PARAMETER DEPENDENT PULL-BACK
OF CLOSED DIFFERENTIAL FORMS
AND INVARIANT INTEGRALS

JEAN MAWHIN

Dedicated to the memory of Olga Ladyzhenskaya

Abstract. We prove, given a closed differential $k$-form $\omega$ in an arbitrary
open set $D \subset \mathbb{R}^n$, and a parameter dependent smooth map $F(\cdot, \lambda)$ from
an arbitrary open set $G \subset \mathbb{R}^m$ into $D$, that the derivative with respect to
$\lambda$ of the pull-back $F(\cdot, \lambda)^* \omega$ is exact in $G$. We give applications to various
theorems in topology, dynamics and hydrodynamics.

1. Introduction

It is well known that a closed differential form (cocycle) on a set $D \subset \mathbb{R}^n$
needs not be exact (coboundary) on $D$ [8], [15]. The converse of Poincaré’s
lemma implies that it is the case if $D$ is simply connected. In recent papers
[9], [10], it has been shown that given a differential $n$-form $\omega$ on $D \subset \mathbb{R}^n$, which
necessarily is a cocycle, the derivative with respect to $\lambda$ of its pull-back $F(\cdot, \lambda)^* \omega$
by a $C^2$ parameter dependent mapping $F(\cdot, \lambda): G \subset \mathbb{R}^n \to D \subset \mathbb{R}^n$ is always a
coboundary. This result allows a simple and complete proof of a lemma on the
invariance of an integral stated and proved in a special case by Tartar [16] and
reproduced in [2]. This lemma was used in [9] to obtain the homotopy invariance
of Brouwer degree, and in [10] to give elementary proofs of various existence and
fixed point theorems.

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In this paper, we want to show that the above mentioned property holds indeed for any \( k \)-cocycle on \( D \subset \mathbb{R}^n \) and any \( C^2 \) parameter dependent mapping \( F(\cdot, \lambda); G \subset \mathbb{R}^m \to D \subset \mathbb{R}^n \) (Theorem 61). The given proof is a lengthy and tedious computation, which is substantially shorter only for \( k = 1 \) and for \( k = n \). For the readers uniquely interested in those situations, we have explicited the proof for \( k = 1 \) (Theorem 4.1) and reproduced, for the sake of completeness, the proof for \( k = n \) given in [10] (Theorem 5.1).

For \( k = 1 \), we give as direct applications simple proofs of the \( n \)-dimensional generalization of a theorem on the invariance of the circulation of a perfect fluid due to Lord Kelvin [17] (see also [6]), and of Cauchy integral theorem for holomorphic functions. For \( k = n - 1 \), Theorem 61 generalizes a result of Hatziafratis and Tsarpalias [3] obtained for the \((n-1)\) solid angle form occurring in the definition of Kronecker’s index. For \( k = n \), we complete the applications given in [10] by an elementary proof of a Poincaré–Krasnosel’skiĭ bifurcation theorem in finite dimension.

In some physical situations, the family of pull-back transformations is parametrized by time and is given by the flow associated to an evolution equation. We show in two classical examples, Liouville’s theorem in dynamics [7] and Helmholtz theorem in hydrodynamics [4] (see also [14]), how those classical results follow from the same type of reasonings (Theorems 7.1 and 8.2). Those results belong of course to Poincaré’s theory of integral invariants (see [12] and [13]), which also can be related to the considerations developed here.

2. Parameter dependent differential forms

We first recall a few elementary facts and results on differential forms [8], [15].

If \( D \subset \mathbb{R}^n \) is open and \( 0 \leq k \leq n \) is an integer, we consider the differential \( k \)-form of class \( C^l \) in \( D \) \((l \geq 0)\)

\[
\omega = \sum_{1 \leq i_1 < \ldots < i_k \leq n} w_{i_1 \ldots i_k} \, dx_{i_1} \wedge \ldots \wedge dx_{i_k},
\]

where the real functions \( w_{i_1 \ldots i_k} \) are of class \( C^l \) on \( D \). If \( G \subset \mathbb{R}^m \) is open and \( T: G \to D \) is of class \( C^1 \), the pull-back \( T^*\omega \) is the differential \( k \)-form in \( G \) defined by

\[
T^*\omega = \sum_{1 \leq i_1 < \ldots < i_k \leq n} (w_{i_1 \ldots i_k} \circ T) \, dt_{i_1} \wedge \ldots \wedge dt_{i_k},
\]

where \( dt_i \) is the differential 1-form on \( G \) defined by \( dt_i = \sum_{j=1}^m \partial_j T_i \, dy_j \). If \( \omega \) is of class \( C^1 \), the exterior differential \( d\omega \) of \( \omega \) is the differential \((k+1)\)-form in \( D \) defined by

\[
d\omega = \sum_{1 \leq i_1 < \ldots < i_k \leq n} dw_{i_1 \ldots i_k} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}.
\]
where \( dw_{i_1 \ldots i_k} = \sum_{j=1}^n \partial_j w_{i_1 \ldots i_k} \, dx_j \). Explicitly, with
\[
1 \leq i_1, \ldots, i_k, j_1, \ldots, j_{k+1}, j \leq n,
\]
we have
\[
d\omega = \sum_{j_1 < \ldots < j_{k+1}} \left[ \sum_{l=1}^{k+1} (-1)^{l-1} \partial_{j_l} w_{j_1 \ldots j_l \ldots j_{k+1}} \right] dx_{j_1} \wedge \ldots \wedge dx_{j_{k+1}},
\]
where the symbol \( \sim \) means that the corresponding term is missing. When \( \omega \) is of class \( C^1 \), \( \omega \) is closed or is a \( k \)-cocycle if \( d\omega = 0 \), which, by the computation above, is equivalent to the set of conditions
\[
(2.1) \sum_{l=1}^{k+1} (-1)^{l-1} \partial_{j_l} w_{j_1 \ldots j_l \ldots j_{k+1}} = 0 \quad (1 \leq j_1 < j_2 < \ldots < j_{k+1} \leq n).
\]

Consider now a parameter dependent differential \( k \)-form in \( D \subset \mathbb{R}^n \)
\[
\mu(\lambda) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} m_{i_1 \ldots i_k}(\cdot, \lambda) \, dx_{i_1} \wedge \ldots \wedge dx_{i_k}
\]
whose coefficients
\[
m_{i_1 \ldots i_k} : D \times [a, b] \to \mathbb{R}, \quad (x, \lambda) \mapsto m_{i_1 \ldots i_k}(x, \lambda)
\]
are of class \( C^1 \) on \( D \times [a, b] \).

