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EXISTENCE RESULTS FOR FIRST AND SECOND ORDER SEMILINEAR IMPULSIVE DIFFERENTIAL INCLUSIONS

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Dedicated to the memory of Professor Olga A. Ladyzhenskaya

Abstract. In this paper we prove existence results for first and second order semilinear impulsive differential inclusions in Banach spaces.

1. Introduction

In this paper, we shall be concerned with the existence of mild solutions for first and second order impulsive semilinear differential inclusions in a real Banach space. First, we consider first order impulsive semilinear differential inclusions of the form,

$$y'(t) \in Ay(t) + F(t, y(t)), \quad \text{a.e. } t \in J,\ t \neq t_k,\ k = 1, \ldots, m, \tag{1.1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{1.2}$$

$$y(0) = a, \tag{1.3}$$

where $J = [0, b]$ and $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = b$ are fixed points of impulses, $F: J \times E \to \mathcal{P}(E)$ is a multivalued map ($\mathcal{P}(E)$ is the family of all nonempty subsets of E), $A: D(A) \subset E \to E$ is the infinitesimal generator of a family of semigroup $T(t)$ such that $t \geq 0$, $a \in E$, $I_k \in C(E, E)$ $(k = 1, \ldots, m)$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^+} y(t_k - h)$

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represent the right and left limits of $y(t)$ at $t = t_k$, respectively and E a real separable Banach space with norm $|\cdot|$.

In Section 4 we study first order impulsive functional differential inclusions of the form

$$y'(t) \in Ay(t) + F(t, y_t), \quad \text{a.e. } t \in J := [0, b],\ t \neq t_k,\ k = 1, \ldots, m, \tag{1.4}$$

$$y(t_k^+) = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{1.5}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{1.6}$$

where $F: J \times \mathcal{D} \to \mathcal{P}(E)$ is a multivalued map, $\mathcal{D} = \{\psi: [-r, 0] \to E \mid \psi$ is continuous everywhere except for a finite number of points s at which $\psi(s)$ and the right limit $\psi(s^+)$ exist and $\psi(s^-) = \psi(s)\}$, $\phi \in \mathcal{D}$, $(0 < r < \infty)$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = b$, $I_k \in C(E, E)$ $(k = 1, \ldots, m)$, E a real separable Banach space with norm $|\cdot|$ and $\mathcal{P}(E)$ is the family of all subsets of E.

For any continuous function y defined on the interval $[-r, b] \setminus \{t_1, \ldots, t_m\}$ and any $t \in J$, we denote by y_t the element of $\mathcal{D}$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0].$$

Here $y_t(\,\cdot\,)$ represents the history of the state from time $t - r$, up to the present time t. For $\psi \in \mathcal{D}$ the norm of ψ is defined by

$$\|\psi\|_{\mathcal{D}} = \sup\{|\psi(\theta)|,\ \theta \in [-r, 0]\}.$$

In Section 5 we consider the problem (1.1)–(1.3) where $A: D(A) \subset E \to E$ is a nondensely defined closed linear operator.

Finally, in Section 6 we consider second order impulsive semilinear differential inclusions.

IVPs (1.1)–(1.3) and (1.4)–(1.6) was studied in the literature under growth conditions on F. For example the IVP (1.1)–(1.3) was studied in [4] under the following growth assumption:

(H) $\|F(t, u)\| := \sup\{|v| : v \in F(t, u)\} \leq p(t)\psi(|u|)$ for almost all $t \in J$ and all $u \in E$, where $p \in L^1(J, \mathbb{R}_+)$ and $\psi: \mathbb{R}_+ \to (0, \infty)$ is continuous and increasing with
$$\int^{\infty} \frac{d\tau}{\psi(\tau)} = \infty.$$

Here by using the ideas in [1] we obtain new results if instead of (H) we assume the existence of a maximal solution to an appropriate problem. Our results in Sections 4 to 6 are new even if the problems have no impulses.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that are used throughout this paper.

Let (X,d) be a metric space. We use the notations:

$$\begin{aligned}
P(X) &= \{Y \in \mathcal{P}(X) : Y \neq \emptyset\},\\
P_{\mathrm{cl}}(X) &= \{Y \in P(X) : Y \text{ is closed}\},\\
P_{\mathrm{b}}(X) &= \{Y \in P(X) : Y \text{ is bounded}\},\\
P_{\mathrm{c}}(X) &= \{Y \in P(X) : Y \text{ is convex}\},\\
P_{\mathrm{cp}}(X) &= \{Y \in P(X) : Y \text{ is compact}\}.
\end{aligned}$$

A multivalued map $G: X \to \mathcal{P}(X)$ is *convex* (*closed*) *valued* if $G(x)$ is convex (closed) for all $x \in X$. G is *bounded on bounded sets* if $G(\mathcal{B}) = \bigcup_{x\in\mathcal{B}} G(x)$ is bounded in X for all $\mathcal{B} \in \mathcal{P}_b(X)$ (i.e. $\sup_{x\in\mathcal{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$).

G is called *upper semi-continuous* (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set $\mathcal{U}$ of X containing $G(x_0)$, there exists an open neighbourhood $\mathcal{V}$ of x_0 such that $G(\mathcal{V}) \subseteq \mathcal{U}$.

G is said to be *completely continuous* if $G(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in \mathcal{P}_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a *fixed point* if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $\operatorname{Fix} G$.

A multivalued map $N: J \to \mathcal{P}_{\mathrm{cl}}(E)$ is said to be *measurable*, if for every $y \in E$, the function $t \mapsto d(y, N(t)) = \inf\{|y - z| : z \in N(t)\}$ is measurable. For more details on multivalued maps see the books of Aubin and Cellina [3], Deimling [7], Górniewicz [10] and Hu and Papageorgiou [13].

Let E be a Banach space and $B(E)$ be the Banach space of linear bounded operators.

Definition 2.1. A semigroup of class (C_0) is a *one parameter family* $\{T(t) : t \geq 0\} \subset B(E)$ satisfying the conditions:

(a) $T(t) \circ T(s) = T(t+s)$, for $t, s \geq 0$,
(b) $T(0) = I$, (the identity operator in E),
(c) the map $t \to T(t)(x)$ is strongly continuous, for each $x \in E$, i.e.

$$\lim_{t\to 0} T(t)x = x \quad \text{for all } x \in E.$$

A semigroup of bounded linear operators $T(t)$, is uniformly continuous if

$$\lim_{t\to 0} \|T(t) - I\| = 0.$$

We note that if a semigroup $T(t)$ is class (C_0), then we have the growth condition:

- $\|T(t)\|_{B(E)} \leq Me^{\beta t}$, for $0 \leq t < \infty$, with some constants $M > 0$ and $\beta \in \mathbb{R}$.

If, in particular $M = 1$ and $\beta = 0$, i.e. $\|T(t)\|_{B(E)} \leq 1$, for $t \geq 0$, then the semigroup $T(t)$ is called a *contraction semigroup* (C_0).

DEFINITION 2.2. Let $T(t)$ be a semigroup of class (C_0) defined on E. The *infinitesimal generator* A of $T(t)$ is the linear operator defined by

$$A(x) = \lim_{h \to 0} \frac{T(h)(x) - x}{h}, \quad \text{for } x \in D(A),$$

where

$$D(A) = \left\{ x \in E : \lim_{h \to 0} \frac{T(h)(x) - x}{h} \ \text{ exists in } E \right\}.$$

PROPOSITION 2.3. *The infinitesimal generator A is a closed linear and densely defined operator in E. If $x \in D(A)$, then $T(t)(x)$ is a C^1-map and*

$$\frac{d}{dt} T(t)(x) = A(T(t)(x)) = T(t)(A(x)) \quad \text{on } [0, \infty).$$

It is well known ([17]) that the operator A generates a C_0 semigroup if A satisfies

(i) $\overline{D(A)} = E$, (D means domain),
(ii) the Hille–Yosida condition that is, there exists $M \geq 0$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$, $\sup\{(\lambda I - \omega)^n |(\lambda I - A)^{-n}| : \lambda > \omega,\ n \in \mathbb{N}\} \leq M$,

where $\rho(A)$ is the resolvent operator set of A and I is the identity operator.

DEFINITION 2.4. The multivalued map $F: J \times E \to \mathcal{P}(E)$ is said to be L^1-*Carathéodory* if:

(a) $t \mapsto F(t, u)$ is measurable for each $u \in E$;
(b) $u \mapsto F(t, u)$ is upper semicontinuous on E for almost all $t \in J$;
(c) For each $\rho > 0$, there exists $h_\rho \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, u)\| := \sup\{|v| : v \in F(t, u)\} \leq h_\rho(t) \quad \text{for all } |u| \leq \rho \text{ and for a.e. } t \in J.$$

3. First order semilinear impulsive differential inclusions

In order to define the solutions of the above problems, we shall consider the spaces $PC(J, E) = \{y: J \to E \mid y(t)$ is continuous everywhere except for some t_k at which $y(t_k^-)$ and $y(t_k^+)$, $k = 1, \ldots, m$ exist and $y(t_k^-) = y(t_k)\}$ and $PC^1(J, E) = \{y: J \to E \mid y(t)$ is continuously differentiable everywhere except for some t_k at which $y'(t_k^-)$ and $y'(t_k^+)$, $k = 1, \ldots, m$ exist and $y'(t_k^-) = y'(t_k)\}$.

Evidently, $PC(J,E)$ is a Banach space with the norm

$$\|y\|_{PC} = \sup\{|y(t)| : t \in J\}.$$

It is also clear that $PC^1(J,E)$ is a Banach space with the norm

$$\|y\|_{PC^1} = \max\{\|y\|_{PC},\ \|y'\|_{PC}\}.$$

We study the existence of solutions for problem (1.1)–(1.3) when the right hand side has convex or nonconvex values. We assume first that $F\colon J\times E \to P(E)$ is a compact and convex valued multivalued map.

