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ALMOST FLAT BUNDLES AND ALMOST FLAT STRUCTURES

Alexander S. Mishchenko — Nicolae Teleman

Abstract. In this paper we discuss some geometric aspects concerning almost flat bundles, notion introduced by Connes, Gromov and Moscovici [2]. Using a natural construction of [1], we present here a simple description of such bundles. For this we modify the notion of almost flat structure on bundles over smooth manifolds and extend this notion to bundles over arbitrary CW-spaces using quasi-connections [3].

Connes, Gromov and Moscovici [2] showed that for any almost flat bundle α over the manifold M, the index of the signature operator with values in α is a homotopy equivalence invariant of M. From here it follows that a certain integer multiple n of the bundle α comes from the classifying space $B\pi_1(M)$. The geometric arguments discussed in this paper allow us to show that the bundle α itself, and not necessarily a certain multiple of it, comes from an arbitrarily large compact subspace $Y \subset B\pi_1(M)$ trough the classifying mapping.

1. Definition of almost flat bundles

Due to [2], the element $\alpha \in K(M)$ over the smooth manifold M is called almost flat bundle if for any $\varepsilon > 0$ there exist two vector bundles ξ, η with linear connections ∇^ξ, ∇^η such that

(1) $\alpha = \xi - \eta \in K(M)$,

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(2) $\|\Theta^\xi\| < \varepsilon$, $\|\Theta^\eta\| < \varepsilon$, where

$$\|\Theta\| = \sup_{x\in M}\{\|\Theta_x(X\wedge Y)\| : \|X\wedge Y\| \leq 1\}, \tag{1.1}$$

and $\Theta_x(X\wedge Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ denotes the curvature form of the connection ∇.

If α is almost flat then the sum $\alpha \oplus \beta$ is again almost flat for any trivial bundle β. This means that without loss of generality one can consider elements from the group $K(M)$ which are represented by vector bundles. In other words, one can consider two sequences of bundles $\xi = \{\xi_k\}$ and $\eta = \{\eta_k\}$ with fixed linear connections ∇^1_k, ∇^2_k such that

$$\xi_k = \eta_k \oplus \alpha.$$

$\dim\alpha = d$, $\dim\xi_k = n_k$, $\dim\eta_n = m_k = n_k - d$. Assume that

$$\lim_{k\to\infty} \|\Theta^i_k\| = 0, \quad i = 1, 2.$$

So, instead of the element α we shall consider a finer structure, which we call the structure of almost flat bundle, consisting of

(1) The sequences of bundles $\xi = \{\xi_k\}$ and $\eta = \{\eta_k\}$ with fixed linear connections ∇^1_k and ∇^2_k with conditions

$$\lim_{k\to\infty} \|\Theta^i_k\| = 0, \quad i = 1, 2, \tag{1.2}$$

(2) The sequence of isomorphisms

$$f_k \colon \xi_k \approx \eta_k \oplus \alpha.$$

The same bundle α may admit several structures of almost flat bundle. We say that the structure of almost flat bundle

$$\mathcal{P} = \{\alpha : \xi = \{\xi_k, \nabla^1_k\},\ \eta = \{\eta_k, \nabla^2_k\},\ f = \{f_k\}\}$$

is equivalent to another one

$$'\mathcal{P} = \{'\alpha : {'\xi} = \{'\xi_k, {'\nabla^1_k}\},\ {'\eta} = \{'\eta_k, {'\nabla^2_k}\},\ {'f} = \{'f_k\}\},$$

if the first one can be obtained from the other by a sequence of such operations:

(1) Passage to subsequences.
(2) Homotopy of linear connection ∇^1_k and ∇^2_k in the class of connections which satisfy the conditions (1.2), and homotopy of isomorphisms $f = \{f_k\}$.
(3) Stabilization of bundles, that is, replacing the pairs $\{\xi = \{\xi_k\}, \nabla^1_k\}$, $\{\eta = \{\eta_k\}, \nabla^2_k\}$, α and isomorphisms $f = \{f_k\}$ by their direct sums with a trivial bundle endowed with trivial connection.

By abusing the terminology, we convene to call the equivalence class of an almost flat bundle, simply, a flat bundle on the manifold M.