**Definition 2.1.** The partial derivative \( \partial_\lambda \mu \) of \( \mu(\lambda) \) with respect to \( \lambda \) is the differential \( k \)-form in \( D \)
\[
\partial_\lambda \mu(\lambda) := \sum_{1 \leq i_1 < \ldots < i_k \leq n} \partial_\lambda m_{i_1 \ldots i_k}(\cdot, \lambda) \, dx_{i_1} \wedge \ldots \wedge dx_{i_k},
\]
It follows easily from this definition that if \( f : D \times [a, b] \to \mathbb{R} \) and
\[
\nu(\lambda) = \sum_{1 \leq j_1 < \ldots < j_l \leq n} n_{j_1 \ldots j_l}(\cdot, \lambda) \, dx_{j_1} \wedge \ldots \wedge dx_{j_l},
\]
are of class \( C^1 \) on \( D \times [a, b] \), then
\[
(2.2) \partial_\lambda [f(\cdot, \lambda)\mu(\lambda)] = \partial_\lambda f(\cdot, \lambda)\mu(\lambda) + f(\cdot, \lambda)\partial_\lambda \mu(\lambda),
\]
\[
(2.3) \partial_\lambda [\mu(\lambda) \wedge \nu(\lambda)] = \partial_\lambda \mu(\lambda) \wedge \nu(\lambda) + \mu(\lambda) \wedge \partial_\lambda \nu(\lambda),
\]
and if \( \mu(\lambda) \) is of class \( C^2 \), then
\[
(2.4) \partial_\lambda [d\mu(\lambda)] = d[\partial_\lambda \mu(\lambda)].
\]
3. Parameter dependent pullback of a differential form

If $D \subset \mathbb{R}^n$ is open and $0 \leq k \leq n$ is an integer, let us consider the differential $k$-form in $D$

$$\omega = \sum_{1 \leq i_1 < \cdots < i_k \leq n} w_{i_1 \ldots i_k} \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$ 

If $G \subset \mathbb{R}^m$ is open and if $F: G \times [a, b] \to D$ is of class $C^2$, we consider for each $\lambda \in [a, b]$ the pull-back $F(\cdot, \lambda)^* \omega$ of $\omega$ by $F(\cdot, \lambda)$

$$(3.1) \quad F(\cdot, \lambda)^* \omega := \sum_{1 \leq i_1 < \cdots < i_k \leq n} (w_{i_1 \ldots i_k} \circ F) (\cdot, \lambda) \, dF_{i_1} \wedge \cdots \wedge dF_{i_k},$$

where we write

$$dF_i = dF_i(\cdot, \lambda) = \sum_{l=1}^m \partial_i F_l(\cdot, \lambda) \, dy_l.$$ 

Notice that, by formula (2.4), we have

$$(3.2) \quad \partial_\lambda (dF_i) = d(\partial_\lambda F_i).$$

**Lemma 3.1.** If the differential $k$-form $\omega$ is of class $C^1$ on $D$, and $F: G \times [a, b] \to D$ is of class $C^2$, then, with $1 \leq i_1, \ldots, i_k \leq n$,

$$(3.3) \quad \partial_\lambda [F(\cdot, \lambda)^* \omega] = \sum_{i_1 < \cdots < i_k} \sum_{j=1}^n (\partial_j w_{i_1 \ldots i_k} \circ F) \partial_\lambda F_j \, dF_{i_1} \wedge \cdots \wedge dF_{i_k}$$

$$+ \sum_{i_1 < \cdots < i_k} (w_{i_1 \ldots i_k} \circ F) \sum_{l=1}^k (-1)^{i-l} d[\partial_\lambda F_{i_l} \wedge \cdots \wedge \hat{dF_{i_l}} \wedge \cdots \wedge dF_{i_k}].$$

**Proof.** Using formulas (2.2) and (3.2), we get, if $\omega$ is of class $C^1$ in $D$, and $1 \leq i_1, \ldots, i_k \leq n, 1 \leq j_1, \ldots, j_{k+1} \leq n$,

$$\partial_\lambda [F(\cdot, \lambda)^* \omega] = \partial_\lambda \left[ \sum_{i_1 < \cdots < i_k} (w_{i_1 \ldots i_k} \circ F) \, dF_{i_1} \wedge \cdots \wedge dF_{i_k} \right]$$

$$= \sum_{i_1 < \cdots < i_k} \partial_\lambda (w_{i_1 \ldots i_k} \circ F) \, dF_{i_1} \wedge \cdots \wedge dF_{i_k}$$

$$+ \sum_{i_1 < \cdots < i_k} (w_{i_1 \ldots i_k} \circ F) \sum_{l=1}^k dF_{i_l} \wedge \cdots \wedge d(\partial_\lambda F_{i_l}) \wedge \cdots \wedge dF_{i_k}$$

$$= \sum_{i_1 < \cdots < i_k} \sum_{j=1}^n (\partial_j w_{i_1 \ldots i_k} \circ F) \partial_\lambda F_j \, dF_{i_1} \wedge \cdots \wedge dF_{i_k}$$

$$+ \sum_{i_1 < \cdots < i_k} (w_{i_1 \ldots i_k} \circ F) \sum_{l=1}^k (-1)^{l-1} d[\partial_\lambda F_{i_l} \wedge \cdots \wedge \hat{dF_{i_l}} \wedge \cdots \wedge dF_{i_k}]. \quad \square$$
4. The case of 1-cocycle

Let the differential 1-form

\( \omega = \sum_{j=1}^{n} w_j \, dx_j \)

be of class \( C^1 \) on \( D \). By formula (2.1), \( \omega \) is a 1-cocycle if and only if

\( \partial_i w_j = \partial_j w_i \quad (1 \leq i < j \leq n) \).

