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EXISTENCE AND NON EXISTENCE OF THE GROUND STATE SOLUTION FOR THE NONLINEAR SCHROEDINGER EQUATIONS WITH $V(\infty) = 0$

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To the memory of Olga Ladyzhenskaya

Abstract. We study the existence of the ground state solution of the problem

$$\begin{cases} -\Delta u + V(x)u = f'(u) & x \in \mathbb{R}^N, \\ u(x) > 0, \end{cases}$$

under the assumption that $\lim_{x\to\infty} V(x) = 0$.

1. Introduction

In recent years, the stationary solutions of the nonlinear Schroedinger equation (NSL)

$$i\,\frac{\partial \psi}{\partial t} = (-\Delta + V(x))\psi - f'(|\psi|)\,\frac{\psi}{|\psi|}$$

have received a lot of attention. In order to find such solutions the following ansatz is done

$$\psi(t,x) = u(x)e^{-i\omega t}$$

and we are led to the study of the following equation:

$$-\Delta u + (V(x) - \omega)u = f'(u). \tag{1.1}$$

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Now we make the following assumptions

$$\lim_{x\to\infty} V(x) = 0, \tag{1.2}$$

$$f(0) = f'(0) = f''(0) = 0 \tag{1.3}$$

which are natural and used in many physical problems. Under these assumptions it is well known that there exist finite energy solutions, provided that $\omega < 0$ and f satisfies suitable assumptions (e.g. $f(u) = |u|^p$, $p \le 2^*$) (see e.g. [12], [4] and [2] and the references therein). In this paper, we are interested to analyze the case $\omega = 0$. Thus we are led to the study of the following problem

$$\begin{cases} -\Delta u + V(x)u = f'(u), & x \in \mathbb{R}^N,\ N \ge 3, \\ F_V(u) < \infty, \\ u(x) > 0, \end{cases} \tag{1.4}$$

where

$$F_V(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V u^2\,dx - \int_{\mathbb{R}^N} f(u)\,dx \tag{1.5}$$

is the energy functional. Berestycki and Lions [9] proved that, if $f(u) = |u|^p$ and $V = 0$, problem (1.4) has no solutions. Actually they proved that, if $V = 0$ a necessary condition for the existence of solutions is that f behaves as $|u|^q$ for u small and $|u|^p$ ($p < 2^* < q$) for u large. For example, the required assumptions are satisfied by the function

$$f(u) = \begin{cases} u^q & \text{if } u \le 1, \\ a + bu + cu^p & \text{if } u \ge 1, \end{cases} \tag{1.6}$$

where a, b and c are constants which make the function $f \in C^2$.

Now we present the main result of this paper: we assume that the function f satisfy (1.3) and the following assumptions:

- there exists $\mu > 2$ such that

$$0 < \mu f(s) \le f'(s)s < f''(s)s^2 \quad \text{for all } s \ne 0, \tag{1.7}$$

- there exist positive numbers c_0, c_2, p, q with $N < p < 2^* < q$ such that

$$\begin{cases} c_0|s|^p \le f(s) & \text{for } |s| \ge 1, \\ c_0|s|^q \le f(s) & \text{for } |s| \le 1, \end{cases} \tag{1.8}$$

$$\begin{cases} |f''(s)| \le c_2|s|^{q-2} & \text{for } |s| \ge 1, \\ |f''(s)| \le c_2|s|^{p-2} & \text{for } |s| \le 1, \end{cases} \tag{1.9}$$

where $2^* = 2N/(N-2)$.

For example the function (1.6) satisfies the above requirements.

We assume that V satisfies the following assumptions:

$$V \in L^{N/2}(\mathbb{R}^N) \cap L^t(\mathbb{R}^N) \quad \text{for some } t > N/2, \tag{1.10}$$

$$\|V^-\|_{L^{N/2}} < S, \tag{1.11}$$

where

$$S = \inf_{u \in \mathcal{D}^{1,2}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{(\int_{\mathbb{R}^N} |u|^{2^*})^{2/2^*}} \quad \text{and} \quad V(x)^- = -\min\{0, V(x)\}.$$

Our first theorem is a non existence result:

Theorem 1.1. *If $V(x) \geq 0$ for every $x \in \mathbb{R}^N$ and $V(x) > 0$ on a set of positive measure then problem* (1.4) *has no ground state solution.*

We recall that a solution of (1.4) is called "ground state" solution if it minimizes the energy on the Nehari manifold

$$\mathcal{N}^V = \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N} |\nabla u|^2 + V u^2 - f'(u) u = 0 \right\}. \tag{1.12}$$

As far as the existence is concerned we have the following:

Theorem 1.2. *If $V(x) \leq 0$ and $V(x) < 0$ on a set of positive measure, then problem* (1.4) *has a ground state solution.*

Remark 1.3. The assumptions of Theorem 1.2 can be weakened requiring that

$$\int_{\mathbb{R}^N} V(x) w(x)^2 \, dx < 0$$

where w is the ground state solution of problem (1.4) with $V = 0$.

Remark 1.4. The assumption (1.2) implies that the solutions of (1.4) do not live in $H^1(\mathbb{R}^N)$. Probably, this is the reason why, in spite of the large literature on the NSE, there are not many results in this direction. As far as we know, the works related to this problem are the folowing: first of all, there is the pioneering mentioned work of Berestycki and Lions in which the case $V = 0$ is analyzed. Moreover, there is a recent paper of Benci and Micheletti [7] where $V = 0$, but the domain is an exterior domain $\Omega \neq \mathbb{R}^N$. Finally, there is a paper of Ambrosetti, Felli and Malchiodi [3] where $f(u)$ is replaced by a function $f(x, u)$ of the type $k(x)|u|^p$ where $k(x) \to 0$ as $|x| \to \infty$.

The plan of the paper is the following: in Section 2 we recall some technical results concerning the appropriate function spaces required by the growth properties of f; the proves of these results are contained in [5], [6], [7], [11]. In Section 3 we prove a "splitting lemma" which is a key ingredient to deal with problems with lack of compactness. This lemma is a variant of a well known result of Struwe [13]; see also [6] and [7] for variants of this lemma related to the space $\mathcal{D}^{1,2}(\mathbb{R}^N)$. In Section 4 we prove our main results.