The trivial almost flat bundle is by definition the equivalence class of the bundle

$$\mathcal{P}^0 = \{\alpha^0 : \xi^0 = \{\xi^0_k, \nabla^{1,0}_k\},\ \eta^0 = \{\eta^0_k, \nabla^{2,0}_k\},\ f^0 = \{f^0_k\}\},$$

where all bundles α^0, ξ^0_k, η^0_k and connections $\nabla^{1,0}_k$, $\nabla^{2,0}_k$ are trivial, and the isomorphisms f^0_k are identical.

The direct sum of almost flat bundles passes to equivalence classes; let $\mathcal{V}ect_{af}(M)$ denote the semigroup of equivalence classes of almost flat bundles with respect to the direct sum operation. The trivial flat bundle is the neutral element in this semigroup.

The corresponding Grothendieck group will be denoted by $\mathcal{K}_{af}(M)$.

2. Almost flatness in terms of transition functions and quasi-connections

The almost flat bundle structure requires that the base of the bundle should be a smooth manifold. In [2] the possibility of extending the notion of almost flat bundles over arbitrary simplicial complexes is stated. We present here one such possible extension.

Although our considerations could be extended even further, for the sake of simplicity, we restrict our attention to simplicial spaces with a fixed simplicial structure.

So, let M be a finite simplicial complex. Let $V(M)$ be the set of its vertices and let

$$S(M) = \coprod_{k=0}^{\dim M} S_k(M), \quad V(M) = S_0(M)$$

be the set of all its simplices. Then each simplex $\sigma \in S_k(M)$of dimension k is determined by the collection of its vertices

$$\sigma \overset{\text{def}}{=} (a_0, \ldots, a_k); \quad a_i \in V(M),\ 0 \le i \le k.$$

Cover M with the atlas $\mathcal{U} = \{U_i\}$ consisting of the stars of its vertices

$$U_i \overset{\text{def}}{=} \operatorname{star}(a_i) = \bigcup (\sigma \in S(M) : a_i \in \sigma)$$

Let ξ be a continuous complex vector bundle of rank n over M with projection mapping p. The bundle ξ, restricted to each chart U_i of the atlas, is trivial; choose such a trivialization

$$\psi_{a_i} : p^{-1}(U_i) \to U_i \times \mathbb{C}^n.$$

Let ∇_i be the obvious flat parallel transport defined by the trivialization ψ_{a_i} over U_i.

These data produce a system of transition functions for the bundle ξ as follows. If $\gamma\colon [0,1] \to M$ is any curve connecting two points x_0 and x_1 on the manifold M,

$$x_0 = \gamma(0), \quad x_1 = \gamma(1). \tag{2.1}$$

and ∇ is any linear connection, let $\tau_\gamma\colon \xi_{x_0} \to \xi_{x_1}$ be the parallel transport defined by the connection ∇ along the curve γ. Then the transition function φ_{ij} is defined on the intersection of two charts $U_{ij} = U_i \cap U_j$ as follows

$$\varphi_{ij}(x) \stackrel{\text{def}}{=} \tau_{[a_i,x]\cup[x,a_j]}\colon \xi_{a_i} \to \xi_{a_j}, \quad x \in U_{ij}. \tag{2.2}$$

In (2.2) the curve $[a_i,x]\cup[x,a_j]$ is the union of the two segments $[a_i,x]$, $[x,a_j]$. This path lies in the star of the minimal simplex containing x; this star contains also the vertices a_i, a_j. The precise formula for φ_{ij} has to employ the local trivializations

$$\psi_{a_i}\colon \xi_{a_i} \to \mathbb{C}^n; \tag{2.3}$$

this means that the precise formula (2.2) for φ_{ij} is (2.3):

$$\varphi_{ij}(x) \stackrel{\text{def}}{=} \psi_{a_i}^{-1} \cdot \tau_{[a_i,x]\cup[x,a_j]} \cdot \psi_{a_j}\colon \mathbb{C}^n \to \xi_{a_i} \to \xi_{a_j} \to \mathbb{C}^n, \quad x \in U_{ij}.$$

