_s = 0 && \text{on } S_1^T, \\ &\chi'_{s,x_3} = 0 && \text{on } S_2^T, \\ &\chi'_s|_{t=0} = \chi'(0) && \text{in } \Omega. \end{aligned}$$

Since we are looking for a solution which is a regularization of a weak solution we use the energy type estimate for the weak solution

$$|v|_{2,\infty,\Omega^t} + |\nabla v|_{2,\Omega^t} \le l_1(t), \quad t \le T.$$

Repeating the considerations leading to (4.17) and (4.43) we obtain

$$\begin{aligned} \|v_s\|_{2,27/16,\Omega^t} &\le G(\gamma_{s1}(t), G_1(t), G_0(0)), \quad t \le T, \\ \|v_s\|_{2+\sigma,\rho,\Omega^t} &\le G(\gamma_{s1}(t), \overline{G}_1(t), \overline{G}_0), \qquad t \le T, \end{aligned} \tag{5.26}$$

where $\gamma_{s1}(t)$ is equal to $\gamma_1(t)$, where $h^{(1)}, h^{(2)}$ are replaced by $\widetilde{h}^{(1)}_s, \widetilde{h}^{(2)}_s$, respectively.

For solutions of problem (5.23) we have

$$\begin{aligned} &\|h^{(1)}_s\|_{2+\beta,\delta,\Omega^t} + \|\nabla q^{(1)}_s\|_{\beta,\delta,\Omega^t} \\ &\qquad \le c(\|v_s \cdot \nabla\widetilde{h}^{(1)}_s\|_{\beta,\delta,\Omega^t} + \|\widetilde{h}^{(1)}_s \cdot \nabla v_s\|_{\beta,\delta,\Omega^t}) + ck_1(t), \end{aligned} \tag{5.27}$$

for $t \le T$. Similarly, for solutions of problem (5.25) we get

$$\begin{aligned} &\|h^{(2)}_s\|_{2+\beta',\delta,\Omega^t} + \|\nabla q^{(2)}_s\|_{\beta',\delta,\Omega^t} \\ &\qquad \le c(\|\widetilde{h}^{(2)}_s \cdot \nabla v_s\|_{\beta',\delta,\Omega^t} + \|\widetilde{h}^{(1)}_s \cdot \nabla\widetilde{h}^{(1)}_s\|_{\beta',\delta,\Omega^t} \\ &\qquad\quad + \|v_s \cdot \nabla\widetilde{h}^{(2)}_s\|_{\beta',\delta,\Omega^t} + \|\gamma_s\|_{1+\beta'-1/\delta,\delta,S_2^t}) + ck_2(t), \end{aligned} \tag{5.28}$$

for $t \le T$. Let us introduce the space

$$\mathfrak{M}_*(\Omega^T) = \bigcap_{i=1}^{3} \mathfrak{M}_i(\Omega^T).$$

In view of theorems of imbedding we obtain from (5.27) and (5.28) the inequalities

$$\|h^{(1)}_s\|_{2+\beta,\delta,\Omega^t} + \|\nabla q^{(1)}_s\|_{\beta,\delta,\Omega^t} \le c\|v_s\|_{2,27/16,\Omega^t}\|\widetilde{h}^{(1)}_s\|_{\mathfrak{M}_*(\Omega^t)} + ck_1(t), \tag{5.29}$$

for $t \le T$, $s = 1, 2$, and

$$\begin{aligned} &\|h^{(2)}_s\|_{2+\beta',\delta,\Omega^t} + \|\nabla q^{(2)}_s\|_{\beta',\delta,\Omega^t} \le c\|v_s\|_{2,27/16,\Omega^t}\|\widetilde{h}^{(2)}_s\|_{\mathfrak{N}_1(\Omega^t)} \\ &\qquad + c\|\widetilde{h}^{(1)}_s\|^2_{\mathfrak{M}_*(\Omega^t)} + c(\|v_s\|_{2+\sigma,\rho,\Omega^t} + 1)\|\widetilde{h}^{(1)}_s\|_{\mathfrak{M}_*(\Omega^t)} + ck_2(t), \end{aligned} \tag{5.30}$$

for $t \le T$, $s = 1, 2$. In view of (5.26) and the definition of $\mathfrak{M}_0(\Omega^T)$ and $\mathfrak{N}_0(\Omega^T)$ we have

$$\|v_s\|_{2,27/16,\Omega^t} + \|v_s\|_{2+\sigma,\rho,\Omega^t} \le \varphi(\|\widetilde{h}_s^{(1)}\|_{\mathfrak{M}_0(\Omega^t)} + \|\widetilde{h}_s^{(2)}\|_{\mathfrak{N}_0(\Omega^t)}, \overline{G}_1(t), \overline{G}_0(0)) \tag{5.31}$$

where $s = 1, 2$, $t \le T$. By (5.29)–(5.31) we get

$$\|h_s^{(1)}\|_{\mathfrak{M}(\Omega^t)} + \|h_s^{(2)}\|_{\mathfrak{N}(\Omega^t)} \le \varphi(\|\widetilde{h}_s^{(1)}\|_{\mathfrak{M}(\Omega^t)}, \widetilde{h}_s^{(2)}\|_{\mathfrak{N}(\Omega^t)}, G_1(t), G_0(0)) + c(k_1(t) + k_2(t)),$$

$s = 1, 2$, $t \le T$. Hence the mapping Φ transforms bounded sets in $\mathfrak{M}(\Omega^T) \times \mathfrak{N}(\Omega^T)$ into bounded set in $\mathfrak{M}(\Omega^T) \times \mathfrak{N}(\Omega^T)$.

Now we shall show the uniform continuity of mapping Φ. For this purpose we introduce

$$\begin{aligned} H^{(i)} &= h_1^{(i)} - h_2^{(i)}, \\ Q^{(i)} &= q_1^{(i)} - q_2^{(i)}, \quad i = 1, 2, \\ V &= v_1 - v_2, \\ K &= \chi_1 - \chi_2. \end{aligned}$$

Then problem for $H^{(1)}$ assumes the form

$$\begin{aligned} &H_{,t}^{(1)} - \operatorname{div} \mathbb{T}(H^{(1)}, Q^{(1)}) = -V \cdot \nabla \widetilde{h}_1^{(1)} - v_2 \cdot \nabla \widetilde{H}^{(1)} \\ &\qquad - \widetilde{H}^{(1)} \cdot \nabla v_1 - \widetilde{h}_2^{(1)} \cdot \nabla V && \text{in } \Omega^T, \\ &\operatorname{div} H^{(1)} = 0 && \text{in } \Omega^T, \\ &H^{(1)} \cdot \overline{n} = 0, \quad \nu \overline{n} \cdot \mathbb{D}(H^{(1)}) \cdot \overline{\tau}_\alpha + \gamma H^{(1)} \cdot \overline{\tau}_\alpha = 0, \quad \alpha = 1, 2, && \text{on } S_1^T, \\ &H_i^{(1)} = 0, \quad H_{3,x_3}^{(1)} = 0, \quad i = 1, 2, && \text{on } S_2^T, \\ &H^{(1)}|_{t=0} = 0 && \text{in } \Omega. \end{aligned} \tag{5.32}$$