2. Notation and preliminary results

We will use the following notations:

- $v_y(x) = v(x+y)$,
- $B_R = \{x \in \mathbb{R}^N : |x| < R\}$,
- $\Gamma_v = \{x \in \mathbb{R}^N : |v(x)| > 1\}$,
- $|A| =$ measure of the subset $A \subset \mathbb{R}^N$,
- $\mathcal{D}^{1,2}(\mathbb{R}^N) =$ completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm:

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{1/2}.$$

The solutions of problem (1.4) are the critical points of the energy functional (1.5) on the manifold (1.12). We set

$$m = \inf_{u \in \mathcal{N}^0} F_0(u) \tag{2.1}$$

where

$$F_0(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} f(u) \, dx \tag{2.2}$$

and

$$m_V = \inf_{u \in \mathcal{N}^V} F_V(u). \tag{2.3}$$

In [9] or in Lemma 3.3 of [7] the existence of a positive and spherically symmetrical minimizer w of (2.1) has been proved. Hence w is a solution to the problem

$$\begin{cases} -\Delta w = f'(w), \\ w \in \mathcal{D}^{1,2}(\mathbb{R}^N). \end{cases} \tag{2.4}$$

We are looking for conditions on V which provide existence or non existence of minimizers of (1.5). The answers to these questions are contained in Theorems 1.1 and 1.2, which substantially relates the existence to the sign of the quantity $\int_{\mathbb{R}^N} V(x)w(x)^2 \, dx$. Indeed, if $\int_{\mathbb{R}^N} V(x)w(x)^2 \, dx < 0$ there exists a ground state solution of problem (1.4), otherwise, if $\int_{\mathbb{R}^N} V(x)w(x)^2 \, dx > 0$ and $V \geq 0$, problem (1.4) has no ground state solution.

Given $p \neq q$, we consider the space $L^p + L^q$ made up of the functions $v\colon \mathbb{R}^N \mapsto \mathbb{R}$ such that

$$v = v_1 + v_2 \quad \text{with } v_1 \in L^p(\mathbb{R}^N),\ v_2 \in L^q(\mathbb{R}^N).$$

$L^p + L^q$ is a Banach space with the norm:

$$\|v\|_{L^p+L^q} = \inf\{\|v_1\|_{L^p} + \|v_2\|_{L^q} : v_1 + v_2 = v\}.$$

It is well known that (see [1]) $L^p + L^q$ coincides with the dual of $L^{p'} \cap L^{q'}$. Then:

$$L^p + L^q = (L^{p'} \cap L^{q'})' \quad \text{with } p' = \frac{p}{p-1},\ q' = \frac{q}{q-1} \tag{2.5}$$

and the following norm is equivalent to the previous one:

$$|||v|||_{L^p+L^q} = \sup_{\varphi\neq 0} \frac{\int v(x)\varphi(x)\,dx}{\|\varphi\|_{L^{p'}} + \|\varphi\|_{L^{q'}}}. \tag{2.6}$$

Actually $L^p + L^q$ is an Orlicz space with N-function (cf. e.g. [1])

$$A(u) = \max\{|u|^p, |u|^q\}.$$

First we recall some inequalities relative to the space $L^p + L^q$ proved in [6] (see also [5]).

Lemma 2.1.

(a) *If $v \in L^p + L^q$, the following inequalities hold:*

$$\max\bigg[\|v\|_{L^q(\mathbb{R}^N-\Gamma_v)} - 1,\ \frac{1}{1+|\Gamma_v|^{1/\tau}}\|v\|_{L^p(\Gamma_v)}\bigg] \le \|v\|_{L^p+L^q} \le \max[\|v\|_{L^q(\mathbb{R}^N-\Gamma_v)},\ \|v\|_{L^p(\Gamma_v)}]$$

when $\tau = pq/(q-p)$.

(b) *Let $\{v_n\} \subset L^p + L^q$ be and set $\Gamma_n = \{x \in \Omega : |v_n(x)| > 1\}$. Then $\{v_n\}$ is bounded in $L^p + L^q$ if and only if the sequences $\{|\Gamma_n|\}$ and $\{\|v_n\|_{L^q(\mathbb{R}^N-\Gamma_n)} + \|v_n\|_{L^p(\Gamma_n)}\}$ are bounded.*

(c) *f' is a bounded map from $L^p + L^q$ into $L^{p/(p-1)} \cap L^{q/(q-1)}$.*

Remark 2.2. By Lemma 2.1(a) we have $L^{2^*} \subset L^p + L^q$ when $2 < p < 2^* < q$. Then, by the Sobolev inequality, we get the continuous embedding:

$$\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L^p + L^q.$$

In order to prove the C^2 regularity of the functional F_V we need the following lemmas proved in [7] (see also [11]):

Lemma 2.3.

(a) *If θ, u are bounded in $L^p + L^q$, then $f''(\theta)u$ is bounded in $L^{p'} \cap L^{q'}$.*

(b) *f'' is a bounded map from $L^p + L^q$ into $L^{p/(p-2)} \cap L^{q/(q-2)}$.*

(c) *f'' is a continuous map from $L^p + L^q$ into $L^{p/(p-2)} \cap L^{q/(q-2)}$.*

(d) *The map $(u, v) \mapsto uv$ from $(L^p + L^q)^2$ in $L^{p/2} + L^{q/2}$ is bounded.*

Lemma 2.4. *The functional F_0 is of class C^2 and it holds*

$$\langle F_0'(u), v\rangle = \int_{\mathbb{R}^N} \nabla u \nabla v - f'(u)v\,dx.$$

Moreover, the Nehari manifold $\mathcal{N}^0$ is of class C^1 and its tangent space is:

$$T_{\mathcal{N}^0}(u) = \bigg\{ v \in \mathcal{D}^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \nabla u \nabla v\,dx - \frac{1}{2}\int_{\mathbb{R}^N} f'(u)v - f''(u)uv\,dx = 0 \bigg\}.$$

Lemma 2.5. *If the sequence $\{u_n\}$ converges to u in L^p+L^q, then the sequence $\{\int_\Omega f'(u_n)u_n\,dx\}$ converges to $\int_\Omega f'(u)u\,dx$.*

Lemma 2.6. *We assume that the sequence $\{u_n\}$ converges to u_0 weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. We set $\psi_n = u_n - u_0$. Then we have:*

(a) $\int_{\mathbb{R}^N} f'(\psi_n)\psi_n\,dx = \int_{\mathbb{R}^N} f'(u_n)u_n\,dx - \int_{\mathbb{R}^N} f'(u_0)u_0\,dx + o(1)$,
(b) $\int_{\mathbb{R}^N} f(\psi_n)\,dx = \int_{\mathbb{R}^N} f(u_n)\,dx - \int_{\mathbb{R}^N} f(u_0)\,dx + o(1)$.

3. The splitting lemma

The aim of this section it to prove a splitting lemma which is the main tool for proving Theorems 1.1 and 1.2.