Notice that if the point x lies on the segment $[a_i,a_j]$, one has

$$\varphi_{ij}(x) = \psi_{a_i}^{-1} \cdot \tau_{[a_i,a_j]} \cdot \psi_{a_j} = \varphi_{ij},$$

i.e. the transition function does not depend on the choice of the point x on the segment. The transition function $\varphi_{ij}(x)$ is in general not constant, in contrast to the case of flat connections in flat bundles. In the general case of smooth connections, the deviation of the transition functions from being constant can be estimated in terms of the curvature tensor

$$\|\varphi_{ij}(x) - \varphi_{ij}\| \le \max_k \|\varphi_{ik}\cdot\varphi_{kj} - \varphi_{ij}\| \le C \cdot \varepsilon,$$

where C estimates the maximal area of the triangles (a_i, a_k, a_j), and ε estimates the curvature norm (1.1).

Therefore, an almost flat bundle over an arbitrary simplicial complex M, consists of

(1) A sequences $\xi = \{\xi_k\}$ and $\eta = \{\eta_k\}$ of vector bundles defined by transition functions

$$\varphi_{ij}^{k,s}(x), \quad x \in U_{ij}, \quad s = 1,2,$$

relative to the atlas $\mathcal{U}$; the transition functions satisfy the condition

$$\lim_{k\to\infty} \sup_{x,y\in U_{ij}} \|\varphi_{ij}^{k,s}(x) - \varphi_{ij}^{k,s}(y)\| = 0, \quad s = 1,2. \tag{2.4}$$

(2) A sequence of isomorphisms

$$f_k\colon \xi_k \approx \eta_k \oplus \alpha. \tag{2.5}$$

2.1. Almost flat quasi-connections. The notion of quasi-connection was introduced in [3]. Here we use a slight version of it which is going to be more suitable for the purpose of describing almost flat bundles.

Let ξ be a continuous vector bundle on a topological space X; consider the diagram involving the two canonical projections

$$\begin{array}{ccc} X \times X & \xrightarrow{\mathrm{pr}_r} & X \\ {\scriptstyle \mathrm{pr}_l}\big\downarrow & & \\ X & & \end{array}$$

We may associate two bundles on the space $X \times X$

$$\begin{array}{ccccccccc} \xi & \longleftarrow & \mathrm{pr}_l^*(\xi) & & & & \mathrm{pr}_r^*(\xi) & \longrightarrow & \xi \\ \big\downarrow & & \big\downarrow & & & & \big\downarrow & & \big\downarrow \\ X & \xleftarrow[\mathrm{pr}_l]{} & X \times X & & = & & X \times X & \xrightarrow[\mathrm{pr}_r]{} & X \end{array} \tag{2.6}$$

By restricting the bundles of the diagram (2.6) to the diagonal $X \xrightarrow{\Delta} X \times X$ we get the commutative diagram of canonical isomorphisms

$$\begin{array}{ccccccccccc} \xi & \longleftarrow & \mathrm{pr}_l^*(\xi) & \longleftarrow & \Delta^*\mathrm{pr}_l^*(\xi) & \xleftrightarrow{\mathrm{id}} & \Delta^*\mathrm{pr}_r^*(\xi) & \longrightarrow & \mathrm{pr}_r^*(\xi) & \longrightarrow & \xi \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ X & \xleftarrow[\mathrm{pr}_l]{} & X \times X & \xleftarrow[\Delta]{} & X & = & X & \xrightarrow[\Delta]{} & X \times X & \xrightarrow[\mathrm{pr}_r]{} & X \end{array}$$

since

$$\Delta \mathrm{pr}_r = \Delta \mathrm{pr}_l = \mathrm{id}.$$

By definition, a quasi-connection in the bundle ξ is a homomorphism

$$\tau\colon \mathrm{pr}_l^*(\xi) \to \mathrm{pr}_r^*(\xi),$$

$$\begin{array}{ccccccc} \xi & \longleftarrow & \mathrm{pr}_l^*(\xi) & \xrightarrow{\tau} & \mathrm{pr}_r^*(\xi) & \longrightarrow & \xi \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ X & \xleftarrow[\mathrm{pr}_l]{} & X \times X & = & X \times X & \xrightarrow[\mathrm{pr}_r]{} & X \end{array}$$

having the property that its restriction to the diagonal is the identity

$$\tau_{|\Delta(X)} = \mathrm{id}\colon \Delta^*\mathrm{pr}_l(\xi) \to \Delta^*\mathrm{pr}_r^*(\xi).$$

In particular, this means that in a neighbourhood of the diagonal τ is an isomorphism.

