Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 25, 2005, 375–390

ON GENERALIZED SOBOLEV ALGEBRAS AND THEIR APPLICATIONS

Séverine Bernard — Silvère P. Nuiro

Abstract. In the last two decades, many algebras of generalized functions have been constructed, particularly the so-called generalized Sobolev algebras. Our goal is to study the latter and some of their main properties. In this framework, we pose and solve a nonlinear degenerated Dirichlet problem with irregular data such as Dirac generalized functions.

1. Introduction

A theoretical study of most of the well-known algebras of generalized functions has pointed out two fundamental structures. The first one is the algebraic structure of a solid factor ring $\mathcal{C}$ of generalized numbers. The second one is the topological structure defined by a family $\mathcal{P}$ of seminorms, on a locally convex linear space E, which is also an algebra. These algebras have been denoted by $\mathcal{A}(\mathcal{C}, E, \mathcal{P})$ and one speaks of $(\mathcal{C}, E, \mathcal{P})$-algebras of generalized objects. The definition covers most of the well-known algebras of generalized functions, as for example, the Colombeau simplified algebra [3], Goursat algebras [13] and asymptotic algebras [4]. On the other hand, special choices for E, $\mathcal{P}$ and $\mathcal{C}$ also allow the introduction of some new algebras. One of them is the so-called Egorov extended algebra, because of the similarity with the Egorov [5] algebra of generalized functions. We have been interested in working within the framework of the

2000 *Mathematics Subject Classification.* 35J70, 46F30, 46E35, 35D05, 35B50.
Key words and phrases. Nonlinear degenerate Dirichlet problem, generalized solution, Sobolev algebra, non positive solution.

so-called generalized Sobolev algebras based on the classical Sobolev spaces. As E is a differential algebra, the main interest of these algebras is to give a framework which is well suitable to solve many non linear differential problems with irregular data. The method is based on the extension of a mapping from $(E_1,\mathcal{P}_1)$ into $(E_2,\mathcal{P}_2)$ to a mapping from $\mathcal{A}(\mathcal{C}_1,E_1,\mathcal{P}_1)$ into $\mathcal{A}(\mathcal{C}_2,E_2,\mathcal{P}_2)$. This method has been introduced, in the framework of asymptotic algebras, by A. Delcroix and D. Scarpalezos [4], and used, in the framework of $(\mathcal{C},E,\mathcal{P})$-algebras, to solve a non linear Dirichlet problem [12] and a non linear Neumann problem [11], both with irregular data by J.-A. Marti and S. P. Nuiro.

In this paper, our goal is to lift up the generalized Sobolev algebras, by giving more clear definitions of all the statements and general results in this framework, in order to work more easily with these algebras. We introduce the first example of ordered generalized Sobolev algebras, which allows us to pose and eventually solve an obstacle problem with irregular data. We also point out some sufficient properties for the existence of an embedding of some space into a generalized Sobolev algebra. In the framework of generalized Sobolev algebra, we are able to solve a non linear degenerated Dirichlet problem [12] with weaker assumptions.

Consider Ω an open bounded domain of $\mathbb{R}^d$ $(d\in\mathbb{N}^*)$ with a lipschitz continuous boundary $\partial\Omega$, we can state this formal problem:

$$\text{(P)}\qquad \begin{cases} -\Delta\Phi(u)+u=f & \text{in } \Omega,\\ u=g & \text{on } \partial\Omega,\end{cases}$$

where f and g are non smooth functions defined on Ω and $\partial\Omega$ respectively, Φ an increasing real-valued differentiable function defined on $\mathbb{R}$ so that Φ' is a continuous bounded function that can vanish on a finite set of discrete points of $\mathbb{R}$. This is a quasilinear diffusion type problem, with non homogeneous Dirichlet condition on the boundary. One can remark that the formal second order differential operator $\mathcal{L}=-\operatorname{div}(\Phi'(\,\cdot\,)\nabla_x)+I_d$ is a degenerated one, because Φ' can vanish. Thus, (P) is a Dirichlet nonlinear elliptic degenerated problem. In order to solve this problem, we introduce an auxiliary problem by using an artificial viscosity regularization depending on a parameter ε.

2. Special types of generalized algebras

2.1. Definitions. Let us, first, state that $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, and $\mathbb{I}=(\mathbb{I}_\varepsilon)_\varepsilon$ where $\mathbb{I}_\varepsilon=1$ for all ε. The generalized algebras constructed from E, a normed $\mathbb{K}$-algebra, are particular case of $(\mathcal{C},E,\mathcal{P})$-algebras [10]–[13].

Consider a subring A of the ring $\mathbb{K}^{]0,1]}$ so that $\mathbb{I}\in A$, and which, as a ring, is solid (with compatible lattice structure) in the following sense:

Definition 2.1. A is said to be *solid* if from $(s_\varepsilon)_\varepsilon\in A$ and $|t_\varepsilon|\le|s_\varepsilon|$ for each $\varepsilon\in\,]0,1]$ it follows that $(t_\varepsilon)_\varepsilon\in A$.