Next we have the problem for $H^{(2)}$,

$$\begin{aligned} &H_{,t}^{(2)} - \operatorname{div} \mathbb{T}(H^{(2)}, Q^{(2)}) = -\widetilde{H}^{(2)} \cdot \nabla v_1 - \widetilde{h}_2^{(2)} \cdot \nabla V \\ &\qquad - 2\widetilde{H}^{(1)} \cdot \nabla \widetilde{h}_1^{(1)} - 2\widetilde{h}_2^{(1)} \cdot \nabla \widetilde{H}^{(1)} - V \cdot \nabla \widetilde{h}_1^{(2)} - v_2 \cdot \nabla \widetilde{H}^{(2)} && \text{in } \Omega^T, \\ &\operatorname{div} H^{(2)} = 0 && \text{in } \Omega^T, \\ &\overline{n} \cdot H^{(2)} = 0, \quad \nu \overline{n} \cdot \mathbb{D}(H^{(2)}) \cdot \overline{\tau}_\alpha + \gamma H^{(2)} \cdot \overline{\tau}_\alpha = 0, \quad \alpha = 1, 2, && \text{on } S_1^T, \\ &H_3^{(2)} = 0, \quad H_{i,x_3}^{(2)} = [V_j \widetilde{h}_{i,x_j}^{(1)} + v_{2j} \widetilde{H}_{i,x_j}^{(1)} - V_j d_{,x_i x_j} \\ &\qquad - d(\widetilde{H}_{3,x_i}^{(1)} - \widetilde{H}_{i,x_3}^{(1)}) + \widetilde{H}_3^{(1)} \widetilde{h}_{1i}^{(1)} + \widetilde{h}_{23}^{(1)} \widetilde{H}_i^{(1)} - \widetilde{H}_3^{(1)} d_{,x_i} \\ &\qquad + \widetilde{H}_j^{(1)} v_{1i,x_j} + \widetilde{h}_{2j}^{(1)} V_{i,x_j} - d_{,x_j} V_{j,x_i}], \quad i = 1, 2, && \text{on } S_2^T, \\ &H^{(2)}|_{t=0} = 0 && \text{in } \Omega. \end{aligned} \tag{5.33}$$

For solutions to problem (5.32) we obtain

$$
\begin{aligned}
(5.34)\quad \|H^{(1)}\|_{2+\beta,\delta,\Omega^t} + \|\nabla Q^{(1)}\|_{\beta,\delta,\Omega^t} &\le c(\|V\cdot\nabla\widetilde{h}_1^{(1)}\|_{\beta,\delta,\Omega^t}\\
&\quad + \|v_2\cdot\nabla\widetilde{H}^{(1)}\|_{\beta,\delta,\Omega^t} + \|\widetilde{H}^{(1)}\cdot\nabla v_1\|_{\beta,\delta,\Omega^t} + \|\widetilde{h}_2^{(1)}\cdot\nabla V\|_{\beta,\delta,\Omega^t})\\
&\le c(\|V\|_{2,r,\Omega^t}\|\widetilde{h}_1^{(1)},\widetilde{h}_2^{(1)}\|_{\mathfrak{M}_*(\Omega^t)} + \|v_1,v_2\|_{2,r,\Omega^t}\|\widetilde{H}^{(1)}\|_{\mathfrak{M}_*(\Omega^t)}),
\end{aligned}
$$

where $t\le T$, $r=27/16$ and

$$
\begin{aligned}
\|\widetilde{h}_1^{(1)},\widetilde{h}_2^{(1)}\|_{\mathfrak{M}_*(\Omega^t)} &= \|\widetilde{h}_1^{(1)}\|_{\mathfrak{M}_*(\Omega^t)} + \|\widetilde{h}_2^{(1)}\|_{\mathfrak{M}_*(\Omega^t)},\\
\|v_1,v_2\|_{2,r,\Omega^t} &= \|v_1\|_{2,r,\Omega^t} + \|v_2\|_{2,r,\Omega^t}.
\end{aligned}
$$

Assume that $\widetilde{h}_s^{(1)},\widetilde{h}_s^{(2)}$, $s=1,2$, belong to a bounded set in $\mathfrak{M}(\Omega^T)\times\mathfrak{N}(\Omega^T)$. Hence there exists a constant A such that

$$
(5.35)\qquad \sum_{s=1}^{2}(\|\widetilde{h}_s^{(1)}\|_{\mathfrak{M}(\Omega^T)} + \|\widetilde{h}_s^{(2)}\|_{\mathfrak{N}(\Omega^T)}) \le A.
$$

Then from (5.34) we obtain

$$
(5.36)\qquad \|H^{(1)}\|_{2+\beta,\delta,\Omega^t} + \|\nabla Q^{(1)}\|_{\beta,\delta,\Omega^t} \le \varphi(A)[\|V\|_{2,r,\Omega^t} + \|\widetilde{H}^{(1)}\|_{\mathfrak{M}_*(\Omega^t)}],
$$

where $t\le T$ and φ is an increasing positive function.

For solutions of problem (5.33) we get

$$
\begin{aligned}
(5.37)\quad \|H^{(2)}\|_{2+\beta',\delta,\Omega^t} + \|\nabla Q^{(2)}\|_{\beta',\delta,\Omega^t} &\le c(\|\widetilde{H}^{(2)}\cdot\nabla v_1\|_{\beta',\delta,\Omega^t}\\
&\quad + \|\widetilde{h}_2^{(2)}\cdot\nabla V\|_{\beta',\delta,\Omega^t} + \|\widetilde{H}^{(1)}\cdot\nabla\widetilde{h}_1^{(1)}\|_{\beta',\delta,\Omega^t}\\
&\quad + \|\widetilde{h}_2^{(1)}\cdot\nabla\widetilde{H}^{(1)}\|_{\beta',\delta,\Omega^t} + \|V\cdot\nabla\widetilde{h}_1^{(2)}\|_{\beta',\delta,\Omega^t}\\
&\quad + \|v_2\cdot\nabla\widetilde{H}^{(2)}\|_{\beta',\delta,\Omega^t} + \|\widetilde{H}^{(1)}\|_{2+\beta',\delta,\Omega^t})\\
&\quad + (\|V\widetilde{h}_{,x}^{(1)}\|_{1+\beta'-1/\delta,S_2^t} + \|v_2\widetilde{H}_{,x}^{(1)}\|_{1+\beta'-1/\delta,S_2^t}\\
&\quad + \|Vd_{,x'x'}\|_{1+\beta'-1/\delta,S_2^t} + \|\widetilde{H}^{(1)}\widetilde{h}^{(1)}\|_{1+\beta'-1/\delta,S_2^t}\\
&\quad + \|\widetilde{H}^{(1)}d_{,x'}\|_{1+\beta'-1/\delta,S_2^t} + \|\widetilde{H}^{(1)}v_{,x}\|_{1+\beta-1/\delta,S_2^t}\\
&\quad + \|\widetilde{h}^{(1)}V_{,x}\|_{1+\beta'-1/\delta,S_2^t} + \|d_{,x'}V_{,x}\|_{1+\beta'-1/\delta,S_2^t}\\
&\le c[\|V\|_{2,r,\Omega^t}\|\widetilde{h}_1^{(1)},\widetilde{h}_2^{(2)}\|_{\mathfrak{N}_1(\Omega^t)} + \|v_1,v_2\|_{2,r,\Omega^t}\|\widetilde{H}^{(2)}\|_{\mathfrak{N}_1(\Omega^t)}\\
&\quad + (\|v_1,v_2\|_{2+\sigma,\rho,\Omega^t} + \|\widetilde{h}_1^{(1)},\widetilde{h}_2^{(1)}\|_{\mathfrak{M}_*(\Omega^t)} + 1)\|\widetilde{H}^{(1)}\|_{\mathfrak{M}_*(\Omega^t)}\\
&\quad + (\|\widetilde{h}_1^{(1)},\widetilde{h}_2^{(1)}\|_{\mathfrak{M}_*(\Omega^t)} + 1)\|V\|_{2+\sigma,\rho,\Omega^t}]\\
&\le \varphi(A)(\|V\|_{2,r,\Omega^t} + \|V\|_{2+\sigma,\rho,\Omega^t} + \|\widetilde{H}^{(2)}\|_{\mathfrak{N}_1(\Omega^t)} + \|\widetilde{H}^{(1)}\|_{\mathfrak{M}_*(\Omega^t)}),
\end{aligned}
$$

where $5/3 < r \le 27/16$.

