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ON MULTIPLE SOLUTIONS OF THE EXTERIOR NEUMANN PROBLEM INVOLVING CRITICAL SOBOLEV EXPONENT

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Dedicated to the memory of Professor Olga A. Ladyzhenskaya

Abstract. In this paper we consider the exterior Neumann problem involving a critical Sobolev exponent. We establish the existence of two solutions having a prescribed limit at infinity.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$. We set $\Omega^c = \mathbb{R}^N - \overline{\Omega}$. We consider the Neumann problem on the exterior domain Ω^c

$$(1_\mu)\qquad \begin{cases} -\Delta u = Q(x)u^{2^*-1} & \text{in } \Omega^c, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega,\ u > 0 \text{ on } \overline{\Omega}^c, \\ \lim\limits_{|x|\to\infty} u(x) = \mu > 0, \end{cases}$$

where $2^* = 2N/(N-2)$, $N \geq 3$, is a critical Sobolev exponent and $\mu > 0$ is a given parameter. We assume that the coefficient Q is locally Hölder continuous on Ω^c, $Q(x) > 0$ on Ω^c and

$$(\mathrm{Q}_1)\qquad Q(x) \leq C|x|^r$$

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for some constant $C > 0$ and $r < -2$ and large $|x|$. More specific conditions on r will be given later. The novelty here is that we consider the exterior Neumann problem with a critical Sobolev exponent and with a prescribed limit at infinity. A similar problem in the case of the Dirichlet problem has been considered in the paper [6]. In the present paper we show the existence of two solutions. The first one is obtained through the method of sub and super-solutions. This solution will be used to translate the variational functional for (1_μ) and then apply the mountain-pass principle to get a second solution.

In this paper we use standard notations. By $D^{1,2}(\Omega^c)$ we denote the Sobolev space defined by

$$D^{1,2}(\Omega^c) = \{u : u \in L^{2^*}(\Omega^c),\ |\nabla u| \in L^2(\Omega^c)\},$$

equipped with the norm

$$\|u\|_{L^{2^*}(\Omega^c)} + \||\nabla u|\|_{L^2(\Omega^c)}.$$

This norm is equivalent to the norm $\||\nabla u|\|_{L^2(\Omega^c)}$ (see [9]). The space $D^{1,2}(\Omega^c)$ is a natural space for the translated variational functional corresponding to problem (1_μ). Let

$$S(\Omega^c) = \inf_{\substack{\phi \in D^{1,2}(\Omega^c)\\ \phi \neq 0}} \frac{\int_{\Omega^c} |\nabla\phi|^2\,dx}{(\int_{\Omega^c} |\phi|^{2^*}\,dx)^{(N-2)/N}}.$$

It is known [11] that if the mean curvature of $\partial\Omega$, when seen from inside of Ω, is negative somewhere, then

$$S(\Omega^c) < \frac{S}{2^{2/N}}, \tag{s}$$

where S is the usual best Sobolev constant, i.e.

$$S = \inf_{\substack{\phi \in D^{1,2}(\mathbb{R}^N)\\ \phi \neq 0}} \frac{\int_{\mathbb{R}^N} |\nabla\phi|^2\,dx}{(\int_{\mathbb{R}^N} |\phi|^{2^*}\,dx)^{(N-2)/N}}.$$

Here $D^{1,2}(\mathbb{R}^N)$ is a Sobolev space defined by

$$D^{1,2}(\mathbb{R}^N) = \{u : u \in L^{2^*}(\mathbb{R}^N),\ |\nabla u| \in L^2(\mathbb{R}^N)\}.$$

Thus if (s) holds, then $S(\Omega^c)$ is achieved. Moreover, if $\Omega = B(0,R)$, or Ω is close to a ball, then $S(\Omega^c) = S/2^{2/N}$ (see [11]).

In a given Banach space X we denote a strong convergence by "$\to$" and weak convergence by "$\rightharpoonup$". We recall that a C^1-functional $\Phi\colon X \to \mathbb{R}$ on a Banach space X satisfies the Palais–Smale condition at level c ($(\mathrm{PS})_c$ condition for short), if each sequence $\{x_m\}$ such that

(∗) $\Phi(x_m) \to c$, and
(∗∗) $\Phi'(x_m) \to 0$ in X^*

is relatively compact in X. Finally, any sequnce satisfying $(*)$ and $(**)$ is called a Palais–Smale sequence at level c (a $(\mathrm{PS})_c$ sequence for short).

The norms in the Lebesgue spaces $L^q(\Omega^c)$ will be denoted by $\|\cdot\|_q$.

2. Minimal solution

In this section we establish the existence of a solution of (1_μ) through the method of sub and super-solutions.

To construct a supersolution we need the solution of the problem

$$\begin{cases} -\Delta w = Q(x) & \text{in } \Omega^c, \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \lim\limits_{|x|\to\infty} w(x) = 0. \end{cases} \tag{2.1}$$

Lemma 2.1. *Problem* (2.1) *has a solution satisfying*

$$0 < w(x) \leq \begin{cases} C|x|^{2-N} & \text{if } r < -N, \\ C|x|^{2-N}\log|x| & \text{if } r = -N, \\ C|x|^{2+r} & \text{if } -N < r < -2, \end{cases} \tag{2.2}$$

for large $|x|$ and some constant $C > 0$.

Proof. Let $m_\circ \in \mathbb{N}$ be such that $\Omega \subset B(0, m_\circ)$. For each $m > m_\circ$ we consider the problem

$$\begin{cases} -\Delta u = Q(x) & \text{in } \Omega^c \cap B(0,m), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u = 0 & \text{on } \partial B(0,m). \end{cases} \tag{1_m}$$

For each $m \geq m_\circ$ problem (1_m) has a solution u_m. We extend u_m by 0 outside $B(0,m)$. By the maximum principle the sequence $\{u_m\}$ is increasing and uniformly bounded. By the Schauder estimates (see [8]) we may assume that $u_m \to w$ in $C^2(\Omega^c \cap B(0,R))$ and $C^1(\overline{\Omega}^c \cap B(0,R))$ for each $R > 0$ large. Obviously $w > 0$ on Ω^c and w satisfies the equation and the boundary condition in (2.1). To show that w satisfies (2.2), we introduce a function $z(x)$ which is a solution of the exterior Dirichlet problem

$$\begin{cases} -\Delta z = Q(x) & \text{in } \Omega^c, \\ z = 0 & \text{on } \partial\Omega, \\ \lim\limits_{|x|\to\infty} z(x) = 0. \end{cases}$$

The function z is positive on Ω^c and satisfies (2.2) (see [6]). Since $\{u_m\}$ are uniformly bounded on Ω^c, there exists a constant $C > 1$ such that $u_m(x) \leq Cz(x)$ for $x \in \partial B(0, m_\circ)$ and $m > m_\circ$. Moreover, $u_m(x) = 0$ for $x \in \partial B(0,m)$ and

$$-\Delta(u_m - Cz) = Q(x) - CQ(x) < 0 \quad \text{on } B(0,m) - B(0,m_\circ).$$

Hence by the maximum principle $u_m \le Cz$ on $\mathbb{R}^N - B(0, m_\circ)$ for every $m > m_\circ$. Letting $m \to \infty$ we get $w(x) \le Cz(x)$ and the result follows. □

Lemma A. *Suppose that*

(H) *$Q\colon \overline{\Omega}^c \to \mathbb{R}$ is locally Hölder continuous, $Q(x) > 0$ and $Q(x) \le c|x|^r$ on $\overline{\Omega}^c$, where $r < -(N+2)/2$ and $c > 0$.*

Then the problems (2.1) *and*

$$\text{(2.1')}\qquad \begin{cases} -\Delta w = Q(x), \ w(x) > 0 & \textit{in } \Omega^c, \\ \dfrac{\partial w}{\partial \nu} = 0 & \textit{on } \partial\Omega, \ w \in D^{1,2}\Omega^c), \end{cases}$$

are equivalent. Moreover, the solution of (2.1) (*or* (2.1')) *exists and is unique.*