Lemma 3.1. *If V satisfies* (1.10) *and* (1.11) *then there exists a constant c depending on $\|V^-\|_{N/2}$ such that*

$$\int_{\mathbb{R}^N} |\nabla u|^2 + Vu^2 \geq c\|u\|_{\mathcal{D}^{1,2}} \quad \textit{for every } u \in \mathcal{D}^{1,2}(\mathbb{R}^n).$$

Proof. It follows at once by the Sobolev embedding theorem. $\square$

Lemma 3.2. *We have* $\inf_{u\in\mathcal{N}^V} \|u\|_{\mathcal{D}^{1,2}} > 0$.

Proof. Let $\{u_n\}$ be a minimizing sequence in $\mathcal{N}^V$. By contradiction, we suppose that u_n converges to 0. We set $t_n = \|u_n\|_{\mathcal{D}^{1,2}}$, hence we can write $u_n = t_n v_n$ where $\|v_n\|_{\mathcal{D}^{1,2}} = 1$. By Remark 2.2 the sequence $\{v_n\}$ is bounded in $L^p + L^q$. Since $u_n \in \mathcal{N}^V$ and $\{t_n\}$ converges to 0, we have

$$\begin{aligned} ct_n = \frac{c}{t_n}\|u_n\|^2_{\mathcal{D}^{1,2}} &\leq \frac{1}{t_n}\int_{\mathbb{R}^N} |\nabla u_n|^2 + Vu_n^2\,dx = \int_{\mathbb{R}^N} f'(t_n v_n)v_n\,dx \\ &\leq c_1 t_n^{q-1}\int_{\mathbb{R}^N\setminus\Gamma_{t_n v_n}} |v_n|^q\,dx + c_1 t_n^{p-1}\int_{\Gamma_{t_n v_n}} |v_n|^p\,dx \end{aligned}$$

$$\begin{aligned}
&\le c_1 t_n^{q-1}\int_{\mathbb{R}^N\setminus\Gamma_{t_n v_n}}|v_n|^q\,dx + c_1 t_n^{p-1}\int_{\Gamma_{v_n}}|v_n|^p\\
&\le c_1 t_n^{q-1}\int_{\mathbb{R}^N\setminus\Gamma_{v_n}}|v_n|^q\,dx\\
&\quad + c_1 t_n^{q-1}\int_{(\mathbb{R}^N\setminus\Gamma_{t_n v_n})\cap\Gamma_{v_n}}\frac{|v_n|^p}{t_n^{q-p}}\,dx + c_1 t_n^{p-1}\int_{\Gamma_{v_n}}|v_n|^p\,dx\\
&\le c_1 t_n^{q-1}\int_{\mathbb{R}^N\setminus\Gamma_{v_n}}|v_n|^q\,dx + 2c_1 t_n^{p-1}\int_{\Gamma_{v_n}}|v_n|^p\,dx.
\end{aligned}$$

Hence we get:

$$c\le c_1 t_n^{q-2}\int_{\mathbb{R}^N\setminus\Gamma_{v_n}}|v_n|^q\,dx + 2c_1 t_n^{p-2}\int_{\Gamma_{v_n}}|v_n|^p\,dx$$

and by Lemma 2.1(b) we get the contradiction. □

Lemma 3.3 (Splitting Lemma). *Let $\{u_n\}\subset\mathcal{N}^V$ be a sequence such that:*

$$\begin{aligned}
F_V(u_n) &\to c \quad \text{as } n\to\infty,\\
F'_V|_{\mathcal{N}^V}(u_n) &\to 0 \quad \text{in } (\mathcal{D}^{1,2}(\mathbb{R}^N))' \text{ as } n\to\infty.
\end{aligned}$$

Then there exist k sequences of points $\{y_n^j\}_{n\in\mathbb{N}}$ $(1\le j\le k)$ with $|y_n^j|\to\infty$ as $n\to\infty$, and $k+1$ sequences of functions $\{u_n^j\}_{n\in\mathbb{N}}$ $(0\le j\le k)$ such that, up to a subsequence:

(a) $u_n(x)=u_n^0(x)+\sum_{j=1}^k u_n^j(x-y_n^j)$,
(b) $u_n^0(x)\to u^0(x)$ *as* $n\to\infty$ *in* $\mathcal{D}^{1,2}(\mathbb{R}^N)$,
(c) $u_n^j(x)\to u^j(x)$ *as* $n\to\infty$ *in* $\mathcal{D}^{1,2}(\mathbb{R}^N)$,

where u^0 is a solution of (1.4) *and u^j $(1\le j\le k)$ is a solution of* (2.4). *Furthermore, when $n\to\infty$:*

$$\|u_n\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}\to\|u^0\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}+\sum_{j=1}^k\|u^j\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}$$

and

$$F_V(u_n)\to F_V(u_0)+\sum_{j=1}^k F_0(u^j).$$

Proof. *Step 1.* The sequence $\{u_n\}$ converges to u^0 weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ (up to a subsequence) and u^0 solves (1.4).

First we see that $\{u_n\}$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Indeed by (1.7), Remark 3.1 and the fact that $u_n\in\mathcal{N}^V$, we have:

$$\begin{aligned}
F_V(u_n) &\ge \frac12\int_{\mathbb{R}^N}(|\nabla u_n|^2+au_n^2)\,dx-\frac1\mu\int_{\mathbb{R}^N}f'(u_n)u_n\,dx\\
&=\left(\frac12-\frac1\mu\right)\int_{\mathbb{R}^N}(|\nabla u_n|^2+Vu_n^2)\,dx\ge c\|u_n\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}.
\end{aligned}\tag{3.1}$$

Since $\{F_V(u_n)\}$ converges, we get the boundness of $\{u_n\}$. Hence we can extract a subsequence $\{u_n\}$ (relabelled) which converges to u^0 weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. We verify that u^0 solves (1.4). We observe that, if $\{u_n\}$ is a Palais–Smale sequence for F_V restricted to the Nehari manifold $\mathcal{N}^V$, then it is also a Palais–Smale sequence for F_V on the whole $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Given $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, we have:

$$\lim_{n\to\infty} \langle F_V'(u_n), \varphi\rangle = \lim_{n\to\infty} \int_{\mathbb{R}^N} [\nabla u_n \nabla\varphi + V u_n \varphi - f'(u_n)\varphi]\, dx = 0. \tag{3.2}$$