To show continuity of transformation Φ we have to find an estimate for $\|V\|_{2,r,\Omega^t}$. For this purpose we consider the problem

$$
(5.38)\qquad
\begin{aligned}
&V_{,t}-\operatorname{div}\mathbb{T}(V,Q)=-V'\cdot\nabla v_1-v_2'\cdot\nabla V-Wh_1^{(1)}-w_2H^{(1)} && \text{in } \Omega^T,\\
&\operatorname{div}V=0 && \text{in } \Omega^T,\\
&V\cdot\overline{n}=0,\quad \overline{n}\cdot\mathbb{T}(V,Q)\cdot\overline{\tau}_\alpha+\gamma V\cdot\overline{\tau}_\alpha=0,\quad \alpha=1,2, && \text{on } S^T,\\
&V|_{t=0}=0 && \text{in } \Omega,
\end{aligned}
$$

where $\gamma|_{S_2}=0$, $V'=(V_1,V_2)$, $W=V_3$, $v_s'=(v_{s1},v_{s2})$, $w_s=v_{s3}$, $s=1,2$.

For solutions of (5.38) we have

$$
(5.39)\qquad \|V\|_{2,r,\Omega^t}+|\nabla Q|_{r,\Omega^t} \le c(|V'\cdot\nabla v_1|_{r,\Omega^t}+|v_2'\cdot\nabla V|_{r,\Omega^t}+|Wh_1^{(1)}|_{r,\Omega^t}+|w_2H^{(1)}|_{r,\Omega^t}).
$$

We bound the first term on the r.h.s. of (5.39) by $c|V|_{5,\Omega^t}\|v_1\|_{2,r,\Omega^t}\equiv I_1$. For $r>5/3$ we obtain by interpolation the inequality

$$
I_1\le \varepsilon_1\|V\|_{2,r,\Omega^t}+c(1/\varepsilon_1)\varphi(\|v_1\|_{2,r,\Omega^t})|V|_{2,\Omega^t}.
$$

The second term on the r.h.s. of (5.39) is estimated by

$$
c|\nabla V|_{5/2,\Omega^t}\|v_2'\|_{2,r,\Omega^t}\equiv I_2.
$$

Hence for $r>5/3$ we can apply some interpolation inequality to get

$$
I_2\le \varepsilon_2\|V\|_{2,r,\Omega^t}+c(1/\varepsilon_2)\varphi(\|v_2'\|_{2,r,\Omega^t})|V|_{2,\Omega^t}.
$$

By the Hölder inequality the third term on the r.h.s. of (5.39) is restricted by

$$
|W|_{\sigma_1,\Omega^t}|h_1^{(1)}|_{\sigma_2,\Omega^t}\equiv I_3,
$$

where $1/\sigma_1+1/\sigma_2=1/r$.

Assuming that $5/r-5/\sigma_1<2$, $5/\delta-5/\sigma_2\le 2+\beta$, which are satisfied for $5/\delta<4+\beta$, we obtain the estimate

$$
I_3\le \varepsilon_3\|V\|_{2,r,\Omega^t}+c(1/\varepsilon_3)\varphi(\|h_1^{(1)}\|_{2+\beta,\delta,\Omega^t})|V|_{2,\Omega^t}.
$$

Finally, by the Hölder inequality the last term on the r.h.s. of (5.39) is bounded by

$$
|w_2|_{\rho_1,\Omega^t}|H^{(1)}|_{\rho_2,\Omega^t}\equiv I_4,
$$

where $1/\rho_1+1/\rho_2=1/r$.

Assuming that ρ_1 is such that $5/r-5/\rho_1=2$ we obtan that $\rho_2=5/2$ and

$$
I_4\le c\|v_2\|_{2,r,\Omega^t}|H^{(1)}|_{5/2,\Omega^t}.
$$

Utilizing the above estimates in the r.h.s. of (5.39) and assuming that $\varepsilon_1-\varepsilon_4$ are sufficiently small we obtain

$$
(5.40)\qquad \|V\|_{2,r,\Omega^t}+|\nabla Q|_{r,\Omega^t}\le \varphi(A)(|V|_{2,\Omega^t}+|H^{(1)}|_{5/2,\Omega^t}).
$$

Now we have to estimate the r.h.s. of (5.40) in terms of $\widetilde{H}^{(1)}$ and $\widetilde{H}^{(2)}$. Multiplying $(5.38)_1$ by V and integrating over Ω yield

$$(5.41)\quad \frac{1}{2}\frac{d}{dt}|V|^2_{2,\Omega}+\nu\|V\|^2_{1,\Omega}\le c(|\nabla v_1|^2_{3,\Omega}+|h_1^{(1)}|^2_{3,\Omega})|V|^2_{2,\Omega}+c|w_2|^2_{3,\Omega}|H^{(1)}|^2_{2,\Omega}.$$

Multiplying $(5.32)_1$ by $H^{(1)}$ and integrating the result over Ω imply

$$\begin{aligned}(5.42)\quad &\frac{1}{2}\frac{d}{dt}|H^{(1)}|^2_{2,\Omega}+\nu\|H^{(1)}\|^2_{1,\Omega}\\ &\le c|\nabla\widetilde{h}_1^{(1)}|^2_{3,\Omega}|V|^2_{2,\Omega}+c|v_2|^2_{6\lambda_1/5,\Omega}|\nabla\widetilde{H}^{(1)}|^2_{6\lambda_2/5,\Omega}\\ &\quad+c|\nabla v_1|^2_{3,\Omega}|\widetilde{H}^{(1)}|^2_{2,\Omega}+c\sup_t|\widetilde{h}_2^{(1)}|^2_{3,\Omega}|\nabla V|^2_{2,\Omega},\end{aligned}$$

where $1/\lambda_1+1/\lambda_2=1$. Adding appropriately (5.41) and (5.42) gives

$$\begin{aligned}(5.43)\quad &\frac{1}{2}\frac{d}{dt}(c_1|V|^2_{2,\Omega}+|H^{(1)}|^2_{2,\Omega})+\nu\bigg(\frac{c_1}{2}\|V\|^2_{1,\Omega}+\|H^{(1)}\|^2_{1,\Omega}\bigg)\\ &\le c(|\nabla v_1|^2_{3,\Omega}+|h_1^{(1)}|^2_{3,\Omega}+|\nabla\widetilde{h}_1^{(1)}|^2_{3,\Omega})|V|^2_{2,\Omega}+c|w_2|^2_{3,\Omega}|H^{(1)}|^2_{2,\Omega}\\ &\quad+c|v_2|^2_{6\lambda_1/5,\Omega}|\nabla\widetilde{H}^{(1)}|^2_{6\lambda_2/5,\Omega}+c|\nabla v_1|^2_{3,\Omega}|\widetilde{H}^{(1)}|^2_{2,\Omega},\end{aligned}$$

where $c_1/2\ge c\sup_t|\widetilde{h}_2^{(1)}|^2_{3,\Omega}$. Integrating (5.43) with respect to time yields

$$\begin{aligned}(5.44)\quad &|V(t)|^2_{2,\Omega}+|H^{(1)}(t)|^2_{2,\Omega}+\nu\int_0^t(\|V(t')\|^2_{1,\Omega}+\|H^{1)}(t')\|^2_{1,\Omega})\,dt'\\ &\le c\exp c(\big|\nabla v_1\big|^2_{3,2,\Omega^t}+\big|h_1^{(1)}\big|^2_{3,2,\Omega^t}+\big|\nabla h_1^{(1)}\big|^2_{3,2,\Omega^t}+\big|v_2\big|^2_{3,2,\Omega^t})\\ &\quad\cdot(\big|v_2\big|^2_{6\lambda_1/5,2\mu_1,\Omega^t}\big|\nabla\widetilde{H}^{(1)}\big|^2_{6\lambda_2/5,2\mu_2,\Omega^t}+\big|\nabla v_1\big|^2_{3,2,\Omega^t}\big|\widetilde{H}^{(1)}\big|^2_{2,\infty,\Omega^t})\equiv J,\end{aligned}$$

where $1/\lambda_1+1/\lambda_2=1$, $1/\mu_1+1/\mu_2=1$. By imbedding we have

$$\begin{aligned}J&\le c\exp(\|v_1\|^2_{2,r,\Omega^t}+\|v_2\|^2_{2,r,\Omega^t}+\|h^{(1)}\|^2_{2+\beta,\delta,\Omega^t})\cdot\\ &\quad\cdot(\|v_2\|^2_{2,r,\Omega^t}|\nabla\widetilde{H}^{(1)}|^2_{2,\Omega^t}+\|v_1\|^2_{2,r,\Omega^t}\big|\widetilde{H}^{(1)}\big|^2_{2,\infty,\Omega^t})\equiv J_1.\end{aligned}$$

