Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 26, 2005, 35–74

HOMOLOGY INDEX BRAIDS IN INFINITE-DIMENSIONAL CONLEY INDEX THEORY

Maria C. Carbinatto — Krzysztof P. Rybakowski

Dedicated to the memory of Olga Ladyzhenskaya

Abstract. We extend the notion of a categorial Conley–Morse index, as defined in [20], to the case based on a more general concept of an index pair introduced in [12]. We also establish a naturality result of the long exact sequence of attractor-repeller pairs with respect to the choice of index triples. In particular, these results immediately give a complete and rigorous existence result for homology index braids in infinite dimensional Conley index theory.

Finally, we describe some general regular and singular continuation results for homology index braids obtained in our recent papers [6] and [7].

1. Introduction

The concept of the categorial Morse index for flows on locally compact spaces is a refinement of Conley index. It was developed by Conley [8] and his students (mainly Kurland [14]). Roughly speaking, the categorial Morse index (or Conley–Morse index) $I(S)$ of a compact isolated invariant set S (relative to a given flow) is a connected simple system and a subcategory of the homotopy category of pointed spaces with objects $(N_1/N_2,[N_2])$ where (N_1,N_2) is an index pair in some compact isolating neighbourhood N of S. The morphisms of $I(S)$ are inclusion or flow induced. Later Franzosa [9]–[11] used a somewhat more general

2000 *Mathematics Subject Classification.* Primary 37B30; Secondary 35B25, 35B20, 37C70.
Key words and phrases. Morse–Conley index theory, homology index braid, continuation properties, singular perturbations.
The research of the first author was partially supported by the grant CNPq–Brazil 302854/2002-9.

concept of an index pair and an ensuing categorial Conley–Morse index, more suitable for applications to Morse-decompositions and homology index braids.

The Conley index theory and the categorial Conley–Morse index were extended by Rybakowski [19], [20] to semiflows on (not necessarily locally compact) metric spaces. The isolating neighbourhoods in that theory are required to satisfy an admissibility condition, making the theory applicable to various classes of evolution equations. The concept of index pairs in this extended theory is analogous to that used in [8] and [14].

Parts of Franzosa's theory of Morse-decompositions and homology index braids were extended by Franzosa and Mischaikow [12] to the setting of [19] and [20]. These authors use a definition of index pairs which is the analogue of Franzosa's definition in the locally compact case.

Motivated by [12] we define in the present paper a categorial Conley–Morse index $\mathcal{C}(S)$ whose objects are generated by index pairs in the sense of [12] (rather than index pairs as defined in [19]). We also establish existence of $\mathcal{C}(S)$ (Propositions 4.1 and 4.2) and prove that certain types of inclusion induced morphisms lie in $\mathcal{C}(S)$ (Propositions 4.4 and 4.5). These results are not only of interest in themselves but they are also needed for a precise definition of long exact sequences of attractor-repeller pairs in the non-locally compact case considered here.

Simplifying slightly the approach of Kurland [16] we also define the category of connected simple systems in a given category $\mathcal{K}$. Moreover, for a given connected simple system $\mathcal{C}$ in $\mathcal{K}$ and a functor Φ from $\mathcal{K}$ to a module category, we define the image module $\hat{\Phi}(\mathcal{C})$ (cf Section 3).

All this allows us, in Section 5, to define the long exact homology sequence of an attractor-repeller pair (A, A^*) in S, associated with a given index triple (N_1, N_2, N_3) for (S, A, A^*). In Theorem 5.1 we prove that this sequence is independent of the choice of (N_1, N_2, N_3).

These results also resolve some technical issues which remained open in the derivation of the homology index braid as outlined in [12] (the hints given on pp. 282–283 of [12] are insufficient for that). In particular, we can now proceed exactly as in [9] and [12] to obtain a precise definition of the homology index braid for a given (partially ordered) Morse-decomposition. This is done in Section 6, in which we also discuss morphisms from one homology index pair to another. In particular, we define inclusion induced morphisms between homology index braids and show that, under a certain nesting property, these morphism are isomorphisms.

In Section 7, which is based on our recent paper [6], we consider a sequence π_n, $n \in \mathbb{N}_0$, of local semiflows on X and a sequence $(\pi_n, S_n, (M_{p,n})_{p\in P})$, $n \in \mathbb{N}_0$, of Morse-decompositions such that $(\pi_n, S_n, (M_{p,n})_{p\in P})$ regularly converges to

$(\pi_0, S_0, (M_{p,0})_{p\in P})$. We state the *nested index filtration theorem* (Theorem 7.3), which immediately implies a general (regular) continuation result for homology index braids and Morse-decompositions (Theorems 7.4 and 7.6). We apply this result to Galerkin approximations of semilinear parabolic equations.

Finally, in Section 8, based on our recent work [7], we state a nested index filtration theorem in the context of singular perturbation problems (Theorem 8.5), which implies a general singular continuation result for homology index braids and connection matrices (Theorem 8.6). We apply this result to reaction-diffusion equations on thin domains.

2. Preliminaries

The purpose of this section is to recall a few concepts from Conley index theory and to establish some preliminary results needed later in this paper. We assume the reader's familiarity with the (infinite-dimensional) Conley index theory, as expounded in the papers [19] and [20] (or the book [21]), and with the papers [9], [11] and [12].

Let X be a topological space. Choose an arbitrary, but fixed point $\overline{p} \notin X$. Let A, Y be subspaces of X. Suppose first that $Y \cap A \neq \emptyset$. Define an equivalence relation on Y by letting $x \sim y$ if and only if $x = y$ or $x, y \in Y \cap A$. We denote by Y/A the quotient space of Y modulo this equivalence relation. We write $[A]$ to denote the equivalence class of any member x of $Y \cap A$. Set-theoretically, $[A] = Y \cap A$. We endow Y/A with the quotient topology.

Now let $Y \cap A = \emptyset$. We endow the set $X' := X \cup \{\overline{p}\}$ with the sum topology, i.e. U is *open in* $X \cup \{\overline{p}\}$ if and only if $U \cap X$ is open in X. Setting $Y' := Y \cup \{\overline{p}\}$, $A' := \{\overline{p}\}$ we define Y'/A' and $[A']$ as above and set $Y/A := Y'/A'$ and $[A] := [A']$. Note that $[A] = \{\overline{p}\}$ this time.

With our choice $\overline{p} \notin X$ the following simple result holds.

Proposition 2.1. *If* $A \subset Y \subset X$ *then the pair* (Y, A) *is uniquely determined by the pointed space* $(Y/A, [A])$.

Proof. If $A \neq \emptyset$, then $A = [A]$ while Y is the union of all equivalence classes of the relation $\sim$, i.e. $Y = \bigcup Y/A$. If $A = \emptyset$, then $Y = \{\, y \in X \mid \{y\} \in Y/A \,\}$. $\square$

Remark. If A, Y are subspaces of a topological space X, we will often denote the pointed space $(Y/A, [A])$ simply by Y/A. This should not lead to confusion.

For the rest of this paper, unless otherwise specified, X is a metric space, π is a local semiflow on X and all (the relevant) concepts are defined *relative to* π.

Suppose that Y is a subset of X. By $\mathrm{Inv}^+_\pi(Y)$, resp. $\mathrm{Inv}^-_\pi(Y)$, resp. $\mathrm{Inv}_\pi(Y)$ we denote the largest positively invariant, resp. negatively invariant, resp. invariant subset of Y. Moreover, let the function $\rho_Y = \rho_{Y,\pi}\colon Y \to \mathbb{R} \cup \{\infty\}$ be given by

(2.1) $\rho_Y(x) := \sup\{\, t \geq 0 \mid x\pi t \text{ is defined and } x\pi[0,t] \subset Y \,\}.$

It is clear that

(2.2) if Y, $Y' \subset X$ and $x \in Y \cap Y'$, then $\rho_{Y \cap Y'}(x) = \min(\rho_Y(x), \rho_{Y'}(x))$.

Y is called *π-admissible* if Y is closed and whenever $(x_n)_n$ and $(t_n)_n$ are such that $t_n \to \infty$ and $x_n\pi\,[0,t_n] \subset Y$ for all $n \in \mathbb{N}$, then the sequence $(x_n\pi t_n)_n$ has a convergent subsequence. We say that π *does not explode in* Y if whenever $x \in X$ and $x\pi t \in Y$ as long as $x\pi t$ is defined, then $x\pi t$ is defined for all $t \in [0,\infty[$. Y is called *strongly π-admissible* if Y is π-admissible and π does not explode in Y.

Let N and Y be subsets of X. The set Y is called *N-positively invariant* if whenever $x \in Y$, $t \geq 0$ are such that $x\pi\,[0,t] \subset N$, then $x\pi\,[0,t] \subset Y$.

Let N, Y_1 and Y_2 be subsets of X. The set Y_2 is called an *exit ramp for N within* Y_1 if whenever $x \in Y_1$ and $x\pi t' \notin N$ for some $t' \in [0,\infty[$, then there exists a $t_0 \in [0,t']$ such that $x\pi\,[0,t_0] \subset N$ and $x\pi t_0 \in Y_2$.

If Y_1 and Y_2 are subsets of X then Y_2 is called an *exit ramp for* Y_1 if Y_2 is an exit ramp for N within Y_1, where $N = Y_1$.

Definition 2.2. Let $B \subset X$ be a closed set and $x \in \partial B$. The point x is called a *strict egress* (respectively *strict ingress*, respectively *bounce-off*) *point of* B, if for every solution $\sigma\colon [-\delta_1,\delta_2] \to X$ of π through x, with $\delta_1 \geq 0$ and $\delta_2 > 0$, the following properties hold:

(a) There exists an $\varepsilon_2 \in \,]0,\delta_2[$ such that $\sigma(t) \notin B$ (respectively $\sigma(t) \in \operatorname{Int}_X(B)$, respectively $\sigma(t) \notin B$), for $t \in \,]0,\varepsilon_2]$.
(b) If $\delta_1 > 0$, then there exists an $\varepsilon_1 \in \,]0,\delta_1[$ such that $\sigma(t) \in \operatorname{Int}_X(B)$ (respectively $\sigma(t) \notin B$, respectively $\sigma(t) \notin B$), for $t \in [-\varepsilon_1,0[$.

The set of all strict egress (respectively strict ingress, respectively bounce-off) points of B is denoted by B^e (respectively B^i, respectively B^b). Moreover, we call $B^- := B^e \cup B^b$ the *exit set of* B and $B^+ := B^i \cup B^b$ the *entrance set of* B. B is called an *isolating block*, if $\partial B = B^e \cup B^i \cup B^b$ and B^- is closed. If B is also an isolating neighbourhood of an invariant set S, then we say that B is an *isolating block for* S.

If B is an isolating block then (B,B^-) is an example of an index pair in B. More generally, we have the following definition.

Definition 2.3. Let N be closed in X. A pair (N_1,N_2) is called an *index pair in* N (*relative to* π) if:

(a) N_1 and N_2 are closed and N-positively invariant subsets of N;
(b) N_2 is an exit ramp for N within N_1;
(c) $\operatorname{Inv}_\pi(N)$ is closed and $\operatorname{Inv}_\pi(N) \subset \operatorname{Int}_X(N_1 \setminus N_2)$.

Definition 2.4. A pair (N_1, N_2) is called a *Franzosa–Mischaikow-index pair* (*or* FM-*index pair*) *for* (π, S) if:

(a) N_1 and N_2 are closed subsets of X with $N_2 \subset N_1$ and N_2 is N_1-positively invariant;
(b) N_2 is an exit ramp for N_1;
(c) S is closed, $S \subset \operatorname{Int}_X(N_1 \setminus N_2)$ and S is the largest invariant set in $\operatorname{Cl}_X(N_1 \setminus N_2)$;

Proposition 2.5 (cf. [12]). *Let (N_1, N_2) be a pair of closed subsets of X with $N_2 \subset N_1$.*

(a) *If S is an isolated invariant set, N_1 is an isolating neighbourhood of S and (N_1, N_2) is an index pair in N_1, then (N_1, N_2) is an* FM-*index pair for (π, S).*
(b) *If (N_1, N_2) is an* FM-*index pair for (π, S) and N is an isolating neighbourhood of S with $N_1 \setminus N_2 \subset N$, then $N_1 \cap N$ is an isolating neighbourhood of S and $(N_1 \cap N, N_2 \cap N)$ is an index pair in $N_1 \cap N$.*

Proposition 2.6. *Let N_1, N_2 and N be closed subsets of X with $N_1 \setminus N_2 \subset N$. Then the inclusion induced map*

$$j\colon (N_1 \cap N)/(N_2 \cap N) \to N_1/N_2$$

is an isomorphism in the category of pointed spaces.

Proof. Proposition I.6.2 in [21] implies that j is a continuous map. Moreover, there is an inclusion induced map (in the sense of Definition I.6.1 in [21]) $k\colon N_1/N_2 \to (N_1 \cap N)/(N_2 \cap N)$ which is also continuous (by Proposition I.6.2 in [21]).

We need to show that k is the inverse of j. Let $z \in (N_1 \cap N)/(N_2 \cap N)$. If $z = [x]$, where $x \in (N_1 \cap N) \setminus (N_2 \cap N)$, then $j(z) = [x] \in N_1/N_2$ and so $k(j(z)) = [x] \in (N_1 \cap N)/(N_2 \cap N)$. Otherwise, $z = [N_2 \cap N]$ and $j(z) = [N_2]$. Thus, $k(j(z)) = [N_2 \cap N]$, since k and j are base-point preserving maps.

Let $z \in N_1/N_2$. If $z = [x]$, where $x \in N_1 \setminus N_2$, then $k(z) = [x] \in (N_1 \cap N)/(N_2 \cap N)$ and $x \in (N_1 \cap N) \setminus (N_2 \cap N)$. Therefore, $j(k(z)) = [x] \in N_1/N_2$. Otherwise, $z = [N_2]$ and $k(z) = [N_2 \cap N]$ and so $j(k(z)) = [N_2]$. □

Definition 2.7. Let S be a compact invariant set and (A, A^*) be an attractor-repeller pair in S, relative to π. A pair (B_1, B_2) is called a *block pair* (*for* (π, S, A, A^*)) if B_1 is an isolating block for A^*, B_2 is an isolating block for A, $B := B_1 \cup B_2$ is an isolating block for S and $B_1 \cap B_2 \subset B_1^- \cap B_2^+$.

If (B_1, B_2) is a block pair then $(B, B_2 \cup B^-, B^-)$ is an example of an FM-index triple:

Definition 2.8. Let S be a compact invariant set and (A, A^*) be an attractor-repeller pair in S relative to π. A triple (N_1, N_2, N_3) with $N_3 \subset N_2 \subset N_1$ is called an FM-*index triple* (*for* (π, S, A, A^*)) if (N_1, N_3) is an FM-index pair for (π, S) and (N_2, N_3) is an FM-index pair for (π, A).

Proposition 2.9 (cf. [12]). *Let* (N_1, N_2, N_3) *be an* FM-*index triple for* (π, S, A, A^*). *Then* (N_1, N_2) *is an* FM-*index pair for* (π, A^*).

For the rest of this paper we fix a (commutative) ring Γ and a Γ-module $\overline{G}$. Given a chain complex C, we denote by $H_q(C)$, $q \in \mathbb{Z}$, the homology of C with coefficients in $\overline{G}$.

Recall (cf. [11]) that a sequence

$$C_1 \xrightarrow{\ i\ } C_2 \xrightarrow{\ p\ } C_3$$

of chain maps is called *weakly exact* if $\ker i = 0$, $p \circ i = 0$ and the map $H_q(\rho): H_q(C_2/\operatorname{im} i) \to H_q(C_3)$ is an isomorphism for each $q \in \mathbb{Z}$. Here, the map $\rho: C_2/\operatorname{im} i \to C_3$ is the (uniquely determined) chain map with $\rho \circ Q = p$, where $Q: C_2 \to C_2/\operatorname{im} i$ is the quotient map.

Given a weakly exact sequence

$$C_1 \xrightarrow{\ i\ } C_2 \xrightarrow{\ p\ } C_3$$

and $q \in \mathbb{Z}$, define $\widehat{\partial}_q: H_q(C_3) \to H_{q-1}(C_1)$ by $\widehat{\partial}_q := \partial_{*q} \circ H_q(\rho)^{-1}$, where $\partial_{*q}: H_q(C_2/\operatorname{im} i) \to H_{q-1}(C_1)$ is the connecting homomorphism in the long exact sequence induced by the short exact sequence

$$0 \longrightarrow C_1 \xrightarrow{\ i\ } C_2 \xrightarrow{\ Q\ } C_2/\operatorname{im} i \longrightarrow 0.$$

Using elementary homology theory we obtain the following result.

Proposition 2.10 (cf. [11]). *Given a weakly exact sequence*

$$C_1 \xrightarrow{\ i\ } C_2 \xrightarrow{\ p\ } C_3$$

the corresponding homology sequence

$$\longrightarrow H_q(C_1) \xrightarrow{H_q(i)} H_q(C_2) \xrightarrow{H_q(p)} H_q(C_3) \xrightarrow{\ \widehat{\partial}_q\ } H_{q-1}(C_1) \longrightarrow$$

is exact. Moreover, given a commutative diagram

$$\begin{tikzcd}
C_1 \arrow[r, "i"] \arrow[d, "f_1"'] & C_2 \arrow[r, "p"] \arrow[d, "f_2"] & C_3 \arrow[d, "f_3"] \\
\widetilde{C}_1 \arrow[r, "\widetilde{i}"'] & \widetilde{C}_2 \arrow[r, "\widetilde{p}"'] & \widetilde{C}_3
\end{tikzcd}$$

of chain maps with weakly exact rows, the induced long homology ladder

$$\begin{tikzcd}
\arrow[r] & H_q(C_1) \arrow[r, "H_q(i)"] \arrow[d, "H_q(f_1)"'] & H_q(C_2) \arrow[r, "H_q(p)"] \arrow[d, "H_q(f_2)"'] & H_q(C_3) \arrow[r, "\widehat{\partial}_q"] \arrow[d, "H_q(f_3)"] & H_{q-1}(C_1) \arrow[r] \arrow[d, "H_{q-1}(f_1)"] & {} \\
\arrow[r] & H_q(\widetilde{C}_1) \arrow[r, "H_q(\widetilde{i})"'] & H_q(\widetilde{C}_2) \arrow[r, "H_q(\widetilde{p})"'] & H_q(\widetilde{C}_3) \arrow[r, "\widetilde{\widehat{\partial}_q}"'] & H_{q-1}(\widetilde{C}_1) \arrow[r] & {}
\end{tikzcd}$$

is commutative.

If Y is a topological space, then $\Delta(Y)$ denotes the singular chain complex (see [23]). If (Y, B) is a topological pair, we define

$$C(Y/B) = C(Y/B, \{[B]\}) := \Delta(Y/B)/\Delta(\{[B]\}).$$

As usual, we set

$$H_q(Y/B) := H_q(C(Y/B)), \quad q \in \mathbb{Z}.$$

Thus,

(2.3) for $q \in \mathbb{Z}$, $H_q(Y/B)$ is the q-th singular homology group of the pair $(Y/B, \{[B]\})$, with coefficients in $\overline{G}$.

Proposition 2.11 (cf. [11] and [12]). *Let (N_1, N_2, N_3) be an* FM*-index triple for (π, S, A, A^*) with $\mathrm{Cl}_X(N_1 \setminus N_3)$ strongly π-admissible. Then the inclusion induced sequence*

$$N_2/N_3 \xrightarrow{\ i\ } N_1/N_3 \xrightarrow{\ p\ } N_1/N_2 \tag{2.4}$$

of pointed spaces induces a weakly exact sequence

$$C(N_2/N_3) \xrightarrow{\ i\ } C(N_1/N_3) \xrightarrow{\ p\ } C(N_1/N_2)$$

of chain maps.