Let \( G \subset \mathbb{R}^m \) be open and \( F: G \times [a,b] \to D, (y,\lambda) \mapsto F(y,\lambda) \) be of class \( C^2 \).

THEOREM 4.1. If \( \omega \) is a 1-cocycle of class \( C^1 \) on \( D \), then

\[ \partial_\lambda [F(\cdot,\lambda)^* \omega] := \partial_\lambda \left[ \sum_{j=1}^{n} (w_j \circ F) \, dF_j \right] = d \left[ \sum_{j=1}^{n} (w_j \circ F) \partial_\lambda F_j \right]. \]

PROOF. We have, using formulas (3.3) and (4.2),

\[
\begin{align*}
\partial_\lambda [F(\cdot,\lambda)^* \omega] &= \sum_{j=1}^{n} \sum_{k=1}^{n} (\partial_\lambda w_j \circ F) \, \partial_\lambda F_k \, dF_j + \sum_{j=1}^{n} (w_j \circ F) \, d(\partial_\lambda F_j) \\
&= \sum_{k=1}^{n} \sum_{j=1}^{n} (\partial_\lambda w_k \circ F) \, \partial_\lambda F_k \, dF_j + \sum_{k=1}^{n} (w_k \circ F) \, d(\partial_\lambda F_k) \\
&= d(\sum_{k=1}^{n} (w_k \circ F) \partial_\lambda F_k) + d(\sum_{j=1}^{n} (w_j \circ F) \partial_\lambda F_j) \\
&= d \left[ \sum_{j=1}^{n} (w_j \circ F) \partial_\lambda F_j \right].
\end{align*}
\]

We now show how Theorem 4.1 imply some classical conservation theorems.

The first result for \( n = 3 \) is due to Lord Kelvin [17], in the context of hydrodynamics of perfect fluids. Recall that the circulation of the differential 1-form \( \omega \) along the 1-simplex \( \varphi: [0,T] \to D \) of class \( C^1 \) is defined by the integral

\[
\int_{\varphi} \omega = \int_0^T \varphi^* \omega = \int_0^T \left[ \sum_{j=1}^{n} u_j(\varphi(s)) \varphi_j'(s) \, ds \right].
\]

\( \varphi \) is called a 1-cycle if \( \varphi(0) = \varphi(T) \).

COROLLARY 4.2. If \( \omega = \sum_{j=1}^{n} w_j \, dx_j \) is a 1-cocycle of class \( C^1 \) on \( D \), and for each \( \lambda \in [a,b] \), \( F(\cdot,\lambda): [0,T] \to D \) is a 1-cycle of class \( C^2 \) in \( D \), then the circulation of \( \omega \) along \( F(\cdot,\lambda) \)

\[
\int_{F(\cdot,\lambda)} \omega = \int_0^T \sum_{j=1}^{n} (w_j \circ F) \, \partial_\lambda F_j(y,\lambda) \, dy
\]
is independent of \( \lambda \) on \([a, b]\).

**Proof.** Using Leibniz’ rule and Theorem 4.1, we obtain

\[
\partial_\lambda \left[ \int_{F(\cdot, \lambda)} \omega \right] = \int_0^T \partial_\lambda \left[ F(\cdot, \lambda)^* \omega \right] = \int_0^T d \left[ \sum_{j=1}^n (w_j \circ F) \partial_\lambda F_j \right] = 0,
\]
as \( F(\cdot, \lambda) \) is a 1-cycle.

**Remark 4.3.** If \( n = 3 \) and if \((w_1, w_2, w_3)\) denotes the field of velocities of the irrotational motion of a perfect fluid, if \( \lambda \) denotes the time and if \( F([a, b], \lambda) \) denotes the time evolution of a closed curve under the motion of the fluid, Corollary 4.2 expresses the constancy of the circulation of the velocity around the closed curve.

A second consequence of Theorem 4.1 is a version of Cauchy’s theorem in complex functions theory [8]. Let \( D \subset \mathbb{C} \) be open, \( f: D \to \mathbb{C} \) holomorphic and let

\[
\Gamma_j: [0, T] \times [a, b] \to D, \quad (y, \lambda) \mapsto \Gamma_j(y, \lambda), \quad (1 \leq j \leq m)
\]
be of class \( C^2 \) and such that

\[
\Gamma_j(T, \lambda) = \Gamma_{j+1}(0, \lambda), \quad (j = 1, \ldots, m - 1), \quad \Gamma_m(T, \lambda) = \Gamma_1(0, \lambda), \quad \lambda \in [a, b].
\]

So, when \( \lambda \) varies, the family of the \( \Gamma_j(\cdot, \lambda) \) represents a continuous deformation of a piecewise-\( C^2 \) 1-cycle in \( D \).

**Corollary 4.4.** The expression

\[
\sum_{j=1}^m \int_{\Gamma_j(\cdot, \lambda)} f(z) \, dz
\]
is independent of \( \lambda \) on \([a, b]\).

**Proof.** We have, using Leibniz rule and Theorem 4.1,

\[
\partial_\lambda \left( \sum_{j=1}^m \int_{\Gamma_j(\cdot, \lambda)} f(z) \, dz \right) = \sum_{j=1}^m \int_0^T \partial_\lambda \left[ \Gamma_j^*(\cdot, \lambda) (f(z) \, dz) \right]
\]

\[
= \sum_{j=1}^m \int_0^T d \left[ (f \circ \Gamma_j)(\cdot, \lambda) \partial_\lambda \Gamma_j \right]
\]

\[
= \sum_{j=1}^m \left[ ((f \circ \Gamma_j)(T, \lambda) \partial_\lambda \Gamma_j(T, \lambda) - (f \circ \Gamma_j)(0, \lambda) \partial_\lambda \Gamma_j(0, \lambda)) \right]
\]

\[
= (f \circ \Gamma_m)(T, \lambda) \partial_\lambda \Gamma_m(T, \lambda) - (f \circ \Gamma_1)(0, \lambda) \partial_\lambda \Gamma_1(0, \lambda) = 0. \quad \square
\]
5. The case of a differential \( n \)-form

Let the differential \( n \)-form
\[
\omega = w \, dx_1 \wedge \ldots \wedge dx_n,
\]
be of class \( C^1 \) in \( D \). Notice that \( \omega \) is always a \( n \)-cocycle in \( D \), as \( d\omega \) is a differential \((n + 1)\)-form in \( \mathbb{R}^n \). Let \( G \subset \mathbb{R}^m \) be open and \( F: G \times [a, b] \to D, (y, \lambda) \mapsto G(y, \lambda) \) be of class \( C^2 \).