Let us start by defining what we mean by a mild solution of problem (1.1)–(1.3). We denote by $J_0 = [0,t_1]$ and for $k = 1,\dots,m$, $J_k = (t_k, t_{k+1}]$.

Definition 3.1. A function $y \in PC(J,E)] \cap AC(J_k,E)$, $0 \le k \le m$, is said to be a *mild solution of* (1.1)–(1.3) if there exist functions $v_k \in L^1(J_k,E)$ such that $v_k(t) \in F(t,y(t))$ a.e. on J_k, $0 \le k \le m$, and

$$y(t) = \begin{cases} T(t)a + \displaystyle\int_0^t T(t-s)v_0(s)\,ds & \text{if } t \in J_0, \\ T(t-t_k)[I_k(y(t_k^-)) + y(t_k^-)] & \\ \qquad + \displaystyle\int_{t_k}^t T(t-s)v_k(s)\,ds & \text{if } t \in J_k,\ \ k = 1,\dots,m. \end{cases}$$

For the multivalued map F and for each $y \in C(J_k,E)$, $0 \le k \le m$, we define $S_{F,y,k}$ by

$$S_{F,y,k} = \{v \in L^1(J_k,E) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in J_k\}.$$

For convenience we write $S_{F,y}$ for $S_{F,y,0}$.

Our first existence result for the IVP (1.1)–(1.3) is the following.

Theorem 3.1. *Assume that:*

(a) *$F\colon J \times E \to \mathcal{P}(E)$ is a L^1-Carathéodory multivalued map;*

(b) *there exist a L^1-Carathéodory function $g\colon J \times [0,\infty) \to [0,\infty)$ such that*

$$\|F(t,u)\| := \sup\{|v| : v \in F(t,u)\} \le g(t,|u|)$$

for almost all $t \in J$ and all $u \in E$;

(c) *$g(t,x)$ is nondecreasing in x for a.e. $t \in J$;*

(d) *the problem*

$$\begin{aligned} v'(t) &= Mg(t,v(t)), \quad \text{a.e. } t \in J_k, \\ v(t_k^+) &= MN_k, \end{aligned}$$

where $t_0^+ = 0$, $N_0 = |a|$, M is as in (e), *$N_k = \sup\{|I_k(x) + x| : x \in [-r_{k-1}(t_k^-), r_{k-1}(t_k^-)],\ k \in \{1,\dots,m\}\}$ has a maximal solution $r_k(t)$, $k = 0,\dots,m$;*

(e) *$A\colon D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$, $t \geq 0$ which is compact for $t > 0$, and there exists a constant $M \geq 1$ such that $\|T(t)\|_{B(E)} \leq M$ for all $t \geq 0$.*

Then the IVP (1.1)–(1.3) *has at least one mild solution.*

Proof. The proof is given in several steps.

Step 1. Consider the problem (1.1)–(1.3) on $J_0 := [0, t_1]$

$$y'(t) \in Ay(t) + F(t, y(t)), \quad \text{a.e. } t \in J_0, \tag{3.1}$$

$$y(0) = a. \tag{3.2}$$

We transform this problem into a fixed point problem. A solution to (3.1)–(3.2) is a fixed point of the operator $G_0\colon C(J_0, E) \to \mathcal{P}(C(J_0, E))$ defined by

$$G_0(y) := \left\{ h \in C(J_0, E) : h(t) = T(t)a + \int_0^t T(t-s)v(s)\,ds,\ v \in S_{F,y} \right\}.$$

We shall show that G_0 is a completely continuous multivalued map, u.s.c. with convex values. The proof will be given in several steps.

Claim 1. *$G_0(y)$ is convex for each $y \in C(J_0, E)$.*

This is obvious, since F has convex values.

Claim 2. *G_0 maps bounded sets into bounded sets in $C(J_0, E)$.*

Indeed, it is enough to show that there exists a positive constant ℓ such that for each $h \in G_0(y)$, $y \in B_q = \{y \in C(J_0, E) : \|y\| = \sup_{t \in J_0} |y(t)| \leq q\}$ one has $\|h\| \leq \ell$. If $h \in G_0(y)$, then there exists $v \in S_{F,y}$ such that for each $t \in J_0$ we have

$$h(t) = T(t)a + \int_0^t T(t-s)v(s)\,ds.$$

Thus, for each $t \in J_0$, we get

$$|h(t)| \leq M|a| + M \int_0^t |v(s)|\,ds \leq M|a| + M\|h_q\|_{L^1},$$

here h_q is chosen as in Definition 2.4. Then for each $h \in G_0(B_q)$ we have

$$\|h\| \leq M|a| + M\|h_q\|_{L^1} := \ell.$$

Claim 3. *G_0 sends bounded sets in $C(J_0, E)$ into equicontinuous sets.*

We consider B_q as in Claim 2 and let $h \in G_0(y)$ for $y \in B_q$. Let $\varepsilon > 0$ be given. Now let $\tau_1, \tau_2 \in J_0$ with $\tau_2 > \tau_1$. We consider two cases $\tau_1 > \varepsilon$ and $\tau_1 \leq \varepsilon$.

Case 1. If $\tau_1 > \varepsilon$ then

$$
\begin{aligned}
|h(\tau_2) - h(\tau_1)| \le & |T(\tau_2)a - T(\tau_1)a| + \int_0^{\tau_1-\varepsilon} |T(\tau_2 - s) - T(\tau_1 - s)||v(s)|\,ds \\
& + \int_{\tau_1-\varepsilon}^{\tau_1} |T(\tau_2 - s) - T(\tau_1 - s)||v(s)|\,ds \\
& + \int_{\tau_1}^{\tau_2} |T(\tau_2 - s)||v(s)|\,ds \\
\le & |T(\tau_2)a - T(\tau_1)a| + M\|T(\tau_2 - \tau_1 + \varepsilon) \\
& - T(\varepsilon)\|_{B(E)} \int_0^{\tau_1-\varepsilon} h_q(s)\,ds \\
& + 2M \int_{\tau_1-\varepsilon}^{\tau_1} h_q(s)\,ds + M \int_{\tau_1}^{\tau_2} h_q(s)\,ds
\end{aligned}
$$

where we have used the semigroup identities

$$T(\tau_2 - s) = T(\tau_2 - \tau_1 + \varepsilon)T(\tau_1 - s - \varepsilon), \quad T(\tau_1 - s) = T(\tau_1 - s - \varepsilon)T(\varepsilon).$$

Case 2. Let $\tau_1 \le \varepsilon$. For $\tau_2 - \tau_1 < \varepsilon$ we get

$$
\begin{aligned}
|h(\tau_2) - h(\tau_1)| \le & |T(\tau_2)a - T(\tau_1)a| \\
& + \int_0^{\tau_2} |T(\tau_2 - s)|h_q(s)\,ds + \int_0^{\tau_1} |T(\tau_2 - s)|h_q(s)\,ds \\
\le & |T(\tau_2)a - T(\tau_1)a| + M \int_0^{2\varepsilon} h_q(s)\,ds + M \int_0^{\varepsilon} h_q(s)\,ds.
\end{aligned}
$$

Note equicontinuity follows since (i). $T(t)$, $t \ge 0$ is a strongly continuous semigroup and (ii). $T(t)$ is compact for $t > 0$ (so $T(t)$ is continuous in the uniform operator topology for $t > 0$).

Let $0 < t \le t_1$ be fixed and let ε be a real number satisfying $0 < \varepsilon < t$. For $y \in B_q$ and $v \in S_{F,y}$ we define

$$
\begin{aligned}
h_\varepsilon(t) &= T(t)a + \int_0^{t-\varepsilon} T(t-s)v(s)\,ds \\
&= T(t)a + T(\varepsilon) \int_0^{t-\varepsilon} T(t-s-\varepsilon)v(s)\,ds.
\end{aligned}
$$

Note

$$\left\{ \int_0^{t-\varepsilon} T(t-s-\varepsilon)v(s)\,ds : y \in B_q \text{ and } v \in S_{F,y} \right\}$$

is a bounded set since

$$\left| \int_0^{t-\varepsilon} T(t-s-\varepsilon)v(s)\,ds \right| \le M \int_0^{t-\varepsilon} h_q(s)\,ds$$

and now since $T(t)$ is a compact operator for $t > 0$, the set $Y_\varepsilon(t) = \{h_\varepsilon(t) : y \in B_q$ and $v \in S_{F,y}\}$ is relatively compact in E for every ε, $0 < \varepsilon < t$. Moreover, for $h = h_0$ we have

$$|h(t) - h_\varepsilon(t)| \leq M \int_{t-\varepsilon}^{t} h_q(s)\, ds.$$

Therefore, the set $Y(t) = \{h(t) : y \in B_q$ and $v \in S_{F,y}\}$ is totally bounded. Hence $Y(t)$ is relatively compact in E.

As a consequence of Claims 2, 3 and the Arzelá–Ascoli theorem we can conclude that $G_0: C(J_0, E) \to \mathcal{P}(C(J_0, E))$ is completely continuous.