By (5.35) we obtain

$$J_1\le\varphi(A)(\big|\widetilde{H}^{1)}\big|^2_{2,\infty,\Omega^t}+|\nabla\widetilde{H}^{(1)}|^2_{2,\Omega^t}).$$

Hence (5.44) takes the form

$$(5.45)\quad \|V\|_{V_2^0(\Omega^t)}+\|H^{(1)}\|_{V_2^0(\Omega^t)}\le\varphi(A)(\big|\widetilde{H}^{(1)}\big|_{2,\infty,\Omega^t}+|\nabla\widetilde{H}^{(1)}|_{2,\Omega^t}).$$

Finally, repeating the proof of Lemma 4.5 for (5.38) yields

$$(5.46)\quad \|V\|_{2+\sigma,\rho,\Omega^t}\le\varphi(A)(\|\widetilde{H}^{(1)}\|_{\mathfrak{M}(\Omega^t)}+\|\widetilde{H}^{(2)}\|_{\mathfrak{N}(\Omega^t)}).$$

Finally, from (5.36), (5.37) (5.40) and (5.45), (5.46) we obtain

$$\|H^{(1)}\|_{\mathfrak{M}(\Omega^T)}+\|H^{(2)}\|_{\mathfrak{N}(\Omega^T)}\le\varphi(A)(\|\widetilde{H}^{(1)}\|_{\mathfrak{M}(\Omega^T)}+\|\widetilde{H}^{(2)}\|_{\mathfrak{N}(\Omega^T)}).$$

This implies continuity of transformation Φ and ends the proof. □

Finally, by the Leray–Schauder fixed point theorem we have

THEOREM 5.4. *Assume that*

(a) $g^{(1)} \in W_\delta^{\beta,\beta/2}(\Omega^T)$, $g^{(2)} \in W_\delta^{\beta',\beta'/2}(\Omega^T)$, $h^{(1)}(0) \in W_\delta^{2+\beta-2/\delta}(\Omega)$, $h^{(2)}(0) \in W_\delta^{2+\beta'-2/\delta}(\Omega)$, $d_{,x'} \in W_\delta^{3+\beta-1/\delta,3/2+\beta/2-1/(2\delta)}(S_2^T)$, $d \in H^{5/2,5/4}(S_2^T)$, $d_{,t} \in W_\delta^{2+\beta-1/\delta,1+\beta/2-1/(2\delta)}(S_2^T)$ $d_{,t} \in L_3(S_2^T)$, $F' \in W_\delta^{1+\beta'-1/\delta,1/2+\beta'/2-1/(2\delta)}(S_2^T)$, $f \in W_\rho^{\sigma,\sigma/2}(\Omega^T)$, $F_3 \in L_{18/13}(\Omega^T)$, $\chi(0) \in L_2(\Omega)$, $v(0) \in W_\rho^{2+\sigma-2/\rho}(\Omega)$, $\beta' < \beta$, $\beta' < \sigma < 1$, $\delta < \rho$, $5/\delta < 3+\beta-\beta'$, $2+\sigma < 5/\rho < 2+\sigma+\beta-\beta'$, $\delta \in (1,2)$, $\beta \in (0,1)$, $3/\delta < 2+\beta$.

(b) *There exists a positive constant A satisfying the inequality*

$$\varphi_0(A, \overline{G}_1, \overline{G}_0, k, k_3, l_3)(\eta_1+\eta_2) + c_0 k \le A, \quad c_0 k < A, \tag{5.47}$$

where φ_0 is a positive increasing function defined by the previous lemmas, c_0 some positive constant, $k_i(T)$, $i=1,2,3$, $k(T) = k_1(T)+k_2(T)$ are defined by (5.11), (5.12)*; $\overline{G}_1 = G_1 + G_2$, $\overline{G}_0 = G_0 + G_0^1$ appear in* (4.17), (4.43)*; η_1, η_2 are expressed by* (3.53) *and* (3.55)*, respectively; l_3 is defined in the assumptions of Lemma* 3.7.

(c) *All quantities in* (5.47) *depend on T. However, for a fixed A we can choose T as large as possible assuming that η_1 and η_2 are sufficiently small.*

Then there exists a local solution to problem (1.1) *in the interval* $[0,T]$ *such that*

$$\sum_{i=1}^{2} (\|h^{(i)}\|_{2+\beta^i,\delta,\Omega^T} + \|\nabla q^{(i)}\|_{\beta^i,\delta,\Omega^T}) \le A, \quad \beta^1 = \beta,\ \beta^2 = \beta',$$
$$\|v\|_{2+\sigma,\rho,\Omega^T} \le G(A, \overline{G}_1(T), \overline{G}_0(0)).$$

6. Global existence

To prove global existence of solutions to problem (1.1) we are going to show that the local existence can be prolonged step by step up to infinity in time. For this purpose we have to show that the initial data which appear in the assumptions of Theorem 1.1 do not increase with time. Since the local existence is proved in the interval $[0,T]$ with T as large as we want, we are able to show by applying decay properties that the norms of initial data at $t = T$ are not larger than at $t = 0$. For this purpose we also use that the following quantities are small in the norms of Theorem 1.1.

(1) derivatives with respect to x_3 of initial velocity and the external force,

(2) derivatives of d_1 and d_2 with respect to $x' = (x_1, x_2)$ and t.

We are going to prove global existence of solutions to problem (1.1) step by step using that the local existence with sufficiently large existence time is shown.

Lemma 6.1. *Assume that there exists a local solution with estimate* (5.13). *Assume that*

$$|d_1-d_2|_{2,S_2}\le ce^{-\delta_1 t},\qquad |\alpha_{,x_3}|_{3,S_2}\le ce^{-\delta_1 t},$$

$$\begin{aligned}&(e^{-\delta_1 T}+\eta_1^2(T))\varphi(A)+c\sup_t|d|^2_{3,S_2}e^{-\delta_1 T}\\ &+\int_0^T(\gamma^2|\alpha|^2_{2,S_1}+\|\alpha_{,t}\|^2_{1,6/5,\Omega}+|f|^2_{6/5,\Omega})\,dt+e^{-\nu T}|w(0)|^2_{2,\Omega}\le|w(0)|^2_{2,\Omega},\end{aligned}\tag{6.1}$$

where T is sufficiently large and $\eta_1(T)$ (see (3.53)*) is sufficiently small. Then*

$$\begin{aligned}&|w(T)|^2_{2,\Omega}\le|w(0)|^2_{2,\Omega},\\ &|w(t)|^2_{2,\Omega}+\nu^2\int_0^t\|w(t')\|^2_{1,\Omega}\,dt'\\ &\le(1+e^{\nu T})[(e^{-\delta,T}+\eta_1^2(T))\varphi(A)+c\sup_t|d|^2_{3,S_2}e^{-\delta_1 T}]\\ &\quad+(1+e^{\nu T})\bigg[\int_0^T(\gamma^2|\alpha^2_{2,S_1}+\|\alpha_{,t}\|^2_{1,6/5,\Omega}+|f|^2_{6/5,\Omega})\,dt+|w(0)|^2_{2,\Omega}\bigg],\end{aligned}\tag{6.2}$$

where $t\le T$.