With this definition of quasi-connections one can define what does it mean that the bundle ξ is ε-flat. Namely, for three points $x, y, z \in X$ consider the composition

$$\tau_{x,y} \cdot \tau_{y,z} \colon \xi_x \to \xi_z.$$

Then the quasi-connection τ is called ε-flat provided the following inequality holds

$$\|\tau_{y,z} \cdot \tau_{x,y} - \tau_{x,z}\| < \varepsilon$$

for arbitrary three points x, y, z belonging to a fixed neighbourhood of the diagonal.

2.2. Construction of quasi-connection by transition functions. It is sufficient to define a quasi-connection only on a neighbourhood of the diagonal. Let then $W_\Delta \subset X \times X$ be such a neighbuorhood. For example, the atlas $\{U_i\}$ defines a neighbourhood of the diagonal by taking

$$W = \bigcup_i (U_i \times U_i).$$

In this case, a quasi-connection can be thought as a family consisting of a quasi-connection on each of the charts

$$\tau_i \colon \mathrm{pr}_l^*(\xi)_{(U_i \times U_i)} \to \mathrm{pr}_r^*(\xi)_{(U_i \times U_i)}, \tag{2.7}$$

subject to the requirement that they agree on the intersections

$$(U_i \times U_i) \cap (U_j \times U_j) = (U_i \cap U_j) \times (U_i \cap U_j).$$

Let $f_i(x, y)$ be a partition of unity which is subordinate to the covering $\{(U_i \times U_i)\}$.

By means of the quasi-connections (2.7) we define the homomorphism

$$\tau(x, y) = \sum_i f_i(x, y) \tau_i(x, y). \tag{2.8}$$

It is clear that τ is a quasi-connection on X.

On the other hand, this new quasi-connection may be analyzed on a separate chart $\{(U_{i_0} \times U_{i_0})\}$ using the coordinate homomorphisms

$$\psi_i \colon p^{-1}(U_i) \to U_i \times V,$$

where $p \colon \xi \to X$ is the projection and V is the fiber. The homeomorphisms ψ_i commute with p, that is they act fiber-wise and hence the homeomorphism ψ_i has the form

$$\psi_i(v) = (p(v), \Psi_i^{p(v)}(v)), \quad \text{where } \Psi_i^x \colon p^{-1}(x) \to V.$$

Define $\sigma_i(x, y) \colon (U_i \times U_i) \times V \to (U_i \times U_i) \times V$ by

$$\sigma_i(x, y) = (\Psi_i^y) \tau_i(x, y) (\Psi_i^x)^{-1}, \quad \tau_i(x, y) = (\Psi_i^y)^{-1} \sigma_i(x, y) (\Psi_i^x).$$

The restriction of (2.8) to the chart $\{(U_{i_0} \times U_{i_0})\}$ may be thought of as *a local quasi-connection form*:

$$\begin{aligned}\tau_{(U_{i_0}\times U_{i_0})}(x,y) &= (\Psi^y_{i_0})\tau(x,y)(\Psi^x_{i_0})^{-1} = (\Psi^y_{i_0})\Big(\sum_i f_i(x,y)\tau_i(x,y)\Big)(\Psi^x_{i_0})^{-1}\\ &= (\Psi^y_{i_0})\Big(\sum_i f_i(x,y)(\Psi^y_i)^{-1}\sigma_i(x,y)(\Psi^x_i)\Big)(\Psi^x_{i_0})^{-1}\\ &= (\varphi_{i,i_0}(y))\Big(\sum_i f_i(x,y)\sigma_i(x,y)\Big)(\varphi_{i_0,i}(x)).\end{aligned}$$

For example, if $\sigma_i(x,y) \equiv \mathrm{id}$, then we get the definition of the canonical quasi-connection associated to the transition functions and partition of unity

$$\tau_{(U_j\times U_j)}(x,y) = \Big(\sum_i f_i(x,y)\varphi_{i,j}(y)\varphi_{j,i}(x)\Big). \tag{2.9}$$