Claim 4. *G_0 has closed graph.*

Let $y_n \to y_*$, $h_n \in G_0(y_n)$ and $h_n \to h_*$. We shall prove that $h_* \in G_0(y_*)$. Now $h_n \in G_0(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that

$$h_n(t) = T(t)a + \int_0^t T(t-s)v_n(s)\, ds, \quad t \in J_0.$$

We must prove that there exists $v_* \in S_{F,y_*}$ such that

$$h_*(t) = T(t)a + \int_0^t T(t-s)v_*(s)\, ds, \quad t \in J_0.$$

Consider the linear continuous operator $\Gamma: L^1(J_0, E) \to C(J_0, E)$ defined by

$$(\Gamma v)(t) = \int_0^t T(t-s)v(s)\, ds.$$

We have $\|(h_n - T(t)a) - (h_* - T(t)a)\| \to 0$ as $n \to \infty$. It follows that $\Gamma \circ S_F$ is a closed graph operator ([16]). Moreover, we have $h_n(t) - T(t)a \in \Gamma(S_{F,y_n})$. Since $y_n \to y_*$, it follows that, for some $v_* \in S_{F,y_*}$,

$$h_*(t) = T(t)a + \int_0^t T(t-s)v_*(s)\, ds, \quad t \in J_0.$$

Claim 5. *The set $\mathcal{M} := \{y \in C(J_0, E) : \lambda y \in G_0(y)$ for some $\lambda > 1\}$ is bounded.*

Let $y \in \mathcal{M}$ be such that $\lambda y \in G_0(y)$ for some $\lambda > 1$. Then there exists $v \in S_{F,y}$ such that

$$y(t) = \lambda^{-1}T(t)a + \lambda^{-1}\int_0^t T(t-s)v(s)\, ds, \quad t \in J_0.$$

This implies by our assumptions that for each $t \in J_0$ we have

$$|y(t)| \leq M|a| + M\int_0^t g(s, |y(s)|)\, ds.$$

Let us take the right-hand side of the above inequality as $v_0(t)$, then we have

$$\begin{gathered} v_0(0) = M|a|, \quad |y(t)| \leq v_0(t), \quad t \in J_0, \\ v_0'(t) = Mg(t, |y(t)|), \quad t \in J_0. \end{gathered}$$

Using the nondecreasing character of g (see Theorem 3.2(d)) we get

$$v_0'(t) \leq Mg(t, v_0(t)), \quad t \in J_0.$$

This implies that ([15, Theorem 1.10.2]) $v_0(t) \leq r_0(t)$ for $t \in J_0$, and hence $|y(t)| \leq b_0' = \sup_{t\in[0,t_1]} r_0(t)$, $t \in J_0$ where b_0' depends only on t_1 and on the function r_0. This shows that $\mathcal{M}$ is bounded.

As a consequence of the Leray–Schauder Alternative for Kakutani maps [11] we deduce that G_0 has a fixed point which is a solution of (3.1)–(3.2). Denote this solution by y_0.

Step 2. Consider now the following problem on $J_1 := (t_1, t_2]$

$$y'(t) \in Ay(t) + F(t, y), \quad \text{a.e. } t \in J_1, \tag{3.3}$$

$$y(t_1^+) = I_1(y_0(t_1^-)) + y_0(t_1^-). \tag{3.4}$$

A solution to (3.3)–(3.4) is a fixed point of the operator

$$G_1: C(J_1, E) \to \mathcal{P}(C(J_1, E))$$

defined by

$$\begin{aligned} G_1(y) := \bigg\{ & h \in C(J_1, E) : \\ & h(t) = T(t - t_1)[I_1(y_0(t_1^-)) + y_0(t_1^-)] + \int_{t_1}^{t} T(t - s)v(s)\,ds,\ v \in S_{F,y,1} \bigg\}. \end{aligned}$$

As in Step 1 we can easily show that G_1 has convex values, is completely continuous and upper semicontinuous. It suffices to show that the set

$$\mathcal{M} := \{y \in C(J_1, E) : \lambda y \in G_1(y) \text{ for some } \lambda > 1\}$$

is bounded. Let $y \in \mathcal{M}$, then $\lambda y \in G_1(y)$ for some $\lambda > 1$. Thus there exists $v \in S_{F,y,1}$ such that

$$y(t) = \lambda^{-1}T(t - t_1)[I_1(y_0(t_1^-)) + y_0(t_1^-)] + \lambda^{-1} \int_{t_1}^{t} T(t - s)v(s)ds, \quad t \in J_1.$$

We note that $|y(t_1^+)| \leq \sup\{|I_1(x) + x| : x \in [-r_0(t_1^-), r_0(t_1^-)]\} = N_1$. Thus, for each $t \in J_1$, we have

$$|y(t)| \leq MN_1 + M \int_{t_1}^{t} g(s, |y(s)|)\,ds.$$

Let us take the right-hand side of the above inequality as $v_1(t)$, then we have

$$v_1(t_1) = MN_1, \quad |y(t)| \leq v_1(t), \quad t \in J_1,$$
$$v_1'(t) = Mg(t, |y(t)|), \quad t \in J_1.$$

Using the nondecreasing character of g (see Theorem 3.2(c)) we get

$$v_1'(t) \leq Mg(t, v_1(t)), \quad t \in J_1.$$

This implies that ([15, Theorem 1.10.2]) $v_1(t) \leq r_1(t)$ for $t \in J_1$, and hence $|y(t)| \leq b_1' = \sup_{t \in J_1} r_1(t)$, $t \in J_1$ where b_1' depends only on b and on the function r_1. This shows that $\mathcal{M}$ is bounded.

As a consequence of the Leray–Schauder Alternative for Kakutani maps [11] we deduce that G_1 has a fixed point which is a solution of (3.3)–(3.4). Denote this solution by y_1.

Step 3. Continue this process and construct solutions $y_k \in C(J_k, E)$ for $k = 2, \ldots, m$ to

$$y'(t) \in Ay(t) + F(t, y(t)), \quad \text{a.e. } t \in J_k, \tag{3.5}$$
$$y(t_k^+) = I_k(y_{k-1}(t_k^-)) + y_{k-1}(t_k^-). \tag{3.6}$$

Then

$$y(t) = \begin{cases} y_0(t) & \text{if } t \in [0, t_1], \\ y_1(t) & \text{if } t \in (t_1, t_2], \\ \dots\dots\dots\dots\dots\dots\dots\dots & \\ y_{m-1}(t) & \text{if } t \in (t_{m-1}, t_m], \\ y_m(t), & \text{if } t \in (t_m, b], \end{cases}$$

is a mild solution of (1.1)–(1.3). □

Next, we study the case where F is not necessarily convex valued. Our approach here is based on the Leray–Schauder Alternative for single valued maps combined with a selection theorem due to Bressan and Colombo [5] for lower semicontinuous multivalued operators with decomposable values.

Theorem 3.3. *Suppose that:*

(a) $F\colon [0, b] \times E \to \mathcal{P}(E)$ *is a nonempty, compact-valued, multivalued map such that:*
   (a1) $(t, u) \mapsto F(t, u)$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable;*
   (a2) $u \mapsto F(t, u)$ *is lower semi-continuous for a.e.* $t \in [0, b]$*;*

(b) *for each* $\rho > 0$*, there exists a function* $h_\rho \in L^1([0, b], \mathbb{R}^+)$ *such that*

$$\|F(t, u)\| = \sup\{|v| : v \in F(t, u)\} \leq h_\rho(t) \quad \textit{for a.e. } t \in [0, b]$$

and for $u \in E$ *with* $|u| \leq \rho$.

In addition suppose (b)–(e) *of Theorem* 3.2 *are satisfied. Then the impulsive initial value problem* (1.1)–(1.3) *has at least one solution.*

PROOF. Assumptions (a) and (b) imply that F is of lower semi-continuous type. Then there exists ([5]) a continuous function $f: PC(J,E) \to L^1([0,b],E)$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in PC(J,E)$, where $\mathcal{F}$ is the Nemitsky operator defined by

$$\mathcal{F}(y) = \{w \in L^1(J,E) : w(t) \in F(t,y(t)) \text{ for a.e. } t \in J\}.$$

Consider the problem

$$y'(t) - Ay(t) = f(y)(t), \quad t \in [0,b],\ t \neq t_k,\ k = 1,\dots,m, \tag{3.7}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \tag{3.8}$$

$$y(0) = a. \tag{3.9}$$

It is obvious that if $y \in PC(J,E)$ is a solution of the problem (3.7)–(3.9), then y is a solution to the problem (1.1)–(1.3).

Consider first the problem (3.7)–(3.9) on $J_0 = [0,t_1]$

$$y'(t) - Ay(t) = f(y)(t), \quad t \in J_0, \tag{3.10}$$

$$y(0) = a. \tag{3.11}$$

Transform the problem (3.10)–(3.11) into a fixed point problem considering the operator $G_0: C(J_0,E) \to C(J_0,E)$ defined by:

$$G_0(y)(t) := T(t)a + \int_0^t T(t-s)f(y)\,ds.$$

We prove that $G_0: C(J_0,E) \to C(J_0,E)$ *is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J_0,E)$. Then there is an integer q such that $\|y_n\| \le q$ for all $n \in \mathbb{N}$ and $\|y\| \le q$, so $y_n \in B_q$ and $y \in B_q$. We have then by the dominated convergence theorem

$$\|G_0(y_n) - G_0(y)\| \le M \sup_{t \in J_0} \bigg[\int_0^t |f(y_n) - f(y)|\,ds\bigg] \to 0.$$

Thus G_0 is continuous. Next we prove that G_0 is completely continuous by proving, as in Theorem 3.2, that G_0 *maps bounded sets into bounded sets in* Ω and G_0 *maps bounded sets into equicontinuous sets of* Ω.