Propositions 2.10 and 2.11 thus imply the following result.

Proposition 2.12. *Let (N_1, N_2, N_3) be an* FM*-index triple for (π, S, A, A^*) with $\mathrm{Cl}_X(N_1 \setminus N_3)$ strongly π-admissible. Then the long sequence*

$$\longrightarrow H_q(N_2/N_3) \xrightarrow{H_q(i)} H_q(N_1/N_3) \xrightarrow{H_q(p)} H_q(N_1/N_2) \xrightarrow{\widehat{\partial}_q} H_{q-1}(N_2/N_3) \longrightarrow \tag{2.5}$$

induced by (2.4) *is exact.*

There is a similar result for Alexander–Spanier cohomology [23]. More precisely, let H^q, $q \in \mathbb{Z}$, denote the q-th Alexander–Spanier cohomology functor with values in $\overline{G}$. If Y and B are closed in X and $B \subset Y$, then the strong excision property of Alexander–Spanier cohomology implies that the quotient map

$$Q = Q_{Y,B} \colon (Y, B) \to (Y/B, \{[B]\})$$

induces a module isomorphism

$$H^q(Q): H^q(Y/B, \{[B]\}) \to H^q(Y, B), \quad q \in \mathbb{Z}.$$

Again we set $H^q(Y/B) := H^q(Y/B, \{[B]\})$ for short. In other words,

(2.6) for $q \in \mathbb{Z}$, $H^q(Y/B)$ is the q-th Alexander–Spanier cohomology group of the pair $(Y/B, \{[B]\})$, with coefficients in $\overline{G}$.

Therefore, given a triple (N_1, N_2, N_3) of closed sets in X with $N_1 \supset N_2 \supset N_3$ we can define, for each $q \in \mathbb{Z}$, the map

$$\widehat{\partial}^q: H^{q+1}(N_2/N_3) \to H^q(N_1/N_2)$$

by

$$\widehat{\partial}^q = H^q(Q_{N_1,N_2})^{-1} \circ \partial^{q*} \circ H^{q+1}(Q_{N_2,N_3}),$$

where $\partial^{q*}: H^{q+1}(N_2, N_3) \to H^q(N_1, N_2)$ is the connecting homomorphism of the exact cohomology sequence for the triple (N_1, N_2, N_3).

From the cohomology sequence of space triples we thus obtain the following result.

Proposition 2.13. *Let (N_1, N_2, N_3) be a triple of closed sets in X with $N_1 \supset N_2 \supset N_3$. Then inclusion induced sequence*

$$N_2/N_3 \xrightarrow{i} N_1/N_3 \xrightarrow{p} N_1/N_2$$

induces a long exact cohomology sequence

$$\longleftarrow H^q(N_2/N_3) \xleftarrow{H^q(i)} H^q(N_1/N_3) \xleftarrow{H^q(p)} H^q(N_1/N_2) \xleftarrow{\widehat{\partial}^q} H^{q+1}(N_2/N_3) \longleftarrow$$

3. Categories of connected simple systems

In this section, simplifying a little the approach by Kurland [16], we will define categories of connected simple subsystems of a given category. We will also define images of connected simple systems under functors with values in a category of modules. These notions are required for a precise development of the categorial Conley–Morse index and the long exact (co)homology sequence of an attractor-repeller pair.

Let $\mathcal{K}$ be a fixed category. The letters $\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{C}''$ will denote subcategories of $\mathcal{K}$ which are connected simple systems.

Given objects A, B in $\mathcal{C}$ and objects A', B' in $\mathcal{C}'$ and $\alpha \in \mathrm{Mor}_{\mathcal{K}}(A, A')$, $\beta \in \mathrm{Mor}_{\mathcal{K}}(B, B')$ we say that *α is related to β in $\mathcal{K}$ relative to* $(\mathcal{C}, \mathcal{C}')$ (and we write $\alpha\varrho_{\mathcal{C},\mathcal{C}'}\beta$ or just $\alpha\varrho\beta$) if and only if the following diagram commutes (in $\mathcal{K}$):

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r, "\beta"'] & B' \end{tikzcd}$$

Here, f (resp. f') are the unique elements of $\mathrm{Mor}_{\mathcal{C}}(A,B)$ (resp. $\mathrm{Mor}_{\mathcal{C}'}(A',B')$). Since f and f' are isomorphisms (in $\mathcal{K}$), it follows that $\alpha\varrho\beta$ implies $\beta\varrho\alpha$. Moreover, the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\mathrm{Id}_A"'] & A \arrow[d, "\mathrm{Id}_A"] \\ A \arrow[r, "\alpha"'] & A \end{tikzcd}$$

commutes, so $\alpha\varrho\alpha$. If the diagrams

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r, "\beta"'] & B' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B \arrow[r, "\beta"] \arrow[d, "g"'] & B' \arrow[d, "g'"] \\ C \arrow[r, "\gamma"'] & C' \end{tikzcd}$$

commute, then so does the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "g\circ f"'] & A' \arrow[d, "g'\circ f'"] \\ C \arrow[r, "\gamma"'] & C' \end{tikzcd}$$

Thus $\alpha\varrho\beta$ and $\beta\varrho\gamma$ imply that $\alpha\varrho\gamma$. It follows that $\varrho = \varrho_{\mathcal{C},\mathcal{C}'}$ is an equivalence relation on the set

$$\Omega(\mathcal{C},\mathcal{C}') := \bigcup\{\mathrm{Mor}_{\mathcal{K}}(A,A') \mid A \in \mathrm{Obj}(\mathcal{C}) \text{ and } A' \in \mathrm{Obj}(\mathcal{C}')\}. \tag{3.1}$$

Given $\alpha \in \Omega(\mathcal{C},\mathcal{C}')$, let $[\alpha] = [\alpha]_{\varrho_{\mathcal{C},\mathcal{C}'}}$ be the equivalence class of α.

We define a category $[\mathcal{K}]$ whose objects are all the subcategories of $\mathcal{K}$ which are connected simple systems. Given objects $\mathcal{C}$ and $\mathcal{C}'$ in $\mathcal{K}$, let $\mathrm{Mor}_{[\mathcal{K}]}(\mathcal{C},\mathcal{C}')$ be the set of all ζ for which there is an $\alpha \in \Omega(\mathcal{C},\mathcal{C}')$ with $\zeta = [\alpha]$.

(In order to make the morphism sets mutually disjoint, as is required in the definition of a category, one should more precisely consider ordered triples $(\zeta,\mathcal{C},\mathcal{C}')$ rather than just ζ to be morphisms from $\mathcal{C}$ to $\mathcal{C}'$. We shall not bother, however.)

Given $\zeta \in \mathrm{Mor}_{[\mathcal{K}]}(\mathcal{C},\mathcal{C}')$ and $\zeta' \in \mathrm{Mor}_{[\mathcal{K}]}(\mathcal{C}',\mathcal{C}'')$ let $\alpha\colon A \to A'$ and $\alpha'\colon C' \to A''$ be such that $\zeta = [\alpha]$ and $\zeta' = [\alpha']$. Let f' be the unique element of $\mathrm{Mor}_{\mathcal{C}'}(A',C')$ and define $\zeta'\circ\zeta := [\alpha'\circ f'\circ\alpha]$. We claim that this definition is independent of the choice of α and α'. In fact, if $\beta\colon B \to B'$ and $\beta'\colon D' \to B''$

are such that $\zeta = [\beta]$ and $\zeta' = [\beta']$ and $g\colon B' \to D'$ is the unique element of $\mathrm{Mor}_{\mathcal{C}'}(B', D')$, then the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d] & A' \arrow[r, "f'"] \arrow[d] & C' \arrow[r, "\alpha'"] \arrow[d] & A'' \arrow[d] \\ B \arrow[r, "\beta"'] & B' \arrow[r, "g'"'] & D' \arrow[r, "\beta'"'] & B''. \end{tikzcd}$$

Here, the vertical arrows are the unique morphisms in the respective connected simple systems. It follows that $(\alpha' \circ f' \circ \alpha)\varrho_{\mathcal{C},\mathcal{C}''}(\beta' \circ g' \circ \beta)$ and so $[\alpha' \circ f' \circ \alpha] = [\beta' \circ g' \circ \beta]$ which proves our claim.

Thus the composition $\zeta \circ \zeta'$ is well-defined and is clearly associative. Indeed, the consideration of sequences of the form

$$A \xrightarrow{\alpha} A' \xrightarrow{f'} C' \xrightarrow{\alpha'} A'' \xrightarrow{f''} C'' \xrightarrow{\alpha''} A'''$$

implies that $(\alpha'' \circ f'' \circ \alpha') \circ f' \circ \alpha = \alpha'' \circ f'' \circ (\alpha' \circ f' \circ \alpha)$. Thus, if $\zeta = [\alpha]$, $\zeta' = [\alpha']$ and $\zeta'' = [\alpha'']$, we have

$$(\zeta'' \circ \zeta') \circ \zeta = [\alpha'' \circ f'' \circ \alpha'] \circ [\alpha] = [(\alpha'' \circ f'' \circ \alpha') \circ f' \circ \alpha]$$

and

$$\zeta'' \circ (\zeta' \circ \zeta) = [\alpha''] \circ [\alpha' \circ f' \circ \alpha] = [\alpha'' \circ f'' \circ (\alpha' \circ f' \circ \alpha)].$$

Hence, $(\zeta'' \circ \zeta') \circ \zeta = \zeta'' \circ (\zeta' \circ \zeta)$. Moreover, the commutativity of the diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{Id}_A"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ B \arrow[r, "\mathrm{Id}_B"'] & B, \end{tikzcd}$$

where f is the unique element of $\mathrm{Mor}_{\mathcal{C}}(A, B)$, shows that $\mathrm{Id}_A \,\varrho\, \mathrm{Id}_B$ and so $[\mathrm{Id}_A] = [\mathrm{Id}_B]$ for any two objects in $\mathcal{C}$. We set $\mathrm{Id}_{\mathcal{C}} := [\mathrm{Id}_A]$, where A is any object in $\mathcal{C}$. Clearly whenever $\zeta \in \mathrm{Mor}_{[\mathcal{K}]}(\mathcal{C}, \mathcal{C}')$, then there are objects A and A' in $\mathcal{C}$ and $\mathcal{C}'$ respectively, such that $\zeta = [\alpha]$, where $\alpha \in \mathrm{Mor}_{\mathcal{K}}(A, A')$. Thus $\mathrm{Id}_A \circ \alpha = \alpha$ and $\alpha \circ \mathrm{Id}_A = \alpha$ so $\mathrm{Id}_{\mathcal{C}'} \circ \zeta = \zeta$ and $\zeta \circ \mathrm{Id}_{\mathcal{C}} = \zeta$. It follows that $\mathrm{Id}_{\mathcal{C}}$ is an identity for the composition in $[\mathcal{K}]$ and so $[\mathcal{K}]$ is, indeed, a category, which we term the *category of connected simple systems in* $\mathcal{K}$.

If $\mathcal{C}$, $\mathcal{C}'$ are objects in $[\mathcal{K}]$ and $\alpha \in \Omega(\mathcal{C}, \mathcal{C}')$ (with $\Omega(\mathcal{C}, \mathcal{C}')$ being defined in (3.1)), then $\zeta := [\alpha]_{\varrho_{\mathcal{C},\mathcal{C}'}}$ is called the *morphism in* $[\mathcal{K}]$ *induced by* α*, relative to* $(\mathcal{C}, \mathcal{C}')$.

Remark. The present definition of the category $[\mathcal{K}]$, while conceptually (hopefully!) simpler, is equivalent to the definition of the category $\mathcal{CSS}(\mathcal{K})$ given in [16] in the sense that $[\mathcal{K}]$ and $\mathcal{CSS}(\mathcal{K})$ are isomorphic categories.

Now, suppose Φ is a covariant functor from $\mathcal{K}$ to the category $\mathrm{Mod}(\Gamma)$ of modules over the (commutative) ring Γ. Let $\mathcal{C}$ be an object of $[\mathcal{K}]$. Let $S = S_{\mathcal{C}}$ be the *disjoint* union of all $\Phi(A)$, where A is an arbitrary object of $\mathcal{C}$. Thus, formally we have

$$S = S_{\mathcal{C}} := \bigcup_{A \in \mathrm{Obj}(\mathcal{C})} \Phi(A) \times \{A\}.$$

On $S = S_{\mathcal{C}}$ define a relation $R = R_{\mathcal{C}}$ as follows:

> $(x, A)R(y, B)$ if and only if $y = \Phi(f)x$, where f is the unique morphism in $\mathcal{C}$ from A to B.

Clearly, R is an equivalence relation on S. Let S/R be the set of equivalence classes of R and $Q = Q_{\mathcal{C}}: S \to S/R$ given by

$$Q((x, A)) = [(x, A)]_R \quad \text{for } (x, A) \in S$$

be the canonical quotient map. In the sequel we write $\widehat{\Phi}(\mathcal{C}) := S/R$.

For each $A \in \mathrm{Obj}(\mathcal{C})$, the map $Q_A = Q_{\mathcal{C},A}: \Phi(A) \to \widehat{\Phi}(\mathcal{C})$ given by $Q_A(x) = Q((x, A))$ for $x \in \Phi(A)$ is easily seen to be bijective. Moreover, if $(x, A)R(y, B)$ and $(\widetilde{x}, A)R(\widetilde{y}, B)$, then $(x +_A \widetilde{x}, A)R(y +_B \widetilde{y}, B)$ and $(\lambda \cdot_A x, A)R(\lambda \cdot_B y, B)$ for every $\lambda \in \Gamma$. Here, for every $C \in \mathrm{Obj}(\mathcal{C})$, $+_C$ (resp. $\cdot_C$) is the addition (resp. scalar multiplication) in the Γ-module $\Phi(C)$. Therefore, there is a unique addition $+ = +_{\mathcal{C}}$ and scalar multiplication $\cdot = \cdot_{\mathcal{C}}$ in $\widehat{\Phi}(\mathcal{C})$ such that for every $A \in \mathrm{Obj}(\mathcal{C})$, the map Q_A is a Γ-module isomorphism. The Γ-module $\widehat{\Phi}(\mathcal{C})$ is called *the image module of* $\mathcal{C}$ *under* Φ.

Now let $\mathcal{C}$ and $\mathcal{C}'$ be objects of $[\mathcal{K}]$ and $A \in \mathrm{Obj}(\mathcal{C})$, $A' \in \mathrm{Obj}(\mathcal{C}')$ be arbitrary. If F is a morphism in $\mathrm{Mod}(\Gamma)$ from $\Phi(A)$ to $\Phi(A')$, then define the map $\langle F \rangle: \widehat{\Phi}(\mathcal{C}) \to \widehat{\Phi}(\mathcal{C}')$ by

$$\langle F \rangle := Q_{\mathcal{C}',A'} \circ F \circ {Q_{\mathcal{C},A}}^{-1}.$$

Then $\langle F \rangle$ is a Γ-module homomorphism. Moreover,

Proposition 3.1. *Suppose A, $B \in \mathrm{Obj}(\mathcal{C})$, A', $B' \in \mathrm{Obj}(\mathcal{C}')$. If the diagram*

$$\begin{array}{ccc} \Phi(A) & \xrightarrow{F} & \Phi(A') \\ {\scriptstyle \Phi(f)}\downarrow & & \downarrow{\scriptstyle \Phi(f')} \\ \Phi(B) & \xrightarrow[G]{} & \Phi(B') \end{array}$$

commutes, then $\langle F \rangle = \langle G \rangle$, where f (resp. f') is the unique morphism in $\mathcal{C}$ (resp. $\mathcal{C}'$) from A to B (resp. from A' to B').

Proof. Let $\eta \in \widehat{\Phi}(\mathcal{C})$ be arbitrary. Then there exist an $x \in \Phi(A)$ and a $y \in \Phi(B)$ such that $\eta = Q\left((x, A)\right) = Q\left((y, B)\right)$. It follows that $y = \Phi(f)x$. Now

$$\langle F \rangle(\eta) = (Q_{A'} \circ F \circ {Q_A}^{-1})(\eta) = Q_{A'}Fx$$

and

$$\langle G\rangle(\eta) = (Q_{B'} \circ G \circ Q_B{}^{-1})(\eta) = Q_{B'}(Gy) = Q_{B'}G(\Phi(f)x) = Q_{B'}(\Phi(f')Fx).$$

Notice that $Q_{B'}(\Phi(f')Fx) = Q((\Phi(f')Fx, B')) = Q((Fx, A')) = Q_{A'}(Fx)$. This implies that $\langle F\rangle(\eta) = \langle G\rangle(\eta)$. The proposition is proved. □

The following result is obvious.

Proposition 3.2. *Let $\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{C}''$ be objects of $[\mathcal{K}]$ and $A \in \mathrm{Obj}(\mathcal{C})$, $A' \in \mathrm{Obj}(\mathcal{C}')$, $A'' \in \mathrm{Obj}(\mathcal{C}'')$ be arbitrary. Let F be a morphism in $\mathrm{Mod}(\Gamma)$ from $\Phi(A)$ to $\Phi(A')$ and F' be a morphism in $\mathrm{Mod}(\Gamma)$ from $\Phi(A')$ to $\Phi(A'')$ then*

$$\langle F' \circ F\rangle = \langle F'\rangle \circ \langle F\rangle.$$

If (F, F') is exact, i.e., $\ker F' = \mathrm{im}\, F$, then so is $(\langle F\rangle, \langle F'\rangle)$.

We call the assignment $F \mapsto \langle F\rangle$ the $\langle\,\cdot\,\rangle$*-operation* (*associated with* Φ).

4. The categorial Conley–Morse index

In this section we will extend the notion of the categorial Morse index from [20] in the sense that index pairs (and quasi-index pairs) will be replaced by FM-index pairs.

Let $\widetilde{\mathcal{K}}$ be the homotopy category of pointed spaces. If π is a local semiflow defined in a metric space X and S is an isolated π-invariant set admitting a strongly π-admissible isolating neighbourhood, then we define a subcategory $\mathcal{C}(S) = \mathcal{C}(\pi, S)$ as follows. The objects of $\mathcal{C}(S)$ are the pointed spaces $(E, p) = (N_1/N_2, [N_2])$, where (N_1, N_2) is an FM-index pair for (π, S) and $\mathrm{Cl}_X(N_1 \setminus N_2)$ is strongly π-admissible.