**Theorem 5.1.** If \( \omega \) is a differential \( n \)-form of class \( C^1 \) in \( D \), then
\[
\partial_\lambda [F^*(\cdot, \lambda)\omega] := \partial_\lambda [(w \circ F) \, dF_1 \wedge \ldots \wedge dF_n] \\
= d \left( (w \circ F) \left( \sum_{j=1}^{n} (-1)^{j-1} \partial_\lambda F_j \, dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n \right) \right).
\]

**Proof.** We have, using formula (3.3)
\[
\partial_\lambda [F^*(\cdot, \lambda)\omega] = \left[ \sum_{j=1}^{n} (\partial_j w \circ F) \, \partial_\lambda F_j \right] \, dF_1 \wedge \ldots \wedge dF_n \\
+ (w \circ F) \left[ \sum_{j=1}^{n} (-1)^{j-1} d(\partial_\lambda F_j \, dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n) \right] \\
= \sum_{j=1}^{n} (-1)^{j-1} (\partial_j w \circ F) \, dF_j \wedge \partial_\lambda F_j \, dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n \\
+ (w \circ F) \left[ \sum_{j=1}^{n} (-1)^{j-1} d(\partial_\lambda F_j \, dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n) \right] \\
= \sum_{j=1}^{n} (-1)^{j-1} \left[ \sum_{k=1}^{n} (\partial_k w \circ F) \, dF_k \right] \\
\wedge \partial_\lambda F_j \, dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n \\
+ (w \circ F) \left[ \sum_{j=1}^{n} (-1)^{j-1} d(\partial_\lambda F_j \, dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n) \right] \\
= d(w \circ F) \wedge \left( \sum_{j=1}^{n} (-1)^{j-1} \partial_\lambda F_j \, dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n \right) \\
+ (w \circ F) \, d \left( \sum_{j=1}^{n} (-1)^{j-1} \partial_\lambda F_j \, dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n \right) \\
= d(w \circ F) \left( \sum_{j=1}^{n} (-1)^{j-1} \partial_\lambda F_j \, dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n \right). \quad \square
\]

Like in the previous section, one deduces from Theorem 5.1 the following invariance result.
Corollary 5.2. Let $\omega = w \, dx_1 \wedge \ldots \wedge dx_n$ be a differential $n$-form of class $C^1$ in the open set $D \subset \mathbb{R}^n$, $G \subset \mathbb{R}^n$ be open and bounded and $F: G \times [a, b] \to D$ be of class $C^2$. If, for each $\lambda \in [a, b]$, one has

\begin{equation}
\text{supp} \, \omega \cap F(\cdot, \lambda)(\partial G) = \emptyset,
\end{equation}

then the integral

\begin{equation}
\int_G F(\cdot, \lambda)^* \omega = \int_G [w \circ F(y, \lambda)] \text{Jac} \, F(y, \lambda) \, dy
\end{equation}

is independent of $\lambda$ on $[a, b]$.

As an application of Corollary 5.2, let us give an elementary proof of a fundamental bifurcation result which can be traced to Poincaré [11] and Krasnovskiĭ [5]. Let $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be continuous and such that $f(0, \lambda) = 0$ for each $\lambda \in \mathbb{R}$, and consider the family of equations

\begin{equation}
f(x, \lambda) = 0.
\end{equation}

Definition 5.3. $(0, \lambda_0)$ is a bifurcation point for (5.4) if

\begin{equation}
(\forall r > 0)(\exists (x, \lambda) \in (B[0, r] \setminus \{0\}) \times [\lambda_0 - r, \lambda_0 + r]) : f(x, \lambda) = 0.
\end{equation}

Theorem 5.4. Let $A: \mathbb{R} \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ be continuous and $R: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be continuous and such that

\begin{equation}
\lim_{x \to 0} \frac{R(x, \lambda)}{\|x\|} = 0,
\end{equation}

uniformly on compact intervals of $\mathbb{R}$. Assume that there exists $a < b$ such that

\begin{equation}
det A(a) \det A(b) < 0.
\end{equation}

Then (5.4) with

\begin{equation}
f(x, \lambda) := A(\lambda)x + R(x, \lambda)
\end{equation}

has a bifurcation point in $\{0\} \times [a, b]$.

Proof. Notice first that if $(0, \lambda_0)$ is not a bifurcation point for (5.4), then there exists $r = r(\lambda_0) > 0$ such that $f(x, \lambda) \neq 0$ for all $x \in B[0, r] \setminus \{0\}$ and all $\lambda \in [\lambda_0 - r, \lambda_0 + r]$. Hence, an easy compactness argument implies that if (5.4) has no bifurcation point in $\{0\} \times [a, b]$, then

\begin{equation}
(\exists r > 0)(\forall x \in B[0, r] \setminus \{0\})(\forall \lambda \in [a, b]) : f(x, \lambda) \neq 0.
\end{equation}

Now, by assumptions, there exists $\alpha > 0$ such that, for all $x \in \mathbb{R}^n$,

\begin{align*}
\|A(\alpha)x\| &\geq \alpha \|x\|, \\
\|A(b)x\| &\geq \alpha \|x\|,
\end{align*}

for all $x \in \mathbb{R}^n$. The proof then proceeds as above.
and there exists \( r_1 \in [0, r] \) such that, for all \( x \in B[0, r_1] \) and all \( \lambda \in [a, b] \), one has

\[
\|R(x, \lambda)\| \leq \frac{\alpha}{2} \|x\|.
\]