Finally, as in Theorem 3.2 we can show that the set

$$\mathcal{E}(G_0) := \{y \in C(J_0,E) : y = \lambda G_0(y) \text{ for some } 0 < \lambda < 1\}$$

is bounded. As a consequence of the Leray–Schauder Alternative for single valued maps we deduce that G_0 has a fixed point y which is a solution to problem (3.10)–(3.11). Denote this solution by y_0. Continue by considering the problem in the interval $J_1 = (t_1, t_2]$ and so on as in Theorem 3.2. We omit the details. □

4. First order semilinear impulsive functional differential inclusions

In this section we consider the problem (1.4)–(1.6). We consider the spaces

$$\begin{aligned} PC([-r,b],E) = \{y\colon [-r,b] \to E : {} & y(t) \text{ is continuous everywhere} \\ & \text{exept for some } t_k \text{ at which } y(t_k^-) \text{ and } y(t_k^+), \\ & k = 1,\dots,m \text{ exist and } y(t_k^-) = y(t_k)\}, \\ PC^1([0,b],E) = \{y\colon [0,b] \to E : {} & y(t) \text{ is continuously differentiable everywhere} \\ & \text{except for some } t_k \text{ at which } y'(t_k^-) \text{ and } y'(t_k^+), \\ & k = 1,\dots,m \text{ exist and } y'(t_k^-) = y'(t_k)\}. \end{aligned}$$

Let $Z = PC([-r,b],E) \cap PC^1([0,b],E)$. Obviously, for any $t \in [0,b]$ and $y \in Z$, we have $y_t \in \mathcal{D}$ and $PC([-r,b],E)$ and Z are Banach spaces with the norms

$$\|y\| = \sup\{|y(t)| : t \in [-r,b]\} \quad \text{and} \quad \|y\|_Z = \|y\| + \|y'\|,$$

where

$$\|y'\| = \sup\{|y'(t)| : t \in [0,b]\}.$$

Definition 4.1. The multivalued map $F\colon J \times \mathcal{D} \to \mathcal{P}(E)$ is said to be *L^1-Carathéodory* if:

(a) $t \mapsto F(t,u)$ is measurable for each $u \in \mathcal{D}$;
(b) $u \mapsto F(t,u)$ is upper semicontinuous on E for almost all $t \in J$;
(c) for each $\rho > 0$, there exists $h_\rho \in L^1(J,\mathbb{R}_+)$ such that

$$\|F(t,u)\| := \sup\{|v| : v \in F(t,u)\} \le h_\rho(t) \text{ for all } \|u\|_{\mathcal{D}} \le \rho \text{ and for a.e. } t \in J.$$

Definition 4.2. A function $y \in PC([-r,b],E)] \cap AC^1(J_k,E)$, $0 \le k \le m$, is said to be a *mild solution of* (1.4)–(1.6) if there exist functions $v_k \in L^1(J_k,E)$ such that $v_k(t) \in F(t,y_t)$ a.e. on J_k, $0 \le k \le m$, and

$$y(t) = \begin{cases} \phi(t) & \text{for } t \in [-r,0], \\ T(t)\phi(0) + \displaystyle\int_0^t T(t-s)v_0(s)\,ds & \text{for } t \in [0,t_1], \\ T(t-t_k)I_k(y(t_k^-)) + \displaystyle\int_{t_k}^t T(t-s)v_k(s)\,ds & \text{for } t \in J_k,\ k = 1,\dots,m. \end{cases}$$

We are now in a position to state and prove our existence results for the IVP (1.4)–(1.6).

Theorem 4.3. *Suppose that:*

(a) $F\colon J \times \mathcal{D} \to \mathcal{P}(E)$ *is a* L^1*-Carathéodory multivalued map;*

(b) *there exist a* L^1*-Carathéodory function* $g\colon J \times [0,\infty) \to [0,\infty)$ *such that*

$$\|F(t,u)\| := \sup\{|v| : v \in F(t,u)\} \le g(t, \|u\|_{\mathcal{D}})$$

for almost all $t \in J$ *and all* $u \in \mathcal{D}$*;*

(c) *the problem*

$$\begin{aligned} v'(t) &= Mg(t, v(t)), \quad a.e.\ t \in J_k, \\ v(t_k^+) &= MN_k + Q_k, \end{aligned}$$

where $t_0^+ = 0$, $N_0 = \|\phi\|_{\mathcal{D}}$, $Q_0 = 0$, $Q_k = \max\{\|\phi\|_{\mathcal{D}}, \sup_{t\in J_0} |r_0(t)|, \dots, \sup_{t\in J_{k-1}} |r_{k-1}(t)|\}$, $k \in \{1,\dots,m\}$, $N_k = \sup\{|I_k(x)| : x \in [-r_{k-1}(t_k^-), r_{k-1}(t_k^-)],\ k \in \{1,\dots,m\}\}$, *has a maximal solution* $r_k(t)$, $k = 0,\dots,m$.

In addition assume that conditions (c) *and* (e) *of Theorem* 3.2 *are satisfied. Then the* IVP (1.4)–(1.6) *has at least one mild solution.*

Proof. The proof is given in several steps.

Step 1. Consider the problem (1.4)–(1.6) on $[-r, t_1]$

$$y'(t) \in Ay(t) + F(t, y_t), \quad t \in J_0 = [0, t_1], \tag{4.1}$$

$$y(t) = \phi(t), \quad t \in [-r, 0]. \tag{4.2}$$

We transform this problem into a fixed point problem. A mild solution to (4.1)–(4.2) is a fixed point of the operator $G_0\colon C([-r,t_1],E) \to \mathcal{P}(C([-r,t_1],E))$ defined by

$$G_0(y) := \left\{ h \in C([-r,t_1],E) : h(t) = \begin{cases} \phi(t) & \text{if } t \in [-r,0], \\ T(t)\phi(0) + \displaystyle\int_0^t T(t-s)v(s)\,ds & \text{if } t \in [0,t_1], \end{cases} \right\}$$

where $v \in S_{F,y} = \{v \in L^1([0,t_1],E) : v(t) \in F(t,y_t) \text{ for a.e. } t \in [0,t_1]\}$.

Claim 1. *$G_0(y)$ is convex for each $y \in C([-r,t_1],E)$.*

This claim is obvious, since F has convex values.

Claim 2. *G_0 sends bounded sets into bounded sets in $C([-r,t_1],E)$.*

Let $B_q := \{y \in C([-r,t_1],E) : \|y\| = \sup_{t\in[-r,t_1]} |y(t)| \le q\}$ be a bounded set in $C([-r,t_1],E)$ and $y \in B_q$, then for each $h \in G_0(y)$ there exists $v \in S_{F,y}$ such that

$$h(t) = T(t)\phi(0) + \int_0^t T(t-s)v(s)\,ds, \quad t \in [0,t_1].$$

Thus for each $t \in [-r, t_1]$ we get

$$|h(t)| \leq M\|\phi\|_{\mathcal{D}} + M \int_0^t |v(s)|\, ds \leq M\|\phi\|_{\mathcal{D}} + M\|h_q\|_{L^1},$$

and consequently $\|h\| \leq M\|\phi\|_{\mathcal{D}} + M\|h_q\|_{L^1} := \ell$.

CLAIM 3. *G_0 sends bounded sets in $C([-r, t_1], E)$ into equicontinuous sets.*

We consider B_q as in Claim 2 and let $h \in G_0(y)$ for $y \in B_q$. Let $\varepsilon > 0$ be given. Now let $\tau_1, \tau_2 \in [0, t_1]$ with $\tau_2 > \tau_1$. We consider two cases $\tau_1 > \varepsilon$ and $\tau_1 \leq \varepsilon$.

Case 1. It $\tau_1 > \varepsilon$ then

$$\begin{aligned}|h(\tau_2) - h(\tau_1)| \leq |T(\tau_2)\phi(0) - T(\tau_1)\phi(0)| + 2M \int_0^{\tau_1-\varepsilon} h_q(s)\, ds \\ + 2M \int_{\tau_1-\varepsilon}^{\tau_1} h_q(s)\, ds + M \int_{\tau_1}^{\tau_2} h_q(s)\, ds.\end{aligned}$$

Case 2. Let $\tau_1 \leq \varepsilon$. For $\tau_2 - \tau_1 < \varepsilon$ we get

$$|h(\tau_2) - h(\tau_1)| \leq |T(\tau_2)\phi(0) - T(\tau_1)\phi(0)| + M \int_0^{2\varepsilon} h_q(s)\, ds + M \int_0^{\varepsilon} h_q(s)\, ds.$$

Note equicontinuity follows since (i). $T(t), t \geq 0$ is a strongly continuous semigroup and (ii). $T(t)$ is compact for $t > 0$ (so $T(t)$ is continuous in the uniform operator topology for $t > 0$).

The equicontinuity for the case $\tau_1 < \tau_2 \leq 0$ follows from the uniform continuity of ϕ on the interval $[-r, 0]$, and for the case $\tau_1 \leq 0 \leq \tau_2$ by combining the previous cases.

Let $0 < t \leq t_1$ be fixed and let ε be a real number satisfying $0 < \varepsilon < t$. For $y \in B_q$ and $v \in S_{F,y}$ we define

$$h_\varepsilon(t) = T(t)\phi(0) + \int_0^{t-\varepsilon} T(t-s)v(s)\, ds = T(t)\phi(0) + T(\varepsilon) \int_0^{t-\varepsilon} T(t-s-\varepsilon)v(s)\, ds.$$

Note

$$\left\{ \int_0^{t-\varepsilon} T(t-s-\varepsilon)v(s)\, ds : y \in B_q \text{ and } v \in S_{F,y} \right\}$$

is a bounded set since

$$\left| \int_0^{t-\varepsilon} T(t-s-\varepsilon)v(s)\, ds \right| \leq M \int_0^{t-\varepsilon} h_q(s)\, ds$$

and now since $T(t)$ is a compact operator for $t > 0$, the set $Y_\varepsilon(t) = \{h_\varepsilon(t) : y \in B_q \text{ and } v \in S_{F,y}\}$ is relatively compact in E for every ε, $0 < \varepsilon < t$. Moreover, for $h = h_0$ we have

$$|h(t) - h_\varepsilon(t)| \leq M \int_{t-\varepsilon}^{t} h_q(s)\, ds.$$

Therefore, the set $Y(t) = \{h(t) : y \in B_q \text{ and } v \in S_{F,y}\}$ is totally bounded. Hence $Y(t)$ is relatively compact in E.

As a consequence of Claims 2, 3 and the Arzelá–Ascoli theorem we can conclude that $G_0: C([-r,t_1],E) \to \mathcal{P}(C([-r,t_1],E))$ is completely continuous.