Given two objects (E, p) and $(\widetilde{E}, \widetilde{p})$ in $\mathcal{C}(S)$, Proposition 2.1 implies that there are unique FM-index pairs (N_1, N_2) and $(\widetilde{N}_1, \widetilde{N}_2)$ for (π, S) with $\mathrm{Cl}_X(N_1 \setminus N_2)$ and $\mathrm{Cl}_X(\widetilde{N}_1 \setminus \widetilde{N}_2)$ strongly π-admissible such that $(E, p) = (N_1/N_2, [N_2])$ and $(\widetilde{E}, \widetilde{p}) = (\widetilde{N}_1/\widetilde{N}_2, [\widetilde{N}_2])$. Let N and $\widetilde{N}$ be arbitrary strongly π-admissible isolating neighbourhoods of S with $N_1 \setminus N_2 \subset N$ and $\widetilde{N}_1 \setminus \widetilde{N}_2 \subset \widetilde{N}$ (e.g. we may take $N = \mathrm{Cl}_X(N_1 \setminus N_2)$ and $\widetilde{N} = \mathrm{Cl}_X(\widetilde{N}_1 \setminus \widetilde{N}_2)$). Then Proposition 2.5 implies that $(N_1 \cap N, N_2 \cap N)$ and $(\widetilde{N}_1 \cap \widetilde{N}, \widetilde{N}_2 \cap \widetilde{N})$ are index pairs in $N_1 \cap N$ and $\widetilde{N}_1 \cap \widetilde{N}$, respectively. Therefore, there is a unique morphism $\tau\colon N_1/N_2 \to \widetilde{N}_1/\widetilde{N}_2$ in $\widetilde{\mathcal{K}}$ making the following diagram commutative in $\widetilde{\mathcal{K}}$.

$$\begin{array}{ccc} (N_1 \cap N)/(N_2 \cap N) & \xrightarrow{\beta} & (\widetilde{N}_1 \cap \widetilde{N})/(\widetilde{N}_2 \cap \widetilde{N}) \\ \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\widetilde{\alpha}} \\ N_1/N_2 & \xrightarrow[\tau]{} & \widetilde{N}_1/\widetilde{N}_2 \end{array} \tag{4.1}$$

Here, α and $\widetilde{\alpha}$ are the homotopy classes of the inclusion induced maps defined in Proposition 2.6 and β is the unique morphism from $(N_1 \cap N)/(N_2 \cap N)$ to $(\widetilde{N}_1 \cap \widetilde{N})/(\widetilde{N}_2 \cap \widetilde{N})$ in the categorial Conley–Morse index $I(\pi, S)$ as defined in [19] or [21].

Proposition 4.1. *The definition of τ is independent of the choice of the sets N and $\widetilde{N}$.*

Proof. Let N' and $\widetilde{N}'$ be some other strongly π-admissible isolating neighbourhoods of S with $N_1 \setminus N_2 \subset N'$ and $\widetilde{N}_1 \setminus \widetilde{N}_2 \subset \widetilde{N}'$.

First we will assume that $N' \subset N$ and $\widetilde{N}' \subset \widetilde{N}$. Then we have the following diagram in $\widetilde{\mathcal{K}}$:

$$\begin{tikzcd}
(N_1 \cap N')/(N_2 \cap N') \arrow[r, "\beta'"] \arrow[d, "\gamma"'] & (\widetilde{N}_1 \cap \widetilde{N}')/(\widetilde{N}_2 \cap \widetilde{N}') \arrow[d, "\widetilde{\gamma}"] \\
(N_1 \cap N)/(N_2 \cap N) \arrow[r, "\beta"'] \arrow[d, "\alpha"'] & (\widetilde{N}_1 \cap \widetilde{N})/(\widetilde{N}_2 \cap \widetilde{N}) \arrow[d, "\widetilde{\alpha}"] \\
N_1/N_2 \arrow[r, "\tau"'] & \widetilde{N}_1/\widetilde{N}_2
\end{tikzcd}$$

Here, γ and $\widetilde{\gamma}$ are inclusion induced maps and β' is the unique morphism from $(N_1 \cap N')/(N_2 \cap N')$ to $(\widetilde{N}_1 \cap \widetilde{N}')/(\widetilde{N}_2 \cap \widetilde{N}')$ in the categorial Conley–Morse index $I(\pi, S)$. Notice that β, γ and $\widetilde{\gamma}$ are also morphisms in $I(\pi, S)$. Therefore, the upper diagram is commutative. Thus, the following diagram also commutes.

$$\begin{tikzcd}
(N_1 \cap N')/(N_2 \cap N') \arrow[r, "\beta'"] \arrow[d, "\alpha\circ\gamma"'] & (\widetilde{N}_1 \cap \widetilde{N}')/(\widetilde{N}_2 \cap \widetilde{N}') \arrow[d, "\widetilde{\alpha}\circ\widetilde{\gamma}"] \\
N_1/N_2 \arrow[r, "\tau"'] & \widetilde{N}_1/\widetilde{N}_2
\end{tikzcd}$$

Therefore, we have proved the proposition in the case $N' \subset N$ and $\widetilde{N}' \subset \widetilde{N}$. The general case can be reduced to this particular one by considering the intersections $N' \cap N$ and $\widetilde{N}' \cap \widetilde{N}$. This completes the proof. □

Using Proposition 4.1 we now define the set of morphisms of $\mathcal{C}(\pi, S)$ from $(E, p) = (N_1/N_2, [N_2])$ to $(\widetilde{E}, \widetilde{p}) = (\widetilde{N}_1/\widetilde{N}_2, [\widetilde{N}_2])$ as the singleton $\{\tau\}$ where τ is defined in (4.1). The morphism composition in $\mathcal{C}(\pi, S)$ is that of $\widetilde{\mathcal{K}}$. With these definitions the following result holds.

Proposition 4.2. *$\mathcal{C}(\pi, S)$ is a subcategory of $\widetilde{\mathcal{K}}$ and a connected simple system.*

Proof. Let (E, p), $(\widetilde{E}, \widetilde{p})$ and (E', p') be objects in $\mathcal{C}(\pi, S)$. It follows from Proposition 2.1 that there are unique FM-index pairs (N_1, N_2), $(\widetilde{N}_1, \widetilde{N}_2)$

and (N_1', N_2') for (π, S) with $N = \mathrm{Cl}_X(N_1 \setminus N_2)$, $\widetilde{N} = \mathrm{Cl}_X(\widetilde{N}_1 \setminus \widetilde{N}_2)$ and $N' = \mathrm{Cl}_X(N_1' \setminus N_2')$ strongly π-admissible isolating neighbourhoods such that $(E, p) = (N_1/N_2, [N_2])$, $(\widetilde{E}, \widetilde{p}) = (\widetilde{N}_1/\widetilde{N}_2, [\widetilde{N}_2])$ and $(E', p') = (N_1'/N_2', [N_2'])$. The following diagram

$$\begin{tikzcd}
(N_1 \cap N)/(N_2 \cap N) \arrow[r, "\beta"] \arrow[d, "\alpha"'] & (\widetilde{N}_1 \cap \widetilde{N})/(\widetilde{N}_2 \cap \widetilde{N}) \arrow[r, "\widetilde{\beta}"] \arrow[d, "\widetilde{\alpha}"] & (N_1' \cap N')/N_2' \cap N') \arrow[d, "\alpha'"] \\
N_1/N_2 \arrow[r, "\tau"'] & \widetilde{N}_1/\widetilde{N}_2 \arrow[r, "\widetilde{\tau}"'] & N_1'/N_2'
\end{tikzcd}$$

shows that the composite of two morphisms in $\mathcal{C}(\pi, S)$ is also a morphism in $\mathcal{C}(\pi, S)$. Moreover, the commutative diagram

$$\begin{tikzcd}
(N_1 \cap N)/(N_2 \cap N) \arrow[r, "\mathrm{Id}"] \arrow[d, "\alpha"'] & (N_1 \cap N)/(N_2 \cap N) \arrow[d, "\alpha"] \\
N_1/N_2 \arrow[r, "\mathrm{Id}"'] & N_1/N_2
\end{tikzcd}$$

shows that the identity morphism $\mathrm{Id}_{(E,p)}$ of $\widetilde{\mathcal{K}}$ lies in $\mathcal{C}(\pi, S)$ for every object (E, p) of $\mathcal{C}(\pi, S)$. Therefore, we have shown that $\mathcal{C}(\pi, S)$ is a subcategory of $\widetilde{\mathcal{K}}$. Since for each two objects (E, p) and $(\widetilde{E}, \widetilde{p})$ of $\mathcal{C}(\pi, S)$ there is exactly one morphism in $\mathcal{C}(\pi, S)$ from (E, p) to $(\widetilde{E}, \widetilde{p})$, we have that $\mathcal{C}(\pi, S)$ is a connected simple system. $\square$

We can now make the following definition.

Definition 4.3. Given an isolated π-invariant set S admitting a strongly π-admissible isolating neighbourhood, set $H_q(\pi, S) := \widehat{\Phi}(\mathcal{C}(\pi, S))$, where $\Phi = H_q$, $q \in \mathbb{Z}$, the q-th singular homology functor with coefficients in $\overline{G}$ (cf. (2.3)). The graded module $(H_q(\pi, S))_{q \in \mathbb{Z}}$ is called the *homology Conley index of S*. If $\Phi = H^q$, where H^q, $q \in \mathbb{Z}$, denotes the q-th Alexander–Spanier cohomology functor (cf. (2.6)), then $(H^q(\pi, S))_{q \in \mathbb{Z}}$, where $H^q(\pi, S) := \widehat{\Phi}(\mathcal{C}(\pi, S))$, $q \in \mathbb{Z}$, is called the *cohomology Conley index of S*.

In the remaining part of this section we will show that certain inclusion induced maps in $\widetilde{\mathcal{K}}$ between objects of $\mathcal{C}(\pi, S)$ are morphisms of $\mathcal{C}(\pi, S)$.

The first result is almost obvious.

Proposition 4.4. *Let (N_1, N_2) and $(\widetilde{N}_1, \widetilde{N}_2)$ be* FM-*index pairs for (π, S) with*

$$N_1 \setminus N_2 = \widetilde{N}_1 \setminus \widetilde{N}_2 \tag{4.2}$$

such that

$$N := \mathrm{Cl}_X(N_1 \setminus N_2) = \mathrm{Cl}_X(\widetilde{N}_1 \setminus \widetilde{N}_2)$$

is strongly π-admissible. Then the inclusion induced map

$$\tau\colon N_1/N_2 \to \widetilde{N}_1/\widetilde{N}_2$$

in $\widetilde{\mathcal{K}}$ lies in $\mathrm{Mor}_{\mathcal{C}(\pi,S)}((N_1/N_2,[N_2]),(\widetilde{N}_1/\widetilde{N}_2,[\widetilde{N}_2]))$ *and so is an isomorphism in $\widetilde{\mathcal{K}}$.*

Proof. By (4.2) there is a commutative diagram

$$\begin{tikzcd} (N_1\cap N)/(N_2\cap N) \arrow[r, "\beta"] \arrow[d, "\alpha"'] & (\widetilde{N}_1\cap N)/(\widetilde{N}_2\cap N) \arrow[d, "\widetilde{\alpha}"] \\ N_1/N_2 \arrow[r, "\tau"'] & \widetilde{N}_1/\widetilde{N}_2 \end{tikzcd} \tag{4.3}$$

and the map β is a morphism of $I(\pi,S)$. (Cf. Definition 9.2 in [21].) The proposition now follows from the definition of $\mathcal{C}(\pi,S)$. $\square$

The next proposition is harder to prove.

Proposition 4.5. *Let (N_1,N_2) and $(\widetilde{N}_1,\widetilde{N}_2)$ be* FM-*index pairs for (π,S) such that $\mathrm{Cl}_X(N_1\setminus N_2)$ and $\mathrm{Cl}_X(\widetilde{N}_1\setminus\widetilde{N}_2)$ are strongly π-admissible. Assume that $(N_1,N_2)\subset(\widetilde{N}_1,\widetilde{N}_2)$. Then the inclusion induced map $N_1/N_2\to\widetilde{N}_1/\widetilde{N}_2$ in $\widetilde{\mathcal{K}}$ lies in* $\mathrm{Mor}_{\mathcal{C}(\pi,S)}((N_1/N_2,[N_2]),(\widetilde{N}_1/\widetilde{N}_2,[\widetilde{N}_2]))$ *and so is an isomorphism in $\widetilde{\mathcal{K}}$.*

The rest of this section is devoted to the proof of Proposition 4.5.

Let N, Y be subsets of X such that $Y\subset N$. For $s\geq 0$, define

(4.4) $Y^{-s}=Y^{-s}(N):=\{\,x\in X\mid$ there is an s', $0\leq s'\leq s$, such that $x\pi s'$ is defined, $x\pi\,[0,s']\subset N$ and $x\pi s'\in Y\,\}$.

Proposition 4.6. *Let $s\in\,]0,\infty[$ and (N_1,N_2) be an* FM-*index pair for (π,S) such that π does not explode in $N_1\setminus N_2$. Then $(N_1,N_2^{-s}(N_1))$ is an* FM-*index pair for (π,S).*

Proof. We need to prove that the conditions of Definition 2.4 are satisfied for the pair $(N_1,N_2^{-s}(N_1))$. We only verify that

(4.5) $N_2^{-s}(N_1)$ is a closed set.

The other conditions are trivial to check. To prove (4.5) let $(x_n)_n$ be a sequence in $N_2^{-s}(N_1)$ such that $x_n\to x$ as $n\to\infty$. Since $x_n\in N_2^{-s}(N_1)$ for all $n\in\mathbb{N}$, it follows that for each $n\in\mathbb{N}$, there exists an $s'_n\in[0,s]$ such that $x_n\pi[0,s'_n]\subset N_1$ and $x_n\pi s'_n\in N_2$. Since $(s'_n)_n$ is a bounded sequence, without loss of generality, we can assume that there exists an $s'\in[0,s]$ such that $s'_n\to s'$ as $n\to\infty$. We need to show that $x\in N_2^{-s}(N_1)$. Now this will certainly be the case if $x\pi s'$ is defined.

First suppose that $\rho_{N_1}(x) = 0$. It follows that there exists a $\tau \in \,]0, s]$ such that $x\pi\tau$ is defined and $x\pi\tau \notin N_1$. Since $x_n \to x$ as $n \to \infty$, it follows that there exists an $n_\tau \in \mathbb{N}$ such that $x_n\pi\tau$ is defined and $x_n\pi\tau \notin N_1$ for all $n \geq n_\tau$. Hence $s_n < \tau$ for all $n \geq n_\tau$. Hence $s' \leq \tau$ and so $x\pi s'$ is defined.

Assume now that $\rho_{N_1}(x) > 0$. We have two cases.

First suppose that $x\pi\,[0, \rho_{N_1}(x)[\, \subset N_1 \setminus N_2$. Since π does not explode in $N_1 \setminus N_2$, it follows that $x\pi\rho_{N_1}(x)$ is defined. Moreover, there exists a $\delta > 0$ such that $x\pi(\rho_{N_1}(x) + \delta)$ is defined and $x\pi(\rho_{N_1}(x) + \delta) \notin N_1$. Hence $s'_n < \rho_{N_1}(x) + \delta$ for all n sufficiently large and so $s' \leq \rho_{N_1}(x) + \delta$. This implies that $x\pi s'$ is defined.

If $x\pi\,[0, \rho_{N_1}(x)[\, \not\subset N_1 \setminus N_2$, there exists a $t_0 \in [0, \rho_{N_1}(x)[$ such that $x\pi t_0 \in N_2$. If $t_0 \leq s$ then $x \in N_2^{-s}(N_1)$ and we are done. If $t_0 > s \geq s'$ then $x\pi s'$ is defined and we are done again.

This proves that $N_2^{-s}(N_1)$ is closed. □

Proposition 4.7. *Let $S \neq \emptyset$ be an isolated invariant set and N be a strongly π-admissible isolating neighbourhood of S. Then there is a $\delta_0 \in \,]0, \infty[$ and for all $\delta \in \,]0, \delta_0]$, there is an isolating block B_δ for S with $B_\delta \subset N$ such that*

(a) *$B_{\delta_2} \subset B_{\delta_1}$, $(B_{\delta_2})^- \subset (B_{\delta_1})^-$ for all δ_2, $\delta_1 \in \,]0, \delta_0]$ with $\delta_2 < \delta_1$;*
(b) *whenever $(\delta_n)_n$ and $(x_n)_n$ are sequences such that $\delta_n \to 0^+$ as $n \to \infty$ and $x_n \in B_{\delta_n}$ for all $n \in \mathbb{N}$, then there is a subsequence of $(x_n)_n$ that converges to an element of $\mathrm{Inv}_\pi^-(N)$.*

Proof. The proposition follows from the proof of Theorem I.5.1 in [21]. □

Lemma 4.8. *Let (N_1, N_2) and $(\widetilde{N}_1, \widetilde{N}_2)$ be* FM*-index pairs for (π, S) such that $\mathrm{Cl}_X(N_1 \setminus N_2)$ and $\mathrm{Cl}_X(\widetilde{N}_1 \setminus \widetilde{N}_2)$ are strongly π-admissible. Then there exist an $s \in [0, \infty[$, an isolating neighbourhood L_1 of S and an index pair (L_1, L_2) in L_1 such that*

$$(L_1, L_2) \subset (N_1, N_2^{-s}) \quad \textit{and} \quad (L_1, L_2) \subset (\widetilde{N}_1, \widetilde{N}_2^{-s}),$$

where $N_2^{-s} = N_2^{-s}(N_1)$ and $\widetilde{N}_2^{-s} = \widetilde{N}_2^{-s}(\widetilde{N}_1)$.

Proof. If $S = \emptyset$, define $L_1 = L_2 = \emptyset$. Let us assume that $S \neq \emptyset$. Define $N := \mathrm{Cl}_X(N_1 \setminus N_2) \cap \mathrm{Cl}_X(\widetilde{N}_1 \setminus \widetilde{N}_2)$. Thus, N is a strongly π-admissible isolating neighbourhood of S. Let $\delta_0 \in \,]0, \infty[$ and $(B_\delta)_{\delta \in]0, \delta_0]}$ be as in Proposition 4.7. We claim that

(4.6) there are an $s_0 \in \,]0, \infty[$ and a $\overline{\delta}_0 \in \,]0, \delta_0]$ such that $(B_\delta)^- \subset N_2^{-s}(N_1) \cap \widetilde{N}_2^{-s}(\widetilde{N}_1)$ for all $s \in [s_0, \infty[$ and $\delta \in \,]0, \overline{\delta}_0]$.

Suppose that (4.6) does not hold. Then there exist sequences $(s_n)_n$, $(\delta_n)_n$ and $(x_n)_n$ such that $s_n \to \infty$ and $\delta_n \to 0^+$ as $n \to \infty$ and for each $n \in \mathbb{N}$, $x_n \in (B_{\delta_n})^- \setminus (N_2^{-s_n}(N_1) \cap \widetilde{N}_2^{-s_n}(\widetilde{N}_1))$. Proposition 4.7 implies that there exists a subsequence of $(x_n)_n$, denoted again by $(x_n)_n$, and an $x \in \mathrm{Inv}_\pi^-(N)$ such that $x_n \to x$ as $n \to \infty$. It follows that $x \in \mathrm{Inv}_\pi^-(N_1) \cap \mathrm{Inv}_\pi^-(\widetilde{N}_1)$.

As $x_n \notin N_2^{-s_n}(N_1) \cap \widetilde{N}_2^{-s_n}(\widetilde{N}_1)$, it follows that for each $n \in \mathbb{N}$, $x_n\pi\,[0, s_n] \subset N_1 \setminus N_2$ or $x_n\pi\,[0, s_n] \subset \widetilde{N}_1 \setminus \widetilde{N}_2$. Since $s_n \to \infty$ as $n \to \infty$, this implies that $x \in \mathrm{Inv}_\pi^+(N_1) \cup \mathrm{Inv}_\pi^+(\widetilde{N}_1)$ (cf. Theorem I.4.5 in [21]) and so $x \in S$. However, $x_n \in (B_{\delta_n})^- \subset (B_{\delta_0})^-$ and this implies $x \in S \cap (B_{\delta_0})^- = \emptyset$ which is a contradiction. This proves (4.6).