Proof. Since

$$Q \in L^{2N/(N+2)}(\Omega^c) \cong (L^{2^*}(\Omega^c))',$$

it follows from the Riesz–Fréchet representation theorem that (2.1') has a unique solution $w_\circ$ in $D^{1,2}(\Omega^c)$. On the other hand the problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega^c, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \lim\limits_{|x|\to\infty} u(x) = 0 \end{cases}$$

has a unique solution $u \equiv 0$ (see [6]). Hence by Lemma 2.1, problem (2.1) has a unique solution, say w_1. Since by Lemma 2.1 $w_1 \in D^{1,2}(\Omega^c)$, $w_1 \equiv w_\circ$. □

Lemma B. *Suppose that the assumption* (H) *holds. Then problems*

$$\begin{cases} -\Delta u = Q(x), \quad u(x) > \mu > 0 & \textit{in } \Omega^c, \\ \dfrac{\partial u}{\partial \nu} = 0 & \textit{on } \partial\Omega^c, \\ \lim_{|x|\to\infty} u(x) = \mu \end{cases}$$

and

$$\begin{cases} -\Delta u = Q(x), \quad \mu > 0 & \textit{on } \Omega^c, \\ \dfrac{\partial u}{\partial \nu} = 0 & \textit{on } \partial\Omega, \ (u - \mu) \in D^{1,2}(\Omega^c) \end{cases}$$

are equivalent and have a unique solutions.

Proof. Define $u = w + \mu$ and apply Lemma A. □

To proceed further we introduce the definition of a subsolution and supersolution of (1_μ).

We say that a function $\phi > 0$ on Ω^c is a supersolution of (1_μ) if $\phi \in C^2(\Omega^c) \cap C^1(\overline{\Omega}^c)$, $-\Delta\phi \ge Q\phi^p$, where $p = 2^* - 1$, on Ω^c, $\partial\phi/\partial\nu = 0$ on $\partial\Omega$ and $\lim_{|x|\to\infty} \phi(x) \ge \mu$.

The definition of a subsolution $\psi > 0$ is obtained by reversing the inequalities in the above definition.

If problem (1_μ) has a subsolution ψ and a supersolution ϕ such that $0 < \psi < \phi$ on Ω^c, then problem (1_μ) has a minimal solution $\underline{u}$ and a maximal solution $\overline{u}$ such that $\psi \leq \underline{u} \leq \overline{u} \leq \phi$ on Ω^c. This can be established by employing a standard monotone iteration technique. First we observe that if w is the solution of (2.1) then the function $w_\mu = \mu + w$ is the unique solution of the following problem

$$\begin{cases} -\Delta u = Q(x) & \text{in } \Omega^c, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \lim\limits_{|x|\to\infty} u(x) = \mu. \end{cases} \tag{2.3}$$

Let $u_0 = \phi$ and for every $j \geq 1$ we define u_j as a solution of the problem

$$\begin{cases} -\Delta u_j = Q(x)u_{j-1}^p & \text{in } \Omega^c, \\ \dfrac{\partial u_j}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \lim\limits_{|x|\to\infty} u_j(x) = \mu. \end{cases}$$

By the maximum principle we have

$$u_j \leq u_{j-1} \leq \ldots \leq u_1 \leq u_0 \quad \text{on } \Omega^c.$$

Similarly, we set $v_0 = \psi$. Let v_j for $j \geq 1$ be a solution of the problem

$$\begin{cases} -\Delta v_j = Q(x)v_{j-1}^p & \text{in } \Omega^c, \\ \dfrac{\partial v_j}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \lim\limits_{|x|\to\infty} v_j(x) = \mu. \end{cases}$$

By the maximum principle we have

$$\psi = v_0 \leq v_1 \leq \ldots \leq v_j \text{ on } \Omega^c.$$

Also, we have $v_j \leq u_j$ on Ω^c. Taking the limits of the sequences $\{v_j\}$ and $\{u_j\}$ we obtain a minimal solution $\underline{u}$ and a maximal solution $\overline{u}$.

To apply the above method, let w_1 be a solution of (2.3) with $\mu = 1$. Then we set $\phi_\mu = \mu w_1$ and $\psi_\mu = \mu$. It is clear that $\psi_\mu < \phi_\mu$ on Ω^c and $\lim_{|x|\to\infty} \phi_\mu(x) = \mu$. We now observe that

$$-\Delta\phi_\mu - Q(x)\phi_\mu^{2^*-1} = \mu Q(x) - Q(x)(\mu w_1)^{2^*-1} = Q(x)\mu(1 - \mu^{2^*-2}w_1^{2^*-1}) \geq 0$$

on Ω^c for μ small, say $0 < \mu \leq \mu_\circ$. Obviously, ψ is a subsolution for (1_μ). By the method of sub and supersolutions problem (1_μ) has a minimal solution u_μ satisfying $\mu \leq u_\mu \leq \phi_\mu$ for $0 < \mu \leq \mu_\circ$.

We let

$$\overline{\mu} = \sup\{\mu > 0 : \text{problem } (1_\mu) \text{ has a solution}\}.$$

Proposition 2.2. *Suppose that the assumption* (H) *holds. Problem* (1_μ) *has a solution for every* $0 < \mu < \overline{\mu}$. *Moreover,* $0 < \overline{\mu} < \infty$ *and there are no solutions for* $\mu > \overline{\mu}$.

Proof. Let $\mu \in (0, \overline{\mu})$. Then there exists $\widetilde{\mu} \in (\mu, \overline{\mu})$ such that problem $(1_{\widetilde{\mu}})$ has a solution $u_{\widetilde{\mu}}$. This solution $u_{\widetilde{\mu}}$ is a supersolution of (1_μ) and $v = \mu$ is a subsolution of (1_μ). Hence problem (1_μ) has a minimal solution u_μ such that $\mu \leq u_\mu \leq u_{\widetilde{\mu}}$. Arguing by contradiction, assume that $\overline{\mu} = \infty$. Then for every $\mu > 0$ there exists a minimal solution u_μ. Letting $v = u_\mu - \mu$, we see that

$$-\Delta v = -\Delta u_\mu^{2^*-1} \geq Q(x)\mu^{2^*-2}(u_\mu - \mu) = Q(x)\mu^{2^*-2} v$$

and $v > 0$ on Ω^c. By Lemma B $v \in D^{1,2}(\Omega^c)$. Hence the first eigenvalue for $-\Delta - Q(x)\mu^{2^*-2}$ is nonnegative. On the other hand for large μ, the first eigenvalue must be negative and we have reached a contradiction. □

3. Properties of minimal solutions

From Lemma B we deduce the following estimate for $u_\mu - \mu$.

Lemma 3.1. *Suppose that the assumption* (H) *holds. Let* u_μ *be the minimal solution of* (1_μ) *from Proposition* 2.2. *Then*

$$0 < u_\mu - \mu \leq \begin{cases} C|x|^{2-N} & \text{if } r < -N, \\ C|x|^{2-N}\log|x| & \text{if } r = -N, \\ C|x|^{2+r} & \text{if } -N < r < -2, \end{cases}$$

for some constant $C > 0$ *and large* $|x|$.

Lemma 3.2. *Suppose* (H) *holds. Further, we assume that* u *is a bounded positive solution of* (1_μ) *such that* $u - \mu \in D^{1,2}(\Omega^c)$. *Then the variational problem*

$$\sigma_\mu = \inf\bigg\{\int_{\Omega^c} |\nabla w|^2\,dx : w \in D^{1,2}(\Omega^c),\ p\int_{\Omega^c} Q(x)u^{p-1}w^2\,dx = 1\bigg\},$$

where $p = 2^* - 1$, *has a minimizer* ψ_μ *satisfying*

$$\begin{cases} -\Delta\psi_\mu = p\sigma_\mu Q(x)u^{p-1}\psi_\mu & \text{in } \Omega^c, \\ \dfrac{\partial\psi_\mu}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

If there exists a bounded positive solution $\overline{u}$ *of* $(1_{\overline{\mu}})$ *with* $\overline{\mu} > \mu$ *and such that* $\overline{u} > u$ *on* Ω^c *and* $\overline{u} - \overline{\mu} \in D^{1,2}(\Omega^c)$, *then* $\sigma_\mu > 1$.