Consequently, for all \( x \in \partial B(0, r_1) \), and all \( \mu \in [0, 1] \), one has

\[
(5.9) \quad \|g_c(x, \mu)\| := \|A(c)x + \mu R(x, c)\| \geq \frac{\alpha}{2} r_1 := \alpha_1, \quad c = a, b.
\]

Now, it follows from relation (5.8) that there exists \( \alpha_2 > 0 \) such that, for all \( x \in \partial B(0, r_1) \) and all \( \lambda \in [a, b] \), one has

\[
\|f(x, \lambda)\| \geq \alpha_1.
\]

Let \( \alpha_3 := \min\{\alpha_1, \alpha_2\} \) and \( w \in C^1(\mathbb{R}^n, \mathbb{R}_+) \) be such that \( \text{supp} w \subset B(0, \alpha_3) \) and

\[
(5.11) \quad \int_{\mathbb{R}^n} w(x) \, dx = 1.
\]

A first application of Corollary 5.2 to the family of pull-backs \( f(\cdot, \lambda), \lambda \in [a, b] \) implies that

\[
(5.12) \quad \int_{B(0, r)} (w \circ f)(y, a) \text{Jac} f(y, a) \, dy = \int_{B(0, r)} (w \circ f)(y, b) \text{Jac} f(y, b) \, dy.
\]

A second application of Corollary 5.2 to the families of pull-backs \( g_a(\cdot, \mu), \quad g_b(\cdot, \mu), \quad \mu \in [0, 1] \) implies that

\[
(5.13) \quad \int_{B(0, r)} (w \circ f)(y, a) \text{Jac} f(y, a) \, dy
\]

\[
= \int_{B(0, r)} (w \circ g_a)(y, 1) \text{Jac} g_a(\cdot, 1) \, dy
\]

\[
= \int_{B(0, r)} (w \circ g_a)(y, 0) \text{Jac} g_a(\cdot, 0) \, dy
\]

\[
= \int_{B(0, r)} (w \circ A(a))(y) \det A(a) \, dy = \text{sign det} A(a).
\]

\[
(5.14) \quad \int_{B(0, r)} (w \circ f)(y, b) \text{Jac} f(y, b) \, dy
\]

\[
= \int_{B(0, r)} (w \circ g_b)(y, 1) \text{Jac} g_b(\cdot, 1) \, dy
\]

\[
= \int_{B(0, r)} (w \circ g_b)(y, 0) \text{Jac} g_b(\cdot, 0) \, dy
\]

\[
= \int_{B(0, r)} (w \circ A(b))(y) \det A(b) \, dy = \text{sign det} A(b).
\]

where we have used the change of variables rule in a multiple integral and condition (5.11). The contradiction follows from relations (5.12)–(5.14) and assumption (5.7). \( \square \)
6. The case of a $k$-cocycle

Let the differential $k$-form

$$\omega = \sum_{1 \leq i_1 < \ldots < i_k \leq n} w_{i_1 \ldots i_k} \, dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

be of class $C^1$ in an open set $D \subset \mathbb{R}^n$. Recall that $\omega$ is a $k$-cocycle if and only if relations (2.1) hold. Let $G \subset \mathbb{R}^n$ be open and let $F: G \times [a, b] \rightarrow \mathbb{R}$ be of class $C^2$.

**Theorem 6.1.** If $\omega$ is a $k$-cocycle in $D$, then, with $1 \leq i_1, \ldots, i_k \leq n$,

$$\partial_\lambda[F(\cdot, \lambda)^* \omega] := \partial_\lambda \left[ \sum_{i_1 < \ldots < i_k} (w_{i_1 \ldots i_k} \circ F) \, dF_{i_1} \wedge \ldots \wedge dF_{i_k} \right]$$

$$= d \left[ \sum_{i_1 < \ldots < i_k} (w_{i_1 \ldots i_k} \circ F) \sum_{j=1}^k (-1)^{j-1} \partial_\lambda F_{j_1} \, dF_{i_1} \wedge \ldots \wedge \hat{dF}_{i_j} \wedge \ldots \wedge dF_{i_k} \right].$$

**Proof.** To simplify some heavy notations in this proof, we write

$$\sum_{l} \text{ for } 1 \leq i_1 < \ldots < i_k \leq n, \quad \sum_{j} \text{ for } 1 \leq j_1 < \ldots < j_{k+1} \leq n$$

and, for $1 \leq i_1, \ldots, i_k \leq n$ and $1 \leq j_1, \ldots, j_{k+1} \leq n$, we set

$$[\hat{dF}_{i_1}] = dF_{i_1} \wedge \ldots \wedge \hat{dF}_{i_1} \wedge \ldots \wedge dF_{i_k}, \quad [\hat{dF}_{j_1}] = dF_{j_1} \wedge \ldots \wedge \hat{dF}_{j_1} \wedge \ldots \wedge dF_{j_{k+1}}.$$

We have, using formula (3.1),

$$\partial_\lambda[F(\cdot, \lambda)^* \omega] = d \left[ \sum_{l} (w_{i_1 \ldots i_k} \circ F) \sum_{l=1}^k (-1)^{l-1} \partial_\lambda F_{i_l} \, [\hat{dF}_{i_l}] \right]$$

$$+ \sum_{l} \sum_{j=1}^n (\partial_j w_{i_1 \ldots i_k} \circ F) \partial_\lambda F_j \, dF_{i_1} \wedge \ldots \wedge dF_{i_k}$$

$$- \sum_{l} d(w_{i_1 \ldots i_k} \circ F) \wedge \left[ \sum_{l=1}^k (-1)^{l-1} \partial_\lambda F_{i_l} \, [\hat{dF}_{i_l}] \right].$$