Fix an $s \in [s_0, \infty[$ and a $\delta \in \,]0, \overline{\delta}_0]$. Define

$$L_1 := B_\delta \quad \text{and} \quad L_2 := (B_\delta)^-.$$

Thus, L_1 is an isolating neighbourhood of S and (L_1, L_2) is an index pair in L_1. Moreover, $L_1 \subset N$ and so, $L_1 \subset N_1$ and $L_1 \subset \widetilde{N}_1$. Inclusion (4.6) implies that $L_2 \subset N_2^{-s}$ and $L_2 \subset \widetilde{N}_2^{-s}$. The proof is complete. □

Proof of Proposition 4.5. Define $N := \mathrm{Cl}_X(N_1 \setminus N_2)$ and $\widetilde{N} := \mathrm{Cl}_X(\widetilde{N}_1 \setminus \widetilde{N}_2)$. Proposition 2.5 implies that $(N_1 \cap N, N_2 \cap N)$ is an index pair in $N_1 \cap N$ and $(\widetilde{N}_1 \cap \widetilde{N}, \widetilde{N}_2 \cap \widetilde{N})$ is an index pair in $\widetilde{N}_1 \cap \widetilde{N}$ and so $(N_1 \cap N, N_2 \cap N)$ and $(\widetilde{N}_1 \cap \widetilde{N}, \widetilde{N}_2 \cap \widetilde{N})$ are FM-index pairs for (π, S).

Lemma 4.8 implies that there are an $s \in [0, \infty[$, an isolating neighbourhood L_1 of S and an index pair (L_1, L_2) in L_1 such that

$$(L_1, L_2) \subset (N_1 \cap N, (N_2 \cap N)^{-s}) \quad \text{and} \quad (L_1, L_2) \subset (\widetilde{N}_1 \cap \widetilde{N}, (\widetilde{N}_2 \cap \widetilde{N})^{-s}),$$

where $(N_2 \cap N)^{-s} = (N_2 \cap N)^{-s}(N_1 \cap N)$ and $(\widetilde{N}_2 \cap \widetilde{N})^{-s} = (\widetilde{N}_2 \cap \widetilde{N})^{-s}(\widetilde{N}_1 \cap \widetilde{N})$.

Consider the following diagram in $\widetilde{\mathcal{K}}$:

$$\begin{tikzcd}
L_1/L_2 \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{Id}"'] & (N_1 \cap N)/(N_2 \cap N)^{-s} \arrow[r, "\alpha_2"] & (N_1 \cap N)/(N_2^{-s} \cap N) \arrow[r, "\alpha_5"] \arrow[d, "\beta_s"] & N_1/N_2^{-s} \arrow[d, "\tau_s"] \\
L_1/L_2 \arrow[r, "\alpha_3"'] & (\widetilde{N}_1 \cap \widetilde{N})/(\widetilde{N}_2 \cap \widetilde{N})^{-s} \arrow[r, "\alpha_4"'] & (\widetilde{N}_1 \cap \widetilde{N})/(\widetilde{N}_2^{-s} \cap \widetilde{N}) \arrow[r, "\alpha_6"'] & \widetilde{N}_1/\widetilde{N}_2^{-s}
\end{tikzcd}$$

where, $N_2^{-s} = N_2^{-s}(N_1)$, $\widetilde{N}_2^{-s} = \widetilde{N}_2^{-s}(\widetilde{N}_1)$ and the morphisms τ_s and α_i, $i \in \{1, \dots, 6\}$, are inclusion induced maps and β_s is the unique morphism from $(N_1 \cap N)/(N_2^{-s} \cap N)$ to $(\widetilde{N}_1 \cap \widetilde{N})/(\widetilde{N}_2^{-s} \cap \widetilde{N})$ lying in $I(\pi, S)$. Since all the morphisms in the left rectangle lie in $I(\pi, S)$, the left rectangle commutes and all the maps are isomorphisms. Hence

$$\alpha_6 \circ \beta_s = \alpha_6 \circ \alpha_4 \circ \alpha_3 \circ \alpha_1^{-1} \circ \alpha_2^{-1}. \tag{4.7}$$

Moreover, the full rectangle, obtained by taking out β_s, is inclusion induced and commutes. Thus

$$\tau_s \circ \alpha_5 \circ \alpha_2 \circ \alpha_1 = \alpha_6 \circ \alpha_4 \circ \alpha_3. \tag{4.8}$$

Equalities (4.7) and (4.8) imply that $\alpha_6 \circ \beta_s = \tau_s \circ \alpha_5$. In other words, the right rectangle also commutes.

By the definition of $\mathcal{C}(\pi, S)$ we thus see that the inclusion induced morphism τ_s is a morphism in $\mathcal{C}(\pi, S)$. Now consider the following diagram of inclusion induced maps:

$$\begin{tikzcd} N_1/N_2 \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & N_1/N_2^{-s} \arrow[d, "\tau_s"] \\ \widetilde{N}_1/\widetilde{N}_2 \arrow[r, "\widetilde{\alpha}"'] & \widetilde{N}_1/\widetilde{N}_2^{-s} \end{tikzcd}$$

It follows that this diagram commutes. If we show that α and $\widetilde{\alpha}$ are morphisms in $\mathcal{C}(\pi, S)$, then it follows that γ is also a morphism in $\mathcal{C}(\pi, S)$ and the proposition is proved. To show that α is a morphism in $\mathcal{C}(\pi, S)$ consider the diagram

$$\begin{tikzcd} (N_1 \cap N)/(N_2 \cap N) \arrow[r] \arrow[d] & (N_1 \cap N)/(N_2^{-s} \cap N) \arrow[d] \\ N_1/N_2 \arrow[r] & N_1/N_2^{-s} \end{tikzcd}$$

Since $N_1 \setminus N_2^{-s} \subset N_1 \setminus N_2 \subset N$, it follows from the definition of $\mathcal{C}(\pi, S)$ and Proposition 4.1 that α is a morphism in $\mathcal{C}(\pi, S)$. Analogously, we prove that $\widetilde{\alpha}$ is a morphism in $\mathcal{C}(\pi, S)$. The proof is complete. $\square$

5. Attractor-repeller pairs and long exact sequences in (co)homology

For the rest of this section let S be a compact π-invariant set and (A, A^*) be an attractor-repeller pair of S relative to π.

Let (N_1, N_2, N_3) be an FM-index triple for (π, S, A, A^*) such that $\mathrm{Cl}_X(N_1 \setminus N_3)$ strongly π-admissible. Then, by Propositions 2.12 and 2.13, the inclusion induced sequence

$$N_2/N_3 \xrightarrow{i} N_1/N_3 \xrightarrow{p} N_1/N_2$$

of pointed spaces induces the long exact sequences

$$\longrightarrow H_q(N_2/N_3) \xrightarrow{H_q(i)} H_q(N_1/N_3) \xrightarrow{H_q(p)} H_q(N_1/N_2) \xrightarrow{\widehat{\partial}_q} H_{q-1}(N_2/N_3) \longrightarrow$$

and

$$\longleftarrow H^q(N_2/N_3) \xleftarrow{H^q(i)} H^q(N_1/N_3) \xleftarrow{H^q(p)} H^q(N_1/N_2) \xleftarrow{\widehat{\partial}^q} H^{q+1}(N_2/N_3) \longleftarrow$$

By Proposition 3.2 and Definition 4.3 we obtain the long exact sequences

$$\longrightarrow H_q(\pi, A) \xrightarrow{\langle H_q(i)\rangle} H_q(\pi, S) \xrightarrow{\langle H_q(p)\rangle} H_q(\pi, A^*) \xrightarrow{\langle\widehat{\partial}_q\rangle} H_{q-1}(\pi, A) \longrightarrow \tag{5.1}$$

and

$$(5.2)\qquad \longleftarrow H^q(\pi,A) \xleftarrow{\langle H^q(i)\rangle} H^q(\pi,S) \xleftarrow{\langle H^q(p)\rangle} H^q(\pi,A^*) \xleftarrow{\langle \widehat{\partial}^q\rangle} H^{q+1}(\pi,A) \longleftarrow$$

The purpose of this section is to show that these sequences are independent of the choice of FM-index triples. More precisely we will prove the following result:

Theorem 5.1. *If* (N_1,N_2,N_3) *and* $(\widetilde{N}_1,\widetilde{N}_2,\widetilde{N}_3)$ *are* FM-*index triples for* (π,S,A,A^*) *with* $\mathrm{Cl}_X(N_1\setminus N_3)$ *and* $\mathrm{Cl}_X(\widetilde{N}_1\setminus\widetilde{N}_3)$ *strongly* π-*admissible, then the diagrams*

$$(5.3)\qquad \begin{array}{ccccccccc}
\longrightarrow & H_q(N_2/N_3) & \xrightarrow{H_q(i)} & H_q(N_1/N_3) & \xrightarrow{H_q(p)} & H_q(N_1/N_2) & \xrightarrow{\widehat{\partial}_q} & H_{q-1}(N_2/N_3) & \longrightarrow \\
& {\scriptstyle H_q(\iota_A)}\downarrow & & {\scriptstyle H_q(\iota_S)}\downarrow & & \downarrow{\scriptstyle H_q(\iota_{A^*})} & & \downarrow{\scriptstyle H_{q-1}(\iota_A)} & \\
\longrightarrow & H_q(\widetilde{N}_2/\widetilde{N}_3) & \xrightarrow[H_q(\widetilde{i})]{} & H_q(\widetilde{N}_1/\widetilde{N}_3) & \xrightarrow[H_q(\widetilde{p})]{} & H_q(\widetilde{N}_1/\widetilde{N}_2) & \xrightarrow[\widetilde{\widehat{\partial}}_q]{} & H_{q-1}(\widetilde{N}_2/\widetilde{N}_3) & \longrightarrow
\end{array}$$

and

$$(5.4)\qquad \begin{array}{ccccccccc}
\longleftarrow & H^q(N_2/N_3) & \xleftarrow{H^q(i)} & H^q(N_1/N_3) & \xleftarrow{H^q(p)} & H^q(N_1/N_2) & \xleftarrow{\widehat{\partial}^q} & H^{q+1}(N_2/N_3) & \longleftarrow \\
& {\scriptstyle H^q(\iota_A)}\uparrow & & {\scriptstyle H^q(\iota_S)}\uparrow & & \uparrow{\scriptstyle H^q(\iota_{A^*})} & & \uparrow{\scriptstyle H^{q+1}(\iota_A)} & \\
\longleftarrow & H^q(\widetilde{N}_2/\widetilde{N}_3) & \xleftarrow[H^q(\widetilde{i})]{} & H^q(\widetilde{N}_1/\widetilde{N}_3) & \xleftarrow[H^q(\widetilde{p})]{} & H^q(\widetilde{N}_1/\widetilde{N}_2) & \xleftarrow[\widetilde{\widehat{\partial}}^q]{} & H^{q+1}(\widetilde{N}_2/\widetilde{N}_3) & \longleftarrow
\end{array}$$

commute, where ι_A *is the unique morphism from* N_2/N_3 *to* $\widetilde{N}_2/\widetilde{N}_3$ *in* $\mathcal{C}(\pi,A)$, ι_S *is the unique morphism from* N_1/N_3 *to* $N_1/\widetilde{N}_3$ *in* $\mathcal{C}(\pi,S)$ *and* ι_{A^*} *is the unique morphism from* N_1/N_2 *to* $\widetilde{N}_1/\widetilde{N}_2$ *in* $\mathcal{C}(\pi,A^*)$.

In view of Theorem 5.1 and Proposition 3.1 we have

$$\langle H_q(i)\rangle = \langle H_q(\widetilde{i})\rangle, \quad \langle H_q(p)\rangle = \langle H_q(\widetilde{p})\rangle \quad \textit{and} \quad \langle\widehat{\partial}_q\rangle = \langle\widetilde{\widehat{\partial}}_q\rangle, \quad q\in\mathbb{Z}.$$

Therefore the sequence (5.1) is indeed independent of the choice of an FM-index triple and exact (by Proposition 3.2). This sequence is called the *homology index sequence of* (π,S,A,A^*). Similarly, we see that sequence (5.2) is independent of the choice of an FM-index triple and exact. This sequence is called the *cohomology index sequence of* (π,S,A,A^*).

The rest of this section is devoted to the proof of Theorem 5.1. We need some preliminary results.

Proposition 5.2. *Let* $s\in\,]0,\infty[$ *and let* (N_1,N_2,N_3) *be an* FM-*index triple for* (π,S,A,A^*) *such that* π *does not explode in* $N_1\setminus N_3$. *Then* $(N_1,N_2\cup N_3^{-s}$,

N_3^{-s}), *where* $N_3^{-s} = N_3^{-s}(N_1)$, *is an* FM-*index triple for* (π, S, A, A^*). *Moreover, the inclusion induced diagram of pointed spaces*

$$\begin{tikzcd} N_2/N_3 \arrow[r] \arrow[d] & N_1/N_3 \arrow[r] \arrow[d] & N_1/N_2 \arrow[d] \\ (N_2 \cup N_3^{-s})/N_3^{-s} \arrow[r] & N_1/N_3^{-s} \arrow[r] & N_1/(N_2 \cup N_3^{-s}) \end{tikzcd}$$

commutes.

Proof. The proof is a simple exercise, using Proposition 4.6. □

Lemma 5.3. *Suppose that* (N_1, N_2, N_3) *and* $(\widetilde{N}_1, \widetilde{N}_2, \widetilde{N}_3)$ *are* FM-*index triples for* (π, S, A, A^*) *with* $\mathrm{Cl}_X(N_1 \setminus N_3)$ *and* $\mathrm{Cl}_X(\widetilde{N}_1 \setminus \widetilde{N}_3)$ *strongly* π-*admissible. Then there exist a* $\tau \in [0, \infty[$ *and an* FM-*index triple* (L_1, L_2, L_3) *for* (π, S, A, A^*) *such that*

(5.5) *both* $(L_1, L_2, L_3) \subset (N_1, N_2 \cup N_3^{-\tau}, N_3^{-\tau})$ *and* $(L_1, L_2, L_3) \subset (\widetilde{N}_1, \widetilde{N}_2 \cup \widetilde{N}_3^{-\tau}, \widetilde{N}_3^{-\tau})$, *where* $N_3^{-\tau} = N_3^{-\tau}(N_1)$ *and* $\widetilde{N}_3^{-\tau} = \widetilde{N}_3^{-\tau}(\widetilde{N}_1)$.

Proof. If $S = \emptyset$, define $L_1 = L_2 = L_3 = \emptyset$. Let us assume that $S \neq \emptyset$. Define $N := \mathrm{Cl}_X(N_1 \setminus N_3) \cap \mathrm{Cl}_X(\widetilde{N}_1 \setminus \widetilde{N}_3)$. Thus, N is a strongly π-admissible isolating neighbourhood of S. Let $\delta_0 \in]0, \infty[$ and $(B_\delta)_{\delta \in]0, \delta_0]}$ be as in Proposition 4.7 (with the present choice of the set N). Proceeding as the proof of Lemma 4.8 we see that there are an $s_0 \in]0, \infty[$ and a $\overline{\delta}_0 \in]0, \delta_0]$ such that

$$(5.6)\qquad (B_\delta)^- \subset N_3^{-s}(N_1) \cap \widetilde{N}_3^{-s}(\widetilde{N}_1) \quad \text{for all } s \in [s_0, \infty[\text{ and } \delta \in]0, \overline{\delta}_0].$$

Fix $s_1 \in [s_0, \infty[$ and define $N_1' := N_1$, $N_2' := N_2 \cup N_3^{-s_1}(N_1)$, $N_3' := N_3^{-s_1}(N_1)$, $\widetilde{N}_1' := \widetilde{N}_1$, $\widetilde{N}_2' := \widetilde{N}_2 \cup \widetilde{N}_3^{-s_1}(\widetilde{N}_1)$ and $\widetilde{N}_3' := \widetilde{N}_3^{-s_1}(\widetilde{N}_1)$. It follows from Proposition 5.2 that (N_1', N_2', N_3') and $(\widetilde{N}_1', \widetilde{N}_2', \widetilde{N}_3')$ are FM-index triples for (π, S, A, A^*). Moreover,

$$(5.7)\qquad (B_\delta)^- \subset N_3' \cap \widetilde{N}_3' \quad \text{for all } \delta \in]0, \overline{\delta}_0].$$

We claim that

(5.8) there exists an $\overline{s}_0 \in]0, \infty[$ and a $\delta_1 \in]0, \overline{\delta}_0]$ such that $B_\delta \cap (N_3' \cup \widetilde{N}_3') \subset N_3'^{-s}(N_1') \cap \widetilde{N_3'}^{-s}(\widetilde{N}_1')$ for all $s \in [\overline{s}_0, \infty[$ and $\delta \in]0, \delta_1]$.

Suppose that (5.8) does not hold. Then there exist sequences $(s_n)_n$, $(\delta_n)_n$ and $(x_n)_n$ such that $s_n \to \infty$ and $\delta_n \to 0^+$ as $n \to \infty$ and for each $n \in \mathbb{N}$, $x_n \in (B_{\delta_n} \cap (N_3' \cup \widetilde{N}_3')) \setminus (N_3'^{-s_n}(N_1') \cap \widetilde{N_3'}^{-s_n}(\widetilde{N}_1'))$.

Proposition 4.7 implies that there exists a subsequence of $(x_n)_n$, denoted again by $(x_n)_n$, and an $x \in \mathrm{Inv}_\pi^-(N)$ such that $x_n \to x$. Hence

$$x \in \mathrm{Inv}_\pi^-(\mathrm{Cl}_X(N_1 \setminus N_3)) \cap \mathrm{Inv}_\pi^-(\mathrm{Cl}_X(\widetilde{N}_1 \setminus \widetilde{N}_3)).$$

Since $x_n \notin N_3'^{-s_n}(N_1') \cap \widetilde{N_3'}^{-s_n}(\widetilde{N}_1')$ for all $n \in \mathbb{N}$, it follows that for each $n \in \mathbb{N}$, $x_n\pi\,[0, s_n] \subset N_1' \setminus N_3'$ or $x_n\pi\,[0, s_n] \subset \widetilde{N}_1' \setminus \widetilde{N_3'}$. Since $s_n \to \infty$ as $n \to \infty$, this implies that $x \in \mathrm{Inv}_\pi^+(\mathrm{Cl}_X(N_1' \setminus N_3')) \cup \mathrm{Inv}_\pi^+(\mathrm{Cl}_X(\widetilde{N}_1' \setminus \widetilde{N}_3'))$ (cf. Theorem I.4.5 in [21]). Since $N_1' = N_1$, $N_3 \subset N_3'$, $\widetilde{N}_1' = \widetilde{N}_1$ and $\widetilde{N}_3 \subset \widetilde{N}_3'$, it follows that $x \in \mathrm{Inv}_\pi^+(\mathrm{Cl}_X(N_1 \setminus N_3)) \cup \mathrm{Inv}_\pi^+(\mathrm{Cl}_X(\widetilde{N}_1 \setminus \widetilde{N}_3))$ and so $x \in S$. We thus obtain that $x \in S \cap (N_3' \cup \widetilde{N}_3') = \emptyset$ which is a contradiction and so our claim is proved.