Proof. The first part of the lemma follows from the fact that the functional $w \in D^{1,2}(\Omega^c) \to \int_{\Omega^c} Q(x)u^{p-1}w^2\,dx$ is weakly sequentially compact. Here we need the assumption (Q_1). We only give the proof of the second part. We set $v = u - \mu$ and $\overline{v} = \overline{u} - \overline{\mu}$. Then

$$-\Delta(\overline{v} - v) = Q(x)(\overline{v} + \overline{\mu})^p - Q(x)(v + \mu)^p = Q(x)(\overline{u}^p - u^p) \geq 0,$$

$\partial(\overline{v}-v)/\partial\nu = 0$ on $\partial\Omega$ and $\overline{v}-v \to 0$ as $|x| \to \infty$. Therefore by the maximum principle $\overline{v} > v$ on Ω^c. We now observe that

$$\begin{cases} -\Delta(\overline{v}-v) = Q(x)(\overline{u}^p - u^p) \geq pQ(x)u^{p-1}(\overline{v}-v+(\overline{\mu}-\mu)) & \text{in } \Omega^c, \\ \dfrac{\partial(\overline{v}-v)}{\partial\nu} = 0, \quad \overline{v}-v \in D^{1,2}(\Omega^c) & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Let $w = \overline{u} - u$. Testing (3.2) with ψ_μ we get

$$\int_{\Omega^c} \nabla\psi_\mu \nabla w \, dx \geq p \int_{\Omega^c} Q(x)u^{p-1}\big(w + (\overline{\mu}-\mu)\big)\psi_\mu \, dx. \tag{3.3}$$

On the other hand since ψ_μ is a solution of (3.1), we get

$$\int_{\Omega^c} \nabla\psi_\mu \nabla w \, dx = p\sigma_\mu \int_{\Omega^c} Q(x)u^{p-1}\psi_\mu w \, dx.$$

Then (3.2) and (3.3) imply that

$$p\sigma_\mu \int_{\Omega^c} Q(x)u^{p-1} w\psi_\mu \, dx > p \int_{\Omega^c} Q(x)u^{p-1} w\psi_\mu \, dx.$$

This shows that $\sigma_\mu > 1$. □

Lemma 3.2 can be applied to a family of minimal solutions $\{u_\mu\}$, $0 < \mu < \overline{\mu}$, since by Lemma B $u_\mu - \mu \in D^{1,2}(\Omega^c)$. Taking in Lemma 3.2 $u = u_\mu$ for $0 < \mu < \overline{\mu}$, we see that the corresponding $\sigma_\mu > 1$. However, Lemma 3.2 cannot be applied to $u_{\overline{\mu}}$. Later we shall show that $\sigma_{\overline{\mu}} = 1$.

Lemma 3.3. *Suppose* (H) *holds. Then there exists a constant $C > 0$ independent of μ such that*

$$\|\nabla(u_\mu - \mu)\|_2 \leq C$$

for every $0 < \mu < \overline{\mu}$.

Proof. Let $v_\mu = u_\mu - \mu$. Then by Lemma B we have

$$\int_{\Omega^c} |\nabla v_\mu|^2 \, dx = \int_{\Omega^c} Q(x)(v_\mu + \mu)^p v_\mu \, dx. \tag{3.4}$$

Applying Lemma 3.2 we get

$$\int_{\Omega^c} |\nabla v_\mu|^2 \, dx \geq p\sigma_\mu \int_{\Omega^c} Q(x)(v_\mu + \mu)^{p-1} v_\mu^2 \, dx \, dx.$$

Combining these two relations we get

$$\begin{aligned} p \int_{\Omega^c} Q(x) v_\mu^{p+1} \, dx &\leq p\sigma_\mu \int_{\Omega^c} Q(x)(v_\mu + \mu)^{p-1} v_\mu^2 \, dx \, dx \\ &\leq \int_{\Omega^c} Q(x)(v_\mu + \mu)^p v_\mu \, dx \\ &= \int_{\Omega^c} Q(x)(v_\mu + \mu)^{p-1} v_\mu^2 \, dx \, dx \\ &\quad + \int_{\Omega^c} Q(x)(v_\mu + \mu)^{p-1} \mu v_\mu \, dx. \end{aligned} \tag{3.5}$$

Hence by the Hölder and Young inequalities, we have for every $\varepsilon > 0$

$$\begin{aligned}(p-1)\int_{\Omega^c} Q(x)(u_\mu + \mu)^{p-1} v_\mu^2\,dx &\le \int_{\Omega^c} Q(x)(v_\mu+\mu)^{p-1}\mu v_\mu\,dx \\ &\le C\bigg(\int_{\Omega^c} Q(x)v_\mu^p\,dx + \int_{\Omega^c} Q(x)v_\mu\,dx\bigg) \\ &\le C\bigg(\int_{\Omega^c} Q(x)\,dx\bigg)^{1/(p+1)}\bigg(\int_{\Omega^c} Q(x)v_\mu^{p+1}\,dx\bigg)^{p/(p+1)} \\ &\quad + C\bigg(\int_{\Omega^c} Q(x)\,dx\bigg)^{p/(p+1)}\bigg(\int_{\Omega^c} Q(x)v_\mu^{p+1}\,dx\bigg)^{1/(p+1)} \\ &\le \varepsilon\int_{\Omega^c} Q(x)v_\mu^{p+1}\,dx + C_\varepsilon\int_{\Omega^c} Q(x)\,dx.\end{aligned}$$

Taking $\varepsilon > 0$ sufficiently, small we derive from this inequality and (3.5) that

$$\int_{\Omega^c} Q(x)v_\mu^{p+1}\,dx \le C\int_{\Omega^c} Q(x)\,dx. \tag{3.6}$$

The desired result follows from (3.4) and (3.6) with the aid of the Hölder inequality. □

We show below that problem (1_μ) is also solvable for $\mu = \overline{\mu}$.

Proposition 3.4. *Suppose* (H) *holds. Then problem* $(1_{\overline{\mu}})$ *has a solution.*

Proof. Let v_μ be the function introduced in the proof of Lemma 3.3. The function v_μ satisfies

$$\begin{cases} -\Delta v_\mu = Q(x)(v_\mu+\mu)^p & \text{in } \Omega^c, \\ \dfrac{\partial v_\mu}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \lim\limits_{|x|\to\infty} v_\mu(x) = 0. \end{cases} \tag{3.7}$$

We commence by showing that

$$\int_{\Omega^c} v_\mu^q\,dx \le C \tag{3.8}$$

for some constant $C > 0$ independent of μ and for all $q \ge 2^*$. Due to the estimates of Lemma 2.1 $\phi_j(v_\mu) \in D^{1,2}(\Omega^c)$, where $\phi_j(t) = t^j$, $j \ge 1$. It follows from Lemma 3.2 that

$$\int_{\Omega^c} |\phi_j'(v_\mu)^2|\nabla v_\mu|^2\,dx \ge p\int_{\Omega^c} Q(x)(v_\mu+\mu)^{p-1}\phi_j(v_\mu)^2\,dx. \tag{3.9}$$

Let $\psi_j(t) = \int_0^t \phi_j'(s)^2\,ds = j^2/(2j-1)t^{2j-1}$. Testing (3.7) with $\psi_j(v_\mu)$ we get

$$\int_{\Omega^c} \psi_j'(v_\mu)|\nabla v_\mu|^2\,dx = \int_{\Omega^c} Q(x)(v_\mu+\mu)^p\psi_j(v_\mu)\,dx. \tag{3.10}$$

We deduce from (3.9) and (3.10) that

$$p\int_{\Omega^c} Q(x)(v_\mu+\mu)^{p-1}v_\mu^{2j}\,dx \le \frac{j^2}{2j-1}\bigg[\int_{\Omega^c} Q(x)(v_\mu+\mu)^{p-1}v_\mu^{2j}\,dx + \int_{\Omega^c} Q(x)(v_\mu+\mu)^{p-1}\mu v_\mu^{2j-1}\bigg].$$