Proof. We use the proof of Lemma 2.2 where the integral

$$-\int_\Omega v\cdot\nabla(w+\delta)\cdot w\,dx=-\int_\Omega v\cdot\nabla ww\,dx-\int_\Omega v\cdot\nabla\delta\cdot w\,dx\equiv I_1+I_2$$

will be estimated in a different way. Integrating by parts yields

$$\begin{aligned}I_1&=-\frac12\int_\Omega v\cdot\nabla w^2\,dx=\frac12\bigg(\int_{S_2(-a)}d_1w^2\,dx'-\int_{S_2(a)}d_2w^2\,dx'\bigg)\\ &=\frac12\int_{S_2}(d_1-d_2)w^2\,dx'+\frac12\int_{S_2}d_2\big(w^2|_{x_3=-a}-w^2|_{x_3=a}\big)\,dx'\equiv I_1^1+I_1^2,\end{aligned}$$

where

$$|I_1^1|\le c|d_1-d_2|_{2,S_2}\|w\|^2_{1,\Omega},$$

and

$$I_1^2=-\frac12\int_{S_2}d_2\,dx'\int_{-a}^a dx_3\partial_{x_3}w^2=-\int_{S_2}d_2\,dx'\int_{-a}^a dx_3wh^{(1)}.$$

Hence,

$$|I_1^2|\le\varepsilon_1|w|^2_{6,\Omega}+c(1/\varepsilon_1)|d_2|^2_{3,S_2}|h^{(1)}|^2_{2,\Omega}.$$

Next, we examine

$$I_2=-\int_\Omega(w+\delta)\cdot\nabla\delta\cdot w\,dx=-\int_\Omega w\cdot\nabla\delta\cdot w\,dx-\int_\Omega\delta\cdot\nabla\delta\cdot w\,dx\equiv I_2^1+I_2^2,$$

where

$$\begin{aligned}|I_2^1|&\le\varepsilon_2|w|^2_{6,\Omega}+c(1/\varepsilon_2)|\nabla\delta|^2_{3,\Omega}|w|^2_{2,\Omega}\le\varepsilon_2|w|^2_{6,\Omega}+c(1/\varepsilon_2)|\alpha_{,x_3}|^2_{3,\Omega}|w|^2_{2,\Omega},\\ |I_2^2|&\le\varepsilon_3|w|^2_{6,\Omega}+c(1/\varepsilon_3)|\delta|^2_{3,\Omega}|\nabla\delta|^2_{2,\Omega}\le\varepsilon_3|w|^2_{6,\Omega}+c(1/\varepsilon_3)|d|^2_{3,S_2}|\alpha_{,x_3}|^2_{2,\Omega}.\end{aligned}$$

In view of the above estimates instead of (2.13) we obtain

$$\begin{aligned}(6.3)\quad \frac{d}{dt}|w|^2_{2,\Omega}+\nu\|w\|^2_{1,\Omega} &\le c|d_1-d_2|_{2,S_2}\|w\|^2_{1,\Omega}+c|d_2|^2_{3,S_2}|h^{(1)}|^2_{2,\Omega}\\ &+c|\alpha_{,x_3}|^2_{3,\Omega}|w|^2_{2,\Omega}+c|d|^2_{3,S_2}|\alpha_{,x_3}|^2_{2,\Omega}\\ &+c(\gamma^2|\alpha|^2_{2,S_1}+\|\alpha_{,t}\|^2_{1,6/5,\Omega}+|f|^2_{6/5,\Omega})\equiv c\Gamma_1^2(t).\end{aligned}$$

From (6.3) we have

$$\frac{d}{dt}(|w|^2_{2,\Omega}e^{\nu t})\le\Gamma_1^2(t)e^{\nu t}. \tag{6.4}$$

To integrate (6.4) with respect to time we use that

$$\begin{aligned}e^{-\nu t}\int_0^t\Gamma_1^2(t')e^{\nu t'}\,dt' &\le(e^{-\delta_1 t}+\eta_1^2(t))\varphi(A)+c\sup_t|d|^2_{3,S_2}e^{-\delta_1 t}\\ &+c\int_0^t(\gamma^2|\alpha|^2_{2,S_1}+\|\alpha_{,t}\|^2_{1,6/5,\Omega}+|f|^2_{6/5,\Omega})\,dt',\end{aligned}$$

which follows from (3.53) expressed in the form $\|h^{(1)}\|_{V_2^0(\Omega^t)}\le\varphi(A)\eta_1(t)$, from $(6.1)_{1,2}$ and the existence of local solution with estimate (5.13). Then (6.4) implies

$$\begin{aligned}(6.5)\quad |w(T)|^2_{2,\Omega} &\le(e^{-\delta_1 T}+\eta_1^2(T))\varphi(A)+c\sup_t|d|^2_{3,S_2}e^{-\delta_1 T}\\ &+\int_0^T(\gamma^2|\alpha|^2_{2,S_1}+\|\alpha_{,t}\|^2_{1,6/5,\Omega}+|f|^2_{6/5,\Omega})\,dt+e^{-\nu T}|w(0)|^2_{2,\Omega}.\end{aligned}$$

Inequality $(6.2)_1$ follows from (6.5) under assumption $(6.1)_3$.

Let $\nu=\nu_1+\nu_2$, $\nu_i>0$, $i=1,2$. Then (6.3) can be written in the form

$$\frac{d}{dt}|w|^2_{2,\Omega}+\nu_1|w|^2_{2,\Omega}+\nu_2\|w\|^2_{1,\Omega}\le c\Gamma_1^2(t).$$

Hence we have

$$\frac{d}{dt}(|w|^2_{2,\Omega}e^{\nu_1 t})+\nu_2\|w\|^2_{1,\Omega}e^{\nu_1 t}\le c\Gamma_1^2(t)e^{\nu_1 t}. \tag{6.6}$$

Integrating (6.6) with respect to time we obtain $(6.2)_2$. This ends the proof. □

In view of (1.7) and $(6.2)_1$ we have

$$|v(T)|_{2,\Omega}\le|w(T)|_{2,\Omega}+c_1|d(T)|_{2,S_1}\le|w(0)|_{2,\Omega}+c_1|d(0)|_{2,S_1}\le c_2|v(0)|_{2,\Omega}.$$

Let

$$\begin{aligned}\Gamma_2^2(t)&=\|v\|^2_{1,\Omega}\|d_{,x'}\|^2_{1,3,S_2}+(1+|d|^2_{3,S_2})\|d_{,x'}\|^2_{1,S_2}\\ &\quad+\|d_{,t}\|^2_{1,6/5,S_2}+|f_3|^2_{4/3,S_2}+|g^{(1)}|^2_{6/5,\Omega},\\ \alpha_1(t)&=|d_1|^6_{3,6,S_2^t}+|\nabla v|^2_{3,2,\Omega^t}.\end{aligned}$$

Lemma 6.2. *Assume that there exists a local solution in the interval* $[0,T]$. *Assume*

$$-\frac{\nu}{2}T+\alpha_1(T)\le 0, \tag{6.7}$$

$$\Gamma_2^2(t)\le\Gamma_2^2(0)e^{-\delta_2 t} \tag{6.8}$$

$$\Gamma_2^2(0)e^{-\delta_2 T+\alpha_1(t)}+|k^{(1)}(0)|_{2,\Omega}^2e^{-\nu T/2}\le|k^{(1)}(0)|_{2,\Omega}^2. \tag{6.9}$$