Fix an $s \in [\bar{s}_0, \infty[$ and a $\delta \in\,]0, \delta_1]$. Define

$$\begin{aligned}
L_1 &:= B_\delta, \\
L_2 &:= (B_\delta \cap (N_2' \cap \widetilde{N}_2')) \cup (B_\delta \cap (N_3' \cup \widetilde{N}_3')), \\
L_3 &:= B_\delta \cap (N_3' \cup \widetilde{N}_3').
\end{aligned}$$

Since $B_\delta \subset N \subset N_1 \cap \widetilde{N}_1$ we obtain that

$$L_1 \subset N_1 \quad \text{and} \quad L_1 \subset \widetilde{N}_1. \tag{5.9}$$

Inclusion (5.8) implies that $L_2 \subset N_2' \cup N_3'^{-s}(N_1')$, $L_2 \subset \widetilde{N}_2' \cup \widetilde{N_3'}^{-s}(\widetilde{N}_1')$, $L_3 \subset N_3'^{-s}(N_1')$ and $L_3 \subset \widetilde{N_3'}^{-s}(\widetilde{N}_1')$. Let $x \in N_3'^{-s}(N_1')$. Thus, there is an $s' \in [0, s]$ such that $x\pi s'$ is defined, $x\pi\,[0, s'] \subset N_1' = N_1$ and $x\pi s' \in N_3'$. Since $x\pi s' \in N_3' = N_3^{-s_1}(N_1)$, it follows that there is an $s'' \in [0, s_1]$ such that $(x\pi s')\pi s''$ is defined, $(x\pi s')\pi\,[0, s''] \subset N_1$ and $(x\pi s')\pi s'' \in N_3$. Thus, $x\pi\,[0, s'' + s'] \subset N_1$ and $x\pi(s'' + s') \in N_3$ with $0 \le s'' + s' \le s + s_1$. In other words, $x \in {N_3}^{-(s+s_1)}(N_1)$. Therefore,

$$L_2 \subset N_2 \cup {N_3}^{-s_1}(N_1) \cup {N_3}^{-(s+s_1)}(N_1) \subset N_2 \cup {N_3}^{-\tau}(N_1), \tag{5.10}$$

where $\tau := s + s_1$. Moreover

$$L_3 \subset N_3'^{-s}(N_1') \subset {N_3}^{-(s+s_1)}(N_1) = {N_3}^{-\tau}(N_1). \tag{5.11}$$

Analogously we obtain that

$$L_2 \subset \widetilde{N}_2 \cup \widetilde{N}_3^{-s_1}(\widetilde{N}_1) \cup \widetilde{N}_3^{-(s+s_1)}(\widetilde{N}_1) \subset \widetilde{N}_2 \cup \widetilde{N}_3^{-\tau}(\widetilde{N}_1) \tag{5.12}$$

and

$$L_3 \subset \widetilde{N_3'}^{-s}(\widetilde{N}_1') \subset \widetilde{N}_3^{-(s+s_1)}(\widetilde{N}_1) = \widetilde{N}_3^{-\tau}(\widetilde{N}_1). \tag{5.13}$$

Inclusions (5.9)–(5.13) imply the inclusions in (5.5).

To finish the proof we need to show that (L_1, L_2, L_3) is an FM-index triple for (π, S, A, A^*).

We claim that (L_1, L_3) is an FM-index pair for (π, S). Indeed, notice that $S \subset \mathrm{Int}_X(L_1)$ and $S \cap (N_3' \cup \widetilde{N}_3') = \emptyset$. Thus, $S \subset \mathrm{Int}_X(L_1 \setminus L_3)$ and so $S \subset \mathrm{Inv}_\pi(\mathrm{Cl}_X(L_1 \setminus L_3))$. On the other hand, $L_1 \setminus L_3 \subset N_1' \setminus N_3'$ and this implies that $\mathrm{Inv}_\pi(\mathrm{Cl}_X(L_1 \setminus L_3)) \subset \mathrm{Inv}_\pi(\mathrm{Cl}_X(N_1' \setminus N_3')) = S$.

Let $x \in L_3$ and $t \geq 0$ be such that $x\pi\,[0,t] \subset L_1$. Hence, $x \in L_1 \cap (N'_3 \cup \widetilde{N}'_3)$ and so $x \in N'_3 \cup \widetilde{N}'_3$. Since N'_3 is N'_1-positively invariant and $\widetilde{N}'_3$ is $\widetilde{N}'_1$-positively invariant, it follows that $x\pi\,[0,t] \subset N'_3 \cup \widetilde{N}'_3$ so $x\pi\,[0,t] \subset L_3$ and so L_3 is L_1-positively invariant.

Let $x \in L_1$ be such that $x\pi t \notin L_1$ for some $t > 0$. Since (B_δ, B_δ^-) is an index pair in B_δ, there is a $t' \in [0,t[$ such that $x\pi\,[0,t'] \subset L_1$ and $x\pi t' \in (B_\delta)^-$. Inclusion (5.7) implies that $x\pi t' \in N'_3 \cap \widetilde{N}'_3$ and so $x\pi t' \in L_3$. Thus L_3 is an exit ramp for L_1. The proof of our claim is complete.

We now claim that (L_2, L_3) is an FM-index pair for (π, A). Note that $A \subset \mathrm{Int}_X(N'_2) \cap \mathrm{Int}_X(\widetilde{N}'_2)$, $A \subset S \subset \mathrm{Int}_X(B_\delta)$ and $A \cap (B_\delta \cap (N'_3 \cup \widetilde{N}'_3)) = \emptyset$ so $A \subset \mathrm{Int}_X(L_2 \setminus L_3)$ and so $A \subset \mathrm{Inv}_\pi(\mathrm{Cl}_X(L_2 \setminus L_3))$. On the other hand, $L_2 \setminus L_3 \subset N'_2 \setminus N'_3$ and this implies that $\mathrm{Inv}_\pi(\mathrm{Cl}_X(L_2 \setminus L_3)) \subset \mathrm{Inv}_\pi(\mathrm{Cl}_X(N'_2 \setminus N'_3)) = A$. Let $x \in L_3$ and $t \geq 0$ be such that $x\pi\,[0,t] \subset L_2$. Thus $x \in N'_3 \cup \widetilde{N}'_3$. Since N'_3 is N'_2-positively invariant and $\widetilde{N}'_3$ is $\widetilde{N}'_2$-positively invariant, it follows that $x\pi\,[0,t] \subset N'_3 \cup \widetilde{N}'_3$. Recall that $L_2 \subset B_\delta$. Thus, $x\pi\,[0,t] \subset L_3$ and so L_3 is L_2-positively invariant. Let $x \in L_2$ be such that $x\pi t \notin L_2$ for some $t > 0$. We need to show that there exists a $t' \in [0,t[$ such that $x\pi\,[0,t'] \subset L_2$ and $x\pi t' \in L_3$. Since $x\pi t \notin L_2$, it follows that $x\pi t \notin B_\delta \cap (N'_2 \cap \widetilde{N}'_2)$ and $x\pi t \notin B_\delta \cap (N'_3 \cup \widetilde{N}'_3)$. Suppose first $x \in B_\delta \cap (N'_2 \cap \widetilde{N}'_2)$. Set $t' := \rho_{B_\delta \cap (N'_2 \cap \widetilde{N}'_2)}(x)$. By (2.1), $x\pi r$ is defined and $x\pi r \in B_\delta \cap (N'_2 \cap \widetilde{N}'_2)$ for all $r \in [0,t'[$. Therefore, we cannot have $t < t'$ so $t' \in [0,t]$. Moreover, $x\pi[0,t'] \subset B_\delta \cap (N'_2 \cap \widetilde{N}'_2)$, since $B_\delta \cap (N'_2 \cap \widetilde{N}'_2)$ is closed. By (2.2) we have that $t' = \rho_{B_\delta}(x)$ or $t' = \rho_{N'_2}(x)$ or $t' = \rho_{\widetilde{N}'_2}(x)$. In the first case it follows that $x\pi t' \in {B_\delta}^- \subset B_\delta \cap (N'_3 \cup \widetilde{N}'_3) = L_3$; in the second case $x\pi t' \in N'_3$ so $x\pi t' \in B_\delta \cap (N'_3 \cup \widetilde{N}'_3) = L_3$ and in the third case $x\pi t' \in \widetilde{N}'_3$ so $x\pi t' \in B_\delta \cap (N'_3 \cup \widetilde{N}'_3) = L_3$.

Suppose now that $x \in B_\delta \cap (N'_3 \cup \widetilde{N}'_3)$. In this case, define $t' := 0$. The proof of the lemma is complete. $\square$

Proof of Theorem 5.1. Let (N_1, N_2, N_3) and $(\widetilde{N}_1, \widetilde{N}_2, \widetilde{N}_3)$ be two FM-index triples for (π, S, A, A^*) with $\mathrm{Cl}_X(N_1 \setminus N_3)$ and $\mathrm{Cl}_X(\widetilde{N}_1 \setminus \widetilde{N}_3)$ strongly π-admissible. Let $s > 0$ and (L_1, L_2, L_3) be an FM-index triple for (π, S, A, A^*) such that the conclusions of Lemma 5.3 holds. Proposition 4.5 and Proposition 5.2 imply that the inclusion induced diagram (5.14) of pointed spaces commutes. Passing to homology in diagram (5.14) we obtain the commutative diagram (5.15) (in which we set $M_2 := N_2 \cup {N_3}^{-s}$ and $\widetilde{M}_2 := \widetilde{N}_2 \cup {\widetilde{N}_3}^{-s}$) made of four long homology ladders. An application of Proposition 4.5 shows that the vertical morphisms in diagram (5.15) are isomorphisms. Thus we can reverse the vertical arrows in the second and fourth ladders. Composing the resulting ladders, we obtain the commutative diagram (5.3), completing the proof in the singular homology case.

(5.14)
$$
\begin{tikzcd}
N_2/N_3 \arrow[r, "i"] \arrow[d] & N_1/N_3 \arrow[r, "p"] \arrow[d] & N_1/N_2 \arrow[d] \\
(N_2 \cup N_3^{-s})/N_3^{-s} \arrow[r] & N_1/N_3^{-s} \arrow[r] & N_1/(N_2 \cup N_3^{-s}) \\
L_2/L_3 \arrow[r] \arrow[u] \arrow[d] & L_1/L_3 \arrow[r] \arrow[u] \arrow[d] & L_1/L_2 \arrow[u] \arrow[d] \\
(\widetilde{N}_2 \cup \widetilde{N}_3^{-s})/\widetilde{N}_3^{-s} \arrow[r] & \widetilde{N}_1/\widetilde{N}_3^{-s} \arrow[r] & \widetilde{N}_1/(\widetilde{N}_2 \cup \widetilde{N}_3^{-s}) \\
\widetilde{N}_2/\widetilde{N}_3 \arrow[r, "\widetilde{i}"'] \arrow[u] & \widetilde{N}_1/\widetilde{N}_3 \arrow[r, "\widetilde{p}"'] \arrow[u] & \widetilde{N}_1/\widetilde{N}_2 \arrow[u]
\end{tikzcd}
$$

(5.15)
$$
\begin{tikzcd}
{} \arrow[r] & H_q(N_2/N_3) \arrow[r, "H_q(i)"] \arrow[d] & H_q(N_1/N_3) \arrow[r, "H_q(p)"] \arrow[d] & H_q(N_1/N_2) \arrow[r, "\widehat{\partial}_q"] \arrow[d] & H_{q-1}(N_2/N_3) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H_q(M_2/N_3^{-s}) \arrow[r] & H_q(N_1/N_3^{-s}) \arrow[r] & H_q(N_1/M_2) \arrow[r] & H_{q-1}(M_2/N_3^{-s}) \arrow[r] & {} \\
{} \arrow[r] & H_q(L_2/L_3) \arrow[r] \arrow[u] \arrow[d] & H_q(L_1/L_3) \arrow[r] \arrow[u] \arrow[d] & H_q(L_1/L_2) \arrow[r] \arrow[u] \arrow[d] & H_{q-1}(L_2/L_3) \arrow[r] \arrow[u] \arrow[d] & {} \\
{} \arrow[r] & H_q(\widetilde{M}_2/\widetilde{N}_3^{-s}) \arrow[r] & H_q(\widetilde{N}_1/\widetilde{N}_3^{-s}) \arrow[r] & H_q(\widetilde{N}_1/\widetilde{M}_2) \arrow[r] & H_{q-1}(\widetilde{M}_2/\widetilde{N}_3^{-s}) \arrow[r] & {} \\
{} \arrow[r] & H_q(\widetilde{N}_2/\widetilde{N}_3) \arrow[r, "H_q(\widetilde{i})"'] \arrow[u] & H_q(\widetilde{N}_1/\widetilde{N}_3) \arrow[r, "H_q(\widetilde{p})"'] \arrow[u] & H_q(\widetilde{N}_1/\widetilde{N}_2) \arrow[r, "\widehat{\widetilde{\partial}}_q"'] \arrow[u] & H_{q-1}(\widetilde{N}_2/\widetilde{N}_3) \arrow[r] \arrow[u] & {}
\end{tikzcd}
$$

The proof for the Alexander–Spanier cohomology is analogous. □

6. Morse decompositions and (co)homology index braids

Recall that a *strict partial order* on a set P is a relation $\prec\ \subset P \times P$ which is irreflexive and transitive. As usual, we write $x \prec y$ instead of $(x, y) \in\ \prec$. The symbol $<$ will be reserved for the less-than-relation on $\mathbb{R}$.

For the rest of this paper, unless specified otherwise, let P be a fixed finite set and $\prec$ be a fixed strict partial order on P.

A set $I \subset P$ is called *a $\prec$-interval* if whenever i, j, $k \in P$, i, $k \in I$ and $i \prec j \prec k$, then $j \in I$. By $\mathcal{I}(\prec)$ we denote the set of all $\prec$-intervals in P. A set I is called *a $\prec$-attracting interval* if whenever i, $j \in P$, $j \in I$ and $i \prec j$, then $i \in I$. By $\mathcal{A}(\prec)$ we denote the set of all $\prec$-attracting intervals in P. Of course, $\mathcal{A}(\prec) \subset \mathcal{I}(\prec)$.

An *adjacent n-tuple of $\prec$-intervals* is a sequence $(I_j)_{j=1}^n$ of pairwise disjoint $\prec$-intervals whose union is a $\prec$-interval and such that, whenever $j < k$, $p \in I_j$

and $p' \in I_k$, then $p' \not\prec p$ (i.e. $p \prec p'$ or else p and p' are not related by $\prec$). By $\mathcal{I}_n(\prec)$ we denote the set of all adjacent n-tuples of $\prec$-intervals.

Let $S \subset X$ be a compact π-invariant set. A family $(M_i)_{i\in P}$ of subsets of S is called a *$\prec$-ordered Morse decomposition of S* if the following properties hold:

(1) The sets M_i, $i \in P$, are closed, π-invariant and pairwise disjoint.
(2) For every full solution σ of π lying in S either $\sigma(\mathbb{R}) \subset M_k$ for some $k \in P$ or else there are k, $l \in P$ with $k \prec l$, $\alpha(\sigma) \subset M_l$ and $\omega(\sigma) \subset M_k$.

Let S be a compact invariant set and $(M_i)_{i\in P}$ be a $\prec$-ordered Morse decomposition of S. If A, $B \subset X$ then the *(π, S)-connection set* $\mathrm{CS}_{\pi,S}(A, B)$ *from A to B* is the set of all points $x \in X$ for which there is a solution $\sigma\colon \mathbb{R} \to S$ of π with $\sigma(0) = x$, $\alpha(\sigma) \subset A$ and $\omega(\sigma) \subset B$.

For an arbitrary $\prec$-interval I set

$$M(I) = M_{\pi,S}(I) = \bigcup_{(i,j)\in I\times I} \mathrm{CS}_{\pi,S}(M_i, M_j).$$

An *index filtration for* $(\pi, S, (M_p)_{p\in P})$ is a family $\mathcal{N} = (N(I))_{I\in\mathcal{A}(\prec)}$ of closed subsets of X such that

(1) for each $I \in \mathcal{A}(\prec)$, the pair $(N(I), N(\emptyset))$ is an FM-index pair for $(\pi, M(I))$,
(2) for each I_1, $I_2 \in \mathcal{A}(\prec)$, $N(I_1 \cap I_2) = N(I_1) \cap N(I_2)$ and $N(I_1 \cup I_2) = N(I_1) \cup N(I_2)$.

$\mathcal{N}$ is called *strongly π-admissible* if $N(P)$ is strongly π-admissible. An existence result for such index filtrations was established in [12].

Let $\mathcal{N}$ be a strongly π-admissible index filtration for $(\pi, S, (M_p)_{p\in P})$. For $J \in \mathcal{I}(\prec)$ the set $M(J)$ is an isolated invariant set and we write

$$H_q(J) = H_q(\pi, J) := H_q(\pi, M(J)), \quad q \in \mathbb{Z}.$$

If $(I, J) \in \mathcal{I}_2(\prec)$, then $(M(I), M(J))$ is an attractor-repeller pair in $M(IJ)$, where $IJ := I \cup J$. Let B be the set of all $p \in P \setminus (IJ)$ for which there is a $p' \in IJ$ with $p \prec p'$. It follows that B, BI, $BIJ \in \mathcal{A}(\prec)$. Moreover, $(N(BIJ), N(BI), N(B))$ is an FM-index triple for $(\pi, M(IJ), M(I), M(J))$ with $\mathrm{Cl}_X(N(BIJ) \setminus N(B))$ strongly π-admissible. The inclusion induced sequence

$$N(BI)/N(B) \xrightarrow{i_{I,J}} N(BIJ)/N(B) \xrightarrow{p_{I,J}} N(BIJ)/N(BI)$$

induces the homology index sequence

$$\longrightarrow H_q(I) \xrightarrow{\langle H_q(i_{I,J})\rangle} H_q(IJ) \xrightarrow{\langle H_q(p_{I,J})\rangle} H_q(J) \xrightarrow{\langle \widehat{\partial}_{I,J,q}\rangle} H_{q-1}(I) \longrightarrow$$

of $(\pi, M(IJ), M(I), M(J))$. Let $(I, J, K) \in \mathcal{I}_3(\prec)$ and define $H := \{p \in P \mid$ there is a $p' \in IJK$ with $p \prec p'\}$. It follows that $H \in \mathcal{A}(\prec)$ and $(H, I, J, K) \in$

$\mathcal{I}_4(\prec)$. Hence, HI, HIJ, $HIJK \in \mathcal{A}(\prec)$. Define $N_1 := N(HIJK)$, $N_2 := N(HIJ)$, $N_3 := N(HI)$ and $N_4 := N(H)$. We obtain the following inclusion induced diagram

$$
\begin{tikzcd}
N_3/N_4 \arrow[dr, "i_1"] \arrow[dd, bend right, "i_4"'] & & \\
& N_2/N_4 \arrow[dl, "i_2"'] \arrow[dr, "p_1"] & \\
N_1/N_4 \arrow[dr, "p_4"] \arrow[dd, bend right, "p_2"'] & & N_2/N_3 \arrow[dl, "i_3"] \\
& N_1/N_3 \arrow[dl, "p_3"] & \\
N_1/N_2 & &
\end{tikzcd}
\tag{6.1}
$$

of pointed spaces. Applying Propositions 2.10 and 2.11 to diagram (6.1) and then using the $\langle\,\cdot\,\rangle$-operation together with Proposition 3.2 we obtain the commutative diagram

$$
\begin{tikzcd}
{} \arrow[d, bend right] & {} \arrow[dl] \arrow[dr] & {} \arrow[d, bend left] \\
H_q(I) \arrow[dr, "\langle H_q(i_1)\rangle"] \arrow[dd, bend right, "\langle H_q(i_4)\rangle"'] & & H_{q+1}(K) \arrow[dl, "\langle\widehat{\partial}_{2,q}\rangle"'] \arrow[dd, bend left, "\langle\widehat{\partial}_{3,q}\rangle"] \\
& H_q(IJ) \arrow[dl, "\langle H_q(i_2)\rangle"'] \arrow[dr, "\langle H_q(p_1)\rangle"] & \\
H_q(IJK) \arrow[dr, "\langle H_q(p_4)\rangle"] \arrow[dd, bend right, "\langle H_q(p_2)\rangle"'] & & H_q(J) \arrow[dl, "\langle H_q(i_3)\rangle"'] \arrow[dd, bend left, "\langle\widehat{\partial}_{1,q}\rangle"] \\
& H_q(JK) \arrow[dl, "\langle H_q(p_3)\rangle"'] \arrow[dr, "\langle\widehat{\partial}_{4,q}\rangle"] & \\
H_q(K) \arrow[dr, "\langle\widehat{\partial}_{2,q}\rangle"] \arrow[dd, bend right, "\langle\widehat{\partial}_{3,q}\rangle"'] & & H_{q-1}(I) \arrow[dl, "\langle H_{q-1}(i_1)\rangle"'] \arrow[dd, bend left, "\langle H_{q-1}(i_4)\rangle"] \\
& H_{q-1}(IJ) \arrow[dl, "\langle H_{q-1}(p_1)\rangle"'] \arrow[dr, "\langle H_{q-1}(i_2)\rangle"] & \\
H_{q-1}(J) \arrow[dr] \arrow[d, bend right] & & H_{q-1}(IJK) \arrow[dl] \arrow[d, bend left] \\
{} & {} & {}
\end{tikzcd}
\tag{6.2}
$$

Since all morphisms in diagram (6.2) are in the long exact homology sequences of the appropriate attractor-repeller pairs, it follows that diagram (6.2) is independent of the choice of an admissible index filtration for $(\pi, S, (M_p)_{p\in P})$. The following concept is thus well defined.