We now choose $j_\circ > 1$, close to 1, so that $j^2/(2j-1) < p$ for every $j \le j_\circ$. Let $p - j^2/(2j-1) = \alpha(j,p) > 0$. We then derive from the above estimate that

$$\begin{aligned}(3.11)\qquad \alpha(j,p)\int_{\Omega^c} Q(x)v_\mu^{p+2j-1}\,dx &\le \alpha(j,p)\int_{\Omega^c} Q(x)(v_\mu+\mu)^{p-1}v_\mu^{2j}\,dx\\ &\le \frac{j^2}{2j-1}\int_{\Omega^c} Q(x)(v_\mu+\mu)^{p-1}\mu v_\mu^{2j-1}\,dx\\ &\le \frac{Cj^2}{2j-1}\bigg[\int_{\Omega^c} Q(x)v_\mu^{p+2j-2}\mu\,dx + \int_{\Omega^c} Q(x)\mu^p v_\mu^{2j-1}\,dx\bigg]\\ &\le C\bigg[\int_{\Omega^c} Q(x)v_\mu^{p+2j-2}\,dx + \int_{\Omega^c} Q(x)v_\mu^{2j-1}\,dx\bigg],\end{aligned}$$

where $C = C(\overline{\mu}, j)$. We now estimate both integrals on the right side of this inequality. By the Hölder and Young inequalities we have for every $\delta > 0$

$$\begin{aligned}\int_{\Omega^c} Q(x)v_\mu^{p+2j-2}\,dx &\le \bigg(\int_{\Omega^c} Q(x)\,dx\bigg)^{1/(p+2j-1)} \cdot \bigg(\int_{\Omega^c} Q(x)v_\mu^{p+2j-1}\,dx\bigg)^{(p+2j-2)/(p+2j-1)}\\ &\le \frac{\delta}{2}\int_{\Omega^c} Q(x)v_\mu^{p+2j-1}\,dx + C(\delta)\int_{\Omega^c} Q(x)\,dx.\end{aligned}$$

For the second integral we have

$$(3.12)\qquad \int_{\Omega^c} Q(x)v_\mu^{2j-1}\,dx \le \frac{\delta}{2}\int_{\Omega^c} Q(x)v_\mu^{p+2j-1}\,dx + C(\delta)\int_{\Omega^c} Q(x).$$

It then follows from (3.11) and the last two estimates

$$(3.13)\qquad \int_{\Omega^c} Q(x)v_\mu^{p+2j-1}\,dx \le C_1(\delta)$$

for some $\delta > 0$ small enough with a constant $C_1(\delta)$ independent of μ. Combining (3.10), (3.12), (3.13) and the Sobolev inequality we get

$$(3.14)\qquad \bigg(\int_{\Omega^c} v_\mu^{j(p+1)}\,dx\bigg)^{(N-2)/N} \le C_1\int_{\Omega^c} Q(x)v_\mu^{p+2j-1}\,dx + C_2$$

for some constant $C_1 > 0$ and $C_2 > 0$ independent of μ. We choose $2N/(N-2) < q \le p + 2j_\circ - 1$ and write it as $q = (p+1)j$ for some $1 < j \le j_\circ$. Therefore we have

$$\int_{\Omega^c} v_\mu^q \, dx \le C \tag{3.15}$$

for some constant C independent of $\mu \in (0, \overline{\mu})$ and for every $p+1 \le q \le p+2j_\circ-1$. We now take $q_\circ = p+1 = 2N/(N-2)$ and $\delta = p+2j_\circ - 1 - 2N/(N-2) > 0$. Testing (3.7) with $v_\mu^{q_\circ-1}$ we get

$$\begin{aligned}\frac{4(q_\circ-1)}{q_\circ}\int_{\Omega^c} |\nabla v_\mu^{q_\circ/2}|^2 \, dx &= \int_{\Omega^c} Q(x)(v_\mu+\mu)^p v_\mu^{q_\circ-1} \, dx \\ &\le C\bigg[\int_{\Omega^c} Q(x) v_\mu^{p+q_\circ-1} \, dx + \int_{\Omega^c} Q(x) v_\mu^{q_\circ-1} \, dx\bigg] \\ &\le C\int_{\Omega^c} Q(x) v_\mu^{p+q_\circ-1} \, dx \\ &\quad + C\bigg(\int_{\Omega^c} Q(x) v_\mu^{p+q_\circ-1} \, dx\bigg)^{(q_\circ-1)/(p+q_\circ-1)} \\ &\quad \cdot \bigg(\int_{\Omega^c} Q(x) \, dx\bigg)^{p/(p+q_\circ-1)} \\ &\le C_1 \int_{\Omega^c} Q(x) v_\mu^{p+q_\circ-1} \, dx + C_2 \int_{\Omega^c} Q(x) \, dx,\end{aligned} \tag{3.16}$$

where $C_1 > 0$ and $C_2 > 0$ are constants independent of μ. Since $q_\circ < q_\circ + p - 1 < p - 1 + q_\circ + 2\delta/N$, we have

$$t^{p-1+q_\circ} \le \varepsilon t^{p-1+q_\circ+2\delta/N} + C_\varepsilon t^{q_\circ}$$

for every $t \ge 0$. Applying (3.15) with $q = p+2j_\circ-1$, we get

$$\begin{aligned}&\int_{\Omega^c} Q(x) v_\mu^{p+q_\circ-1} \, dx \le \varepsilon \int_{\Omega^c} Q(x) v_\mu^{p-1+q_\circ+2\delta/N} \, dx + C_\varepsilon \\ &\le \varepsilon \bigg(\int_{\Omega^c} Q(x)(v_\mu^{q_\circ})^{(p+1)/2} \, dx\bigg)^{2/(p+1)} \bigg(\int_{\Omega^c} Q(x) v_\mu^{(p-1+2\delta/N)N/2} \, dx\bigg)^{2/N} + C \\ &\le \varepsilon C \int_{\Omega^c} Q(x)(v_\mu)^{q_\circ(p+1)/2} \, dx + C_1 \le \varepsilon C_2 \int_{\Omega^c} |\nabla v_\mu^{q_\circ/2}|^2 \, dx + C_3.\end{aligned}$$

This combined with (3.16) gives

$$\int_{\Omega^c} |\nabla v_\mu^{q_\circ/2}|^2 \, dx \le C$$

for some $C > 0$ independent of μ. By the Sobolev inequality we get

$$\int_{\Omega^c} v_\mu^{q_\circ^2/2} \, dx \le C$$

and the result follows by iteration. □

It follows from (3.8) that $Q(v_\mu + \mu) \in L^q(\Omega^c)$ for every $q \ge p+1$. Therefore using the L^p estimates up to the boundary [1] and the interior L^p estimates ([8, Theorem 9.11]), we show as in [6] that up to a subsequence, $v_\mu \to v$ as $\mu \to \overline{\mu}$ in $C^1(\overline{\Omega}^c \cap B(0,R))$ for all $R > 0$. Due to Lemma 3.3 we can also assume that $v \in D^{1,2}(\Omega^c)$ and v is a weak solution of

$$\begin{cases} -\Delta v = Q(x)(v+\overline{\mu})^p & \text{in } \Omega^c, \\ \dfrac{\partial v_\mu}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

By the results of the next section $\lim_{|x|\to\infty} v(x) = 0$. Thus $v + \overline{\mu}$ is a solution of problem $(1_{\overline{\mu}})$. The solution $u_{\overline{\mu}}$ is unique. Indeed, let $\widetilde{u}_{\overline{\mu}}$ be another solution of $(1_{\overline{\mu}})$. Since $\widetilde{u}_{\overline{\mu}}$ is a supersolution of (1_μ) for $\mu < \overline{\mu}$, we see that $\widetilde{u}_{\overline{\mu}} > u_\mu$ for $\mu < \overline{\mu}$. Consequently, $\widetilde{u}_{\overline{\mu}} \geq u_{\overline{\mu}}$. We now show that $\sigma_{\overline{\mu}} = 1$. Otherwise, applying the implicit function theorem to the operator $F(v,\mu) = -\Delta v + Q(x)(v^\mu)^p$ as a mapping from $D^{1,2}(\Omega^c) \times [0,\infty)$ into $D^{1,2}(\Omega^c)$, we deduce the existence of a positive solution v for every μ in a small interval $(\overline{\mu}-\delta, \overline{\mu}+\delta)$. By the results of the next section these solutions have limit equal to 0 as $|x| \to \infty$. Clearly, this contradicts the definition of $\overline{\mu}$. Repeating the argument from p. 216 of [6] we show that $\widetilde{u}_{\overline{\mu}} = u_{\overline{\mu}}$.