Now

$$\sum_{l} \sum_{j=1}^n (\partial_j w_{i_1 \ldots i_k} \circ F) \partial_\lambda F_j \, dF_{i_1} \wedge \ldots \wedge dF_{i_k}$$

$$= \sum_{l} \sum_{j<i_1} (\partial_j w_{i_1 \ldots i_k} \circ F) \partial_\lambda F_j \, dF_{i_1} \wedge \ldots \wedge dF_{i_k}$$

$$+ \sum_{l} (\partial_{i_1} w_{i_1 \ldots i_k} \circ F) \partial_\lambda F_{i_1} \, dF_{i_1} \wedge \ldots \wedge dF_{i_k}$$

$$+ \sum_{l} \sum_{i_1<j<i_2} (\partial_j w_{i_1 \ldots i_k} \circ F) \partial_\lambda F_j \, dF_{i_1} \wedge \ldots \wedge dF_{i_k}.$$
Renaming the indices, we obtain

\[ + \sum_{I} (\partial_{I} w_{i_1 \ldots i_k} \circ F) \partial_{\lambda} F_{i_1} d F_{i_2} \wedge \ldots \wedge d F_{i_k} + \ldots \]

\[ + \sum_{I} \sum_{i_{k-1} < j < i_{k}} (\partial_{j} w_{i_1 \ldots i_k} \circ F) \partial_{\lambda} F_{i_1} d F_{i_2} \wedge \ldots \wedge d F_{i_k} \]

\[ + \sum_{I} (\partial_{I} w_{i_1 \ldots i_k} \circ F) \partial_{\lambda} F_{i_1} d F_{i_2} \wedge \ldots \wedge d F_{i_k} \]

\[ + \sum_{I} \sum_{i_k < j} (\partial_{j} w_{i_1 \ldots i_k} \circ F) \partial_{\lambda} F_{i_1} d F_{i_2} \wedge \ldots \wedge d F_{i_k}. \]

Grouping the terms of similar nature and renaming the multi-indices, we obtain

(6.3) \[ \sum_{I} \sum_{j=1}^{n} (\partial_{j} w_{i_1 \ldots i_k} \circ F) \partial_{\lambda} F_{j} d F_{i_1} \wedge \ldots \wedge d F_{i_k} \]

\[ = \sum_{I} \sum_{j=1}^{k+1} (\partial_{j} w_{j_1 \ldots j_{k+1}} \circ F) \partial_{\lambda} F_{j_1} [d F_{j_2}] \]

\[ + \sum_{I} \sum_{j=1}^{k} (\partial_{j} w_{i_1 \ldots i_k} \circ F) \partial_{\lambda} F_{i_1} d F_{i_2} \wedge \ldots \wedge d F_{i_k}. \]

On the other hand, we have

\[ \sum_{I} d(w_{i_1 \ldots i_k} \circ F) \wedge \left[ \sum_{l=1}^{k} (-1)^{l-1} \partial_{\lambda} F_{i_l} [d F_{i_l}] \right] \]

\[ = \sum_{I} \sum_{j} (\partial_{j} w_{i_1 \ldots i_k} \circ F) d F_{j} \wedge \left[ \sum_{l=1}^{k} (-1)^{l-1} \partial_{\lambda} F_{i_l} [d F_{i_l}] \right] \]

\[ = \sum_{I} \sum_{j=1}^{k} (-1)^{l-1} \left( \sum_{j<i_{l+1}} + \sum_{i_{l+1} < j<i_{l+2}} + \ldots + \sum_{i_{l-1} < j<i_{l}} \right) (\partial_{j} w_{i_1 \ldots i_k} \circ F) \partial_{\lambda} F_{i_l} [d F_{i_l}] \]

\[ + \sum_{I} \sum_{l=1}^{k} (-1)^{l-1} (\partial_{j} w_{i_1 \ldots i_k} \circ F) \partial_{\lambda} F_{i_l} [d F_{i_l}] \]

\[ + \sum_{I} \sum_{l=1}^{k} (-1)^{l-1} \left( \sum_{i_{l+1} < j<i_{l+2}} + \ldots + \sum_{i_{k} < j} \right) (\partial_{j} w_{i_1 \ldots i_k} \circ F) \partial_{\lambda} F_{i_l} [d F_{i_l}]. \]

Renaming the indices, we obtain

\[ \sum_{I} d(w_{i_1 \ldots i_k} \circ F) \wedge \left[ \sum_{l=1}^{k} (-1)^{l-1} \partial_{\lambda} F_{i_l} [d F_{i_l}] \right] \]

\[ = \sum_{j_2 < \ldots < j_{k+1}} \sum_{l=1}^{k} (-1)^{l-1} \sum_{j_1 < j_2} (\partial_{j_1} w_{j_1 j_2 \ldots j_{k+1}} \circ F) \partial_{\lambda} F_{j_1} [d F_{j_{1+1}}] \]

\[ + \sum_{j_1 < j_2 < \ldots < j_{k+1}} \sum_{l=1}^{k} (-1)^{l-1} \sum_{j_1 < j_2 < j_3} (\partial_{j_2} w_{j_1 j_2 j_3 \ldots j_{k+1}} \circ F). \]
Using relations (2.1), this implies that
\[
\partial \lambda F_{j+1} \left( -1 \right) [d\hat{F}_{j+1}] + \ldots
\]
\[
+ \sum_{j_1 < \ldots < j_i < \ldots < j_{k+1}} \sum_{l=1}^k (-1)^{l-1} \sum_{j_{l-1} < j_l < j_{l+1}} \left( \partial_{j_l} \hat{w}_{j_1 \ldots j_i \ldots j_{k+1}} \circ F \right)
\]
\[
- \partial \lambda F_{j+1} \left( -1 \right) [d\hat{F}_{j+1}]
\]
\[
+ \sum_{l=1}^k \sum_{i=1}^k (-1)^{l-1} \left( \partial_{j_{l+1}} \hat{w}_{j_1 \ldots j_i \ldots j_{k+1}} \circ F \right) \partial \lambda F_{j+1} \left( -1 \right) [d\hat{F}_{j+1}]
\]
\[
+ \sum_{j_1 < \ldots < j_{k+1}} \sum_{l=1}^k (-1)^{l-1} \sum_{j_{l-1} < j_l < j_{l+1}} \left( \partial_{j_{l+1}} \hat{w}_{j_1 \ldots j_i \ldots j_{k+1}} \circ F \right) \partial \lambda F_{j+1} \left( -1 \right) [d\hat{F}_{j+1}]
\]
\[
= \sum_{l=1}^k (-1)^{l-1} \sum_{j=1}^k \sum_{s=1}^l (-1)^{s-1} \left( \partial_{j_s} \hat{w}_{j_1 \ldots j_{s-1} \ldots j_{k+1}} \circ F \right) \partial \lambda F_{j+1} \left[ d\hat{F}_{j_{i+1}} \right]
\]
\[
+ \sum_{j=1}^k \sum_{s=1}^{k+1} (-1)^{s-1} \left( \partial_{j_s} \hat{w}_{j_1 \ldots j_{s-1} \ldots j_{k+1}} \circ F \right) \partial \lambda F_{j+1} \left[ d\hat{F}_{j_s} \right]
\]
\[
+ \sum_{l=1}^k (-1)^{l-1} \sum_{j=1}^k \sum_{s=1}^{k+1} \left( \partial_{j_s} \hat{w}_{j_1 \ldots j_{s-1} \ldots j_{k+1}} \circ F \right) \partial \lambda F_{j+1} \left[ d\hat{F}_{j_{i+1}} \right]
\]
\[
- \sum_{l=1}^k \sum_{j=1}^{k+1} (-1)^{2l+1} \left( \partial_{j_{l+1}} \hat{w}_{j_1 \ldots j_{l-1} \ldots j_{k+1}} \circ F \right) \partial \lambda F_{j+1} \left[ d\hat{F}_{j_{l+1}} \right].
\]