Then

$$|h^{(1)}(T)|_{2,\Omega}^2\le|h^{(1)}(0)|_{2,\Omega}^2+c_1\sup_t|d_{,x'}|_{2,S_2}^2, \tag{6.10}$$

$$\begin{aligned}|h^{(1)}(t)|_{2,\Omega}^2\le{}&e^{-\delta_2 t+\alpha_1(t)}\Gamma_2^2(0)+|h^{(1)}(0)|_{2,\Omega}^2e^{-\nu_1 t+\alpha_1(t)}\\&+c_1\sup_t|d_{,x'}|_{2,S_2}^2(1+e^{-\nu_1 t+\alpha_1(t)}),\end{aligned} \tag{6.11}$$

$$\begin{aligned}\int_0^t\|h^{(1)}(t')\|_{1,\Omega}^2\,dt'\le{}&e^{(\nu_1-\delta_2)t+\alpha_1(t)}\Gamma_2^2(0)+|h^{(1)}(0)|_{2,\Omega}^2e^{\alpha_1(t)}\\&+c_1\int_0^t\|d_{,x'}(t')\|_{1,S_2}^2\,dt'+c_1\sup_t|d_{,x'}|_{2,S_2}^2e^{\alpha_1(t)}.\end{aligned} \tag{6.12}$$

Proof. From (3.34) we obtain the inequality

$$\frac{d}{dt}|k^{(1)}|_{2,\Omega}^2+\nu\|k^{(1)}\|_{1,\Omega}^2\le c(|d_1|_{3,S_2}^6+|\nabla v|_{3,\Omega}^2)|k^{(1)}|_{2,\Omega}^2+c\Gamma_2^2(t). \tag{6.13}$$

From (6.13) we have also

$$\frac{d}{dt}|k^{(1)}|_{2,\Omega}^2+\nu|k^{(1)}|_{2,\Omega}^2\le c(|d_1|_{3,S_2}^6+|\nabla v|_{3,\Omega}^2)|k^{(1)}|_{2,\Omega}^2+c\Gamma_2^2(t)$$

which implies

$$\frac{d}{dt}(|k^{(1)}|_{2,\Omega}^2e^{\nu t-\alpha_1(t)})\le c\Gamma_2^2(t)e^{\nu t-\alpha_1(t)}. \tag{6.14}$$

Integrating (6.14) with respect to time yields

$$|k^{(1)}(t)|_{2,\Omega}^2\le e^{-\nu t+\alpha_1(t)}\int_0^t\Gamma_2^2(t')e^{\nu t'-\alpha_1(t')}\,dt'+|k^{(1)}(0)|_{2,\Omega}^2e^{-\nu t+\alpha_1(t)}.$$

In view of (6.7) and (6.8) the above inequality implies

$$|k^{(1)}(T)|_{2,\Omega}^2\le\Gamma_2^2(0)e^{-\delta_2 T+\alpha_1(T)}+|k^{(1)}(0)|_{2,\Omega}^2e^{-\nu T/2},$$

which in view of (6.9) gives

$$|k^{(1)}(T)|_{2,\Omega}^2\le|k^{(1)}(0)|_{2,\Omega}^2. \tag{6.15}$$

In view of (3.23) and Lemma 3.5 we obtain from (6.15) inequality (6.10).

Imposing $\nu=\nu_1+\nu_2$, $\nu_i>0$, $i=1,2$, we obtain from (6.13) the inequality

$$|k^{(1)}(t)|^2_{2,\Omega}+\nu_2 e^{-\nu_1 t+\alpha_1(t)}\int_0^t \|k^{(1)}(t')\|^2_{1,\Omega}e^{\nu_1 t'-\alpha_1(t')}\,dt' \le e^{-\nu_1 t+\alpha_1(t)}\int_0^t \Gamma_2^2(t')e^{\nu_1 t'-\alpha_1(t')}\,dt'+|k^{(1)}(0)|^2_{2,\Omega}e^{-\nu_1 t+\alpha_1(t)}. \tag{6.16}$$

Utilizing (6.8), (3.23) and Lemma 3.5 we get from (6.16) inequalities (6.11) and (6.12). This concludes the proof. □

Let

$$\begin{aligned}
\alpha_2(t)&=|d_1|^6_{3,6,S_2^t}+\int_0^t(\|v\|^2_{1,3,\Omega}+|\nabla h^{(1)}|^2_{2,\Omega})\,dt',\\
\Gamma_3^2(t)&=(\|v\|^2_{1,\Omega}+|\nabla h^{(1)}|^2_{2,\Omega})\|d_{,x'}\|^2_{2,3,S_2},\\
\Gamma_4^2(t)&=(1+|d|^2_{3,S_2})\|d_{,x'}\|^2_{2,3/2,S_2}+\|d_{,t}\|^2_{2,6/5,S_2}\\
&\quad+|g^{(1)}|^2_{6/5,\Omega}+|g^{(2)}|^2_{6/5,\Omega}+|f_3|^2_{4/3,S_2}+|F'|^2_{4/3,S_2},\\
\Gamma_5^2(t)&=|v|^2_{4,S_2}|h^{(1)}_{,x'}|^2_{2,S_2}+|v|^2_{4,S_2}\|d_{,x'}\|^2_{1,S_2}+|d|^2_{\infty,S_2}|h^{(1)}_{,x}|^2_{4/3,S_2}\\
&\quad+|h^{(1)}|^2_{4,S_2}|h^{(1)}|^2_{2,S_2}+|h^{(1)}|^2_{4,S_2}|d_{,x'}|^2_{2,S_2}\\
&\quad+|h^{(1)}|^2_{4,S_2}|v_{,x'}|^2_{2,S_2}+\|d_{,x'}\|^2_{1,S_2}|v_{,x}|^2_{2,S_2}.
\end{aligned}$$

Lemma 6.3. *Assume that*

$$-\nu T+c\alpha_2(T)\le-\frac{\nu}{2}T, \tag{6.17}$$

$$\Gamma_6^2(t)\equiv\|d_{,x'}(t)\|^2_{2,3,S_2}+\|d_{,t}(t)\|^2_{2,6/5,S_2}+|g^{(1)}(t)|^2_{6/5,\Omega}+|g^{(2)}(t)|^2_{6/5,\Omega} +|f_3(t)|^2_{4/3,S_2}+|F'(t)|^2_{4/3,S_2}+|h^{(1)}|^2_{2,\Omega}\le\Gamma_6^2(0)e^{-\delta_3 t}. \tag{6.18}$$

Then

$$|h^{(1)}(T)|^2_{2,\Omega}+|h^{(2)}(T)|^2_{2,\Omega}\le e^{-\delta_3 T+c\alpha_2(T)}\Gamma_6^2(0)\varphi(A) +e^{-\nu T/2}(|h^{(1)}(0)|^2_{2,\Omega}+|h^{(2)}(0)|^2_{2,\Omega})+c_1\sup_{t\le T}\|d_{,x'}\|^2_{1,S_2}, \tag{6.19}$$

$$\int_0^t(\|h^{(1)}(t')\|^2_{1,\Omega}+\|h^{(2)}(t')\|^2_{1,\Omega})\,dt'\le e^{(\nu_1-\delta_3)t+c\alpha_2(t)}\Gamma_6^2(0)\varphi(A) +e^{c\alpha_2(t)}(|h^{(1)}(0)|^2_{2,\Omega}+|h^{(2)}(0)|^2_{2,\Omega})+c_1\sup_t\|d_{,x'}\|^2_{1,S_2}e^{c\alpha_2(t)}. \tag{6.20}$$