4. Application of the mountain-pass principle

For every $\mu \in (0, \overline{\mu})$ we consider the problem

$$(4.1)\qquad \begin{cases} -\Delta v = Q(x)((v+u_\mu)^{2^*-1} - u_\mu^{2^*-1}) & \text{in } \Omega^c, \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ v > 0 & \text{on } \Omega^c, \\ \lim\limits_{|x|\to\infty} v(x) = 0, \end{cases}$$

where u_μ is a minimal solution of (1_μ). If v_μ is a solution of (4.1), then $U_\mu = v_\mu + u_\mu$ is a solution of (1_μ). A solution of (4.1) will be found as a critical point of the functional

$$\begin{aligned} J_\mu(v) = {} & \frac{1}{2}\int_{\Omega^c} |\nabla v|^2\,dx - \frac{1}{2^*}\int_{\Omega^c} Q(x)(u_\mu+v^+)^{2^*}\,dx \\ & + \frac{1}{2^*}\int_{\Omega^c} Q(x)u_\mu^{2^*}\,dx + \int_{\Omega^c} Q(x)u_\mu^{2^*-1}v^+\,dx \end{aligned}$$

for $v \in D^{1,2}(\Omega^c)$. It is easy to show that J_μ is a C^1-functional and we have

$$\langle J'_\mu(v), \phi\rangle = \int_{\Omega^c} [\nabla v \nabla\phi - Q(x)\big((u_\mu+v^+)^{2^*-1} - u_\mu^{2^*-1}\big)]\phi\,dx$$

for every $\phi \in D^{1,2}(\Omega^c)$. To show that the functional J_μ has a mountain-pass structure, we need the following inequality: let $p > 2$, then for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that, for every $s \geq 0$,

$$(4.2)\qquad (u_\mu+s)^p - u_\mu^p - pu_\mu^{p-1}s \leq \varepsilon u_\mu^{p-1}s + C_\varepsilon s^p.$$

Lemma 4.1. *There exist $\alpha > 0$ and $\rho > 0$ such that $J_\mu(v) \geq \alpha > 0$ for $v \in D^{1,2}(\Omega^c)$ with $\|\nabla v\|_2 = \rho$.*

Proof. We write J_μ in the form

$$\begin{aligned}J_\mu(v) = \frac{1}{2}\int_{\Omega^c} |\nabla v|^2\,dx - \frac{2^*-1}{2}\int_{\Omega^c} Q(x)u_\mu^{2^*-2}\big(v^+\big)^2\,dx \\ - \int_{\Omega^c}\int_0^{v^+} Q(x)[(u_\mu+s)^{2^*-1} - u_\mu^{2^*-1} - (2^*-1)u_\mu^{2^*-2}s]\,ds\,dx.\end{aligned}$$

Applying (4.2) with $p = 2^* - 1$ we get

$$\begin{aligned}J_\mu(v) \geq \frac{1}{2}\int_{\Omega^c}\Big[|\nabla v|^2 - (2^*-1)Q(x)u_\mu^{2^*-2}(v^+)^2\Big]\,dx \\ - \int_{\Omega^c} Q(x)\bigg[\frac{\varepsilon}{2}u_\mu^{2^*-2}(v^+)^2 + C_\varepsilon\frac{(v^+)^{2^*}}{2^*}\bigg]\,dx.\end{aligned}$$

Hence by Lemma 3.2 we have

$$J_\mu(v) \geq \frac{1}{2}\bigg(1 - \frac{2^*-1-\varepsilon}{\sigma_\mu(2^*-1)}\bigg)\int_{\Omega^c}|\nabla v|^2\,dx - \frac{C_\varepsilon}{2^*}\int_{\Omega^c} Q(x)\frac{\big(v^+\big)^{2^*}}{2^*}\,dx.$$

An application of the Sobolev inequality completes the proof. □

In Propositions 4.2 and 4.3, below, we examine the (PS) sequences of the functional J_μ.

Proposition 4.2. *Let $\{v_m\} \subset D^{1,2}(\Omega^c)$ be a $(\mathrm{PS})_c$ sequence for J_μ. Then $\{v_m\}$ is bounded in $D^{1,2}(\Omega^c)$.*

Proof. We compute

$$\begin{aligned}(4.3)\quad J_\mu(v_m) &- \frac{1}{2}\langle J_\mu'(v_m), v_m\rangle \\ &= -\frac{1}{2^*}\int_{\Omega^c} Q(x)(u_\mu + v_m^+)^{2^*}\,dx + \frac{1}{2^*}\int_{\Omega^c} Q(x)u_\mu^{2^*}\,dx \\ &\quad + \int_{\Omega^c} Q(x)u_\mu^{2^*-1}v_m^+\,dx + \frac{1}{2}\int_{\Omega^c} Q(x)(u_\mu+v_m^+)^{2^*-1}v_m\,dx \\ &\quad - \frac{1}{2}\int_{\Omega^c} Q(x)u_\mu^{2^*-1}v_m\,dx \\ &= \frac{1}{N}\int_{\Omega^c} Q(x)(u_\mu+v_m^+)^{2^*}\,dx - \frac{1}{2}\int_{\Omega^c} Q(x)(u_\mu+v_m^+)^{2^*-1}v_m^-\,dx \\ &\quad - \frac{1}{2}\int_{\Omega^c} Q(x)(u_\mu+v_m^+)^{2^*-1}u_\mu\,dx + \frac{1}{2^*}\int_{\Omega^c} Q(x)u_\mu^{2^*}\,dx \\ &\quad + \int_{\Omega^c} Q(x)u_\mu^{2^*-1}v_m^+\,dx - \frac{1}{2}\int_{\Omega^c} Q(x)u_\mu^{2^*-1}v_m\,dx \\ &= \frac{1}{N}\int_{\Omega^c} Q(x)(u_\mu+v_m^+)^{2^*}\,dx - \frac{1}{2}\int_{\Omega^c} Q(x)(u_\mu+v_m^+)^{2^*-1}u_\mu\,dx\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2}\int_{\Omega^c} Q(x)u_\mu^{2^*-1}v_m^- \, dx + \frac{1}{2^*}\int_{\Omega^c} Q(x)u_\mu^{2^*}\, dx \\
&+ \int_{\Omega^c} Q(x)u_\mu^{2^*-1}v_m^+ \, dx - \frac{1}{2}\int_{\Omega^c} Q(x)u_\mu^{2^*-1}v_m \, dx \\
= & \frac{1}{N}\int_{\Omega^c} Q(x)(u_\mu + v_m^+)^{2^*} dx - \frac{1}{2}\int_{\Omega^c} Q(x)(u_\mu + v_m^+)^{2^*-1}u_\mu \, dx \\
&+ \frac{1}{2}\int_{\Omega^c} Q(x)u_\mu^{2^*-1}v_m^+ \, dx + \frac{1}{2^*}\int_{\Omega^c} Q(x)u_\mu^{2^*}\, dx.
\end{aligned}
$$

Given $\delta > 0$ we choose $C(\delta) > 0$ that

$$
\int_{\Omega^c} Q(x)(u_\mu + v_m^+)^{2^*-1}u_\mu \, dx \le \delta \int_{\Omega^c} Q(x)(u_\mu + v_m^+)^{2^*}\, dx + C(\delta)\int_{\Omega^c} Q(x)u_\mu^{2^*}\, dx.
$$