By of Lemma 2.3(a), since $0 < \theta < 1$, we get

$$\int_{\mathbb{R}^N} [f'(u_n) - f'(u^0)]\varphi\, dx = \int_{\mathrm{supp}(\varphi)} f''(\theta u_n + (1-\theta)u^0)(u_n - u^0)\varphi\, dx \to 0$$

as $n \to \infty$, because $u_n \to u_0$ strongly in $L^p(\omega)$ for ω bounded subset of $\mathbb{R}^N$. Then

$$\int_{\mathbb{R}^N} \nabla u_n \nabla\varphi + V u_n\varphi - f'(u_n)\varphi\, dx \to \int_{\mathbb{R}^N} \nabla u^0 \nabla\varphi + V u^0\varphi - f'(u^0)\varphi\, dx \tag{3.3}$$

as $n \to \infty$. Hence u^0 solves (1.4) and $u^0 \in \mathcal{N}^V$. Now we set

$$\psi_n(x) = u_n(x) - u^0(x), \tag{3.4}$$

so $\psi_n \rightharpoonup 0$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

Step 2. The following equalities hold:

$$\|\psi_n\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = \|u_n\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} - \|u_0\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} + o(1), \tag{3.5}$$

$$\begin{aligned} F_0(\psi_n) &= F_0(u_n) - F_0(u^0) + o(1), \\ F_V(\psi_n) &= F_V(u_n) - F_V(u^0) + o(1). \end{aligned} \tag{3.6}$$

We show that

$$\int_{\mathbb{R}^N} V(x)\psi_n^2(x)\, dx \to 0 \quad \text{as } n \to \infty. \tag{3.7}$$

In fact, given $\varepsilon > 0$ we take $R > 0$ such that

$$\bigg[\int_{\mathbb{R}^N\setminus B_R} V^{N/2}(x)\, dx\bigg]^{2/N} < \varepsilon. \tag{3.8}$$

Thus we have

$$\begin{aligned} \int_{\mathbb{R}^N} V(x)\psi_n^2(x)\, dx &= \int_{B_R} V(x)\psi_n^2(x)\, dx + \int_{\mathbb{R}^N - B_R} V(x)\psi_n^2(x)\, dx \\ &\le \|V\|_{L^t(B_R)} \|\psi_n\|^2_{L^{2t'}(B_R)} + \|V\|_{L^{N/2}(\mathbb{R}^N - B_R)} \|\psi_n\|^2_{L^{2^*}}. \end{aligned} \tag{3.9}$$

By the fact that $\|\psi_n\|_{L^{2t'}(B_R)} \to 0$ because $2 < 2t' < 2^*$, by (3.8) and (3.9) we get (3.7). By (3.7), (3.5) and Lemma 2.6(a) we get the claim.

Step 3. Assume $\psi_n \not\to 0$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ (otherwise we have the claim). We show that there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ with $|y_n| \to \infty$ as $n \to \infty$ and $\psi_n(x + y_n) \rightharpoonup u^1(x)$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

Since $u_n, u^0 \in \mathcal{N}^V$, by (3.5) and by Lemma 2.6(a) we have:

$$
\begin{aligned}
(3.10)\quad \|\psi_n\|^2_{\mathcal{H}^{1,2}(\mathbb{R}^N)} + o(1) &= \|u_n\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} - \|u^0\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} + \|\psi_n\|^2_{L^2} \\
&= \int_{\mathbb{R}^N} f'(u_n)u_n\,dx - \int_{\mathbb{R}^N} f'(u^0)u^0\,dx + \int_{\mathbb{R}^N} V(u_n^2 - (u^0)^2)\,dx + \|\psi_n\|^2_{L^2} \\
&= \int_{\mathbb{R}^N} f'(\psi_n)\psi_n\,dx + o(1) + \|\psi_n\|^2_{L^2} \\
&\le c_1(\|\psi_n\|^p_{L^p(\Gamma_n)} + \|\psi_n\|^q_{L^q(\mathbb{R}^N - \Gamma_n)}) + \|\psi_n\|^2_{L^2(\mathbb{R}^N)} + o(1)
\end{aligned}
$$

because $\int_{\mathbb{R}^N} V(u_n^2 - (u^0)^2)\,dx \to 0$ (the proof is analog to (3.7)). Here $\Gamma_n = \{x : |\psi_n(x)| > 1\}$. Now we decompose $\mathbb{R}^N$ into N-dimensional hypercubes Q_i, having length L of the side. This length will be suitably chosen. We set:

$$
Q^+_{i,n} = Q_i \cap \Gamma_n, \quad Q^-_{i,n} = Q_i \cap (\mathbb{R}^N - \Gamma_n).
$$

Thus we have:

$$
\begin{aligned}
(3.11)\quad c_1[\|\psi_n\|^p_{L^p(\Gamma_n)} &+ \|\psi_n\|^q_{L^q(\mathbb{R}^N - \Gamma_n)}] + \|\psi_n\|^2_{L^2} \\
&\le c_1 \sum_i \Big[\|\psi_n\|^p_{L^p(Q^+_{i,n})} + \|\psi_n\|^q_{L^q(Q^-_{i,n})} + \|\psi_n\|^2_{L^2(Q_i)}\Big] \\
&\le c_1 \sum_i \Big[\|\psi_n\|^p_{L^p(Q^+_{i,n})} + \|\psi_n\|^2_{L^p(Q^-_{i,n})}\|\psi_n\|^{(p-2)q/p}_{L^q(Q^-_{i,n})} \\
&\qquad + \mathrm{L}^{N(p-2)/p}\|\psi_n\|^2_{L^p(Q_i)}\Big] \\
&\le c_1(d_n + \mathrm{L}^{N(p-2)/p})\|\psi_n\|^2_{\mathcal{H}^{1,2}(\mathbb{R}^N)}
\end{aligned}
$$

where

$$
d_n = \sup_i \Big\{ \max\Big[\|\psi_n\|^{p-2}_{L^p(Q^+_{i,n})}, \|\psi_n\|^{(p-2)q/q}_{L^q(Q^-_{i,n})}\Big]\Big\}.
$$

We choose L such that $c_1\mathrm{L}^{N(p-2)/q} < 1$, so by (3.10) and (3.11) we get $d_n \not\to 0$ when $n \to \infty$. So there exists $\alpha > 0$ and a sequence $\{i_n\} \subset \mathbb{N}$ such that the following inequality holds:

$$
(3.12)\qquad \alpha < \max\Big\{\|\psi_n\|^{p-2}_{L^p(Q^+_{i,n})}, \|\psi_n\|^{(p-2)q/p}_{L^q(Q^-_{i,n})}\Big\}.
$$