Using relations (2.1), this implies that
\[
\sum_{l=1}^k \sum_{i=1}^k (-1)^{l-1} \partial \lambda F_{l \hat{F}_{l+1}} \left[ d\hat{F}_{l+1} \right] = (-1)^{k-1} \sum_{j=1}^k \sum_{s=1}^{k+1} (-1)^{s-1} \left( \partial_{j_s} \hat{w}_{j_1 \ldots j_{s-1} \ldots j_{k+1}} \circ F \right) \partial \lambda F_{j+1} \left[ d\hat{F}_{j_{i+1}} \right]
\]
Regrouping the terms, we find

\[
\sum_{I} d(w_{i_1 \ldots i_k} \circ F) \wedge \left[ \sum_{l=1}^{k} (-1)^{l-1} \partial_{\lambda} F_{i_l} [\overline{dF_{i_l}}] \right] = \sum_{J} (\partial_{i_1 \ldots i_k} w_{j_1 \ldots j_{k+1}} \circ F) \partial_{\lambda} F_{j_1} dF_{i_1} \wedge \ldots \wedge dF_{i_k}.
\]

Comparing formulas (6.3) and (6.4) finishes the proof. \(\square\)

An interesting consequence of Theorem 6.1 is the following result on the invariance of an integral. For the differential \(k\)-form

\[
\omega = \sum_{1 \leq i_1 < \ldots < i_k \leq n} w_{i_1 \ldots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k},
\]

define the support of \(\omega\) by

\[
\text{supp} \omega = \bigcup_{1 \leq i_1 < \ldots < i_k \leq n} \text{supp} w_{i_1 \ldots i_k}.
\]

**Corollary 6.2.** Let \(\omega\) be a differential \(k\)-cocycle of class \(C^1\) in the open set \(D \subset \mathbb{R}^n\), \(G \subset \mathbb{R}^k\) be open and bounded and \(F: G \times [a, b] \to D\) be of class \(C^2\). If, for each \(\lambda \in [a, b]\), one has

\[
(6.5) \quad \text{supp} \omega \cap F(\cdot, \lambda)(\partial G) = \emptyset,
\]

then the integral

\[
(6.6) \quad \int_{G} F(\cdot, \lambda)^* \omega
\]

is independent of \(\lambda\) on \([a, b]\).
Proof. Using Leibniz rule, Theorem 6.1 and Stokes theorem, we get, with

\[ \alpha = \sum_{i_1 < \ldots < i_k} (w_{i_1 \ldots i_k} \circ F) \sum_{j=1}^k (-1)^{i_j-1} \partial \lambda F_{i_j} dF_{i_1} \wedge \ldots \wedge dF_{i_{j-1}} \wedge \ldots \wedge dF_{i_k}, \]

\[ \partial \lambda \left[ \int_G F(\cdot, \lambda)^* \omega \right] = \int_G \partial \lambda [F(\cdot, \lambda)^* \omega] = \int_G d\alpha = \int_{\partial G} \alpha = 0. \]

\[ \square \]

7. Liouville theorem

Let \( v: \mathbb{R}^n \to \mathbb{R}^n \) be of class \( C^1 \) and, for each \( y \in \mathbb{R}^n \), let \( x: [0, T] \times \mathbb{R}^n \to \mathbb{R}^n, (t, y) \mapsto x(t, y) \) be the unique solution of the Cauchy problem

\[ \frac{dx}{dt} = v(t, x), \quad x(0) = y, \]

so that, for each \( (t, y) \in [0, T] \times \mathbb{R}^n \), we have

\[ \partial_t x(t, y) = v[t, x(t, y)]. \]

If \( \omega = dy_1 \wedge \ldots \wedge dy_n \) is the volume \( n \)-form, then, for each \( t \in [0, T] \),

\[ [x(t, \cdot)]^* \omega = dx_1(t, \cdot) \wedge \ldots \wedge dx_n(t, \cdot) = \text{Jac} x(t, \cdot)(y) dy_1 \wedge \ldots \wedge dy_n, \]

where, for each fixed \( t \in [0, T] \), \( \text{Jac} x(t, \cdot) \) is the Jacobian of \( x(t, \cdot) \). For each fixed \( t \), \( \text{div} v(t, \cdot) = \sum_{j=1}^n \partial_j v_j(t, x) \). The following result can be traced to Liouville [7] (see also [1]).