On the intersection of two charts $(U_j \times U_j) \cap (U_k \times U_k)$ the quasi-connection satisfies the relations

$$\begin{aligned}\tau_{(U_j\times U_j)}(x,y) &= (\Psi^y_j)\tau(x,y)(\Psi^x_j)^{-1}\\ &= (\Psi^y_j)(\Psi^y_k)^{-1}\tau_{(U_k\times U_k)}(x,y)(\Psi^x_k)(\Psi^x_j)^{-1}\\ &= \varphi_{k,j}(y)\tau_{(U_k\times U_k)}(x,y)\varphi_{j,k}(x).\end{aligned} \tag{2.10}$$

2.2.1. Almost flat transition functions. Suppose that the transition functions are ε-almost flat, that is

$$\|\varphi_{jk}(x)\varphi_{kj}(y) - \mathrm{id}\| \le \varepsilon, \quad x, y \in U_j \cap U_k.$$

Then from (2.9) one gets

$$\begin{aligned}\|\tau_{(U_j\times U_j)}(x,y) - \mathrm{id}\| &= \Big(\sum_i f_i(x,y)\|\varphi_{i,j}(y)\varphi_{j,i}(x) - \mathrm{id}\|\Big)\\ &\le \Big(\sum_i f_i(x,y)\Big)\varepsilon = \varepsilon.\end{aligned} \tag{2.11}$$

Viceversa, if the quasi-connection satisfies (2.11) on a chart atlas with the transition functions $\varphi_{jk}(x)$, from (2.10) we get

$$\begin{aligned}&\|\varphi_{j,k}(y)\varphi_{k,j}(x) - \mathrm{id}\|\\ &\le \|\varphi_{j,k}(y)\tau_{(U_j\times U_j)}(x,y)\varphi_{k,j}(x) - \mathrm{id}\| + \|\varphi_{j,k}(y)(\tau_{(U_j\times U_j)}(x,y) - \mathrm{id})\varphi_{k,j}(x)\|\\ &= \|\tau_{(U_k\times U_k)}(x,y) - \mathrm{id}\| + \|\varphi_{j,k}(y)(\tau_{(U_j\times U_j)}(x,y) - \mathrm{id})\varphi_{k,j}(x)\|\\ &\le \|\tau_{(U_k\times U_k)}(x,y) - \mathrm{id}\| + \|\varphi_{j,k}(y)\| \cdot \|\tau_{(U_j\times U_j)}(x,y) - \mathrm{id}\| \cdot \|\varphi_{k,j}(x)\|\\ &= \|\tau_{(U_k\times U_k)}(x,y) - \mathrm{id}\| + \|\tau_{(U_j\times U_j)}(x,y) - \mathrm{id}\| \le 2\varepsilon.\end{aligned}$$

Here we have assumed that the matrices $\varphi_{k,j}(x)$ are unitary and hence $\|\varphi_{k,j}(x)\| = 1$. The condition (2.11) does depend on the choice of the transition functions.

2.2.2. Equivalence between transition functions and quasi-connections. Given the transition functions $\varphi_{ij}(x)$, by means of (2.9) we have constructed the quasi-connection $\tau(x,y)$. Successively, using (2.1), to this quasi-connection, we can associate the transition functions Φ_{ij}:

$$\Phi_{ij}(x) = \tau(a_j,x)^{-1} \cdot \tau(a_i,x) \colon \xi_{a_i} \to \xi_{a_j}.$$

These transition functions involve the coordinate homomorphisms

$$\Psi_i^x \colon \xi_x \longrightarrow \xi_{a_i} \xrightarrow{f_i} V, \quad \Psi_i^x = f_i \tau(a_i,x)^{-1}.$$

The next proposition establishes the mutual reciprocity between the two constructions

PROPOSITION 2.1. $\varphi_{ij}(x) \equiv \Phi_{ij}(x)$.