Now we call y_{i_n} the center of the hypercube Q_{i_n}. If $\{y_{i_n}\}$ were bounded, by passing to a subsequence, we should find that y_{i_m} would be in the same Q_j so they coincide. Since $\|\psi_n\|_{\mathcal{H}^{1,2}(Q_j)}$ is bounded, then (up to a subsequence) $\{\psi_n\}$

converges to ψ strongly in $L^p(Q_j)$ and weakly in $\mathcal{H}^{1,2}(Q_j)$. We have $\psi \neq 0$. Indeed if $\|\psi_n\|_{L^p(Q_j)} \to 0$, then

$$\|\psi_n\|_{L^p(Q_j^+)} \to 0 \quad \text{and} \quad \int_{Q_j^-} |\psi_n|^q \, dx \leq \int_{Q_j^-} |\psi_n|^p \, dx \to 0 \quad \text{as } n \to \infty \tag{3.13}$$

and (3.13) contradicts (3.12). But the fact that $\psi_n \to \psi \neq 0$ weakly in $\mathcal{H}^{1,2}(Q_j)$ contradicts the fact that $\psi_n \rightharpoonup 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Concluding $|y_{i_n}| \to \infty$. Now we call u^1 the weak limit in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ of the sequence $\{\psi_n(\,\cdot\, + y_{i_n})\}$. Arguing as before in the hypercube $\overline{Q}$ centered at the origin, we can conclude that $u^1 \neq 0$.

Step 4. u^1 is a weak solution of $-\Delta u^1 = f'(u^1)$.

First we prove that

$$\int_{\mathbb{R}^N} V(x)\psi_n(x)\varphi(x)\, dx \to 0 \quad \text{as } n \to \infty, \tag{3.14}$$

uniformly for $\|\varphi\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \leq c_5$. Indeed we have:

$$\begin{aligned}
&\int_{\mathbb{R}^N} V(x)\psi_n(x)\varphi(x)\, dx \\
&\quad = \int_{\mathbb{R}^N - B_R} V(x)\psi_n(x)\varphi(x)\, dx + \int_{B_R} V(x)\psi_n(x)\varphi(x)\, dx \\
&\quad \leq \|V\|_{L^t(B_R)}\|\psi_n\|_{L^{2t'}(B_R)}\|\varphi\|_{L^{2t'}(B_R)} \\
&\quad\quad + \|V\|_{L^{N/2}(\mathbb{R}^N - B_R)}\|\psi_n\|_{L^{2^*}}\|\varphi\|_{L^{2^*}} \\
&\quad \leq [\|V\|_{L^t}\|\psi_n\|_{L^{2t'}(B_R)}|B_R|^{(2^*-2+1)/2^*} \\
&\quad\quad + \|V\|_{L^{N/2}(\mathbb{R}^N - B_R)}\|\psi_n\|_{L^{2^*}}]\|\varphi\|_{L^{2^*}}.
\end{aligned} \tag{3.15}$$

Since $\|V\|_{L^{N/2}(\mathbb{R}^N - B_R)} \to 0$ as $R \to \infty$ and $\|\psi_n\|_{L^{2t'}(B_R)} \to 0$ as $n \to \infty$, by (3.15) we get (3.14).

Now we prove that for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} [\nabla\psi_n(x+y_n)\nabla\varphi(x) - f'(\psi_n(x+y_n))\varphi(x)]\, dx \to 0 \tag{3.16}$$

as $n \to \infty$. By (3.14) and Lemma 2.3(a) we have:

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla\psi_n(x+y_n)\nabla\varphi(x) - f'(\psi_n(x+y_n))\varphi(x)\, dx \\
&\quad = \int_{\mathbb{R}^N} \nabla\psi_n(x)\nabla\varphi(x-y_n) - f'(\psi_n(x))\varphi(x-y_n)\, dx \\
&\quad = \int_{\mathbb{R}^N} [f'(u_n) - f'(u^0) - f'(\psi_n)]\varphi(x-y_n)\, dx \\
&\quad\quad - \int_{\mathbb{R}^N} V(x)\psi_n(x)\varphi(x-y_n)\, dx + o(1) \\
&\quad = \int_{B_R} [f'(u^0+\psi_n) - f'(u^0)]\varphi(x-y_n)\, dx
\end{aligned} \tag{3.17}$$

$$
\begin{aligned}
&+ \int_{\mathbb{R}^N \setminus B_R} [f'(u^0 + \psi_n) - f'(\psi_n)]\varphi(x - y_n)\,dx \\
&- \int_{\mathbb{R}^N \setminus B_R} f'(u^0)\varphi(x - y_n)\,dx + \int_{B_R} f'(\psi_n)\varphi(x - y_n)\,dx + o(1) \\
\le{}& \|[f''(u^0 + \theta\psi_n) - f''(\theta\psi_n)]\varphi(\,\cdot\, - y_n)\|_{L^{p'}(\mathbb{R}^N)}\gamma_{n,R} \\
&+ \|[f''(\psi_n + \theta u^0) - f''(\theta u^0)]\varphi(\,\cdot\, - y_n)\|_{L^{p'} \cap L^{q'}} M_R + o(1),
\end{aligned}
$$

where $\gamma_{n,R} = \|\psi_n\|_{L^p(B_R)}$, $M_R = \|u^0\|_{L^p + L^q(\mathbb{R}^N \setminus B_R)}$ and $0 < \theta < 1$. Since $M_R \to 0$ as $R \to \infty$ and, given R, $\gamma_{n,R} \to 0$ as $n \to \infty$, by (3.17) we get (3.16). On the other hand, by Lemma 2.1(c), it is easy to see that:

$$
\int_{\mathbb{R}^N} \nabla\psi_n(x + y_n)\nabla\varphi(x) - f'(\psi_n(x + y_n))\varphi(x)\,dx \to \int_{\mathbb{R}^N} \nabla u^1 \nabla\varphi - f'(u^1)\varphi\,dx.
$$

So we get the claim.

Step 5. The conclusion.