Taking $\delta > 0$ small and using the fact that $\{v_m\}$ is a $(\mathrm{PS})_c$ sequence we deduce from (4.3) that there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$
\int_{\Omega^c} Q(x)(u_\mu + v_m^+)^{2^*}\, dx \le C_1 + C_2\|\nabla v_m\|_2 \tag{4.4}
$$

for every $m \ge 1$. On the other hand we have

$$
\begin{aligned}
J_\mu(v_m) - &\frac{1}{2^*}\langle J'_\mu(v_m), v_m\rangle = \frac{1}{N}\int_{\Omega^c} |\nabla v_m|^2 \, dx \\
&+ \frac{1}{2^*}\int_{\Omega^c} Q(x)(u_\mu + v_m^+)^{2^*-1}(v_m - v_m^+ - u_\mu)\, dx + \frac{1}{2^*}\int_{\Omega^c} Q(x)u_\mu^{2^*}\, dx \\
&+ \int_{\Omega^c} Q(x)u_\mu^{2^*-1}v_m^+ \, dx - \frac{1}{2^*}\int_{\Omega^c} Q(x)u_\mu^{2^*-1}v_m \, dx \\
= &\frac{1}{N}\int_{\Omega^c} |\nabla v_m|^2 \, dx - \frac{1}{2^*}\int_{\Omega^c} Q(x)(u_\mu + v_m^+)^{2^*-1}u_\mu \, dx \\
&- \frac{1}{2^*}\int_{\Omega^c} Q(x)(u_\mu + v_m^+)^{2^*-1}v_m^- \, dx \\
&+ \frac{1}{2^*}\int_{\Omega^c} Q(x)u_\mu^{2^*}\, dx + \int_{\Omega^c} Q(x)u_\mu^{2^*-1}v_m^+ \, dx - \frac{1}{2^*}\int_{\Omega^c} Q(x)u_\mu^{2^*-1}v_m \, dx \\
= &\frac{1}{N}\int_{\Omega^c} |\nabla v_m|^2 \, dx - \frac{1}{2^*}\int_{\Omega^c} Q(x)(u_\mu + v_m^+)^{2^*-1}u_\mu \, dx \\
&+ \frac{1}{2^*}\int_{\Omega^c} Q(x)u_\mu^{2^*}\, dx + \Big(1 - \frac{1}{2^*}\Big)\int_{\Omega^c} Q(x)u_\mu^{2^*-1}v_m^+ \, dx \\
\ge &\frac{1}{N}\int_{\Omega^c} |\nabla v_m|^2 \, dx - \frac{1}{2^*}\int_{\Omega^c} Q(x)(u_\mu + v_m^+)^{2^*-1}u_\mu \, dx.
\end{aligned}
$$

From this we derive, using the Young inequality, that

$$
\|\nabla v_m\|_2^2 \le C_3 \int_{\Omega^c} Q(x)(u_\mu + v_m^+)^{2^*}\, dx + C_4\|\nabla v_m\|_2 + C_5. \tag{4.5}
$$

The fact that $\{v_m\}$ is a bounded sequence in $D^{1,2}(\Omega^c)$ is a consequence of (4.4) and (4.5). □

To proceed further we set

$$Q_m = \max_{x\in\partial\Omega} Q(x) \quad \text{and} \quad Q_M = \max_{x\in\Omega^c} Q(x).$$

These two quantities play an essential role in finding an energy level of the functional J_μ below which the Palais–Smale condition holds (see also [4] and [5]).

Proposition 4.3. *Suppose that*

$$J_\mu(v_m) \to c < \min\bigg(\frac{S^{N/2}}{2NQ_m^{(N-2)/2}}, \frac{S^{N/2}}{NQ_M^{(N-2)/2}}\bigg), \quad c > 0, \tag{4.6}$$

and

$$J'_\mu(v_m) \to 0 \quad \text{in } D^{-1,2}(\Omega^c). \tag{4.7}$$

Then the sequence $\{v_m\}$ has a subsequence converging weakly in $D^{1,2}(\Omega^c)$ to a non zero limit.

Proof. Since by Proposition 4.2 $\{v_m\}$ is bounded in $D^{1,2}(\Omega^c)$, we may assume that $v_m \rightharpoonup v$ in $D^{1,2}(\Omega^c)$ and $v_m \to v$ in $L^p(\Omega^c)\cap B(0,R))$ for each $2 \le p < 2^*$ and $R > 0$ with $\Omega \subset B(0,R)$. Testing (4.7) with $\phi = v_m^-$ we get that

$$\int_{\Omega^c} |\nabla v_m^-|^2\,dx = o(1).$$

Therefore we may assume that $v_m \ge 0$ on Ω^c. We now show that $v \not\equiv 0$. Arguing, by contradiction assume that $v = 0$ on Ω^c. We must have $v_m \not\to 0$ in $D^{1,2}(\Omega^c)$ because $c > 0$. Hence the sequence $\{v_m\}$ must concentrate. It cannot concentrate at infinity since $Q(x) \to 0$ as $|x| \to \infty$. Therefore the concentration occurs either on $\partial\Omega$ or inside Ω. By the P. L. Lions concentration-compactness principle (see [10]), there exist sequences of points $\{x_j\} \subset \mathbb{R}^N$ and numbers $\{\nu_j\}$, $\{\mu_j\} \subset (0,\infty)$ such that

$$|v_m|^{2^*} \overset{*}{\rightharpoonup} \sum_j \nu_j\delta_j \quad \text{and} \quad |\nabla v_m|^2 \overset{*}{\rightharpoonup} \sum_j \mu_j\delta_j$$

in $\mathcal{M}$, where $\mathcal{M}$ is a space of measures, moreover

$$\begin{aligned} S\nu_j^{2/2^*} &\le \mu_j \quad \text{if } x_j \in \Omega,\\ S\frac{\nu_j^{2/2^*}}{2^{2/N}} &\le \mu_j \quad \text{if } x_j \in \partial\Omega. \end{aligned}$$

From (4.7) we deduce that $\mu_j \le Q(x_j)\nu_j$ for every j. If $\nu_j > 0$ and $x_j \in \Omega$, then $\nu_j \ge S^{N/2}/Q(x_j)^{N/2}$ and if $x_j \in \partial\Omega$, then $\nu_j \ge S^{N/2}/(2Q(x_j)^{N/2})$. By the

Brézis–Lieb lemma (see [3]) we have

$$\begin{aligned} J_\mu(v_m) - \frac{1}{2}\langle J'_\mu(v_m), v_m\rangle &= \frac{1}{N}\int_{\Omega^c} Q(x)(u_\mu+v_m)^{2^*}\,dx \\ &\quad - \frac{1}{2}\int_{\Omega^c} Q(x)(u_\mu+v_m)^{2^*-1}u_\mu\,dx + \frac{1}{2^*}\int_{\Omega^c} Q(x)u_\mu^{2^*}\,dx + o(1) \\ &= \frac{1}{N}\int_{\Omega^c} Q(x)u_\mu^{2^*}\,dx + \frac{1}{N}\int_{\Omega^c} Q(x)v_m^{2^*}\,dx \\ &\quad - \frac{1}{2}\int_{\Omega^c} Q(x)u_\mu^{2^*}\,dx + \frac{1}{2^*}\int_{\Omega^c} Q(x)u_\mu^{2^*}\,dx + o(1) \\ &= \frac{1}{N}\int_{\Omega^c} Q(x)v_m^{2^*}\,dx + o(1) \\ &= \frac{1}{N}\sum_{x_j\in\partial\Omega} Q(x_j)\nu_j + \frac{1}{N}\sum_{x_j\in\Omega} Q(x_j)\nu_j + o(1) \\ &\geq \frac{1}{N}\sum_{x_j\in\partial\Omega}\frac{S^{N/2}}{Q(x_j)^{(N-2)/2}} + \frac{1}{N}\sum_{x_j\in\Omega}\frac{S^{N/2}}{Q(x_j)^{(N-2)/2}} + o(1). \end{aligned}$$

If $Q_M > 2^{2/(N-2)}Q_m$, then letting $m\to\infty$ we derive that $c \geq S^{N/2}/(NQ_M^{(N-2)/2})$ and if $Q_M \leq 2^{2/(N-2)}Q_m$, then $c \geq S^{N/2}/(2NQ_m^{(N-2)/2})$. In both cases we get a contradiction. □

Lemma 4.4. *There exists $\psi_\circ \in D^{1,2}(\Omega^c)$ such that $\|\nabla\psi_\circ\|_2 > \rho$ and $J_\mu(\psi_\circ) < 0$, where $\rho > 0$ is a constant from Lemma 4.1.*

Proof. Let $\phi_\circ \in D^{1,2}(\Omega^c)$ and $\phi_\circ > 0$ on Ω^c. We then have for $\psi_\circ = t\phi_\circ$

$$\begin{aligned} J_\mu(t\phi_\circ) &\leq \frac{t^2}{2}\int_{\Omega^c} |\nabla\phi_\circ|^2\,dx - \frac{t^{2^*}}{2^*}\int_{\Omega^c} Q(x)\phi_\circ^{2^*}\,dx \\ &\quad + \frac{1}{2^*}\int_{\Omega^c} Q(x)u_\mu^{2^*}\,dx + t\int_{\Omega^c} Q(x)u_\mu^{2^*-1}\phi_\circ\,dx < 0 \end{aligned}$$

for $t > 0$ sufficiently large. □

To apply the mountain-pass principle we define

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_\mu(\gamma(t)),$$

where $\Gamma = \{\gamma : \gamma \in C([0,1], D^{1,2}(\Omega^c)),\ \gamma(0) = 0,\ \gamma(1) = \psi_\circ\}$.