PROOF. In fact,

$$\tau_j(x,y) = \Big(\sum_i f_i(x,y)\varphi_{i,j}(y)\varphi_{j,i}(x)\Big). \tag{2.12}$$

If $U_i = \operatorname{star}(a_i)$, then

$$\Phi_{ij}(x) = \tau^{-1}(a_j,x)\tau(a_i,x) \colon \xi_{a_i} \to \xi_{a_j}. \tag{2.13}$$

In the local form, on the chart U_j, the equation (2.13) can be represented as

$$\Phi_{ij}(x) = \varphi_{ij}(a_j)\tau_i^{-1}(a_j,x)\tau_i(a_i,x). \tag{2.14}$$

From (2.14) one gets

$$\varphi_{ji}(a_j)\Phi_{ij}(x) = \tau_i^{-1}(a_j,x)\tau_i(a_i,x)$$

or

$$\tau_i(a_j,x)\varphi_{ji}(a_j)\Phi_{ij}(x) = \tau_i(a_i,x).$$

Using (2.12), one has

$$\Big(\sum_k f_k(a_j,x)\varphi_{ki}(x)\varphi_{ik}(a_j)\Big)\varphi_{ik}(a_j)\Phi_{ij}(x) = \Big(\sum_k f_k(a_i,x)\varphi_{k,i}(x)\varphi_{i,k}(a_i)\Big). \tag{2.15}$$

Assume, that the atlas has the property that the point $a_i \in U_i$ does not belong to any other chart. This condition is not essentially limitative since it holds for the stars of the simplicial structure. In this case the equality (2.15) can be simplified:

$$f_j(a_j,x)\varphi_{ji}(x)\varphi_{ij}(a_j)\varphi_{ji}(a_j)\Phi_{ij}(x) = f_i(a_i,x)\varphi_{i,i}(x)\varphi_{i,i}(a_i)$$

or

$$f_j(a_j,x)\varphi_{ji}(x)\Phi_{ij}(x) = f_i(a_i,x)\varphi_{i,i}(x)\varphi_{i,i}(a_i) = \mathrm{id},$$

or

$$\varphi_{ji}(x)\Phi_{ij}(x) = \mathrm{id}.$$

In other words

$$\Phi_{ij}(x) = \varphi_{ij}(x). \qquad \square$$

The proposition (2.1) shows that our attention can be focused onto the transition functions constructed by means the quasi-connection by formula (2.13), or more precisely, by the formula (2.14). Then, from (2.14) we can express easily the property of ε-almost flatness of the bundle in terms of quasi-connections. One has

$$\begin{aligned} \varphi_{ij}(x) &= \varphi_{ij}(a_j)\tau_i^{-1}(a_j,x)\tau_i(a_i,x),\\ \varphi_{ij}(a_i) &= \varphi_{ij}(a_j)\tau_i^{-1}(a_j,a_i),\\ \varphi_{ij}(a_j) &= \varphi_{ij}(a_j)\tau_i(a_i,a_j), \end{aligned}$$

that is

$$\begin{aligned} \varphi_{ji}(a_j)\varphi_{ij}(x) &= \tau_i^{-1}(a_j,x)\tau_i(a_i,x),\\ \tau_i(a_j,a_i) &= \varphi_{ji}(a_i)\varphi_{ij}(a_j),\\ \tau_i(a_i,a_j) &= \mathrm{id}. \end{aligned}$$

From (2.12) one has

$$\begin{aligned} \tau_i(a_i,x) &= \left(\sum_k f_k(a_i,x)\varphi_{k,i}(x)\varphi_{i,k}(a_i)\right)\\ &= f_i(a_i,x)\varphi_{i,i}(x)\varphi_{i,i}(a_i) = \mathrm{id},\\ \tau_i(x,a_j) &= \left(\sum_k f_k(x,a_j)\varphi_{k,i}(a_j)\varphi_{i,k}(x)\right)\\ &= f_j(x,a_j)\varphi_{j,i}(a_j)\varphi_{i,j}(x) = \varphi_{j,i}(a_j)\varphi_{i,j}(x). \end{aligned}$$

Hence,

$$\tau_i^{-1}(a_j,x) = \tau_i(x,a_j).$$

This is in agreement with the following definition:

Definition 2.2. We say that the quasi-connection is ε-almost flat if for any chart U_i and point $x \in U_i$ the following inequalities hold:

$$\|\tau(a_i, a_j) - \tau(x, a_j)\tau(a_i, x)\| \le \varepsilon, \tag{2.16}$$

where $x \in \operatorname{star}(a_i)$, $a_j \in \overline{\operatorname{star}(a_i)}$.