By iterating this procedure, we obtain sequences $\{\psi_m^j(x) = \psi_n^{j-1}(x + y_n^{i-1}) - u^{i-1}(x)\}$ and sequences of points $\{y_n^i\}$ $(i \ge 2)$ such that $|y_n^i| \to \infty$ and $\psi_n^j(x + y_n^i) \rightharpoonup u^i(x)$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ (as $n \to \infty$) where $u^j \ne 0$ is a solution of (2.4). Furthermore, by induction:

$$
\begin{aligned}
\text{(3.18)} \qquad 0 &< \|\psi_n^j\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = \|\psi_n^{j-1}\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} - \|u^{j-1}\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} + o(1) \\
&= \|u_n\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} - \|u^0\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} - \sum_{i=1}^{j-1} \|u^i\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} + o(1),
\end{aligned}
$$

$$
\text{(3.19)} \qquad F_0(\psi_n^j) = F_0(\psi_n^{j-1}) - F_0(u^{j-1}) + o(1) = F_0(\psi^1) - \sum_{i=1}^{j-1} F_0(u^i) + o(1).
$$

By (3.7) and (3.6) we have $F_0(\psi_n^1) + o(1) = F_V(\psi_n^1) = F_V(u_n) - F_V(u^0) + o(1)$. Thus, by (3.19) we have:

$$
\text{(3.20)} \qquad F_0(\psi_n^j) = F_V(u_n) - F_V(u^0) + \sum_{i=1}^{j-1} F_0(u^i) + o(1).
$$

By Lemma 3.2 we have:

$$
\text{(3.21)} \qquad 0 < \inf_{v \in \mathcal{N}^V} \|v\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \le \|u^j\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}.
$$

By (3.16), (3.18) and (3.19) we get that the iteration must terminate at some index k. Finally,

- if $k = 0$, we have $u_m^0(x) = u_m(x)$,

- if $k > 0$, we have

$$\begin{aligned} u_n^k(x) &= \psi_n^k(x + y_n^k), \\ u_n^i(x) &= \psi_n^i(x + y_n^i) - \sum_{j=i+1}^{k} u_n^j(x - y_n^j), \quad 1 \le i \le k-1, \\ u_n^0(x) &= u_n(x) - \sum_{j=i}^{k} u_n^j(x - y_n^j). \end{aligned}$$

In this way we get the claim. □

4. The main result

Now we are ready to study the functional F_V on the manifold $\mathcal{N}^V$.

Lemma 4.1.

(a) *F_V is of class C^2;*
(b) *$\mathcal{N}^V(\mathbb{R}^N)$ is a C^1 manifold;*
(c) *for any given $u \in \mathcal{D}^{1,2} \setminus \{0\}$, there exists a unique real number $t_u^V > 0$ such that $ut_u^V \in \mathcal{N}^V$ and $F_V(t_u^V u)$ is the maximum for the function $t \mapsto F_V(tu)$, $t \ge 0$;*
(d) *the function $(V, u) \mapsto t_u^V$ defined on the set $\{V \in L^{N/2} : \|V\|_{N/2} < S\} \times \mathcal{D}^{1,2} \setminus \{0\}$ is of class C^1.*

Proof. (a) It is an easy generalization of Proposition 2.4.

(b) Since the functional F_V is of class C^2, by (f_1) we have for $u \in \mathcal{N}^V$

$$\begin{aligned} (4.1)\qquad &\int_{\mathbb{R}^N} 2|\nabla u|^2 + 2Vu^2 - f'(u)u - f''(u)u^2\,dx \\ &\quad= \int_{\mathbb{R}^N} |\nabla u|^2 + Vu^2 - f''(u)u^2\,dx = \int_{\mathbb{R}^N} f'(u)u - f''(u)u^2\,dx < 0. \end{aligned}$$

Given $u \neq 0$ we set, for $t \ge 0$,

$$g_u(t) = F_V(tu) = \int_{\mathbb{R}^N} \frac{t^2}{2}(|\nabla u|^2 + Vu^2) - f(tu)\,dx.$$

We have

$$\begin{aligned} g_u'(t) &= \int_{\mathbb{R}^N} t|\nabla u|^2 + Vtu^2 - uf'(tu)\,dx, \\ g_u''(t) &= \int_{\mathbb{R}^N} |\nabla u|^2 + Vu^2 - u^2 f''(tu)\,dx. \end{aligned}$$

By hypothesis (f_1), if $\phi_u'(\overline{t}) = 0$ we have

$$\overline{t}^2 \phi_u''(t) = \int_{\mathbb{R}^N} tuf'(tu) - \overline{t}^2 u^2 f''(tu)\,dx < 0$$

then $\bar{t}$ is a maximum point for g_u. Futhermore $0 = g_u(0) = g'_u(0)$ and $g''_u(0) > 0$ then 0 is a local minimum point for g_u. By (1.8), for $t \geq 1$, we have

$$\begin{aligned}
g_u(t) &\leq \int_{\mathbb{R}^N} \frac{t^2}{2}(|\nabla u|^2 + Vu^2)\,dx \\
&\quad - c_0 \int_{\{|tu|\leq 1\}} |tu|^q\,dx - c_0 \int_{\{|tu|>1\}} |tu|^p\,dx \\
&\leq \int_{\mathbb{R}^N} \frac{t^2}{2}(|\nabla u|^2 + Vu^2)\,dx - c_0 \int_{\{|tu|>1\}} |tu|^p\,dx \\
&\leq \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + Vu^2\,dx - c_0 t^p \int_{\{|u|>1\}} |u|^p\,dx.
\end{aligned} \tag{4.2}$$

The last quantity diverges negatively as $t \to \infty$ since $p > 2$ and the claim follows.

(d) We consider the following operator of class C^1:

$$K(t, V, u) = t \int_{\mathbb{R}^N} |\nabla u|^2 + Vu^2\,dx - \int_{\mathbb{R}^N} f'(tu)u\,dx \tag{4.3}$$

Here $t \in \mathbb{R}^+$, $V \in L^{N/2}$ with $\|V\|_{N/2} < S$ and $u \in \mathcal{D}^{1,2}$. If $K(t_0, V_0, u_0) = 0$ with $t_0 > 0$ and $u_0 \neq 0$, then $t_0 u_0 \in \mathcal{N}^{V_0}$ and, by (1.7) we have:

$$\begin{aligned}
K'_t(t_0, V_0, u_0) &= \int_{\mathbb{R}^N} |\nabla u_0|^2 + V_0 u_0^2\,dx - \int_{\mathbb{R}^N} f''(t_0 u_0) u_0^2\,dx \\
&= \int_{\mathbb{R}^N} \frac{f'(t_0 u_0)}{t_0} u_0 - f''(t_0 u_0) u_0^2\,dx < 0.
\end{aligned}$$

By the implicit function theorem there exists a C^1 function

$$(V, u) \mapsto t(V, u) = t_u^V$$

such that $ut_u^V \in \mathcal{N}^V$ and

$$\langle t'_V(\overline{V}, \overline{u}), V \rangle = -\frac{\bar{t} \int_{\mathbb{R}^N} V\overline{u}^2\,dx}{\int_{\mathbb{R}^N} f'(\bar{t}\overline{u})\overline{u}/\bar{t} - f''(\bar{t}\overline{u})\overline{u}^2\,dx} \tag{4.4}$$

where $\bar{t} = t(\overline{V}, \overline{u}) = t_{\overline{u}}^{\overline{V}}$. $\square$

LEMMA 4.2. *Let w be the ground state solution of* (2.4), *then*

(a) *there exist $t_1 > 0$, $t_2 > 0$, $R(V) > 0$ such that $t_1 \leq t_{w_y}^V \leq t_2$ for $|y| > R(V)$ where $t_{w_y}^V$ is defined in Lemma* 4.1.