Claim 4. *G_0 has closed graph.*

Let $y_n \to y_*$, $h_n \in G_0(y_n)$ and $h_n \to h_*$. We shall prove that $h_* \in G_0(y_*)$. Now $h_n \in G_0(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that

$$h_n(t) = T(t)\phi(0) + \int_0^t T(t-s)v_n(s)\,ds, \quad t \in [0,t_1].$$

We must prove that there exists $v_* \in S_{F,y_*}$ such that

$$h_*(t) = T(t)\phi(0) + \int_0^t T(t-s)v_*(s)\,ds, \quad t \in [0,t_1].$$

Consider the linear continuous operator $\Gamma: L^1([0,t_1],E) \to C([0,t_1],E)$ defined by

$$(\Gamma v)(t) = \int_0^t T(t-s)v(s)\,ds.$$

We have $\|(h_n - T(t)\phi(0)) - (h_* - T(t)\phi(0))\| \to 0$ as $n \to \infty$.

It follows that $\Gamma\circ S_F$ is a closed graph operator ([16]). Also from the definition of Γ we have that $h_n(t) - T(t)\phi(0) \in \Gamma(S_{F,y_n})$. Since $y_n \to y_*$, it follows that

$$h_*(t) = T(t)\phi(0) + \int_0^t T(t-s)v_*(s)\,ds, \quad t \in [0,t_1]$$

for some $v_* \in S_{F,y_*}$.

Claim 5. *The set $\mathcal{M} := \{y \in C([-r,t_1],E) : \lambda y \in G_0(y)$ for some $\lambda > 1\}$ is bounded.*

Let $y \in \mathcal{M}$ be such that $\lambda y \in G_0(y)$ for some $\lambda > 1$. Then there exists $v \in S_{F,y}$ such that

$$y(t) = \lambda^{-1}T(t)\phi(0) + \lambda^{-1}\int_0^t T(t-s)v(s)\,ds, \quad t \in [0,t_1].$$

This implies by our assumptions that for each $t \in [0,t_1]$ we have

$$|y(t)| \le M\|\phi\|_{\mathcal{D}} + M\int_0^t g(s,\|y_s\|_{\mathcal{D}})\,ds, \quad t \in [0,t_1].$$

We consider the function μ_0 defined by

$$\mu_0(t) = \sup\{|y(s)| : -r \le s \le t\}, \quad 0 \le t \le t_1.$$

Let $t^* \in [-r,t]$ be such that $\mu_0(t) = |y(t^*)|$. If $t^* \in [0,t_1]$, by the previous inequality we have for $t \in [0,t_1]$ (note $t^* \le t$)

$$\mu_0(t) \le M\|\phi\|_{\mathcal{D}} + M\int_0^t g(s,\mu_0(s))\,ds.$$

If $t^* \in [-r,0]$ then $\mu_0(t) = \|\phi\|_{\mathcal{D}}$ and the previous inequality holds. Let us take the right-hand side of the above inequality as $v_0(t)$, then we have

$$\begin{gathered} v_0(0) = M\|\phi\|_{\mathcal{D}}, \quad \mu_0(t) \le v_0(t), \quad t \in [0,t_1], \\ v_0'(t) = Mg(t,\mu_0(t)), \quad t \in [0,t_1]. \end{gathered}$$

Using the nondecreasing character of g we get

$$v_0'(t) \le g(t, v_0(t)), \quad t \in [0,t_1].$$

This implies that ([15, Theorem 1.10.2]) $v_0(t) \le r_0(t)$ for $t \in [0,t_1]$, and hence $\mu_0(t) \le b_0 = \sup_{t\in[0,t_1]} r_0(t)$, $t \in [0,t_1]$. Thus

$$\sup\{|y(t)| : -r \le t \le t_1\} \le b_0' := \max\{\|\phi\|_{\mathcal{D}}, b_0\},$$

where b_0 depends only on b and on the function r_0. This shows that $\mathcal{M}$ is bounded.

As a consequence of the Leray–Schauder Alternative for Kakutani maps [11] we deduce that G_0 has a fixed point which is a solution of (4.1)–(4.2). Denote this solution by y_0.

Step 2. Consider now the following problem on $J_1 := (t_1,t_2]$

$$y'(t) \in Ay(t) + F(t,y_t), \quad t \in J_1, \tag{4.3}$$

$$y(t_1^+) = I_1(y_0(t_1^-)). \tag{4.4}$$

A solution to (4.3)–(4.4) is a fixed point of the operator

$$G_1 \colon C(J_1,E) \to \mathcal{P}(C(J_1,E))$$

defined by

$$G_1(y) := \bigg\{ h \in C(J_1,E) : h(t) = T(t-t_1)I_1(y_0(t_1^-)) + \int_{t_1}^t T(t-s)v(s)\,ds, v \in S_{F,y,1} \bigg\},$$

where $S_{F,y,1} = \{v \in L^1([t_1,t_2],E) : v(t) \in F(t,y_t)$ for a.e. $t \in J_1\}$; here we assume $y(s) = y_0(s)$ if $s \in [-r,t_1]$.

As in Step 1 we can easily show that G_1 has convex values, is completely continuous and upper semicontinuous. It suffices to show that the set

$$\mathcal{M} := \{y \in C(J_1,E) : \lambda y \in G_1(y) \text{ for some } \lambda > 1\}$$

is bounded.

Let $y \in \mathcal{M}$, then $\lambda y \in G_1(y)$ for some $\lambda > 1$. Thus there exists $v \in S_{F,y,1}$ such that

$$y(t) = \lambda^{-1} T(t-t_1) I_1(y_0(t_1^-)) + \lambda^{-1} \int_{t_1}^t T(t-s) v(s)\, ds, \quad t \in J_1.$$

Thus for each $t \in J_1$ we have

$$|y(t)| \le MN_1 + M \int_{t_1}^t g(s, \|y_s\|_{\mathcal{D}})\, ds, \quad t \in J_1.$$

We consider the function μ_1 defined by

$$\mu_1(t) = \sup\{|y(s)| : t_1 - r \le s \le t\}, \quad t \in J_1.$$

Let $t^* \in [t_1 - r, t]$ be such that $\mu_1(t) = |y(t^*)|$. If $t^* \in [t_1 - r, t_1]$ then $\mu_1(t) \le \sup_{s \in [t_1 - r, t_1]} |y(s)| \le Q_1$. If $t^* \in (t_1, t]$ then for $t \in J_1$ we have

$$\mu_1(t) \le MN_1 + M \int_{t_1}^t g(s, \mu_1(s))\, ds.$$

Thus in both cases we have for $t \in J_1$ that

$$\mu_1(t) \le MN_1 + Q_1 + M \int_{t_1}^t g(s, \mu_1(s))\, ds.$$

Let us take the right-hand side of the above inequality as $v_1(t)$, and then we have

$$v_1(t_1) = MN_1 + Q_1, \quad \mu_1(t) \le v_1(t), \quad t \in J_1$$

and

$$v_1'(t) = Mg(t, \mu_1(t)), \quad t \in J_1.$$

Using the nondecreasing character of g we get

$$v_1'(t) \le Mg(t, v_1(t)), \quad t \in J_1.$$

This implies that ([15, Theorem 1.10.2]) $v_1(t) \le r_1(t)$ for $t \in J_1$, and hence $|y(t)| \le b_1' = \sup_{t \in J_1} r_1(t)$, $t \in J_1$, where b_1' depends only on b and on the function r_1. This shows that $\mathcal{M}$ is bounded.

As a consequence of the Leray–Schauder Alternative for Kakutani maps [11] we deduce that G_1 has a fixed point which is a solution of (4.3)–(4.4). Denote this solution by y_1.

Step 3. Continue this process and construct solutions $y_k \in C(J_k, E)$ for $k = 2, \dots, m$ to

$$y'(t) \in Ay(t) + F(t, y_t), \quad \text{a.e. } t \in J_k, \tag{4.5}$$

$$y(t_k^+) = I_k(y_{k-1}(t_k^-)). \tag{4.6}$$

Then

$$y(t) = \begin{cases} y_0(t) & \text{if } t \in [-r, t_1], \\ y_1(t) & \text{if } t \in (t_1, t_2], \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ y_{m-1}(t) & \text{if } t \in (t_{m-1}, t_m], \\ y_m(t) & \text{if } t \in (t_m, b] \end{cases}$$

is a mild solution of (1.4)–(1.6). □

For the lower semicontinuous case we state without proof the following result.

THEOREM 4.4. *Suppose that the following conditions*

(a) $F\colon [0,b] \times \mathcal{D} \to \mathcal{P}(E)$ *is a nonempty, compact-valued, multivalued map such that:*
   (a1) $(t,u) \mapsto F(t,u)$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable;*
   (a2) $u \mapsto F(t,u)$ *is lower semi-continuous for a.e.* $t \in [0,b]$*;*

(b) *for each* $\rho > 0$*, there exists a function* $h_\rho \in L^1([0,b], \mathbb{R}^+)$ *such that*

$$\|F(t,u)\| = \sup\{|v| : v \in F(t,u)\} \leq h_\rho(t) \quad \textit{for a.e. } t \in [0,b]$$

and for $u \in \mathcal{D}$ *with* $\|u\|_{\mathcal{D}} \leq \rho$*;*

are satisfied. In addition suppose (c), (e) *of Theorem* 3.2 *and* (b), (c) *of Theorem* 4.3 *hold. Then the* IVP (1.4)–(1.6) *has at least one solution.*

5. Semilinear evolution inclusion with nondense domain

In Theorem 3.2 the operator A was densely defined. However, as indicated in [6], we sometimes need to deal with nondensely defined operators. For example, when we look at a one-dimensional heat equation with Dirichlet conditions on $[0,1]$ and consider $A = \partial^2/\partial x^2$ in $C([0,1],\mathbb{R})$ in order to measure the solutions in the sup-norm, then the domain,

$$D(A) = \{\phi \in C^2([0,1],\mathbb{R}) : \phi(0) = \phi(1) = 0\},$$

is not dense in $C([0,1],\mathbb{R})$ with the sup-norm. See [6] for more examples and remarks concerning nondensely defined operators. We can extend the results for problem (1.1)–(1.3) in the case where A is nondensely defined. The basic tool for this study is the theory of integrated semigroups.