In particular, for ε-almost flat quasi-connection and arbitrary 2-simplex $\sigma = (a_i, a_j, a_k)$ one has:

$$\|\tau(a_i, a_j) - \tau(a_k, a_j)\tau(a_i, a_k)\| \le \varepsilon.$$

Proposition 2.3 (This proposition was stated in [2, p. 273] for ε-flat bundles). *Let M be a simply connected simplicial space. There exists a constant $S(M) > 0$, which depends only on the simplicial structure of the space M, such that if $\varepsilon \cdot S(M) \ll 1$, then any ε-almost flat quasi-connection of the vector bundle ξ generates a trivialization of ξ.*

Proof. Fix a vertex $a_0 \in \Gamma$. Consider a tree $\Gamma \subset M$, consisting of all vertices of M and some edges, such that any vertex $a \in M$ is connected with a_0 by a unique path $\gamma = \gamma_a = (a_0, \dots, a_{k-1}, a_k = a)$, which belongs to the tree Γ. Put

$$T(a) = T_{a_k} \stackrel{\text{def}}{=} \tau(a_k, a_{k-1})\tau(a_{k-1}, a_{k-2}) \cdots \tau(a_2, a_1)\tau(a_1, a_0). \tag{2.17}$$

The homomorphism $T(a)$ defined by (2.17) is defined on all vertices $a_i \in V(M)$ of the simplicial space M:

$$T(a)\colon \xi_{a_0} \to \xi_a.$$

We intend to extend the function $T(a)$ to the whole space M. Let $x \in M$ be an arbitrary point; suppose it belongs to the simplex $\sigma = (a_0, a_1, \dots, a_k)$ and suppose it is represented in barycentric coordinates by

$$x = \sum_{i=0}^{k} \lambda_i a_i, \quad \lambda_i \ge 0, \quad \sum \lambda_i = 1.$$

Put

$$T(x) = \sum_{i=0}^{k} \lambda_i(\tau(a_i, x) \cdot T(a_i))\colon \xi_{a_0} \to \xi_x.$$

In order for the operator $T(x)$ to be invertible, it would be sufficient to keep small the distances between the operators $T(a_i)$ and $T(a_j)$ for all vertices of the simplex σ. The pair of vertices a_i and a_j generates the closed path $l(a_i, a_j) = \gamma_{a_j}^{-1} \cdot [a_i, a_j] \cdot \gamma_{a_i}$, which on the account of the simply-connectedness of the space M can be represented as a composition of n_{ij} defining relations in the co-representation of fundamental group of M. Denote by $S(M)$ the maximum of numbers n_{ij}. Provided that $\varepsilon \cdot S(M) \ll 1$ one has

$$\|\tau(a_i, x) \cdot T(a_i) - \tau(a_j, x) \cdot T(a_j)\| \ll 1,$$

and hence the operator (2.18) is invertible.

The continuous function $T(x)$ provides a trivialization of the bundle ξ. □

Remark 2.4. Moreover, the bundle ξ can be extended to a trivial bundle over the cone $CM = (M \times I)/(M \times \{1\}$, along with the extension of the coordinate transition functions so that they will satisfy the condition of ε'-almost flatness for some $\varepsilon' > 0$. Similarly, the quasi-connection τ can be extended over the cone CM, seen as a simplicial space, so that the condition (2.16) holds.

3. Extension of almost flat bundles over compact subspaces of $B\pi_1(M)$

Connes, Gromov and Moscovici [2] proved that for any almost flat bundle α on the manifold M, the higher signature $\operatorname{sign}_x(M)$, $x = \operatorname{ch}\alpha$, does depend only on the homotopy equivalence class of the manifold M.