Theorem 4.5.

(a) *Let $Q_M \leq 2^{2/(N-2)}Q_m$. Suppose that $|Q(x) - Q(y)| = o(|x-y|)$ for x close to y, $Q(y) = Q_m$ and with the mean curvature $H(y) < 0$ when viewed from inside Ω. Then problem (4.1) has a solution.*

(b) *Let $Q_M > 2^{2/(N-2)}Q_m$. Suppose that $|Q(x) - Q(y)| = o(|x-y|^{N-2})$ for x close to y with $Q(y) = Q_M$. Then problem (4.1) has a solution.*

Proof. Since

$$\int_{\Omega^c} Q(x) \int_0^{v+} [(u_\mu + s)^{2^*-1} - u_\mu^{2^*-1} - s^{2^*-1}]\, ds\, dx \geq 0,$$

we have

$$\begin{aligned} J_\mu(v) &= \frac{1}{2} \int_{\Omega^c} |\nabla v|^2\, dx - \frac{1}{2^*} \int_{\Omega^c} Q(x)(u_\mu + v^+)^{2^*}\, dx \\ &\quad + \frac{1}{2^*} \int_{\Omega^c} Q(x) u_\mu^{2^*}\, dx + \int_{\Omega^c} Q(x) u_\mu^{2^*-1} v^+\, dx \\ &= \frac{1}{2} \int_{\Omega^c} |\nabla v|^2\, dx - \frac{1}{2^*} \int_{\Omega^c} Q(x) \big(v^+\big)^{2^*}\, dx \\ &\quad - \int_{\Omega^c} Q(x) \int_0^{v+} [(u_\mu + s)^{2^*-1} - u_\mu^{2^*-1} - s^{2^*-1}]\, ds\, dx \\ &\leq \frac{1}{2} \int_{\Omega^c} |\nabla v|^2\, dx - \frac{1}{2^*} \int_{\Omega^c} Q(x) \big(v^+\big)^{2^*}\, dx. \end{aligned}$$

Hence

$$\begin{aligned} \max_{t\geq 0} J_\mu(tv) &\leq \max_{t\geq 0} \bigg(\frac{t^2}{2} \int_{\Omega^c} |\nabla v|^2\, dx - \frac{t^{2^*}}{2^*} \int_{\Omega^c} Q(x) \big(v^+\big)^{2^*}\, dx \bigg) \\ &= \frac{1}{N} \frac{(\int_{\Omega^c} |\nabla v|^2\, dx)^{N/2}}{(\int_{\Omega^c} Q(x)(v^+)^{2^*}\, dx)^{(N-2)/2}}. \end{aligned} \tag{4.8}$$

(a) We consider the case $Q_M \leq 2^{2/(N-2)} Q_m$. Let

$$U_{\varepsilon,y}(x) = \varepsilon^{-(N-2)/2} U\bigg(\frac{x-y}{\varepsilon}\bigg), \quad \varepsilon > 0,\ y \in \mathbb{R}^N$$

$$\text{where } U(x) = \frac{[N(N-2)]^{(N-2)/2}}{(N(N-2) + |x|^2)^{(N-2)/2}}.$$

This function, called an instanton, has a property

$$\int_{\mathbb{R}^N} |\nabla U_{\varepsilon,y}|^2\, dx = \int_{\mathbb{R}^N} U_{\varepsilon,y}^{2^*}\, dx = S^{N/2}.$$

Moreover, it is known that

$$\frac{\int_{\Omega^c} |\nabla v|^2\, dx}{(\int_{\Omega^c} U_{\varepsilon,y}^{2^*}\, dx)^{2/2^*}} = \frac{S}{2^{2/N}} + \begin{cases} A_N H(y) \varepsilon \log(1/\varepsilon) + O(\varepsilon) & \text{for } N = 3, \\ A_N H(y) \varepsilon + O(\varepsilon^2 \log(1/\varepsilon) & \text{for } N = 4, \\ A_N H(y) \varepsilon + O(\varepsilon^2) & \text{for } N = 5, \end{cases}$$

where $A_N > 0$ is a constant depending on N. This estimate can be obtained from the corresponding estimate on a bounded domain by truncation (see [2], [11], [9]). Substituting $v = U_{\varepsilon,y}$ in (4.8) and using the above estimate together with our assumption Q we get the following estimate for the mountain-pass level

$$c < \frac{S^{N/2}}{2NQ_m^{N-2/2}}.$$

(b) If $Q_M > 2^{2/(N-2)} Q_m$, we take $U_{\varepsilon,y}$ with $Q(y) = Q_M$. We then have

$$\int_{\Omega^c} |\nabla U_{\varepsilon,y}|^2 \, dx = \int_{\mathbb{R}^N} |\nabla U|^2 \, dx - \int_{\Omega} |\nabla U_{\varepsilon,y}|^2 \, dx \leq S^{N/2} - C_1 \varepsilon^{N-2}$$

for some constant $C_1 > 0$ and

$$\begin{aligned} \int_{\Omega^c} Q(x) U_{\varepsilon,y}^{2^*} \, dx &= \int_{\Omega^c} Q_M U_{\varepsilon,y}^{2^*} \, dx + \int_{\Omega^c} \big(Q(x) - Q_M\big) U_{\varepsilon,y}^{2^*} \, dx \\ &= S^{N/2} Q_M + o\big(\varepsilon^{N-2}\big). \end{aligned}$$

Using the last two relations in (4.8) we see that

$$c < \frac{S^{N/2}}{N Q_M^{N-2/2}}. \qquad \square$$

5. Main result

To use a solution u of problem (4.1) to construct a second solution of (1_μ) we have to show that $\lim_{|x|\to\infty} u(x) = 0$. This will be accomplished by using the Moser iteration technique. In Proposition 5.1 below, we use some ideas from the proof of Theorem 8.17 in [8].

Proposition 5.1. *Suppose that* $\lim_{|x|\to\infty} Q(x) = 0$ *and* $Q \in L^{N/2}(\Omega^c)$. *Let* $u \in D^{1,2}(\Omega^c)$ *be a positive solution of* (4.1). *Then there exists* $R > 0$ *such that for every* $B(x_\circ, 2) \subset (|x| > R)$ *we have*

$$\sup_{B(x_\circ,1)} u(x) \leq C \bigg(\int_{B(x_\circ,2)} u^{2^*} \, dx \bigg)^{1/2^*},$$

where a constant C *depends on* u *but is independent of* $x_\circ$.