We begin with some notations and recall some needed preliminaries. Consider the space,

$$\begin{aligned} \Omega = \{y\colon [0,b] \to E : y_k \in C(J_k, E),\ k = 0, \dots, m \text{ and there exist} \\ y(t_k^-) \text{ and } y(t_k^+),\ k = 1, \dots, m \text{ with } y(t_k^-) = y(t_k)\}, \end{aligned}$$

which is a Banach space with the norm

$$\|y\|_\Omega = \max\{\|y_k\|_{J_k},\ k = 0, \dots, m\},$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}]$, $k = 0, \dots, m$, and $\|y_k\|_{J_k} = \sup_{t \in J_k} \|y_k(t)\|$. Set

$$\Omega' = \Omega \cap C(J, \overline{D(A)}).$$

Definition 5.1 ([2]). Let E be a Banach space. An integrated semigroup is a family of operators $(S(t))_{t \geq 0}$ of bounded linear operators $S(t)$ on E with the following properties:

(a) $S(0) = 0$;
(b) $t \to S(t)$ is strongly continuous;
(c) $S(s)S(t) = \int_0^s (S(t + r) - S(r))\, dr$, for all $t, s \geq 0$.

If A is the generator of an integrated semigroup $(S(t))_{t\geq 0}$ which is locally Lipschitz, then from [2], $S(\cdot)x$ is continuously differentiable if and only if $x \in \overline{D(A)}$. In particular $S'(t)x := dS(t)x/dt$ defines a bounded operator on the set $E_1 := \{x \in E : t \to S(t)x$ is continously differentiable on, $[0, \infty)\}$ and $(S'(t))_{t\geq 0}$ is a C_0 semigroup on $\overline{D(A)}$. Here and hereafter, we assume that A satisfies the Hille–Yosida condition.

Let $(S(t))_{t\geq 0}$, be the integrated semigroup generated by A, then one has the following. We note that, since A satisfies the Hille–Yosida condition, $\|S'(t)\|_{B(E)} \leq Me^{\omega t}$, $t \geq 0$, where M and ω are from the Hille–Yosida condition (see [14]).

Theorem 5.2 ([14]). *Let $f\colon [0, b] \to E$ be a continuous function. Then for $y_0 \in \overline{D(A)}$, there exists a unique continuous function $y\colon [0, b] \to E$ such that*

(a) $\int_0^t y(s)\, ds \in D(A)$ *for* $t \in [0, b]$,
(b) $y(t) = y_0 + A \int_0^t y(s)\, ds + \int_0^t f(s)\, ds$, $t \in [0, b]$,
(c) $|y(t)| \leq Me^{\omega t}(|y_0| + \int_0^t e^{-\omega s}|f(s)|\, ds)$, $t \in [0, b]$.

Moreover, y satisfies the following variation of constant formula:

$$y(t) = S'(t)y_0 + \frac{d}{dt}\int_0^t S(t - s)f(s)ds, \quad t \geq 0. \tag{5.1}$$

Let $B_\lambda = \lambda R(\lambda, A) := \lambda(\lambda I - A)^{-1}$. Then ([14]) for all $x \in \overline{D(A)}$, $B_\lambda x \to x$ as $\lambda \to \infty$. Also from the Hille–Yosida condition (with $n = 1$) it easy to see that $\lim_{\lambda\to\infty} |B_\lambda x| \leq M|x|$, since

$$|B_\lambda| = |\lambda(\lambda I - A)^{-1}| \leq \frac{M\lambda}{\lambda - \omega}.$$

Thus $\lim_{\lambda\to\infty} |B_\lambda| \leq M$. Also if y satisfies (5.1), then

$$y(t) = S'(t)y_0 + \lim_{\lambda\to\infty}\int_0^t S'(t - s)B_\lambda f(s)\, ds, \quad t \geq 0. \tag{5.2}$$

Definition 5.3. We say that $y: J \to E$ is an integral solution of (1.1)–(1.3) if

(a) $y \in \Omega$,
(b) $\int_0^t y(s)\,ds \in D(A)$ for $t \in J$,
(c) there exist functions $v_k \in L^1(J_k, E)$, $0 \le k \le m$, such that $v_k(t) \in F(t, y(t))$ a.e. in $J_k, 0 \le k \le m$ and

$$y(t) = \begin{cases} S'(t)a + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)v_0(s)\,ds & \text{if } t \in J_0, \\ S'(t-t_k)I_k(y(t_k^-)) & \\ \qquad + \dfrac{d}{dt}\displaystyle\int_{t_k}^t S(t-s)v_k(s)\,ds, & \text{if } t \in J_k,\ k = 1, \ldots, m. \end{cases}$$

Theorem 5.4. *Assume that conditions* (a)–(c) *of Theorem* 3.2 *hold and in addition suppose that the following conditions are satisfied:*

(a) *A satisfies the Hille–Yosida condition;*
(b) *the operator $S'(t)$ is compact in $\overline{D(A)}$ whenever $t > 0$;*
(c) *$a \in \overline{D(A)}$;*
(d) *the problem*

$$\begin{aligned} v'(t) &= M^* e^{-\omega t} g(t, v(t)), \quad \text{a.e. } t \in J_k, \\ v(t_k^+) &= M^* N_k, \quad M^* = M \max\{e^{\omega b}, 1\}. \end{aligned}$$

where $t_0^+ = 0$, $N_0 = |a|$, $N_k = \sup\{|I_k(x) + x| : x \in [-r_{k-1}(t_k^-), r_{k-1}(t_k^-)]$, $k \in \{1, \ldots, m\}\}$ has a maximal solution $r_k(t), k = 0, 1, \ldots, m$.

Then the IVP (1.1)–(1.3) *has at least one integral solution on J.*

Proof. The proof is given in several steps.

Step 1. Consider the problem (1.1)–(1.3) on $J_0 := [0, t_1]$,

$$y'(t) \in Ay(t) + F(t, y(t)), \quad \text{a.e. } t \in J_0, \tag{5.3}$$

$$y(0) = a, \tag{5.4}$$

and transform this problem into a fixed point problem by considering the operator $G_0: C(J_0, E) \to \mathcal{P}(C(J_0, E))$ defined by

$$G_0(y) := \left\{ h \in C(J_0, E) : h(t) = S'(t)a + \frac{d}{dt}\int_0^t S(t-s)v(s)\,ds,\ v \in S_{F,y} \right\}.$$

We shall show that G_0 is a completely continuous multivalued map, u.s.c. with convex values. The proof will be given in several steps.

Claim 1. *$G_0(y)$ is convex for each $y \in C(J_0, E)$.*

This is obvious, since F has convex values.

Claim 2. *G_0 maps bounded sets into bounded sets in $C(J_0, E)$.*

Indeed, it is enough to show that there exists a positive constant ℓ such that for each $h \in G_0(y)$, $y \in B_q = \{y \in C(J_0, E) : \|y\| \leq q\}$ one has $\|h\| \leq \ell$. If $h \in G_0(y)$, then there exists $v \in S_{F,y}$ such that for each $t \in J_0$ we have

$$h(t) = S'(t)a + \frac{d}{dt}\int_0^t S(t-s)v(s)\,ds.$$

Thus for each $t \in J_0$ we get

$$|h(t)| \leq Me^{\omega t}|a| + Me^{\omega t}\int_0^t e^{-\omega s}|v(s)|\,ds \leq Me^{\omega b}|a| + Me^{\omega b}\int_0^t e^{-\omega s}h_q(s)\,ds;$$

here h_q is chosen as in Definition 2.4. Then for each $h \in G_0(B_q)$ we have

$$\|h\| \leq Me^{\omega b}|a| + Me^{\omega b}\int_0^{t_1} e^{-\omega s}h_q(s)\,ds := \ell.$$

Claim 3. *G_0 sends bounded sets in $C(J_0, E)$ into equicontinuous sets.*

We consider B_q as in Claim 2 and let $h \in G_0(y)$ for $y \in B_q$. Let $\varepsilon > 0$ be given. Now let $\tau_1, \tau_2 \in [0, t_1]$ with $\tau_2 > \tau_1$. We consider two cases $\tau_1 > \varepsilon$ and $\tau_1 \leq \varepsilon$.

Case 1. It $\tau_1 > \varepsilon$ then

$$\begin{aligned}
|h(\tau_2) - h(\tau_1)| \leq{} & |S'(\tau_2)a - S'(\tau_1)a| \\
& + \left|\lim_{\lambda\to\infty}\int_0^{\tau_1-\varepsilon}[S'(\tau_2 - s) - S'(\tau_1 - s)]B_\lambda v(s)\,ds\right| \\
& + \left|\lim_{\lambda\to\infty}\int_{\tau_1-\varepsilon}^{\tau_1}[S'(\tau_2 - s) - S'(\tau_1 - s)]B_\lambda v(s)\,ds\right| \\
& + \left|\lim_{\lambda\to\infty}\int_{\tau_1}^{\tau_2} S'(\tau_2 - s)B_\lambda v(s)\,ds\right| \\
\leq{} & |S'(\tau_2)a - S'(\tau_1)a| \\
& + M^*\|S'(\tau_2 - \tau_1 + \varepsilon) - S'(\varepsilon)\|_{B(E)}\int_0^{\tau_1-\varepsilon} e^{-\omega s}h_q(s)\,ds \\
& + 2M^*\int_{\tau_1-\varepsilon}^{\tau_1} e^{-\omega s}h_q(s)\,ds + M^*\int_{\tau_1}^{\tau_2} e^{-\omega s}h_q(s)\,ds.
\end{aligned}$$

Case 2. Let $\tau_1 \leq \varepsilon$. For $\tau_2 - \tau_1 < \varepsilon$ we get

$$|h(\tau_2)-h(\tau_1)| \leq |S'(\tau_2)a-S'(\tau_1)a|+M^*\int_0^{2\varepsilon} e^{-\omega s}h_q(s)\,ds+M^*\int_0^{\varepsilon} e^{-\omega s}h_q(s)\,ds.$$

Note equicontinuity follows since (i). $S'(t)$ for $t \geq 0$ is a strongly continuous semigroup and (ii). $S'(t)$ is compact for $t > 0$ (so $S'(t)$ is continuous in the uniform operator topology for $t > 0$).