This means that the cohomology class $x = \operatorname{ch}\alpha \in H^*(M; Q)$ lies in the image of the natural mapping f_M^*, where $f_M\colon M \to B\pi_1(M)$ is the classifying mapping. We may assume that the mapping f_M is an embedding if, for example, the classifying space $B\pi_1(M)$ is realized by adding cells to M so that all homotopy groups of degree ≥ 2 will be killed. In other words, for any compact subspace $Y \subset B\pi_1(M)$, $f_M(M) \subset Y$, there exists an integer n and a bundle $\beta \in \mathrm{K}(Y)$ such that

$$n\alpha = f_M^*(\beta),$$

that is

$$n\alpha \in \operatorname{Im}(f_M^*).$$

The geometric arguments discussed above allow us to show that the bundle α itself, and not necessarily a certain multiple n of it, comes from an arbitrarily large compact subspace $Y \subset B\pi_1(M)$ through the classifying mapping. In other words, we are going to show that any almost flat bundle α can be extended to a bundle β over an arbitrary compact CW-subspace $Y \subset B\pi_1(M)$.

However we do not know if it is possible to choose this extension β to be almost flat.

Theorem 3.1. *Let M be a connected, non simply connected, compact* CW-*complex with $\pi_1(M) = \pi$ and let $f_M\colon M \to B\pi$ be a continuous embedding inducing an isomorphism on the fundamental groups. Let α be an almost flat bundle* *on M. Then for any compact* CW*-subspace $Y \subset B\pi$, $f_M(M) \subset Y$ there exists a bundle β on the space Y such that*

$$\alpha = f_M^*(\beta).$$

Proof. Let α be an almost flat bundle on M. According to the definition (2.4), (2.5) this means that there are two sequences $\xi = \{\xi_k\}$ and $\eta = \{\eta_k\}$ with fixed coordinate charts U_i and transition functions

$$\varphi_{ij}^{k,s}(x), \quad x \in U_{ij}, \quad s = 1, 2,$$

such that

$$\lim_{k\to\infty} \sup_{x,y\in U_{ij}} \|\varphi_{ij}^{k,s}(x) - \varphi_{ij}^{k,s}(y)\| = 0, \quad s = 1, 2,$$

$$f_k\colon \xi_k \approx \eta_k \oplus \alpha.$$

Let Y be constructed from M by attaching a finite number of cells. We are going to show that the bundle α can be extended to the next cell attached to the previous space. Since the fundamental group of the space M coincides with the fundamental group of $B\pi$, all attached cells have dimension ≥ 3. In other words, the space Y appears as the end of an increasing sequence of CW-complexes

$$M = Y_0 \subset Y_1 \subset \ldots \subset Y_N = Y,$$

where each space Y_k has the form

$$Y_k = Y_{k-1} \cup_\varphi \mathrm{D}^s,$$

with attaching mapping

$$\varphi\colon \mathrm{S}^{(s-1)} \to Y_{(k-1)}, \quad s \geq 3.$$

Suppose that the almost flat bundle α has already been extended as an almost flat bundle over Y_{k-1}; let (ξ_k and η_k) be the corresponding pair of ε-flat extended bundles.

Consider the next attaching mappings φ above. Then $\varphi^*(\alpha)$ is an almost flat bundle over the sphere $\mathrm{S}^{(s-1)}$. As the image of the mapping φ is compact in Y_{k-1}, the induced bundles $\varphi^*(\xi_l)$, $\varphi^*(\eta_l)$ are $K(\varphi)\varepsilon_l$-flat bundles,where $\varepsilon_l \to 0$ and $K(\varphi)$ is a constant which depends on the attaching function φ. The base space of these bundles, $\mathrm{S}^{(s-1)}$, is a simply connected sphere. Therefore, if ε_l is sufficiently small, Proposition 2.3 implies that these bundles are trivial. Therefore, both of them can be extended over the cone of the mapping φ, i.e. over the attached cell. The Remark 2.4 states that the extended bundles $\varphi^*(\xi_l)$, $\varphi^*(\eta_l)$ over the cone are $K'(\varphi)\varepsilon_l$-flat bundles, where the constant $K'(\varphi)$ may be evaluated.

After a finite number of steps the desired extension will be performed. □

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Alexander S. Mishchenko
Department of Mathematics
Moscow State University
119899 Leninskie Gory
Moscow, RUSSIA

E-mail address: asmish@mech.math.msu.su

Nicolae Teleman
Dipartimento di Scienze Matematica
Universita Politecnica delle Marche
60161 Ancona, ITALIA

E-mail address: teleman@dipmat.inivpm.it