**Theorem 7.1.** For each \( t \in [0, T] \), we have

\[ \partial_t [x(t, \cdot)]^* \omega = [x(t, \cdot)]^* [\text{div} v(t, \cdot) \ dy_1 \wedge \ldots \wedge dy_n] \]

or equivalently

\[ \partial_t [dx_1(t, \cdot) \wedge \ldots \wedge dx_n(t, \cdot)] = \text{div} v[t, x(t, \cdot)] \ dx_1(t, \cdot) \wedge \ldots \wedge dx_n(t, \cdot), \]

or equivalently

\[ \partial_t \text{Jac} x(t, y) = \text{div} v[t, x(t, y)] \text{Jac} x(t, y). \]
Proof. Using formulas (3.2) and (7.2), we get
\[ \partial_t ([x(t, \cdot)]^* \omega) = \partial_t [dx_1(t, \cdot) \wedge \ldots \wedge dx_n(t, \cdot)] \]
\[ = \sum_{j=1}^n dx_1(t, \cdot) \wedge \ldots \wedge d[\partial_t x_j(t, \cdot)] \wedge \ldots \wedge dx_n(t, \cdot) \]
\[ = \sum_{j=1}^n dx_1(t, \cdot) \wedge \ldots \wedge dv_j[t, x_j(t, \cdot)] \wedge \ldots \wedge dx_n(t, \cdot) \]
\[ = \sum_{j=1}^n dx_1(t, \cdot) \wedge \ldots \wedge \left[ \sum_{k=1}^n d[\partial_k v_j[t, x_j(t, \cdot)]] dx_k(t, \cdot) \right] \wedge \ldots \wedge dx_n(t, \cdot) \]
\[ = [x(t, \cdot)]^* [\text{div} v(t, \cdot) dy_1 \wedge \ldots \wedge dy_n] \]
\[ = \text{div} v[t, x(t, \cdot)] dx_1(t, \cdot) \wedge \ldots \wedge dx_n(t, \cdot) \]
\[ = \text{div} v[t, x(t, \cdot)] \text{Jac} x(t, \cdot) dy_1 \wedge \ldots \wedge dy_n. \]
and the three formulas easily follow. \( \square \)

8. Helmholtz theorem

We present here an \( n \)-dimensional version of Helmholtz theorem in hydrodynamics \[4\], \[6\], \[11\]. Let
\[ x: [0, T] \times \mathbb{R}^n \to \mathbb{R}^n, \quad (t, y) \mapsto x(t, y) \]
be of class \( C^2 \). For \( n = 3 \), in the hydrodynamics setting, it represents the position at time \( t \) of a particule located at \( y \) for \( t = 0 \) (Lagrange’s notations). Let
\[ u: [0, T] \times \mathbb{R}^n \to \mathbb{R}^n, \quad (t, x) \mapsto u(t, x) \]
be of class \( C^1 \). For \( n = 3 \), in the hydrodynamics setting, it represents the velocity of a point of the fluid located in \( x \) at time \( t \) (Euler’s notations). Consequently, we have, for all \( (t, y) \in [0, T] \times \mathbb{R}^n \),
\[ u[t, x(t, y)] = \partial_t x(t, y), \]
Assume that there exists a function \( \psi: [0, T] \times \mathbb{R}^n \to \mathbb{R} \) of class \( C^1 \) such that, for all \( (t, y) \in [0, T] \times \mathbb{R}^n \), one has
\[ \frac{d}{dt} [u[t, x(t, y)]] = \nabla_x \psi[t, x(t, y)]. \]
For \( n = 3 \), in the hydrodynamics setting, those are the equations of motion of the fluid, under the assumption that the external forces depend upon a potential and that the density depends only of the pressure.
Lemma 8.1. For each $t \in [0,T]$, one has

$$(8.5) \quad \partial_t \left\{ [x(t, \cdot)]^* \left[ \sum_{j=1}^{n} u_j(t, \cdot) dy_j \right] \right\} = \partial_t \left[ \sum_{j=1}^{n} u_j[t, x(t, \cdot)] dx_j(t, \cdot) \right] = d \left[ \psi(t, \cdot) + \frac{1}{2} \sum_{j=1}^{n} u_j^2[t, x(t, \cdot)] \right].$$

Proof. Using formulations (3.2), (8.3) and (8.4), we get

$$\partial_t \left[ \sum_{j=1}^{n} u_j[t, x(t, \cdot)] dx_j(t, \cdot) \right]$$

$$= \sum_{j=1}^{n} \left[ \frac{d}{dt} \{ u_j[t, x(t, \cdot)] \} dx_j(t, \cdot) + u_j[t, x(t, \cdot)] \partial_t [dx_j(t, \cdot)] \right]$$

$$= \sum_{j=1}^{n} \left[ \partial_j \psi[t, x(t, \cdot)] dx_j(t, \cdot) + u_j[t, x(t, \cdot)] d \{ \partial_t x_j(t, \cdot) \} \right]$$

$$= d \psi(t, \cdot) + \sum_{j=1}^{n} u_j(t, x(t, \cdot)) dx_j(t, \cdot)$$

$$= d \left[ \psi(t, \cdot) + \frac{1}{2} \sum_{j=1}^{n} u_j^2[t, x(t, \cdot)] \right]. \quad \square$$

Let $\gamma: [a,b] \to \mathbb{R}^n$ be a 1-cycle of class $C^2$ (i.e. $\gamma(0) = \gamma(1)$), so that, for each fixed $t \in [0,T]$, $x(t, \gamma(\cdot))$ is the 1-cycle of class $C^2$ which is the image of $\gamma([a,b])$ at time $t$ under the motion of the fluid. Let us consider now the circulation of the velocity field along $x(t, \gamma(\cdot))$,

$$(8.6) \quad C(t) := \int_{x(t, \gamma(\cdot))} \sum_{j=1}^{n} u_j dy_j.$$ 

Theorem 8.2. The integral (8.6) is constant on $[0,T]$.

Proof. We have, from Leibniz' rule and formula (8.5),

$$C'(t) = \int_{0}^{T} \partial_t \left[ \sum_{j=1}^{n} u_j[t, x(t, \gamma(s))] dx_j[t, \gamma(s)] \right]$$

$$= \int_{0}^{T} d \left[ \psi[t, \gamma(s)] + \sum_{j=1}^{n} \frac{u_j^2[t, x(t, \gamma(s))]}{2} \right] = 0. \quad \square$$
References


Jean Mawhin
Université Catholique de Louvain
Département de mathématique
chemin du cyclotron, 2
B-1348 Louvain-la-Neuve, BELGIUM
E-mail address: mawhin@math.ucl.ac.be

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