(b) $t_{w_y}^V \to 1$ *as* $|y| \to \infty$.

PROOF. *Step 1.* We claim that, given a, there exist $t_2 > 0$ and $R(V) > 0$ such that

$$t_{w_y}^V \leq t_2 \quad \text{for } |y| > R(V).$$

First we observe that

$$g^V_{w_y}(t) \doteq F_V(tw_y) = \frac{t^2}{2}\|w_y\|^2_{\mathcal{D}^{1,2}} + t^2\int_{\mathbb{R}^N} Vw_y^2\,dx - \int_{\mathbb{R}^N} f(tw_y)\,dx \tag{4.5}$$
$$= \frac{t^2}{2}\|w\|^2_{\mathcal{D}^{1,2}} + t^2\int_{\mathbb{R}^N} Vw_y^2\,dx - \int_{\mathbb{R}^N} f(tw)\,dx$$
$$= g^0_w(t) + t^2\int_{\mathbb{R}^N} Vw_y^2\,dx.$$

Following the proof of Lemma 4.1 there exists $t_2 > 0$ such that $g^0_w(t_2) < 0$. Now we consider the last integral in the previous equation. We recall that, if $|y_n| \to \infty$ then w_{y_n} converges weakly to 0 in $\mathcal{D}^{1,2}$. Hence, for a fixed $R > 0$, we have:

$$\bigg|\int_{\mathbb{R}^N} Vw_{y_n}^2\,dx\bigg| \le \int_{B_R} |V|w_{y_n}^2\,dx + \int_{\mathbb{R}^N\setminus B_R} |V|w_{y_n}^2\,dx$$
$$\le \|V\|_{L^t(B_R)}\|w_{y_n}\|^2_{L^{2t'}(B_R)} + \|V\|_{L^{\frac{N}{2}}(\mathbb{R}^N\setminus B_R)}\|w_{y_n}\|^2_{L^{2^*}(\mathbb{R}^N)}.$$

Since $\|V\|_{L^{N/2}(\mathbb{R}^N\setminus B_R)} \to 0$ as $R \to \infty$ and $\|w_{y_n}\|_{L^{2t'}(B_R)} \to 0$ as $n \to \infty$ because $2t' < 2^*$, we get that:

$$\int_{\mathbb{R}^N} Vw_y^2\,dx \to 0 \quad \text{as } |y| \to \infty. \tag{4.6}$$

Hence, if R is big enough we have that $g^V_{w_y} < 0$ and, consequently, $t^V_{w_y} \le t_2$.

Step 2. Let be $M = \max\limits_{0\le t\le t_2} g^0_w(t) = g^0_w(t^*)$. By (4.6) there exists $\overline{R}(V) > 0$ such that, if $|y| > \overline{R}(V)$ then $|t_2\int_{\mathbb{R}^N} Vw_y^2\,dx| \le M/3$. Hence, by (4.5), we have that $g^V_{w_y}(t^V_{w_y}) \ge g^V_{w_y}(t^*) \ge g^0_w(t^*) - M/3 = 2/3M$. But $g^0_w(0) = 0$ and g^0_w is continuous, hence, there exists $t_1 > 0$ such that $g^0_w(t) < M/3$ if $t \in [0,t_1]$ and $|y| > \overline{R}(V)$. It follows that $g^V_{w_y}(t) < g^0_w(t) + M/3 < 2M/3$ if $t \in [0,t_1]$, hence $t^V_{w_y} > t_1$.

Step 3. We claim that $|t^V_{w_y} - 1| \to 0$ as $|y| \to \infty$. By Lemma 4.1 and recalling that, by definition of w it results $t^0_{w_y} = t^0_w = 1$, we have:

$$|t^V_{w_y} - 1| = |t^V_{w_y} - t^0_{w_y}| = \langle t'_V(\theta V, w_y), V\rangle = L(\theta V, w_y)\int_{\mathbb{R}^N} Vw_y^2\,dx$$

where

$$L(\theta V, w_y) = \frac{\bar t}{\int_{\mathbb{R}^N} f''(\bar t w_y)w_y^2 - f'(\bar t w_y)w_y/\bar t\,dx} = \frac{\bar t}{\int_{\mathbb{R}^N} f''(\bar t w)w^2 - f'(\bar t w)w/\bar t\,dx}$$

$\bar t = t^{\theta V}_{w_y}$ and $0 < \theta < 1$. By Steps 1 and 2 we get $t_1 \le \bar t \le t_2$ for every y such that $|y| > R(V)$ and for every $0 < \theta < 1$. Since the function

$$t \mapsto \int_{\mathbb{R}^N} f''(tw)w^2 - \frac{f'(tw)w}{t}\,dx$$

is continuous and strictly positive for $t > 0$, its minimum on $[t_1, t_2]$ is positive. Then $L(\theta V, w_y)$ is bounded and, by (4.6), we have the claim. $\square$

LEMMA 4.3. *For every $V \in L^{N/2}$, it holds $m_V \leq m$.*

PROOF. Since $w_y t^V_{w_y} \in \mathcal{N}^V$ we have

$$\begin{aligned}
|F_V(w_y t^V_{w_y}) - m| &= |F_V(w_y t^V_{w_y}) - F_0(w_y)| \\
&\leq |F_0(w_y t^V_{w_y}) - F_0(w_y)| + (t^V_{w_y})^2 \int_{\mathbb{R}^N} |V| w_y^2 \, dx \\
&\leq ((t^V_{w_y})^2 - 1)\|w\|_{\mathcal{D}^{1,2}} + \int_{\mathbb{R}^N} |f(w_y t^V_{w_y}) - f(w_y)| \, dx + (t^V_{w_y})^2 \int_{\mathbb{R}^N} |V| w_y^2 \, dx \\
&\leq ((t^V_{w_y})^2 - 1)\|w\|_{\mathcal{D}^{1,2}} \\
&\quad + |t^V_{w_y} - 1| \int_{\mathbb{R}^N} |f'((\theta t^V_{w_y} + 1 - \theta) w) w| \, dx + (t^V_{w_y})^2 \int_{\mathbb{R}^N} |V| w_y^2 \, dx
\end{aligned}$$

where $0 < \theta < 1$. Since $(\theta t^V_{w_y} + 1 - \theta) w$ is bounded in $L^p + L^q$, by of Lemmas 2.1(c), 4.2 and (4.6) we have:

$$|F_V(w_y t^V_{w_y}) - m| \to 0 \quad \text{as } |y| \to \infty$$

thus we have $m_V \leq m$. $\square$

LEMMA 4.4. *For every V satisfying* (1.10) *and* (1.11)*, and w minimizer of* (2.1)*, it holds:*