DEFINITION 6.1 ([9], [12]). The collection of all the homology indices

$$H_q(\pi, M(J)), \quad q \in \mathbb{Z},\ J \in \mathcal{I}(\prec),$$

and all the maps $\langle H_q(i_{I,J})\rangle$, $\langle H_q(p_{I,J})\rangle$ and $\langle\widehat{\partial}_{I,J,q}\rangle$, $(I,J) \in \mathcal{I}_2(\prec)$ is called *the homology index braid of* $(\pi, S, (M_p)_{p\in P})$. We denote it by $\mathcal{H}(\pi, S, (M_p)_{p\in P})$.

For the rest of this section assume that, for $i = 1, 2$, π_i is a local semiflow on the metric space X_i, S_i is an isolated invariant set and $(M_{p,i})_{p\in P}$ is a $\prec$-ordered Morse decomposition of S_i, relative to π_i. Write $M_i(I) = M_{\pi_i,S_i}(I)$, $H_i(I) = H(\pi_i, M_i(I))$ and $\mathcal{H}_i := \mathcal{H}(\pi_i, S_i, (M_{p,i})_{p\in P})$, for $i = 1, 2$ and $I \in \mathcal{I}(\prec)$.

Suppose $\theta := (\theta(J))_{J\in\mathcal{I}(\prec)}$ is a family $\theta(J): H_1(J) \to H_2(J)$, $J \in \mathcal{I}(\prec)$, of Γ-module homomorphisms such that, for all $(I, J) \in \mathcal{I}_2(\prec)$, the diagram

$$\tag{6.3}\begin{array}{ccccccccccc}
\longrightarrow & H_{1,q}(I) & \xrightarrow{\langle H_q(i_{I,J})\rangle} & H_{1,q}(IJ) & \xrightarrow{\langle H_q(p_{I,J})\rangle} & H_{1,q}(J) & \xrightarrow{\langle\widehat{\partial}_{I,J}\rangle} & H_{1,q-1}(I) & \longrightarrow \\
 & \Big\downarrow{\scriptstyle \theta_q(I)} & & \Big\downarrow{\scriptstyle \theta_q(IJ)} & & \Big\downarrow{\scriptstyle \theta_q(J)} & & \Big\downarrow{\scriptstyle \theta_{q-1}(I)} & \\
\longrightarrow & H_{2,q}(I) & \xrightarrow[\langle H_q(i_{I,J})\rangle]{} & H_{2,q}(IJ) & \xrightarrow[\langle H_q(p_{I,J})\rangle]{} & H_{2,q}(J) & \xrightarrow[\langle\widehat{\partial}_{I,J}\rangle]{} & H_{2,q-1}(I) & \longrightarrow
\end{array}$$

commutes. Then we say that θ is a *morphism from* $\mathcal{H}_1$ *to* $\mathcal{H}_2$ and we write $\theta: \mathcal{H}_1 \to \mathcal{H}_2$. If each $\theta(J)$ is an isomorphism, then we say that θ is an *isomorphism* and that $\mathcal{H}_1$ and $\mathcal{H}_2$ are *isomorphic* homology index braids.

Remark 6.2. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic homology index braids, then, by Proposition 1.5 in [10], $\mathcal{H}_1$ and $\mathcal{H}_2$ determine the same collection of connection matrices and the same collection of C-connection matrices.

We will now introduce an important class of morphisms between homology index braids. Let $\mathcal{N}_i = (N_i(I))_{I\in\mathcal{A}(\prec)}$ be a strongly π_i-admissible index filtration for $(\pi_i, S_i, (M_{p,i})_{p\in P})$, $i = 1, 2$. Assume the *nesting* property

$$N_1(I) \subset N_2(I), \quad I \in \mathcal{A}(\prec).$$

For $J \in \mathcal{I}(\prec)$ choose I, $K \in \mathcal{A}(\prec)$ with $(I, J) \in \mathcal{I}_2(\prec)$ and $K = IJ$. Then, for $i = 1, 2$, $(N_i(K), N_i(I))$ is an FM-index pair for $M_i(J)$, relative to π_i. The inclusion induced map $\alpha: N_1(K)/N_1(I) \to N_2(K)/N_2(I)$ induces a homomorphism

$$\theta(J) = \theta_{\mathcal{N}_1,\mathcal{N}_2}(J): H(\pi_1, M_1(J)) \to H(\pi_2, M_2(J))$$

defined by

$$\theta_q(J) := \langle H_q(\alpha)\rangle, \quad q \in \mathbb{Z}.$$

Of course, this homomorphism depends on the choice of $\mathcal{N}_i$, $i = 1, 2$, but we claim that

(6.4) $\theta_q(J)$, $q \in \mathbb{Z}$, is independent of the choice of I and K.

In fact, if I' and $K' \in \mathcal{A}(\prec)$ are such that $(I', J) \in \mathcal{I}_2(\prec)$ and $K' = I'J$ then property (2) of index filtrations implies that $N_i(K) \setminus N_i(I) = N_i(K') \setminus N_i(I')$, $i = 1, 2$, (see Proposition 3.5 in [9] and its proof, which is also valid in our case) so there is an inclusion induced, commutative, diagram of pointed spaces

$$\begin{tikzcd} N_1(K)/N_1(I) \arrow[r, "\alpha"] \arrow[d] & N_2(K)/N_2(I) \arrow[d] \\ N_1(K')/N_1(I') \arrow[r, "\alpha'"'] & N_2(K')/N_2(I') \end{tikzcd}$$

By Proposition 4.4, the (homotopy classes of the) vertical maps are morphisms of $\mathcal{C}(\pi_i, M_i(J))$, $i = 1, 2$. Thus, passing to homology and using Proposition 3.2 we see that

$$\langle H_q(\alpha)\rangle = \langle H_q(\alpha')\rangle, \quad q \in \mathbb{Z},$$

which is exactly what we claim.

We write

$$\theta_{\mathcal{N}_1,\mathcal{N}_2} = (\theta_{\mathcal{N}_1,\mathcal{N}_2}(J))_{J\in\mathcal{I}(\prec)}.$$

We also claim that $\theta_{\mathcal{N}_1,\mathcal{N}_2}\colon \mathcal{H}_1 \to \mathcal{H}_2$. In fact, let $(I, J) \in \mathcal{I}_2(\prec)$ and let B be the set of all $p \in P \setminus (IJ)$ for which there is a $p' \in IJ$ with $p \prec p'$. It follows that B, BI, $BIJ \in \mathcal{A}(\prec)$. Setting, for $i = 1$, 2, $N_{1,i} = N_i(BIJ)$, $N_{2,i} = N_i(BI)$ and $N_{3,i} = N_i(B)$ we see that $(N_{1,i}, N_{2,i}, N_{3,i})$ is an FM-index triple for $(\pi_i, M_i(IJ), M_i(I), M_i(J))$ with $\mathrm{Cl}_X(N_{1,i} \setminus N_{3,i})$, for $i = 1$, 2, strongly π_i-admissible. Moreover, by Propositions 2.10 and 2.11, the inclusion induced diagram

$$\begin{tikzcd} N_{2,1}/N_{3,1} \arrow[r, "i"] \arrow[d] & N_{1,1}/N_{3,1} \arrow[r, "p"] \arrow[d] & N_{1,1}/N_{2,1} \arrow[d] \\ N_{2,2}/N_{3,2} \arrow[r, "i"'] & N_{1,2}/N_{3,2} \arrow[r, "p"'] & N_{1,2}/N_{2,2} \end{tikzcd}$$

induces the commutative long homology ladder

$$\begin{tikzcd} \to H_q(N_{2,1}/N_{3,1}) \arrow[r, "H_q(i)"] \arrow[d, "\gamma'_q"] & H_q(N_{1,1}/N_{3,1}) \arrow[r, "H_q(p)"] \arrow[d, "\gamma_q"] & H_q(N_{1,1}/N_{2,1}) \arrow[r, "\widehat{\partial}_q"] \arrow[d, "\gamma''_q"] & H_{q-1}(N_{2,1}/N_{3,1}) \to \arrow[d, "\gamma'_{q-1}"] \\ \to H_q(N_{2,2}/N_{3,2}) \arrow[r, "H_q(i)"'] & H_q(N_{1,2}/N_{3,2}) \arrow[r, "H_q(p)"'] & H_q(N_{1,2}/N_{2,2}) \arrow[r, "\widehat{\partial}_q"'] & H_{q-1}(N_{2,2}/N_{3,2}) \to \end{tikzcd} \tag{6.5}$$

Applying the $\langle\,\cdot\,\rangle$-operation to (6.5) and noting that, for all $q \in \mathbb{Z}$, $\theta_q(I) = \langle\gamma'_q\rangle$, $\theta_q(IJ) = \langle\gamma_q\rangle$ and $\theta_q(J) = \langle\gamma''_q\rangle$ (in view of (6.4)) we obtain from Proposition 3.2 a commutative diagram of the form (6.3). This proves our second claim.

We call $\theta := (\theta(J))_{J\in\mathcal{I}(\prec)}$ the *inclusion induced morphism from $\mathcal{H}_1$ to $\mathcal{H}_2$.*

We now obtain the following result:

Proposition 6.3. *For $i = 1, 2$ let*

$$\mathcal{N}_i = (N_i(I))_{I\in\mathcal{A}(\prec)} \quad \textit{and} \quad \widetilde{\mathcal{N}}_i = (\widetilde{N}_i(I))_{I\in\mathcal{A}(\prec)}$$

be strongly π_i-admissible index filtrations for $(\pi_i, S_i, (M_{p,i})_{p\in P})$. Assume the nesting property

$$N_1(I) \subset N_2(I) \subset \widetilde{N}_1(I) \subset \widetilde{N}_2(I), \quad I \in \mathcal{A}(\prec). \tag{6.6}$$

Then the inclusion induced morphism $\theta_{\mathcal{N}_1,\mathcal{N}_2}$ is an isomorphism.

Proof. Let $J \in \mathcal{I}(\prec)$ be arbitrary, $a := \theta_{\mathcal{N}_1,\mathcal{N}_2}(J)$, $b := \theta_{\mathcal{N}_2,\widetilde{\mathcal{N}}_1}(J)$ and $c := \theta_{\widetilde{\mathcal{N}}_1,\widetilde{\mathcal{N}}_2}(J)$. Then $b \circ a$ and $c \circ b$ are isomorphisms, being induced by maps lying in the same connected simple system (the categorial Conley–Morse index of $(\pi_1, M_1(J))$ and $(\pi_2, M_2(J))$, respectively). It follows that a, b and c are isomorphisms. This proves the proposition. □

Remark 6.4. Analogous definitions and results hold for Alexander–Spanier cohomology with the obvious modifications.

7. Regular continuation of (co)homology index braids

Let π_n, $n \in \mathbb{N}_0$, be local semiflows on the metric space X. We say that the sequence $(\pi_n)_{n\in\mathbb{N}}$ *converges* π_0 and we write $\pi_n \to \pi_0$ if whenever $x_n \to x_0$ in X, $t_n \to t_0$ in $[0, \infty[$ and $x_0\pi_0 t_0$ is defined, then $x_n\pi_n t_n$ is defined for all n large enough and $x_n\pi_n t_n \to x_0\pi_0 t_0$ in X.

Given $Y \subset X$ we say that Y *is $(\pi_n)_n$-admissible* if Y is closed and whenever $(x_n)_n$ and $(t_n)_n$ are such that $t_n \to \infty$, $x_n\pi_n t_n$ is defined and $x_n\pi_n [0, t_n] \subset Y$ for all $n \in \mathbb{N}$, then the sequence $(x_n\pi_n t_n)_n$ has a convergent subsequence.

The following continuation result for Morse decompositions was established in [5].

Theorem 7.1 (cf. Corollaries 3.5 and 3.6 in [5]). *Let π_n, where $n \in \mathbb{N}_0$, be local semiflows on X and $\widetilde{N}$ be a closed subset of X which is strongly π_n-admissible for every $n \in \mathbb{N}_0$. Moreover, assume that*

(7.1) *$\pi_n \to \pi_0$ and $\widetilde{N}$ is $(\pi_{n_m})_m$-admissible for every subsequence $(\pi_{n_m})_m$ of $(\pi_n)_n$.*

Suppose that $S_0 := \mathrm{Inv}_{\pi_0}(\widetilde{N}) \subset \mathrm{Int}_X(\widetilde{N})$ and $(M_{p,0})_{p\in P}$ is a $\prec$-ordered Morse decomposition of S_0 relative to π_0. For each $p \in P$, let $\Xi_p \subset \widetilde{N}$ be closed in X and such that $M_{p,0} = \mathrm{Inv}_{\pi_0}(\Xi_p) \subset \mathrm{Int}_X(\Xi_p)$. (Such sets Ξ_p, $p \in P$, always exist.) For $n \in \mathbb{N}$ and $p \in P$ set $S_n := \mathrm{Inv}_{\pi_n}(\widetilde{N})$ and $M_{p,n} := \mathrm{Inv}_{\pi_n}(\Xi_p)$. Then there is an $\overline{n} \in \mathbb{N}$ such that whenever $n \geq \overline{n}$ and $p \in P$ then $S_n \subset \mathrm{Int}_X(\widetilde{N})$, $M_{p,n} \subset \mathrm{Int}_X(\Xi_p)$ and the family $(M_{p,n})_{p\in P}$ is a $\prec$-ordered Morse decomposition of S_n relative to π_n.

Remark 7.2. It follows from [5, Theorem 3.3 and the proof of Corollary 3.5] that Theorem 7.1 remains valid if we replace assumption (7.1) by the following weaker assumption:

(7.2) Whenever $(n_m)_m$ is a sequence in $\mathbb{N}$ with $n_m \to \infty$ and, for every $m \in \mathbb{N}$, u_m is a full solution of π_{n_m} lying in $\widetilde{N}$, then there is a subsequence $(u_{m_k})_k$ of $(u_m)_m$ and a full solution u_0 of π_0 such that $u_{m_k}(t) \to u_0(t)$ as $k \to \infty$, uniformly for t lying in compact subset of $\mathbb{R}$.

However, we require the stronger assumption (7.1) in Theorem 7.3 below.

We can now state the *nested index filtration theorem* proved in [6]:

Theorem 7.3 (cf. Theorem 3.4 in [6]). *Assume the hypotheses* (*and thus also the conclusions*) *of Theorem* 7.1 *and let $\overline{n}$ be as in that theorem. Then there is an $n_1 \geq \overline{n}$ such that for every $n \in \mathbb{N}_0$ with $n = 0$ or $n \geq n_1$ there exist strongly π_n-admissible index filtrations $\mathcal{N}_n = (N_n(I))_{I \in \mathcal{A}(\prec)}$ and $\widetilde{\mathcal{N}}_n = (\widetilde{N}_n(I))_{I \in \mathcal{A}(\prec)}$ for $(\pi_n, S_n, (M_{p,n})_{p\in P})$ such that the following* nesting property *holds:*

$$N_n(I) \subset N_0(I) \subset \widetilde{N}_n(I) \subset \widetilde{N}_0(I) \quad \textit{for all } n \geq n_1 \textit{ and } I \in \mathcal{A}(\prec). \tag{7.3}$$

Theorem 7.3, Proposition 6.3 and Remark 6.2 immediately imply the following *continuation result for homology index braids and connection matrices*.

Theorem 7.4 (cf. Theorem 3.5 in [6]). *Under the hypotheses of Theorem* 7.3 *the homology index braids $\mathcal{H}(\pi_0, S_0, (M_{p,0})_{p\in P})$ and $\mathcal{H}(\pi_n, S_n, (M_{p,n})_{p\in P}))$, $n \geq n_1$, are isomorphic and determine the same collection of connection matrices and the same collection of C-connection matrices.*

Let us make the following definition.

Definition 7.5. Let Λ be a metric space. A family $(\pi_\lambda, S_\lambda, (M_{p,\lambda})_{p\in P})_{\lambda\in\Lambda}$ is called *$\mathcal{S}$-continuous* if for every $\lambda_0 \in \Lambda$ there is a neighbourhood W_{λ_0} of λ_0 in Λ and there are closed subsets N_{λ_0}, $\Xi_{p,\lambda_0} \subset N_{\lambda_0}$, $p \in P$, of X such that for every $\lambda \in W_{\lambda_0}$, π_λ is a local semiflow on X, S_λ is a (compact) π_λ-invariant set, $(M_{p,\lambda})_{p\in P}$ is a Morse decomposition of S_λ, relative to π_λ, N_{λ_0} is a strongly π_λ-admissible isolating neighbourhood of S_λ and, for $p \in P$, Ξ_{p,λ_0} is an isolating neighbourhood of $M_{p,\lambda}$, relative to π_λ. Moreover, whenever $\lambda_n \to \lambda_0$ in W_{λ_0} then $\pi_{\lambda_n} \to \pi_{\lambda_0}$ and N_{λ_0} is $(\pi_{\lambda_n})_n$-admissible.

We can now state our second *continuation result for homology index braids and connection matrices* proved in [6].

Theorem 7.6 (cf. Theorem 3.7 in [6]). *Let Λ be a metric space and let $(\pi_\lambda, S_\lambda, (M_{p,\lambda})_{p\in P})_{\lambda\in\Lambda}$ be an $\mathcal{S}$-continuous family. Then for every $\lambda \in \Lambda$ the homology index braid $\mathcal{H}_\lambda := \mathcal{H}(\pi_\lambda, S_\lambda, (M_{p,\lambda})_{p\in P})$ is defined and for every $\lambda_0 \in \Lambda$ there is a neighbourhood W of λ_0 in Λ such that $\mathcal{H}_\lambda$ is isomorphic to $\mathcal{H}_{\lambda_0}$ for every $\lambda \in W$. In particular, if Λ is connected, then $\mathcal{H}_{\lambda_1}$ and $\mathcal{H}_{\lambda_2}$ are isomorphic* *for all λ_1, $\lambda_2 \in \Lambda$ and determine the same collection of connection matrices and the same collection of C-connection matrices.*

Remark 7.7. Both Definition 7.5 and Theorem 7.6 can be generalized to topological spaces Λ satisfying the first countability axiom, because that is all we use in the proof of Theorem 7.6.