Let $0 < t \le t_1$ be fixed and let ε be a real number satisfying $0 < \varepsilon < t$. For $y \in B_q$ and $v \in S_{F,y}$ we define

$$\begin{aligned} h_\varepsilon(t) &= S'(t)a + \lim_{\lambda\to\infty} \int_0^{t-\varepsilon} S'(t-s)B_\lambda v(s)\,ds \\ &= S'(t)a + S'(\varepsilon) \lim_{\lambda\to\infty} \int_0^{t-\varepsilon} S'(t-s-\varepsilon)B_\lambda v(s)\,ds. \end{aligned}$$

Note

$$\bigg\{ \lim_{\lambda\to\infty} \int_0^{t-\varepsilon} S'(t-s-\varepsilon)B_\lambda v(s)\,ds : y \in B_q \text{ and } v \in S_{F,y} \bigg\}$$

is a bounded set since

$$\bigg| \lim_{\lambda\to\infty} \int_0^{t-\varepsilon} S'(t-s-\varepsilon)B_\lambda v(s)\,ds \bigg| \le M^* \int_0^{t-\varepsilon} e^{-\omega s} h_q(s)\,ds$$

and now since $S'(t)$ is a compact operator for $t > 0$, the set $Y_\varepsilon(t) = \{h_\varepsilon(t) : y \in B_q \text{ and } v \in S_{F,y}\}$ is relatively compact in E for every ε, $0 < \varepsilon < t$. Moreover, for $h = h_0$ we have

$$|h(t) - h_\varepsilon(t)| \le M \int_{t-\varepsilon}^t e^{-\omega s} h_q(s)\,ds.$$

Therefore, the set $Y(t) = \{h(t) : y \in B_q \text{ and } v \in S_{F,y}\}$ is totally bounded. Hence $Y(t)$ is relatively compact in E.

As a consequence of Claims 2, 3 and the Arzelá–Ascoli theorem we can conclude that $G_0 \colon C(J_0, E) \to \mathcal{P}(C(J_0, E))$ is completely continuous.

Claim 4. *G_0 has closed graph.*

Let $y_n \to y_*$, $h_n \in G_0(y_n)$ and $h_n \to h_*$. We shall prove that $h_* \in G_0(y_*)$. Now $h_n \in G_0(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that

$$h_n(t) = S'(t)a + \lim_{\lambda\to\infty} \int_0^t S'(t-s)B_\lambda v_n(s)\,ds, \quad t \in J_0.$$

We must prove that there exists $v_* \in S_{F,y_*}$ such that

$$h_*(t) = S'(t)a + \lim_{\lambda\to\infty} \int_0^t S'(t-s)B_\lambda v_*(s)\,ds, \quad t \in J_0.$$

Consider the linear continuous operator $\Gamma \colon L^1(J_0, E) \to C(J_0, E)$ defined by

$$(\Gamma v)(t) = \lim_{\lambda\to\infty} \int_0^t S'(t-s)B_\lambda v(s)\,ds.$$

We have $\|(h_n - S'(t)a) - (h_* - S'(t)a)\| \to 0$ as $n \to \infty$. It follows that $\Gamma \circ S_F$ is a closed graph operator ([16]). Moreover, we have $h_n(t) - S'(t)a \in \Gamma(S_{F,y_n})$. Since $y_n \to y_*$, it follows that, for some $v_* \in S_{F,y_*}$,

$$h_*(t) = S'(t)a + \lim_{\lambda\to\infty} \int_0^t S'(t-s)B_\lambda v_*(s)\,ds, \quad t \in J_0.$$

Claim 5. *The set* $\mathcal{M} := \{y \in C(J_0, E) : \lambda y \in G_0(y),\ \text{for some } \lambda > 1\}$ *is bounded.*

Let $y \in \mathcal{M}$ be such that $\lambda y \in G_0(y)$ for some $\lambda > 1$. Then

$$y(t) = \lambda^{-1} S'(t) y_0 + \lambda^{-1} \lim_{\lambda \to \infty} \int_0^t S'(t-s) B_\lambda v(s)\, ds.$$

Thus

$$\begin{aligned} |y(t)| &\leq M e^{\omega t} |y_0| + M e^{\omega t} \int_0^t e^{-\omega s} g(s, |y(s)|)\, ds \\ &\leq M^* |y_0| + M^* \int_0^t e^{-\omega s} g(s, |y(s)|)\, ds. \end{aligned}$$

Let us take the right-hand side of the above inequality as $v_0(t)$, then we have

$$\begin{aligned} v_0(0) &= M^* |y_0|, \quad |y(t)| \leq v(t), \quad t \in J_0, \\ v_0'(t) &= M^* e^{-\omega t} g(t, |y(t)|), \qquad t \in J_0. \end{aligned}$$

Using the nondecreasing character of g we get

$$v_0'(t) \leq M^* e^{-\omega t} g(t, v(t)), \quad t \in J_0.$$

This implies that ([15, Theorem 1.10.2]) $v_0(t) \leq r_0(t)$ for $t \in J$, and hence $|y(t)| \leq b' = \sup_{t \in [0, t_1]} r_0(t)$, $t \in J_0$ where b' depends only on b and on the function r_0. This shows that $\mathcal{M}$ is bounded.

As a consequence of the Leray–Schauder Alternative for Kakutani maps [11] we deduce that G_0 has a fixed point which is a solution of (5.3)–(5.4). Denote this solution by y_0. Proceed as in Theorem 3.2. We omit the details. □

We state also without proof a result concerning the lower semicontinuous case for nondensely defined operators and also the two results for functional differential inclusions.

Theorem 5.5. *Assume that the conditions* (b), (c) *of Theorem* 3.2, (a), (b) *of Theorem* 3.3 *and* (a)–(d) *of Theorem* 5.4 *are satisfied. Then the impulsive initial value problem* (1.1)–(1.3) *has at least one solution.*

Theorem 5.6. *Assume that the conditions* (a)–(c) *of Theorem* 4.3 (*with* M *replaced by* $M^* e^{-\omega t}$) (a), (b) *of Theorem* 5.4 *and* (c) *of Theorem* 3.2 *are satisfied. Then, if* $\phi(0) \in \overline{D(A)}$, *the impulsive initial value problem* (1.4)–(1.6) *has at least one solution.*

Theorem 5.7. *Assume that the conditions* (a), (b) *of Theorem* 4.3, (c) *of Theorem* 3.2, (b), (c) *of Theorem* 4.3 (*with* M *replaced by* $M^* e^{-\omega t}$), (a), (b) *of Theorem* 5.4 *are satisfied. Then, if* $\phi(0) \in \overline{D(A)}$, *the impulsive initial value problem* (1.4)–(1.6) *has at least one solution.*

6. Second order semilinear impulsive differential inclusions

In this section we study second order impulsive semilinear evolution inclusions of the form,

$$y''(t) \in Ay(t) + F(t, y(t)), \quad \text{a.e. } t \in J,\ t \neq t_k,\ k = 1, \ldots, m, \tag{6.1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{6.2}$$

$$\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{6.3}$$

$$y(0) = a, \quad y'(0) = \eta, \tag{6.4}$$

where F, I_k and a are as in problem (1.1)–(1.3), A is the infinitesimal generator of a family of cosine operators $\{C(t) : t \geq 0\}$, $\overline{I}_k \in C(E, E)$ and $\eta \in E$.

We say that a family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(E)$ is a *strongly continuous cosine family* if

(i) $C(0) = I$,
(ii) $C(t + s) + C(t - s) = 2C(t)C(s)$, for all $s, t \in \mathbb{R}$,
(iii) the map $t \mapsto C(t)(x)$ is strongly continuous, for each $x \in E$.

The strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, is defined by

$$S(t)(x) = \int_0^t C(s)(x)\, ds, \quad x \in E,\ t \in \mathbb{R}. \tag{6.5}$$

The infinitesimal generator $A: E \to E$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$A(x) = \frac{d^2}{dt^2} C(t)(x)\bigg|_{t=0}.$$

For more details on strongly continuous cosine and sine families, we refer the reader to the books of Goldstein [9], Hekkila and Lakshmikantham [12] and Fattorini [8].

Proposition 6.1 ([19]). *Let $C(t)$, $t \in \mathbb{R}$ be a strongly continuous cosine family in E. Then:*

(a) *there exist constants $M_1 \geq 1$ and $\omega \geq 0$ such that $|C(t)| \leq M_1 e^{\omega |t|}$ for all $t \in \mathbb{R}$;*
(b) $|S(t_1) - S(t_2)| \leq M_1 |\int_{t_2}^{t_1} e^{\omega |s|}\, ds|$ *for all $t_1, t_2 \in \mathbb{R}$.*

Definition 6.2. A function $y(\cdot): J \to E$ is said to be a *mild solution of* (6.1)–(6.4) if there exist $v_k \in L^1(J_k, E)$ such that $v_k(t) \in F(t, y(t))$ a.e. on $J_k, 0 \leq$

$k \leq m$, $y(0) = a$, $y'(0) = \eta$ and

$$y(t) = \begin{cases} C(t)a + S(t)\eta + \displaystyle\int_0^t S(t-s)v_0(s)\,ds, & \text{if } t \in J_0, \\ C(t-t_k)I_k(y(t_k^-)) + S(t-t_k)\overline{I}_k(y(t_k^-)) & \\ \qquad + \displaystyle\int_{t_k}^t S(t-s)v_k(s)\,ds, & \text{if } t \in J_k. \end{cases}$$

THEOREM 6.3. *Assume* (a)–(c) *of Theorem* 3.2 *and the conditions*

(a) *$A: D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in J\}$, and there exist constants $M_2 \geq 1$, and $N \geq 1$ such that $\|C(t)\|_{B(E)} \leq M_2$, $\|S(t)\|_{B(E)} \leq N$ for all $t \in \mathbb{R}$;*