Proof. Let $\varepsilon > 0$ be fixed and set $p = 2^* - 1$. We choose a constant $C_\varepsilon > 0$ such that

$$(u + u_\mu)^p - u_\mu^p \leq (p + \varepsilon) u_\mu^{p-1} u + C_\varepsilon u^p$$

for every $x \in \Omega^c$. Then

$$-\Delta u \leq d(x) u \quad \text{on } \Omega^c, \tag{5.1}$$

where $d(x) = Q(x)(p + \varepsilon) u_\mu^{p-1} + C_\varepsilon Q(x) u^{p-1}$. Let $\eta \in C_0^1(\Omega^c)$ with supp $\subset (|x| > R)$, where $R > 0$ is large and will be determined later. Taking $w = \eta^2 u^\beta$, $\beta > 0$, as a test function in (5.1) we obtain

$$\beta \int_{\Omega^c} \eta^2 u^{\beta-1} |\nabla u|^2 \, dx + 2 \int_{\Omega^c} \eta \nabla\eta \nabla u u^\beta \, dx \leq \int_{\Omega^c} d(x) \eta^2 u^{\beta+1} \, dx. \tag{5.2}$$

We now use the inequality

$$\bigg| 2 \int_{\Omega^c} \eta \nabla\eta \nabla u u^\beta \, dx \bigg| \leq \frac{\beta}{2} \int_{\Omega^c} \eta^2 |\nabla u|^2 u^{\beta-1} \, dx + \frac{2}{\beta} \int_{\Omega^c} |\nabla\eta|^2 u^{\beta+1} \, dx,$$

which inserted into (5.2) gives

$$\frac{\beta}{2}\int_{\Omega^c}\eta^2u^{\beta-1}|\nabla u|^2\,dx\le\int_{\Omega^c}\bigg(d(x)\eta^2+\frac{2}{\beta}|\nabla\eta|^2\bigg)u^{\beta+1}\,dx. \tag{5.3}$$

We set $w=u^{(\beta+1)/2}$ in (5.3) and we obtain

$$\int_{\Omega^c}\eta^2|\nabla w|^2\,dx\le\frac{(\beta+1)^2}{2\beta}\int_{\Omega^c}\bigg(d(x)\eta^2+\frac{2}{\beta}|\nabla\eta|^2\bigg)w^2\,dx. \tag{5.4}$$

We now estimate $\int_{\Omega^c}d(\eta w)^2\,dx$

$$\begin{aligned}\int_{\Omega^c}d(\eta w)^2\,dx&=\int_{\Omega^c}Q(p+\varepsilon)u_\mu^{p-1}(\eta w)^2\,dx+C_\varepsilon\int_{\Omega^c}Qu^{p-1}(\eta w)^2\,dx\\&\le(p+\varepsilon)\|u_\mu\|_\infty^{p-1}\bigg(\int_{\operatorname{supp}\,\eta}Q^{N/2}\,dx\bigg)^{2/N}\|\eta w\|_{2^*}^2\\&\quad+C_\varepsilon Q_{M,R}\bigg(\int_{\Omega^c}u^{2^*}\,dx\bigg)^{2/N}\|\eta w\|_{2^*}^2,\end{aligned}$$

where $Q_{M,R}=\sup_{|x|>R}Q(x)$. Setting

$$M(R)=(p+\varepsilon)\|u_\mu\|_\infty^{p-1}\bigg(\int_{\operatorname{supp}\,\eta}Q^{N/2}\,dx\bigg)^{2/N}+C_\varepsilon Q_{M,R}\bigg(\int_{\Omega^c}u^{2^*}\,dx\bigg)^{2/N},$$

we rewrite the above inequality as

$$\int_{\Omega^c}d(\eta w)^2\,dx\le M(R)\|\eta w\|_{2^*}^2. \tag{5.5}$$

Also, we have

$$\begin{aligned}\bigg(\int_{\Omega^c}(\eta w)^{2^*}\,dx\bigg)^{(N-2)/N}&\le S^{-1}\int_{\Omega^c}|\nabla(\eta w)|^2\,dx\\&=S^{-1}\int_{\Omega^c}(\eta^2|\nabla w|^2+w^2|\nabla\eta|^2+2\eta w\nabla\eta\nabla w)\,dx\\&\le 2S^{-1}\int_{\Omega^c}(\eta^2|\nabla w|^2+w^2|\nabla\eta|^2)\,dx.\end{aligned} \tag{5.6}$$

Inserting (5.5) into (5.4) we obtain

$$\int_{\Omega^c}\eta^2|\nabla w|^2\,dx\le\frac{(\beta+1)^2}{2\beta}M(R)\|(\eta w)\|_{2^*}^2+\frac{(\beta+1)^2}{\beta^2}\int_{\Omega^c}|\nabla\eta|^2w^2\,dx.$$

Combining the last inequality with (5.6) we get

$$\begin{aligned}\bigg(1-S^{-1}\frac{(\beta+1)^2}{\beta}M(R)\bigg)\bigg(\int_{\Omega^c}(\eta w)^{2^*}\,dx\bigg)^{(N-2)/N}&\\\le 2S^{-1}\bigg(1+\frac{(\beta+1)^2}{\beta^2}\bigg)\int_{\Omega^c}|\nabla\eta|^2w^2\,dx.&\end{aligned}$$

We choose $R > 0$ so that

$$1 - S^{-1}\frac{(\beta+1)^2}{\beta}M(R) = \frac{1}{2}.$$

Thus

$$\left(\int_{\Omega^c} (\eta w)^{2^*}\,dx\right)^{(N-2)/N} \leq A\int_{\Omega^c} |\nabla\eta|^2 w^2\,dx, \tag{5.7}$$

with $A = 4S^{-1}(1 + (\beta-1)^2/\beta^2)$. We now make the following choice of η: $\eta(x) = 1$ in $B(x_\circ, r_1)$, $\eta(x) = 0$ in $\Omega^c - B(x_\circ, r_2)$, $|\nabla\eta(x)| \leq 2/(r_2 - r_1)$ in Ω^c, $1 \leq r_1 < r_2 < 3$. It is assumed that $B(x_\circ, 3) \subset (|x| > R)$. Then (5.7) takes form

$$\left(\int_{B(x_\circ,r_1)} w^{2^*}\,dx\right)^{(N-2)/2N} \leq \frac{A_1}{r_2 - r_1}\left(\int_{B(x_\circ,r_2)} w^2\,dx\right)^{1/2}, \tag{5.8}$$

with $A_1 = 2\sqrt{A}$. We set $\gamma = \beta + 1$, $\chi = N/(N-2)$. Then we get from (5.8)

$$\left(\int_{B(x_\circ,r_1)} u^{\gamma\chi}\,dx\right)^{1/(\gamma\chi)} \leq \left(\frac{A_1}{r_2 - r_1}\right)^{2/\gamma}\left(\int_{B(x_\circ,r_2)} u^{\gamma}\,dx\right)^{1/\gamma}. \tag{5.9}$$

To iterate this inequality we take $s_m = 1 + 2^{-m}$, $m = 0, 1, \ldots$ By a simple induction argument we get

$$\left(\int_{B(x_\circ,s_m)} u^{\chi^m\gamma}\,dx\right)^{1/(\gamma\chi^m)} \leq A_1^{(2/\gamma)\sum_{j=0}^{m-1}(1/\chi^j)} 2^{(2/\gamma)\sum_{j=0}^{m}(j+1)/\chi^j}\left(\int_{B(x_\circ,s_0}u^\gamma\,dx\right)^{1/\gamma}$$

for each $m > 1$. This inequality implies

$$\left(\int_{B(x_\circ,1)} u^{\chi^m\gamma}\,dx\right)^{1/(\gamma\chi^m)} \leq A_1^{(2/\gamma)\sum_{j=0}^{m-1}(1/\chi^j)} 2^{(2/\gamma)\sum_{j=0}^{m}(j+1)/\chi^j}\left(\int_{B(x_\circ,2)}u^\gamma\,dx\right)^{1/\gamma}.$$

We now choose $\gamma = \beta + 1 = 2^*$. Letting $m \to \infty$ the result follows. □

It follows from Proposition 5.1 that $\lim_{|x|\to\infty} u(x) = 0$. By the maximum principle, since

$$Q(x)(u + u_\mu)^p - Q(x)u_\mu^p > 0$$

we get $u(x) \geq C_1|x|^{2-N}$ for some constant $C_1 > 0$ and large $|x|$.

If (H) holds, then assumptions of Proposition 5.1 are satisfied.

Theorem 5.2. *Suppose* (H) *holds. Then problem* (1_μ) *has at least two solutions.*

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