Theorems 7.4 and 7.6 refine the corresponding homotopy invariance results for the (infinite dimensional) Conley index established in [19] (or [21]). The convergence and admissibility assumptions make these results applicable to various classes of parameter dependent evolution equations (e.g. parabolic or damped hyperbolic equations on bounded domains and even some parabolic equations on unbounded domains, see the recent paper [17]).

In [6] we show that homology index braids for certain types of parabolic equations are isomorphic to the corresponding homology index braids of their (sufficiently high dimensional) Galerkin approximations. In the rest of this section we will describe these results.

For the rest of this section let X be a real Hilbert space and $A\colon D(A) \subset X \to X$ be a positive selfadjoint operator with compact resolvent. Let $(\phi_\nu)_{\nu\in\mathbb{N}}$ be a complete X-orthonormal basis of X consisting of eigenfunctions of A. Let $P_n\colon X \to X$ be the orthogonal projection of X onto the subspace spanned by the first n eigenfunctions. Moreover, set $Q_n := I - P_n$ where I is the identity map on X. Note that A is sectorial on X and so it generates a family $(X^\alpha)_{\alpha\in[0,\infty[}$ of fractional power spaces. Given $\alpha \in [0,1[$ and a locally Lipschitzian map $g\colon X^\alpha \to X$ we denote by π_g the local semiflow on X^α generated by the abstract parabolic equation (see [13])

$$\dot u = -Au + g(u), \quad u \in X^\alpha.$$

The following result has been proved in [22] (see Theorem 4.3 and Proposition 4.4 in [22]).

Proposition 7.8. *Let $f\colon X^\alpha \to X$ be Lipschitzian on bounded subsets of X^α. For $n \in \mathbb{N}$ and $\tau \in [0,1]$ let $f_{n,\tau}\colon X^\alpha \to X$ be defined by*

$$f_{n,\tau}(u) = (1-\tau)f(u) + \tau P_n f(P_n u), \quad u \in X^\alpha.$$

Let $N \subset X^\alpha$ be bounded and closed. Furthermore, let $(n_m)_m$ be a sequence in $\mathbb{N}$ with $n_m \to \infty$ and $(\tau_m)_m$ be an arbitrary sequence in $[0,1]$. For every $m \in \mathbb{N}$ let u_m be a full solution of $\pi_{f_{n_m,\tau_m}}$ lying in N. Then there is a sequence $(m_k)_k$ with $m_k \to \infty$ and there is a full solution u of π_f lying in N such that $u_{m_k}(t) \to u(t)$ in X^α, uniformly for t lying in compact subsets of $\mathbb{R}$.

In [6] we prove the following continuation results for Morse decompositions and homology index braids (see Corollaries 3.9 and 3.10 in [6]).

Corollary 7.9. *Let* $f\colon X^\alpha \to X$ *and* $f_{n,\tau}$, $n \in \mathbb{N}$, $\tau \in [0,1]$, *be as in Proposition* 7.8. *Let* N *be bounded and closed in* X^α *with* $S := \mathrm{Inv}_{\pi_f}(N) \subset \mathrm{Int}_{X^\alpha}(N)$. *Moreover, let* $(M_p)_{p\in P}$ *be a* $\prec$*-ordered Morse decomposition of* S*, relative to* π_f*. For each* $p \in P$ *let* $\Xi_p \subset N$ *be closed in* X^α *such that* $M_p = \mathrm{Inv}_{\pi_f}(\Xi_p) \subset \mathrm{Int}_{X^\alpha}(\Xi_p)$*. For* $n \in \mathbb{N}$, $\tau \in [0,1]$ *and* $p \in P$ *define* $S_{n,\tau} = \mathrm{Inv}_{\pi_{f_{n,\tau}}}(N)$ *and* $M_{p,n,\tau} = \mathrm{Inv}_{\pi_{f_{n,\tau}}}(\Xi_p)$*. Then there is an* $n_0 \in \mathbb{N}$ *so that whenever* $n \geq n_0$ *and* $\tau \in [0,1]$*, then* $S_{n,\tau} \subset \mathrm{Int}_{X^\alpha}(N)$, $M_{p,n,\tau} \subset \mathrm{Int}_{X^\alpha}(\Xi_p)$, $p \in P$*, and the family* $(M_{p,n,\tau})_{p\in P}$ *is a Morse decomposition of* $S_{n,\tau}$*, relative to* $\pi_{f_{n,\tau}}$*.*

Corollary 7.10. *Let* n_0 *be as in Corollary* 7.9*. Then for* $n \geq n_0$ *and* $\tau \in [0,1]$ *the homology index braid of* $(\pi_{f_{n,\tau}}, S_{n,\tau}, (M_{p,n,\tau})_{p\in P})$ *is isomorphic to the homology index braid of* $(\pi_f, S, (M_p)_{p\in P})$*.*

Given $n \in \mathbb{N}$ and f as in Proposition 7.8 we may consider the local semiflow $\pi'_n = \pi'_{f,n}$ generated on the finite dimensional space $Y_n := P_n(X^\alpha) = P_n(X)$ by the ordinary differential equation

$$\dot{u} = -Au + P_n f(P_n u), \quad u \in Y_n. \tag{7.4}$$

The local semiflow π'_n is the *n-Galerkin approximation of* π_f.

Moreover, let $\pi''_n = \pi''_{f,n}$ be the semiflow generated on $Z_n := Q_n(X^\alpha)$ by the evolution equation

$$\dot{u} = -Au, \quad u \in Z_n. \tag{7.5}$$

If $f_n := f_{n,1} = P_n \circ f \circ P_n$ then, by Proposition 4.2 in [22] and its proof, the space Y_n is positively invariant relative to the local semiflow π_{f_n} and every bounded π_{f_n}-invariant set is included in Y_n and is π'_n-invariant. Moreover, every π'_n-invariant set is π_{f_n}-invariant. Setting

$$S_n := S_{n,1} \quad \text{and} \quad M_{p,n} := M_{p,n,1}, \quad p \in P,$$

we thus see that, whenever $n \geq n_0$, then S_n is a compact π'_n-invariant set and $(M_{p,n})_{p\in P}$ is a Morse decomposition of S_n, relative to π'_n. Moreover,

$$M_{\pi_{f_n},S_n}(I) = M_{\pi'_n,S_n}(I) =: M_n(I), \quad I \in \mathcal{I}(\prec).$$

Choose an arbitrary strongly π'_n-admissible index filtration $\mathcal{N}'_n = (N'_n(I))_{I\in\mathcal{A}(\prec)}$ for $(\pi'_n, S_n, (M_{p,n})_{p\in P})$. (Strong π'_n-admissibility means, in this finite-dimensional case, simply that $N'(P)$ is bounded in Y_n.) Let $B = B_n$ be the closed unit ball in Z_n. Since $|u\pi''_n t|_{Z_n} \leq e^{-\beta_n t}|u|_{Z_n}$ for some $\beta_n \in {]0,\infty[}$ and all $u \in Z_n$ and $t \in [0,\infty[$ it follows that, relative to π''_n, B is an isolating block for $\{0\}$ with empty exit set, so in particular, B is positively invariant.

We define $N_n(I) := N'_n(I) + B \cong N'_n(I) \times B$, $I \in \mathcal{A}(\prec)$. It is now a simple exercise to show that $\mathcal{N}_n = (N_n(I))_{I\in\mathcal{A}(\prec)}$ is a strongly π_{f_n}-admissible index filtration for $(\pi_{f_n}, S_n, (M_{p,n})_{p\in P})$. Since $N'_n(I) \subset N_n(I)$ for $I \in \mathcal{A}(\prec)$

there is an inclusion induced morphism $\theta_{\mathcal{N}'_n,\mathcal{N}_n} = (\theta_{\mathcal{N}'_n,\mathcal{N}_n}(J))_{J\in\mathcal{I}(\prec)}$ from the homology index braid $\mathcal{H}'_n$ of $(\pi'_n, S_n, (M_{p,n})_{p\in P})$ to the homology index braid $\mathcal{H}_n$ of $(\pi_{f_n}, S_n, (M_{p,n})_{p\in P})$. We claim that $\theta_{\mathcal{N}'_n,\mathcal{N}_n}$ is an isomorphism. In fact, let $J \in \mathcal{I}(\prec)$ be arbitrary. Choose I, $K \in \mathcal{A}(\prec)$ with $(I,J) \in \mathcal{I}_2(\prec)$ and $K = IJ$. Let $\phi: N'_n(K)/N'_n(I) \to N_n(K)/N_n(I)$ be inclusion induced and $\psi: N_n(K)/N_n(I) \to N'_n(K)/N'_n(I)$ be induced by the canonical projection $y+z \mapsto y$ of $X^\alpha = Y_n \oplus Z_n$ onto Y_n. It follows that $\psi \circ \phi$ is the identity on $N'_n(K)/N'_n(I)$ while $\phi\circ\psi$ is homotopic to the identity on $N_n(K)/N_n(I)$ via the homotopy $N_n(K)/N_n(I) \times [0,1] \to N_n(K)/N_n(I)$ induced by the homotopy $X^\alpha \times [0,1] \to X^\alpha$, $(y+z,\tau) \mapsto y + (1-\tau)z$. The homotopy axiom for singular homology now implies that the map

$$\theta_{\mathcal{N}'_n,\mathcal{N}_n}(J): H(\pi'_n, M_n(J)) \to H(\pi_{f_n}, M_n(J))$$

(which is induced by ϕ) is an isomorphism.

Using Corollary 7.10 we have established the following homology index braid continuation for the problem (cf. Theorem 3.11 in [6]).

Theorem 7.11. *If $n_0 \in \mathbb{N}$ is as in Corollary 7.10, then, for all $n \geq n_0$, the homology index braids of $(\pi_f, S, (M_p)_{p\in P})$ and $(\pi'_n, S_n, (M_{p,n})_{p\in P})$ are isomorphic so their share the same connection matrices and the same C-connection matrices.*

8. Singular continuation of (co)homology index braids

In this section we will state the *Singular nested index filtration theorem* and the *Singular continuation principle for homology index braids and connection matrices* proved in [7].

Let (X_0, d_0) be a metric space. Let $\varepsilon_0 \in\,]0,\infty[$ and for each $\varepsilon \in\,]0,\varepsilon_0]$ let $(Y_\varepsilon, d_\varepsilon)$ be a metric space and $\theta_\varepsilon \in Y_\varepsilon$ be a distinguished point of Y_ε. The open ball in Y_ε of center in v and radius $\beta > 0$ is denoted by $B_\varepsilon(v,\beta)$. Analogously, $B_\varepsilon[v,\beta]$ is the closed ball in Y_ε of center in $v \in Y_\varepsilon$ and radius $\beta > 0$.

For each $\varepsilon \in\,]0,\varepsilon_0]$ define the set $Z_\varepsilon := X_0 \times Y_\varepsilon$. Endow Z_ε with the metric

$$\Gamma_\varepsilon((u,v),(u',v')) := \max\{\, d_0(u,u'), d_\varepsilon(v,v')\} \quad \text{for } (u,v),\ (u',v') \in Z_\varepsilon.$$

Given a subset V of X_0, $\beta > 0$ and $\varepsilon \in\,]0,\varepsilon_0]$ define the 'inflated' subsets $[V]_{\varepsilon,\beta}$ of Z_ε as follows:

$$[V]_{\varepsilon,\beta} := \{(u,v) \in Z_\varepsilon \mid u \in V \text{ and } v \in \mathrm{Cl}_{Y_\varepsilon}(B_\varepsilon(\theta_\varepsilon,\beta))\}.$$

Let $\varepsilon \in\,]0,\varepsilon_0]$, π_ε (resp. π_0) be a local semiflow on Z_ε (resp. on X_0), S_ε (resp. S_0) be an isolated invariant set relative to π_ε (resp. π_0) and $(M_{p,\varepsilon})_{p\in P}$ (resp. $(M_{p,0})_{p\in P}$) be a Morse decomposition of S_ε (resp. S_0) relative to π_ε

(resp. π_0). Let $\mathcal{N}_\varepsilon = (N_\varepsilon(I))_{I\in\mathcal{A}(\prec)}$ be a strongly π_ε-admissible index filtration for $(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p\in P})$ and $\mathcal{N}_0 = (N_0(I))_{I\in\mathcal{A}(\prec)}$ be a strongly π_0-admissible index filtration for $(\pi_0, S_0, (M_{p,0})_{p\in P})$.

Let $\eta \in \,]0,\infty[$ and suppose that the following *singular nesting property* holds:

$$N_\varepsilon(I) \subset [N_0(I)]_{\varepsilon,\eta} \quad \text{for all } I \in \mathcal{A}(\prec).$$

For $J \in \mathcal{I}(\prec)$ choose I, $K \in \mathcal{A}(\prec)$ with $(I,J) \in \mathcal{I}_2(\prec)$ and $K = IJ$. Then $(N_\varepsilon(K), N_\varepsilon(I))$ is an FM-index pair for $M_\varepsilon(J)$, relative to π_ε and $(N_0(K), N_0(I))$ is an FM-index pair for $M_0(J)$, relative to π_0. The composition of the inclusion induced map from $N_\varepsilon(K)/N_\varepsilon(I)$ to $[N_0(K)]_{\varepsilon,\eta} \,/\, [N_0(I)]_{\varepsilon,\eta}$ followed by the map from $[N_0(K)]_{\varepsilon,\eta} \,/\, [N_0(I)]_{\varepsilon,\eta}$ to $N_0(K)/N_0(I)$ induced by the projection onto the first factor induces, via the $\langle\,\cdot\,\rangle$-operation, a homomorphism

$$[\Theta]_{\varepsilon,\eta,\mathcal{N}_\varepsilon,\mathcal{N}_0}(J)\colon H(\pi_\varepsilon, M_\varepsilon(J)) \to H(\pi_0, M_0(J)).$$

Of course, this homomorphism depends on the choice of $\varepsilon \in \,]0,\varepsilon_0]$, $\eta \in \,]0,\infty[$, $\mathcal{N}_\varepsilon$ and $\mathcal{N}_0$, but we claim that

(8.1) $\quad [\Theta]_{\varepsilon,\eta,\mathcal{N}_\varepsilon,\mathcal{N}_0}(J)$ is independent of the choice of I and K.

In fact, if I' and $K' \in \mathcal{A}(\prec)$ are such that $(I',J) \in \mathcal{I}_2(\prec)$ and $K' = I'J$ then property (2) of index filtrations implies that $N_\varepsilon(K) \setminus N_\varepsilon(I) = N_\varepsilon(K') \setminus N_\varepsilon(I')$ and $N_0(K) \setminus N_0(I) = N_0(K') \setminus N_0(I')$ (see Proposition 3.5 in [9] and its proof, which is also valid in our case). It follows that

$$[N_0(K)]_{\varepsilon,\eta} \setminus [N_0(I)]_{\varepsilon,\eta} = [N_0(K')]_{\varepsilon,\eta} \setminus [N_0(I')]_{\varepsilon,\eta}$$

and so there is an inclusion induced, commutative, diagram of pointed spaces

$$\begin{array}{ccc}
N_\varepsilon(K)/N_\varepsilon(I) & \longrightarrow & [N_0(K)]_{\varepsilon,\eta} \,/\, [N_0(I)]_{\varepsilon,\eta} \\
\big\downarrow & & \big\downarrow \\
N_\varepsilon(K')/N_\varepsilon(I') & \longrightarrow & [N_0(K')]_{\varepsilon,\eta} \,/\, [N_0(I')]_{\varepsilon,\eta}
\end{array}$$

Moreover, the diagram

$$\begin{array}{ccc}
[N_0(K)]_{\varepsilon,\eta} \,/\, [N_0(I)]_{\varepsilon,\eta} & \longrightarrow & N_0(K)/N_0(I) \\
\big\downarrow & & \big\downarrow \\
[N_0(K')]_{\varepsilon,\eta} \,/\, [N_0(I')]_{\varepsilon,\eta} & \longrightarrow & N_0(K')/N_0(I')
\end{array}$$

commutes, where the vertical maps are inclusion induced and the horizontal maps are projection induced. Composing these two diagrams we obtain a commutative diagram

$$\begin{tikzcd} N_\varepsilon(K)/N_\varepsilon(I) \arrow[r, "\alpha"] \arrow[d] & N_0(K)/N_0(I) \arrow[d] \\ N_\varepsilon(K')/N_\varepsilon(I') \arrow[r, "\alpha'"'] & N_0(K')/N_0(I') \end{tikzcd}$$

where, by Proposition 4.4, the (homotopy classes of the) vertical maps are are morphisms in $\mathcal{C}(\pi_\varepsilon, M_\varepsilon(J))$ (resp. in $\mathcal{C}(\pi_0, M_0(J))$). Thus, passing to homology and using Proposition 3.2 we see that

$$\langle H_q(\alpha)\rangle = \langle H_q(\alpha')\rangle, \quad q \in \mathbb{Z},$$

which is exactly what we claim. We write

$$[\Theta]_{\varepsilon,\eta,\mathcal{N}_\varepsilon,\mathcal{N}_0} = ([\Theta]_{\varepsilon,\eta,\mathcal{N}_\varepsilon,\mathcal{N}_0}(J))_{J\in\mathcal{I}(\prec)}.$$

We also claim that $[\Theta]_{\varepsilon,\eta,\mathcal{N}_\varepsilon,\mathcal{N}_0}\colon \mathcal{H}_\varepsilon \to \mathcal{H}_0$. In fact, let $(I,J) \in \mathcal{I}_2(\prec)$ and let B be the set of all $p \in P \setminus (IJ)$ for which there is a $p' \in IJ$ with $p \prec p'$. It follows that B, BI, $BIJ \in \mathcal{A}(\prec)$. Setting $N_{1,\varepsilon} = N_\varepsilon(BIJ)$, $N_{2,\varepsilon} = N_\varepsilon(BI)$ and $N_{3,\varepsilon} = N_\varepsilon(B)$ and $N_{1,0} = N_0(BIJ)$, $N_{2,0} = N_0(BI)$ and $N_{3,0} = N_0(B)$ we see that $(N_{1,\varepsilon}, N_{2,\varepsilon}, N_{3,\varepsilon})$ is an FM-index triple for $(\pi_\varepsilon, M_\varepsilon(IJ), M_\varepsilon(I), M_\varepsilon(J))$ and $(N_{1,0}, N_{2,0}, N_{3,0})$ is an FM-index triple for $(\pi_0, M_0(IJ), M_0(I), M_0(J))$. Moreover, composing the inclusion induced commutative diagram

$$\begin{tikzcd} N_{2,\varepsilon}/N_{3,\varepsilon} \arrow[r, "i"] \arrow[d] & N_{1,\varepsilon}/N_{3,\varepsilon} \arrow[r, "p"] \arrow[d] & N_{1,\varepsilon}/N_{2,\varepsilon} \arrow[d] \\ {[N_{2,0}]_{\varepsilon,\eta} / [N_{3,0}]_{\varepsilon,\eta}} \arrow[r, "i"'] & {[N_{1,0}]_{\varepsilon,\eta} / [N_{3,0}]_{\varepsilon,\eta}} \arrow[r, "p"'] & {[N_{1,0}]_{\varepsilon,\eta} / [N_{2,0}]_{\varepsilon,\eta}} \end{tikzcd}$$

with the inclusion and projection induced commutative diagram

$$\begin{tikzcd} {[N_{2,0}]_{\varepsilon,\eta} / [N_{3,0}]_{\varepsilon,\eta}} \arrow[r, "i"] \arrow[d] & {[N_{1,0}]_{\varepsilon,\eta} / [N_{3,0}]_{\varepsilon,\eta}} \arrow[r, "p"] \arrow[d] & {[N_{1,0}]_{\varepsilon,\eta} / [N_{2,0}]_{\varepsilon,\eta}} \arrow[d] \\ N_{2,0}/N_{3,0} \arrow[r, "i"'] & N_{1,0}/N_{3,0} \arrow[r, "p"'] & N_{1,0}/N_{2,0} \end{tikzcd}$$

yields a commutative diagram

$$\begin{tikzcd} N_{2,\varepsilon}/N_{3,\varepsilon} \arrow[r, "i"] \arrow[d] & N_{1,\varepsilon}/N_{3,\varepsilon} \arrow[r, "p"] \arrow[d] & N_{1,\varepsilon}/N_{2,\varepsilon} \arrow[d] \\ N_{2,0}/N_{3,0} \arrow[r, "i"'] & N_{1,0}/N_{3,0} \arrow[r, "p"'] & N_{1,0}/N_{2,0} \end{tikzcd}$$

which induces the following long homology ladder

$$
(8.2)\qquad
\begin{array}{ccccccccc}
\to H_q(N_{2,\varepsilon}/N_{3,\varepsilon}) & \xrightarrow{H_q(i)} & H_q(N_{1,\varepsilon}/N_{3,\varepsilon}) & \xrightarrow{H_q(p)} & H_q(N_{1,\varepsilon}/N_{2,\varepsilon}) & \xrightarrow{\widehat{\partial}_q} & H_{q-1}(N_{2,\varepsilon}/N_{3,\varepsilon}) \to \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\to H_q(N_{2,0}/N_{3,0}) & \xrightarrow[H_q(i)]{} & H_q(N_{1,0}/N_{3,0}) & \xrightarrow[H_q(p)]{} & H_q(N_{1,0}/N_{2,0}) & \xrightarrow[\widehat{\partial}_q]{} & H_{q-1}(N_{2,0}/N_{3,0}) \to
\end{array}
$$

Applying the $\langle\,\cdot\,\rangle$-operation to diagram (8.2) and using (8.1) together with Proposition 3.2 we obtain a commutative diagram of the form (6.3). This proves our second claim.