Proof. From (3.49) we have

$$\begin{aligned}
&\frac{d}{dt}(|k^{(1)}|^2_{2,\Omega}+|k^{(2)}|^2_{2,\Omega})+\nu(\|k^{(1)}\|^2_{1,\Omega}+\|k^{(2)}\|^2_{1,\Omega})\\
&\le c(|d_1|^6_{3,S_2}+\|v\|^2_{1,3,\Omega}+|\nabla h^{(1)}|^2_{3,\Omega})(|k^{(1)}|^2_{2,\Omega}+|k^{(2)}|^2_{2,\Omega})\\
&\quad+c(\Gamma_3^2(t)+\Gamma_4^2(t)+\Gamma_5^2(t)).
\end{aligned}\tag{6.21}$$

From (6.21) we have

$$\frac{d}{dt}[(|k^{(1)}|_{2,\Omega}^2+|k^{(2)}|_{2,\Omega}^2)e^{\nu t-c\alpha_2(t)}]\le c(\Gamma_3^2(t)+\Gamma_4^2(t)+\Gamma_5^2(t))e^{\nu t-c\alpha_2(t)}. \tag{6.22}$$

Integrating (6.22) with respect to time yields

$$\begin{aligned}|k^{(1)}(T)|_{2,\Omega}^2 &+|k^{(2)}(T)|_{2,\Omega}^2\\ &\le e^{-\nu T+c\alpha_2(T)}\int_0^T(\Gamma_3^2(t)+\Gamma_4^2(t)+\Gamma_5^2(t))e^{\nu t-c\alpha_2(t)}\,dt\\ &\quad+e^{-\nu T+c\alpha_2(T)}(|k^{(1)}(0)|_{2,\Omega}^2+|k^{(2)}(0)|_{2,\Omega}^2).\end{aligned} \tag{6.23}$$

By the interpolation inequalities we get

$$\begin{aligned}\Gamma_5^2(t)\le(\|v\|_{1,S_2}^2&+|h^{(1)}|_{4,S_2}^2+|d|_{\infty,S_2}^2+|d_{,x'}|_{2,S_2}^2)\\ &\cdot(\|h^{(1)}\|_{2+\beta,\rho,\Omega}^{2\theta}|h^{(1)}|_{2,\Omega}^{2(1-\theta)}+|h^{(1)}|_{2,\Omega}^2)+\|v\|_{1,S_2}^2\|d_{,x'}\|_{1,S_2}^2,\end{aligned}$$

where $3/\rho<2+\beta$, $1/\theta=1-\varkappa$, $\varkappa=3/(\rho(2+\beta))$. Assuming

$$|h^{(1)}(t)|_{2,\Omega}\le e^{-\delta' t}|h^{(1)}(0)|_{2,\Omega}$$

we have (6.18).

In view of (6.17) and (6.18) we obtain from (6.23) the inequality

$$\begin{aligned}|k^{(1)}(T)|_{2,\Omega}^2+|k^{(2)}(T)|_{2,\Omega}^2&\le e^{-\delta_3 T+c\alpha_2(T)}\Gamma_6^2(0)\\ &\cdot\int_0^T[\|v\|_{1,\Omega}^2+|\nabla h^{(1)}|_{2,\Omega}^2+|d|_{3,S_2}^2+(\|v\|_{1,S_2}^2+|h^{(1)}|_{4,S_2}^2\\ &+|d|_{\infty,S_2}^2+|d_{,x'}|_{2,S_2}^2)\|h^{(1)}\|_{2+\beta,\rho,\Omega}^{2\theta}+\|v\|_{1,S_2}^2]\,dt'\\ &+e^{-\nu/2T}(|k^{(1)}(0)|_{2,\Omega}^2+|k^{(2)}(0)|_{2,\Omega}^2)\\ &\le e^{-\delta_3T+c\alpha_2(T)}\Gamma_6^2(0)\varphi(A)+e^{-\nu/2T}(|k^{(1)}(0)|_{2,\Omega}^2+|k^{(2)}(0)|_{2,\Omega}^2).\end{aligned} \tag{6.24}$$

In view of Lemmas 3.5 and 3.8 inequality (6.24) implies (6.19).

Assuming $\nu=\nu_1+\nu_2$, $\nu_i>0$, $i=1,2$, we obtain from (6.21) the inequality

$$\begin{aligned}|k^{(1)}(t)|_{2,\Omega}^2&+|k^{(2)}(t)|_{2,\Omega}^2+\nu_2e^{-\nu_1t+c\alpha_2(t)}\int_0^t(\|k^{(1)}(t')\|_{1,\Omega}^2\\ &+\|k^{(2)}(t')\|_{1,\Omega}^2)e^{\nu_1t'-c\alpha_2(t')}\,dt'\\ &\le e^{-\nu_1t+c\alpha_2(t)}\int_0^t(\Gamma_3^2(t')+\Gamma_4^2(t')+\Gamma_5^2(t'))e^{\nu_1t'-c\alpha_2(t')}\,dt'\\ &+(|k^{(1)}(0)|_{2,\Omega}^2+|k^{(2)}(0)|_{2,\Omega}^2)e^{-\nu_1t+c\alpha_2(t)},\quad t\in[0,T].\end{aligned} \tag{6.25}$$

Utilizing Lemmas 3.5 and 3.8 in (6.25) implies (6.20). □

Finally we prove global existence

THEOREM 6.4. *Let the assumptions of Theorem* 5.4 *for a sufficiently large* T *hold. Let the assumptions of Lemmas* 6.1–6.5 *be satisfied. Then the local solution determined by Theorem* 5.5 *can be prolonged infinitely step by step.*

PROOF. From Lemmas 6.2 and 6.3 we have

$$|h^{(i)}(T)|_{2,\Omega} \le e^{-\delta_0 T}|h^{(i)}(0)|_{2,\Omega}, \quad i=1,2, \quad \delta_0>0. \tag{6.26}$$

Since the local solution to problem (1.1) in the interval $[0,T]$ satisfies the estimates

$$\begin{aligned} \|h^{(1)}\|_{2+\beta,\delta,\Omega^T} &\le A,\\ \|h^{(2)}\|_{2+\beta',\delta,\Omega^T} &\le A, \end{aligned} \tag{6.27}$$

we obtain by interpolation the inequalities

$$\begin{aligned} \|h^{(1)}\|_{\sigma_1,\delta,\Omega^T} &\le \varphi(A)e^{-\delta_* T}, \quad \sigma_1<2+\beta,\\ \|h^{(2)}\|_{\sigma_2,\delta,\Omega^T} &\le \varphi(A)e^{-\delta_* T}, \quad \sigma_2<2+\beta',\ \delta_*>0. \end{aligned} \tag{6.28}$$

Let $\zeta(t)$ be a smooth function such that $\zeta(t)=1$ for $t\in[T-t_1,T]$ and $\zeta(t)=0$ for $t\le T-2t_1$. Then $\widetilde{h}^{(i)}=h^{(i)}\zeta$, $i=1,2$, are solutions to the problems

$$\begin{aligned} &\widetilde{h}^{(1)}_{,t}-\operatorname{div}\mathbb{T}(\widetilde{h}^{(1)},\widetilde{q}^{(1)})=-v\cdot\nabla\widetilde{h}^{(1)}-\widetilde{h}^{(1)}\cdot\nabla v+\widetilde{g}^{(1)}+h^{(1)}\dot{\zeta}, && \text{in } \Omega^T,\\ &\operatorname{div}\widetilde{h}^{(1)}=0 && \text{in } \Omega^T,\\ &\overline{n}\cdot\widetilde{h}^{(1)}=0,\quad \nu\overline{n}\cdot\mathbb{D}(\widetilde{h}^{(1)})\cdot\overline{\tau}_\alpha+\gamma\widetilde{h}^{(1)}\cdot\overline{\tau}_\alpha=0,\quad \alpha=1,2, && \text{on } S_1^T,\\ &\widetilde{h}^{(1)}_i=-\widetilde{d}_{,x_i},\quad \widetilde{h}^{(1)}_{3,x_3}=\Delta'\widetilde{d} && \text{on } S_2^T,\\ &\widetilde{h}^{(1)}|_{t=T-2t_1}=0 && \text{in } \Omega, \end{aligned} \tag{6.29}$$