(a) *if $V(x) \leq 0$ for every $x \in \mathbb{R}^N$ and $V(x) < 0$ on a set of positive measure then $m_V < m$,*
(b) *if $\int_{\mathbb{R}^N} V(x) w(x)^2 \, dx < 0$ then $m_V < m$,*
(c) *if $V(x) \geq 0$ for every $x \in \mathbb{R}^N$ and $V(x) > 0$ on a set of positive measure then $m_V = m$.*

PROOF. (a), (b). By Lemma 4.1(b) there exists $t^V_w > 0$ such that $w t^V_w \in \mathcal{N}^V$. Then we have

$$\begin{aligned}
0 = K(t^V_w, V, w) &= t^V_w \int_{\mathbb{R}^N} |\nabla w|^2 + V w^2 \, dx - \int_{\mathbb{R}^N} f'(w t^V_w) w \, dx \\
&= \langle F'_0(w t^V_w), w \rangle + t^V_w \int_{\mathbb{R}^N} V w^2 \, dx.
\end{aligned}$$

Because $w > 0$ we have $\int_{\mathbb{R}^N} V w^2 \, dx < 0$ and $\langle F'_0(w t^V_w), w \rangle > 0$. Hence, by Lemma 4.1(b) we get $t^V_w < t^0_w = 1$. Let us observe that by (1.7) the function $s \mapsto \int_{\mathbb{R}^N} (1/2) f'(sw) sw - f(sw) \, dx$ is strictly increasing, then, remembering that $t^V_w w \in \mathcal{N}^V$, we have:

$$\begin{aligned}
F_V(t^V_w w) &= \int_{\mathbb{R}^N} \frac{1}{2} f'(t^V_w w) t^V_w w - f(t^V_w w) \, dx \\
&< \int_{\mathbb{R}^N} \frac{1}{2} f'(w) w - f(w) \, dx = F_0(w) = m.
\end{aligned} \tag{4.7}$$

It follows that $m_V < m$.

(c) By Lemma 4.1(b), for every $u \in \mathcal{N}^0$ there exist $t_u^V > 0$ such that $ut_u^V \in \mathcal{N}^V$. Then we have:

$$0 = K(t_u^V, V, u) = \langle F_0'(t_u^V u), u\rangle + t_u^V \int_{\mathbb{R}^N} V u^2\, dx.$$

Since $V \geq 0$ we have that $\int_{\mathbb{R}^N} V u^2 \geq 0$ and $\langle F_0'(t_u^V u), u\rangle \leq 0$. Hence, for Lemma 4.1(b) we get $t_u^V \geq 1$ and $t_u^V = 1$ if $\int_{\mathbb{R}^N} V u^2 = 0$. Since $t_u^V u \in \mathcal{N}^V$ and $u \in \mathcal{N}^0$, like in inequality (4.7), we have:

$$F_V(t_u^V u) = \int_{\mathbb{R}^N} \frac{1}{2} f'(t_u^V u) t_u^V u - f(t_u^V u)\, dx \geq \int_{\mathbb{R}^N} \frac{1}{2} f'(u)u - f(u)\, dx = F_0(u).$$

Hence $m \leq m_V$ and, by Lemma 4.3, we get $m_V = m$. □

Now we are ready to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. We suppose that there exists $v \in \mathcal{N}^V$ such that $m_V = F_V(v)$. We know that $\int_{\mathbb{R}^N} V(x)v(x)^2 \geq 0$. If $\int_{\mathbb{R}^N} V(x)v(x)^2 = 0$ then, since $V(x) \geq 0$, it will be $V(x)v(x) = 0$ almost everywhere in $\mathbb{R}^N$. Thus v solves the equation:

$$-\Delta v = f'(v) \quad \text{in } \mathbb{R}^N.$$

Without loss of generality we can take f even and, consequently we can assume $v \geq 0$. Hence $f'(v) > 0$ and, by the strong maximum principle, we get $v > 0$ in $\mathbb{R}^N$ and this gives a contradiction, since, where $V(x) > 0$ it must be $v = 0$. Thus, it results $\int_{\mathbb{R}^N} V(x)v(x)^2\, dx > 0$,

$$0 = K(1, V, v) = \langle F_0'(v), v\rangle + \int_{\mathbb{R}^N} V(x)v(x)^2\, dx$$

and, consequently $\langle F_0'(v), v\rangle < 0$. Then, by Lemma 4.1(b), we get $t_v^0 < t_v^V = 1$. Now we recall that, by (1.7), the function $s \mapsto \int_{\mathbb{R}^N} (1/2)f'(sv)sv - f(sv)\, dx$ is strictly increasing, so we have

$$F_0(vt_v^0) = \int_{\mathbb{R}^N} \frac{1}{2} f'(t_v^0 v)t_v^0 v - f(t_v^0 v)\, dx < \int_{\mathbb{R}^N} \frac{1}{2} f'(v)v - f(v)\, dx = F_V(v) = m_V$$

and we get a contradiction because, by Lemma 4.4(c), $m_V = m$. □

Proof of Theorem 1.2. The claim follows from the splitting lemma. Indeed, let $\{u_n\} \subset \mathcal{N}^V$ be a minimizing sequence for F_V. By Ekeland variational principle, we can suppose that $F_V'|_{\mathcal{N}^V}(u_n) \to 0$ in $\mathcal{D}^{1,2}$. Now, we can apply Lemma 3.3 to the sequence $\{u_n\}$ to obtain

$$u_n(x) = u_n^0 + \sum_{j=1}^{k} u_n^j(x - y_n^j)$$

with $\lim_{n\to\infty} |y_n^j| = \infty$, $\{u_n^0\}$ converging strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ to u_0 solution of (1.4) and $\{u_n^j\}$ converging strongly in $\mathcal{D}^{1,2}$ to u^j solution of (2.4) for every

$j \in \{1, \dots, k\}$. Hence, since $m_a < m$ it has to be $k = 1$, $u_n^1 = 0$ and u_0 is a minimum point for F_V. $\square$

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