We call $[\Theta]_{\varepsilon,\eta,\mathcal{N}_\varepsilon,\mathcal{N}_0}$ a *singular inclusion induced morphism from $\mathcal{H}_\varepsilon$ to $\mathcal{H}_0$.*

We now obtain the following analogue of Proposition 6.3.

Proposition 8.1. *Let ε, $(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p\in P})$ and $(\pi_0, S_0, (M_{p,0})_{p\in P})$ be as above. Suppose that there are $\widetilde{\rho}$, $\widetilde{\eta} \in\ ]0,\infty[$ are such that $\mathrm{Cl}_{Y_\varepsilon}(B_\varepsilon(\theta_\varepsilon,\widetilde{\rho}))$ and $\mathrm{Cl}_{Y_\varepsilon}(B_\varepsilon(\theta_\varepsilon,\widetilde{\eta}))$ are contractible to θ_ε. Let $\mathcal{N}_0 = (N_0(I))_{I\in\mathcal{A}(\prec)}$ and $\widetilde{\mathcal{N}}_0 = (\widetilde{N}_0(I))_{I\in\mathcal{A}(\prec)}$ be strongly π_0-admissible index filtrations for $(\pi_0, S_0, (M_{p,0})_{p\in P})$, $\mathcal{N}_\varepsilon = (N_\varepsilon(I))_{I\in\mathcal{A}(\prec)}$ and $\widetilde{\mathcal{N}}_\varepsilon = (\widetilde{N}_\varepsilon(I))_{I\in\mathcal{A}(\prec)}$ be strongly π_ε-admissible index filtrations for $(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p\in P})$ and such that the following* singular nesting property *holds:*

$$(8.3)\qquad N_\varepsilon(I) \subset [N_0(I)]_{\varepsilon,\widetilde{\rho}} \subset \widetilde{N}_\varepsilon(I) \subset [\widetilde{N}_0(I)]_{\varepsilon,\widetilde{\eta}} \quad \textit{for all } I\in\mathcal{A}(\prec).$$

Then $\mathcal{H}(\pi_0, S_0, (M_p)_{p\in P})$ and $\mathcal{H}(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p\in P}))$ are isomorphic.

In the next two definitions, introduced in [3], $(\pi_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ is a family such that, for every $\varepsilon\in\ ]0,\varepsilon_0]$, π_ε is a local semiflow on Z_ε. Moreover, π_0 is a local semiflow on X_0.

Definition 8.2. With the notation introduced above, we say that the family $(\pi_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ *converges singularly to* π_0 if whenever $(\varepsilon_n)_n$ and $(t_n)_n$ are sequences of positive numbers such that $\varepsilon_n\to 0$, $t_n\to t_0$ as $n\to\infty$, for some $t_0\in[0,\infty[$ and whenever $u_0\in X_0$ and $w_n\in Z_{\varepsilon_n}$ are such that $\Gamma_{\varepsilon_n}(w_n,(u_0,\theta_{\varepsilon_n}))\to 0$ as $n\to\infty$ and $u_0\pi_0 t_0$ is defined, then there exists an $n_0\in\mathbb{N}$ such that for all $n\ge n_0$, $w_n\pi_{\varepsilon_n}t_n$ is defined and $\Gamma_{\varepsilon_n}(w_n\pi_{\varepsilon_n}t_n,(u_0\pi_0 t_0,\theta_{\varepsilon_n}))\to 0$ as $n\to\infty$.

Definition 8.3. Let β be a positive number and N be a closed subset of X_0. We say that N is a *singularly strongly admissible set with respect to β and the family* $(\pi_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ if the following conditions are satisfied:

(a) N is a strongly π_0-admissible set;
(b) for each $\varepsilon\in\ ]0,\varepsilon_0]$ the set $[N]_{\varepsilon,\beta}$ is strongly π_ε-admissible;
(c) whenever $(\varepsilon_n)_n$ and $(t_n)_n$ are sequences of positive numbers such that $\varepsilon_n\to 0$, $t_n\to\infty$ as $n\to\infty$ and whenever $w_n\in Z_{\varepsilon_n}$ is such that $w_n\pi_{\varepsilon_n}[0,t_n]\subset[N]_{\varepsilon_n,\beta}$, then there exist a $u_0\in N$ and a subsequence

of the sequence $(w_n\pi_{\varepsilon_n}t_n)_n$ of endpoints, denoted again by $(w_n\pi_{\varepsilon_n}t_n)_n$, such that $\Gamma_{\varepsilon_n}(w_n\pi_{\varepsilon_n}t_n,(u_0,\theta_{\varepsilon_n}))\to 0$ as $n\to\infty$.

The following singular continuation result for Morse decompositions was established in [5].

Theorem 8.4 (cf. Corollaries 4.14 and 4.15 in [5]). *Assume $(\pi_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ is a family of local semiflows that converges singularly to the local semiflow π_0, $\beta\in]0,\infty[$ and $\widetilde N$ is a singularly strongly admissible set with respect to β and $(\pi_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$. Moreover, suppose that $S_0:=\mathrm{Inv}_{\pi_0}(\widetilde N)$ relative to π_0 and $(M_{p,0})_{p\in P}$ is a $\prec$-ordered Morse decomposition of S_0 relative to π_0. For each $p\in P$, let $\Xi_p\subset\widetilde N$ be closed in X_0 and such that $M_{p,0}=\mathrm{Inv}_{\pi_0}(\Xi_p)\subset\mathrm{Int}_{X_0}(\Xi_p)$.* (*Such sets Ξ_p, $p\in P$, always exist.*)

Let $\eta\in]0,\beta]$. For $\varepsilon\in]0,\varepsilon_0]$ and $p\in P$ set $S_\varepsilon:=\mathrm{Inv}_{\pi_\varepsilon}([\widetilde N]_{\varepsilon,\eta})$ and $M_{p,\varepsilon}:=\mathrm{Inv}_{\pi_\varepsilon}([\Xi_p]_{\varepsilon,\eta})$. Then there is an $\widetilde\varepsilon\in]0,\varepsilon_0]$ such that for every $\varepsilon\in]0,\widetilde\varepsilon]$ and $p\in P$, $S_\varepsilon\subset\mathrm{Int}_{Z_\varepsilon}([\widetilde N]_{\varepsilon,\eta})$ and the family $(M_{p,\varepsilon})_{p\in P}$ is a $\prec$-ordered Morse decomposition of S_ε relative to π_ε.

We can now state the *singular nested index filtration principle* established in [7].

Theorem 8.5 (cf. Theorem 3.9 in [7]). *Assume the hypotheses* (*and thus also the conclusions*) *of Theorem* 8.4 *and let $\widetilde\varepsilon>0$ be as in that theorem. Let $\widetilde\beta_0\in]0,\beta[$ be fixed. Then there are $\widetilde\rho$, $\widetilde\eta\in]0,\widetilde\beta_0]$ and an $\varepsilon_{\mathrm c}\in]0,\widetilde\varepsilon]$ such that for every $\varepsilon\in[0,\varepsilon_{\mathrm c}]$ there exist strongly π_ε-admissible index filtrations $\mathcal N_\varepsilon=(N_\varepsilon(I))_{I\in\mathcal A(\prec)}$ and $\widetilde{\mathcal N}_\varepsilon=(\widetilde N_\varepsilon(I))_{I\in\mathcal A(\prec)}$ for $(\pi_\varepsilon,S_\varepsilon,(M_{p,\varepsilon})_{p\in P})$ such that the following* singular nesting property *holds:*

(8.4) $N_\varepsilon(I)\subset[N_0(I)]_{\varepsilon,\widetilde\rho}\subset\widetilde N_\varepsilon(I)\subset[\widetilde N_0(I)]_{\varepsilon,\widetilde\eta}$ *for all* $I\in\mathcal A(\prec)$ *and* $\varepsilon\in]0,\varepsilon_{\mathrm c}]$.

Theorem 8.5, Proposition 8.1 and Remark 6.2 immediately imply the following *Singular continuation principle for homology index braids and connection matrices*.

Theorem 8.6 (cf. Theorem 3.10 in [7]). *Assume the hypotheses of Theorem 8.5. Suppose that there exists an $\beta_0>0$ such that for all $\varepsilon\in]0,\varepsilon_0]$ and all $\eta\in]0,\beta_0]$ the set $\mathrm{Cl}_{Y_\varepsilon}(B_\varepsilon(\theta_\varepsilon,\eta))$ is contractible to θ_ε. Then there exists an $\varepsilon_{\mathrm c}\in]0,\widetilde\varepsilon]$ such that the homology index braids $\mathcal H(\pi_0,S_0,(M_p)_{p\in P})$ and $\mathcal H(\pi_\varepsilon,S_\varepsilon,(M_{p,\varepsilon})_{p\in P}))$, $\varepsilon\in]0,\varepsilon_{\mathrm c}]$, are isomorphic and determine the same collection of connection matrices and the same collection of C-connection matrices.*

Theorem 8.6 refines the corresponding singular Conley index continuation principle established in [3].

We will now briefly illustrate Theorem 7.4 by applying it to the thin domain problem considered in [18] and [1]. More extensive applications will appear in a subsequent publication. We assume the reader's familiarity with [18] and [1] and only recall some of the relevant notations and definitions.

Let M and N be positive integers. Write (x,y) for a generic point of $\mathbb{R}^M\times\mathbb{R}^N$. Let Ω be an arbitrary nonempty bounded domain in $\mathbb{R}^M\times\mathbb{R}^N$ with Lipschitz boundary and let $\varepsilon>0$ be arbitrary. Define the symmetric bilinear form $a_\varepsilon\colon H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$ by

$$a_\varepsilon(u,v):=\int_\Omega\left(\nabla_x u\cdot\nabla_x v+\frac{1}{\varepsilon^2}\nabla_y u\cdot\nabla_y v\right)dx\,dy$$

and let b be the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_{L^2(\Omega)}$. Let $A_\varepsilon\colon D(A_\varepsilon)\subset H^1(\Omega)\to L^2(\Omega)$ be the linear operator generated by the pair (a_ε,b). We define on $H^1(\Omega)$ the scalar product

$$(u,v)_\varepsilon:=a_\varepsilon(u,v)+b(u,v),\quad u,v\in H^1(\Omega)$$

and the corresponding norm

$$|u|_\varepsilon:=(a_\varepsilon(u,u)+|u|^2_{L^2(\Omega)})^{1/2},\quad u\in H^1(\Omega)$$

which is equivalent to the usual norm on $H^1(\Omega)$.

We also define the "limit" space $H^1_s(\Omega)$ by

$$H^1_s(\Omega)=\{u\in H^1(\Omega)\mid\nabla_y u=0\}.$$

Note that $H^1_s(\Omega)$ is a closed linear subspace of $H^1(\Omega)$ so $H^1_s(\Omega)$ is a Hilbert space under the usual scalar product of $H^1(\Omega)$.

Furthermore, define the space $L^2_s(\Omega)$ to be the closure of the set $H^1_s(\Omega)$ in $L^2(\Omega)$. It follows that $L^2_s(\Omega)$ is a Hilbert space under the scalar product of $L^2(\Omega)$.

Now let $a_0\colon H^1_s(\Omega)\times H^1_s(\Omega)\to\mathbb{R}$ be the "limit" bilinear form defined by

$$a_0(u,v):=\int_\Omega\nabla u\cdot\nabla v\,dx\,dy=\int_\Omega\nabla_x u\cdot\nabla_x v\,dx\,dy.$$

Finally, let b_0 be the restriction of the scalar product b to $L^2_s(\Omega)\times L^2_s(\Omega)$. Denote by A_0 the operator generated by the pair (a_0,b_0).

Now let $\varepsilon_0\in\,]0,1]$ be arbitrary and $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ be a family satisfying hypothesis (A1) introduced in Definition 2.6 in [1]. For $\varepsilon\in\,]0,\varepsilon_0]$ let π_ε be the local semiflow on $H^1(\Omega)$ generated by the solutions of the evolution equation

$$\dot u=A_\varepsilon u+f_\varepsilon(u).$$

Moreover, let π_0 be the local semiflow on $H^1_s(\Omega)$ generated by the solutions of the evolution equation

$$\dot u=A_0 u+f_0(u).$$

We will need the following singular convergence result proved in [1].

Proposition 8.7 (cf. Corollary 2.15 in [1] and its proof). *Let $(\varepsilon_n)_n$ be an arbitrary sequence of positive numbers convergent to zero. Moreover let $t \in [0,\infty[$ and $(t_n)_n$ be a sequence in $[0,\infty[$ converging to t. Finally, let $u_0 \in H^1_s(\Omega)$ and $(u_n)_n$ be a sequence in $H^1(\Omega)$ such that $|u_n - u_0|_{\varepsilon_n} \to 0$ $n \to \infty$. Assume that $u_0\pi_0 t$ is defined. Then, for all $n \in \mathbb{N}$ large enough, $u_n\pi_{\varepsilon_n}t_n$ is defined and*

$$|u_n\pi_{\varepsilon_n}t_n - u_0\pi_0 t|_{\varepsilon_n} \to 0 \quad \textit{as } n \to \infty.$$

For all $\varepsilon \in]0,\varepsilon_0]$ set $\theta_\varepsilon := 0 \in H^1(\Omega)$ and let $Q_\varepsilon: H^1(\Omega) \to H^1(\Omega)$ be the orthogonal projector onto $H^1_s(\Omega)$ with respect to the scalar product $(\,\cdot\,,\,\cdot\,)_\varepsilon$. Let $X_0 := H^1_s(\Omega)$ be endowed with the usual norm of $H^1(\Omega)$ and d_0 be the corresponding metric on X_0. Moreover, let $Y_\varepsilon := (I - Q_\varepsilon)(H^1(\Omega))$ be endowed with the norm $|\cdot|_\varepsilon$ and let d_ε be the corresponding metric on Y_ε. Set $Z_\varepsilon := X_0 \times Y_\varepsilon \cong H^1(\Omega)$ and note that the norm

$$\|(u,v)\|_\varepsilon := \max\{|u|_{H^1(\Omega)}, |v|_\varepsilon\}, \quad (u,v) \in X_0 \times Y_\varepsilon,$$

is equivalent to the norm $|\cdot|_\varepsilon$ on $H^1(\Omega)$ with constants independent of $\varepsilon \in]0,\varepsilon_0]$. Let Γ_ε be the metric on Z_ε generated by the norm $\|\cdot\|_\varepsilon$.

The remarks just made imply that, for every $\varepsilon \in]0,\varepsilon_0]$, π_ε is a local semiflow on Z_ε and π_0 is a local semiflow on X_0, while Proposition 8.7 just says that $(\pi_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ singularly converges to π_0.

Now an application of Lemma 2.21 in [1] shows that whenever $\beta > 0$ and N is closed and bounded in X_0 then N is singularly admissible with respect to β and the family $(\pi_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$.

It is clear that for all $\varepsilon \in]0,\varepsilon_0]$ and all $\beta \in]0,\infty[$ the set $\mathrm{Cl}_{Y_\varepsilon}(B_\varepsilon(\theta_\varepsilon,\beta))$ is contractible to θ_ε.

We thus obtain the following corollary of Theorems 8.4 and 8.5.

Theorem 8.8. *Let β be a positive number and $N \subset H^1_s(\Omega)$ be closed and bounded. Suppose that $(M_p)_{p\in P}$ is a $\prec$-ordered Morse decomposition of $S_0 := \mathrm{Inv}_{\pi_0}(N)$ relative to π_0. For each $p \in P$, let $\Xi_p \subset N$ be closed in X_0 and such that*

$$M_p = \mathrm{Inv}_{\pi_0}(\Xi_p) \subset \mathrm{Int}_{X_0}(\Xi_p).$$

Moreover, for every $I \in I(\prec)$, let $\Xi_I \subset N$ be closed in X_0 and such that

$$M_{\pi_0,S_0}(I) = \mathrm{Inv}_{\pi_0}(\Xi_I) \subset \mathrm{Int}_{X_0}(\Xi_I).$$

For $\varepsilon \in]0,\varepsilon_0]$ and $p \in P$ set $M_p(\varepsilon) := \mathrm{Inv}_{\pi_\varepsilon}([\Xi_p]_{\varepsilon,\beta})$. Then there is an $\widetilde{\varepsilon} \in]0,\varepsilon_0]$ such that for every $\varepsilon \in]0,\widetilde{\varepsilon}]$ the family $(M_p(\varepsilon))_{p\in P}$ is a $\prec$-ordered Morse decomposition of $S_\varepsilon := \mathrm{Inv}_{\pi_\varepsilon}([N]_{\varepsilon,\beta})$ relative to π_ε. Moreover,

$$M_I(\varepsilon) := M_{\pi_\varepsilon,S_\varepsilon} = \mathrm{Inv}_{\pi_\varepsilon}([\Xi_I]_{\varepsilon,\beta}) \subset \mathrm{Int}_{Z_\varepsilon}([\Xi_I]_{\varepsilon,\beta}), \quad I \in \mathcal{I}(\prec).$$

Finally, the homology index braids $\mathcal{H}(\pi_0, S_0, (M_p)_{p\in P})$ *and* $\mathcal{H}(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p\in P}))$ *are isomorphic and determine the same collection of connection matrices and the same collection of* C*-connection matrices.*

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Manuscript received October 5, 2004

Maria C. Carbinatto
Departamento de Matemática
ICMC-USP Caixa Postal 668
13.560-970 São Carlos SP, BRAZIL

E-mail address: mdccarbi@icmc.usp.br

Krzysztof P. Rybakowski
Universität Rostock
Fachbereich Mathematik
Universitätsplatz 1
18055 Rostock, GERMANY

E-mail address: krzysztof.rybakowski@mathematik.uni-rostock.de