and

$$\begin{aligned} &\widetilde{h}^{(2)}_{,t}-\operatorname{div}\mathbb{T}(\widetilde{h}^{(2)},\widetilde{q}^{(2)})=-\widetilde{h}^{(2)}\cdot\nabla v-2h^{(1)}\cdot\nabla\widetilde{h}^{(1)}\\ &\qquad -v\cdot\nabla\widetilde{h}^{(2)}+\widetilde{g}^{(2)}+\widetilde{h}^{(2)}\dot{\zeta} && \text{in } \Omega^T,\\ &\operatorname{div}\widetilde{h}^{(2)}=0 && \text{in } \Omega^T,\\ &\overline{n}\cdot\widetilde{h}^{(2)}=0,\quad \nu\overline{n}\cdot\mathbb{D}(\widetilde{h}^{(2)})\cdot\overline{\tau}_\alpha+\gamma\widetilde{h}^{(2)}\cdot\overline{\tau}_\alpha=0,\quad \alpha=1,2, && \text{on } S_1^T,\\ &\widetilde{h}^{(2)}_3=\Delta'\widetilde{d}, && \text{on } S_2^T,\\ &\widetilde{h}^{(2)}_{i,x_3}=-\frac{2}{\nu}d_{,x_it}\zeta+3\Delta'\widetilde{d}_{,x_i}+\frac{1}{\nu}\widetilde{F}'_i\\ &\qquad+\frac{1}{\nu}[v_j(\widetilde{h}^{(1)}_i-\widetilde{d}_{,x_i})_{,x_j}+d(\widetilde{h}^{(1)}_{i,x_3}-\widetilde{h}^{(1)}_{3,x_i})\\ &\qquad+h^{(1)}_3(\widetilde{h}^{(1)}_i-\widetilde{d}_{,x_i})+\widetilde{h}^{(1)}_jv_{i,x_j}-\widetilde{d}_{,x_j}v_{j,x_i}] && \text{on } S_2^T,\\ &\widetilde{h}^{(2)}|_{t=T-2t_1}=0 && \text{in } \Omega. \end{aligned} \tag{6.30}$$

Assuming that t_1 is small comparing to T, using the decays

$$\|\widetilde{g}^{(i)}\|_{\beta,\delta,\Omega^T}\le a_ie^{-\delta_* T},\quad i=1,2,$$

exponential decays of norms od $d_{,x'}$ and (6.26), (6.27), we obtain from problems (6.29) and (6.30) the decays

$$\|\widetilde{h}^{(1)}\|_{2+\beta,\delta,\Omega^T} \le \varphi(A)e^{-\delta'_* T}, \qquad \|\widetilde{h}^{(2)}\|_{2+\beta',\delta,\Omega^T} \le \varphi(A)e^{-\delta'_* T},$$

where $\delta'_* > 0$. Hence for sufficiently large T we obtain

$$\|h^{(i)}(T)\|_{2+\beta^i-2/\delta,\delta,\Omega} \le \|h^{(i)}(0)\|_{2+\beta^i-2/\delta,\delta,\Omega}, \quad i = 1,2,$$

where $\beta^1 = \beta$, $\beta^2 = \beta'$. Therefore local existence in the interval $[T, 2T]$ can be proved with the same initial data as for $[0,T]$. This implies global existence. This ends the proof. □

Finally we present a global existence proof much less explicit than the previous one but implying existence of much more general global solution

Theorem 6.5. *Let the assumptions of Theorem* 5.4 *be satisfied. Then there exists a sequence* $\{t_i\}_{i=0}^{\infty}$ *increasing to infinity such that the local solution determined by Theorem* 5.4 *exists in the each interval* $[t_i, t_{i+1}]$, $i = 0, 1, \dots,$ *where* $t_0 = 0$.

Proof. Assume that we have proved the existence of local solution with sufficiently large existence time T. Then we have

$$\begin{aligned} \int_0^T |v(t)|^2_{2,\Omega}\,dt &\le c, & \int_0^T \|v(t)\|^\rho_{2+\sigma,\rho,\Omega}\,dt &\le c, \\ \int_0^T |h^{(i)}(t)|^2_{2,\Omega}\,dt &\le c, & \int_0^T \|h^{(i)}(t)\|^\delta_{2+\beta^i,\delta,\Omega}\,dt &\le 0, \quad i = 1,2. \end{aligned}$$

Then there exists $T_* < T$ sufficiently large and there exists $t_* \in [T_*, T]$ such that

$$|v(t_*)|_{2,\Omega}, \quad |h^{(i)}(t_*)|_{2,\Omega}, \quad \|v(t_*)\|_{2+\sigma,\rho,\Omega}, \quad \|h^{(i)}(t_*)\|_{2+\beta^i,\delta,\Omega}, \quad i = 1,2$$

are so small that

$$\begin{aligned} |v(t_*)|_{2,\Omega} &\le |v(0)|_{2,\Omega}, \\ |h^{(i)}(t_*)|_{2,\Omega} &\le |h^{(i)}(0)|_{2,\Omega}, \\ \|v(t_*)\|_{2+\sigma-2/\rho,\rho,\Omega} &\le \|v(0)\|_{2+\sigma-2/\rho,\rho,\Omega}, \\ \|h^{(i)}(t_*)\|_{2+\beta^i-2/\delta,\delta,\Omega} &\le \|h^{(i)}(0)\|_{2+\beta^i-2/\delta,\delta,\Omega}, \quad i = 1,2,\ \beta^1 = \beta,\ \beta^2 = \beta'. \end{aligned}$$

Then we can prove the existence of the local solutions in the interval $[T, t_* + T]$. Hence in this way global existence follows. □

References

[1] W. Alame, *On existence of solutions for the nonstationary Stokes system with slip boundary conditions in general Sobolev–Slobodetskiĭ and Besov spaces*, Banach Center Publ. **70** (2005).

[2] O. V. Besov, V. P. Il'in, S. M. Nikolskij, *Integral Represntations of Functions and Theorems of Imbedding*, Moscow, 1975. (in Russian)

[3] T. Kobayashi, W. M. Zajączkowski, *On global motion of a compressible barotropic viscous fluid with boundary slip conditions*, Appl. Math **26** (1999), no. 2, 159–194.

[4] O. A. Ladyzhenskaya, V. A. Solonnikov, *On some problems of vector analysis and generalized boundary value problems for the Navier–Stokes equations*, Zap. Nauchn. Sem. LOMI **59** (1976), 81–116. (in Russian)

[5] O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Uraltseva, *Linear and Quasilinear Equations of Parabolic Type*, Nauka, Moscow, 1967. (in Russian)

[6] W. M. Zajączkowski, *Long time existence of regular solutions to Navier–Stokes equations in cylindrical domains under boundary slip conditions*, Studia Math. **169** (2005), no. 3, 243–285.

[7] ______, *Global existence of axially symmetric solutions to Navier–Stokes equations with large angular component of velocity*, Colloq. Math. **100** (2004), no. 2, 243–263.

[8] ______, *Global special regular solutions to the Navier–Stokes equationsin a cylindrical domain without the axis of symmetry*, Topol. Methods Nonlinear Anal. **24** (2004), 69–105.

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Wojciech M. Zajączkowski
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-956 Warsaw, POLAND
and
Institute of Mathematics and Cryptology
Military University of Technology
Kaliskiego 2
00-908 Warsaw, POLAND

E-mail address: wz@impan.gov.pl