(b) (b1) *for each bounded $B_0 \subseteq C(J_0, E)$, and $t \in J_0$ the set*

$$\left\{C(t)a + S(t)\eta + \int_0^t S(t-s)v(s)\,ds,\ v \in S_{F,B_0}\right\}$$

is relatively compact in E, where $S_{F,B_0} = \cup\{S_{F,y,1} : y \in B_0\}$;

(b2) *for each bounded $B_k \subseteq C(J_k, E)$, $a_1, a_2 \in \mathbb{R}$, $k = 1, \ldots, m$, and $t \in J_k$ the set*

$$\left\{C(t-t_k)a_1 + S(t-t_k)a_2 + \int_{t_k}^t S(t-s)v(s)\,ds,\ v \in S_{F,B_k}\right\}$$

is relatively compact in E, where $S_{F,B_k} = \cup\{S_{F,y,k} : y \in B_k\}$;

(c) *the problem*

$$\begin{aligned} v'(t) &= Mg(t, v(t)), \quad \text{a.e. } t \in J_k, \\ v(t_k^+) &= N_k, \end{aligned}$$

where $t_0^+ = 0$, $N_0 = M_2|a| + N|\eta|$, $N_k = \sup\{|I_k(x) + x| : x \in [-r_{k-1}(t_k^-), r_{k-1}(t_k^-)],\ k \in \{1, \ldots, m\}\}$, has a maximal solution $r_k(t)$, $k = 0, \ldots, m$,

are satisfied. Then the IVP (6.1)–(6.4) *has at least one mild solution.*

PROOF. The proof is given in several steps.

Step 1. Consider the problem (6.1)–(6.4) on $J_0 := [0, t_1]$

$$y''(t) \in Ay(t) + F(t, y(t)), \quad \text{a.e. } t \in J_0, \tag{6.6}$$

$$y(0) = a, \quad y'(0) = \eta. \tag{6.7}$$

We transform this problem into a fixed point problem. A solution to (6.6)–(6.7) is a fixed point of the operator $G_0: C(J_0, E) \to \mathcal{P}(C(J_0, E))$ defined by

$$G_0(y) := \left\{h \in C(J_0, E) : h(t) = C(t)a + S(t)\eta + \int_0^t S(t-s)v(s)\,ds,\ v \in S_{F,y}\right\}.$$

We shall show that G_0 is a completely continuous multivalued map, u.s.c. with convex values. The proof will be given in several steps.

Claim 1. *$G_0(y)$ is convex for each $y \in C(J_0, E)$.*

This is obvious, since F has convex values.

Claim 2. *G_0 maps bounded sets into bounded sets in $C(J_0, E)$.*

Indeed, it is enough to show that there exists a positive constant ℓ such that for each $h \in G_0(y)$, $y \in B_q = \{y \in C(J_0, E) : \|y\| \leq q\}$ one has $\|h\| \leq \ell$. If $h \in G_0(y)$, then there exists $v \in S_{F,y}$ such that for each $t \in J_0$ we have

$$h(t) = C(t)a + S(t)\eta + \int_0^t S(t-s)v(s)\, ds.$$

Thus for each $t \in J_0$ we get

$$|h(t)| \leq M_2|a| + N|\eta| + N\int_0^t |v(s)|\, ds \leq M_2|a| + N|\eta| + N\|h_q\|_{L^1};$$

here h_q is chosen as in Definition 2.4. Then for each $h \in G_0(B_q)$ we have

$$\|h\| \leq M_2|a| + N|\eta| + N\|h_q\|_{L^1} := \ell.$$

Claim 3. *$G_)$ sends bounded sets in $C(J_0, E)$ into equicontinuous sets.*

We consider B_q as in Claim 2 and we fix $\tau_1, \tau_2 \in J_0$ with $\tau_2 > \tau_1$. For $y \in B_q$, we have using Proposition 6.1

$$\begin{aligned}
|h(\tau_2) - h(\tau_1)| &\leq |C(\tau_2)a - C(\tau_1)a| + |S(\tau_2)\eta - S(\tau_1)\eta| \\
&\quad + \int_0^{\tau_1} |[S(\tau_2 - s) - S(\tau_1 - s)]v(s)\, ds + \int_{\tau_1}^{\tau_2} |S(\tau_2 - s)|v(s)\, ds \\
&\leq |C(\tau_2)a - C(\tau_1)a| + |S(\tau_2)\eta - S(\tau_1)\eta| \\
&\quad + \int_0^{\tau_1} \int_{\tau_1 - s}^{\tau_2 - s} e^{\omega x}\, dx\, v(s)\, ds + N\int_{\tau_1}^{\tau_2} h_q(s)\, ds \\
&\leq |C(\tau_2)a - C(\tau_1)a| + |S(\tau_2)\eta - S(\tau_1)\eta| \\
&\quad + e^{\omega b}(\tau_2 - \tau_1)\int_0^{\tau_1} h_q(s)\, ds + N\int_{\tau_1}^{\tau_2} h_q(s)\, ds.
\end{aligned}$$

As a consequence of Claims 2, 3, Theorem 6.3(b1) and the Arzelá-Ascoli theorem we can conclude that G_0 is completely continuous.

Claim 4. *G_0 has closed graph.*

The proof is similar to that of Theorem 3.2 and we omit the details.

Claim 5. *The set $\mathcal{M} := \{y \in C(J_0, E) : \lambda y \in G_0(y),$ for some $\lambda > 1\}$ is bounded.*

Let $y \in \mathcal{M}$ be such that $\lambda y \in G_0(y)s$ for some $\lambda > 1$. Then there exists $v \in S_{F,y}$ such that for each $t \in J$

$$y(t) = \lambda^{-1}C(t)a + \lambda^{-1}S(t)\eta + \lambda^{-1}\int_0^t S(t-s)v(s)\,ds.$$

This implies that for each $t \in J_0$ we have

$$|y(t)| \le M_2|a| + N|\eta| + N\int_0^t g(s,|y(s)|)\,ds.$$

Let us take the right-hand side of the above inequality as $v_0(t)$, then we have

$$\begin{gathered} v_0(0) = M_2|a| + N|\eta|, \quad |y(t)| \le v_0(t), \quad t \in J_0, \\ v_0'(t) = Ng(t,|y(t)|), \quad t \in J_0. \end{gathered}$$

Using the nondecreasing character of g we get $v_0'(t) \le Ng(t, v_0(t))$, $t \in J_0$. This implies that ([15, Theorem 1.10.2]) $v_0(t) \le r_0(t)$ for $t \in J_0$, and hence $|y(t)| \le b_0' = \sup_{t\in[0,t_1]} r_0(t)$, $t \in J_0$ where b_0' depends only on b and on the function r_0. Consequently the set of solutions is a priori bounded.

As a consequence of the Leray–Schauder Alternative for Kakutani maps ([11] or [10]) we deduce that G_0 has a fixed point which is a solution of (6.6)–(6.7). Denote this solution by y_0.

Step 2. Consider now the problem on $J_1 := (t_1, t_2]$ and so on as in Theorem 3.2. We omit the details. □

For the lower semicontinuous case we state without proof the following result.

Theorem 6.4. *Assume that the conditions* (b), (c) *of Theorem* 3.2, (a), (b) *of Theorem* 3.3, *and* (a)–(c) *of Theorem* 6.3 *are satisfied. Then the impulsive initial value problem* (6.1)–(6.4) *has at least one solution.*

It is clear that we can extend the above results to semilinear functional differential inclusions.

References

[1] R. Agarwal, L. Górniewicz and D. O'Regan, *Aronszain type results for Volterra equations and inclusions*, Topol. Methods Nonlinear Anal. **23** (2004), 149–159.

[2] W. Arendt, *Vector valued Laplace transforms and Cauchy problems*, Israel J. Math. **59** (1987).

[3] J. P. Aubin and A. Cellina, *Differential Inclusions*, Springer–Verlag, Birkhauser, New York, 1984.

[4] M. Benchohra and S. K. Ntouyas, *On first order impulsive semilinear differential inclusions*, Comm. Appl. Anal. **7** (2003), 349–358.

[5] A. Bressan and G. Colombo, *Extensions and selections of maps with decomposable values*, Studia Math. **90** (1988), 69–86.

[6] G. Da Prato and E. Sinestrari, *Differential operators with non-dense domains*, Ann. Scuola. Norm. Sup. Pisa Sci. **14** (1987), 285–344.

[7] K. Deimling, *Multivalued Differential Equations*, Walter De Gruyter, Berlin, New York, 1992.

[8] H. O. Fattorini, *Second Order Linear Differential Equations in Banach Spaces*, North-Holland Mathematics Studies, vol. 108, North-Holland, Amsterdam, 1985.

[9] J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford Univ. Press, New York, 1985.

[10] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings, Mathematics and its Applications*, Kluwer Academic Publishers, Dordrecht, 1999.

[11] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer–Verlag, New York, 2003.

[12] S. Heikkila and V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Marcel Dekker, New York, 1994.

[13] Sh. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis*, Volume I: Theory, Kluwer, Dordrecht, Boston, London, 1997.

[14] H. Kellerman and M. Hieber, *Integrated semigroups*, J. Funct. Anal. **84** (1989), 160–180.

[15] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities*, vol. I, Academic Press, New York, 1969.

[16] A. Lasota and Z. Opial, *An application of the Kakutani–Ky Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys. **13** (1965), 781–786.

[17] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer–Verlag, New York, 1983.

[18] C. Travis and G. Webb, *Existence and stability for partial functional differential equations*, Trans. Amer. Math. Soc. **200** (1974), 395–418.

[19] ______, *Cosine families and abstract nonlinear second order differential equations*, Acta Math. Hungarica **32** (1978), 75–96.

[20] ______, *An abstract second order semilinear Volterra integrodifferential equation*, SIAM J. Math. Anal. **10** (1979), 412–424.

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