We also consider an ideal I_A of A which is solid as well, and so that

$$\lim_{\varepsilon\to 0} r_\varepsilon = 0 \quad \text{for all } (r_\varepsilon)_\varepsilon \in I_A. \tag{2.1}$$

Then, we introduce the factor ring $\mathcal{C} = A/I_A$, which is called a *ring of generalized numbers.*

Definition 2.2. Let E be a normed algebra. We shall call N-generalized algebra all factor algebra

$$\mathcal{A}(\mathcal{C}, E) = \mathcal{H}_A(E)/\mathcal{I}_{I_A}(E),$$

where

$$\mathcal{H}_A(E) = \{(u_\varepsilon)_\varepsilon \in E^{]0,1]} : (\|u_\varepsilon\|_E)_\varepsilon \in A^+\},$$

and

$$\mathcal{I}_{I_A}(E) = \{(u_\varepsilon)_\varepsilon \in E^{]0,1]} : (\|u_\varepsilon\|_E)_\varepsilon \in I_A^+\},$$

when $\|\cdot\|_E$ is the norm on E, $A^+ = \{(r_\varepsilon)_\varepsilon \in A : \text{for all } \varepsilon > 0,\ r_\varepsilon \in \mathbb{R}_+\}$ and $I_A^+ = \{(r_\varepsilon)_\varepsilon \in I_A : \text{for all } \varepsilon > 0,\ r_\varepsilon \in \mathbb{R}_+\}$. Its ring of generalized numbers is defined as the ring

$$\mathcal{H}_A(\mathbb{K})/\mathcal{I}_{I_A}(\mathbb{K}) = \mathcal{C} = A/I_A.$$

Remark 2.3. We remark that the notation is $\mathcal{A}(\mathcal{C}, E)$ instead of $\mathcal{A}(\mathcal{C}, E, \mathcal{P})$ since the family $\mathcal{P}$ is reduced to one single element. The algebra $\mathcal{A}(\mathcal{C}, E)$ is also a vector space on the field $\mathbb{K}$.

Example 2.4.

$$I_A = \{r = (r_\varepsilon)_\varepsilon \in \mathbb{R}^{]0,1]} : \text{for all } k \in \mathbb{N}^*,\ |r_\varepsilon| = O(\varepsilon^k)\}$$

and

$$A = \{r = (r_\varepsilon)_\varepsilon \in \mathbb{R}^{]0,1]} : \text{there exists } k \in \mathbb{Z},\ |r_\varepsilon| = O(\varepsilon^k)\},$$

we obtain a polynomial growth type N-generalized algebra.

Example 2.5. We take

$$I_A = \{r = (r_\varepsilon)_\varepsilon \in \mathbb{R}^{]0,1]} : \text{there exists } \varepsilon_0 \in]0,1],\ \text{for all } \varepsilon \in]0,\varepsilon_0],\ r_\varepsilon = 0\},$$

and $A = \mathbb{R}^{]0,1]}$. With such A and I_A, we obtain another N-generalized algebra.

Example 2.6. When E is a Sobolev algebra (that is, for example, on the form $W^{m+1,p}(\Omega) \cap W^{m,\infty}(\Omega)$, with $m \in \,]0,\infty[$, $p \in [1,\infty[$ and Ω an open subset of $\mathbb{R}^d$ ($d \in \mathbb{N}^*$)), respectively a Banach algebra, we will speak about generalized Sobolev algebra, respectively generalized Banach algebra, instead of N-generalized algebra.

2.2. Embeddings and weak equalities. In the following paragraph, we are going to show a way to embed E into $\mathcal{A}(\mathcal{C}, E)$.

Proposition 2.7. *The mapping i_0 defined on E by*

$$i_0(u) = \mathrm{cl}(u\mathbb{I}_\varepsilon)_\varepsilon \quad \text{for all } u \in E$$

is linear and one-to-one from E into $\mathcal{A}(\mathcal{C}, E)$.

Proof. For every $u \in E$, we have: $(\|u\mathbb{I}_\varepsilon\|_E)_\varepsilon = \|u\|_E \mathbb{I}$. Furthermore, as $\|u\|_E \in \mathbb{K}$ and $\mathbb{I} \in A$, there exists $\lambda \in \mathbb{N}$ so that

$$\|u_\varepsilon\|_E \leq \lambda \mathbb{I}_\varepsilon \quad \text{for all } \varepsilon$$

and obviously $\lambda\mathbb{I} \in A^+$. As a consequence of the solid property which implies that $(u_\varepsilon)_\varepsilon \in \mathcal{H}_A(E)$, we have $i_0(u) \in \mathcal{A}(\mathcal{C}, E)$. It can easily be proved that i_0 is linear and one-to-one. □

Definition 2.8. The mapping i_0 from E into $\mathcal{A}(\mathcal{C}, E)$, defined in Proposition 2.7, will be the so-called trivial embedding of E into $\mathcal{A}(\mathcal{C}, E)$.

We can also embed some topological vector space into $\mathcal{A}(\mathcal{C}, E)$. Let $(G, \mathcal{T})$ be a Hausdorff topological vector space so that there exists a continuous linear mapping j from $(E, \|\cdot\|_E)$ into $(G, \mathcal{T})$.

Definition 2.9. $T \in G$ and $U = \mathrm{cl}(u_\varepsilon)_\varepsilon \in \mathcal{A}(\mathcal{C}, E)$ are $(G, \mathcal{T})$-associated if

$$j(u_\varepsilon) \to T \quad \text{in } (G, \mathcal{T}), \text{ as } \varepsilon \to 0.$$

It will be denoted by $U \overset{G,\mathcal{T}}{\sim} T$.

Remark 2.10. This definition does not depend on the chosen representative of U. Indeed, let $(e_\varepsilon)_\varepsilon \in \mathcal{I}_{I_A}(E)$. Therefore, $\lim_{\varepsilon\to 0} \|e_\varepsilon\|_E = 0$, which means that $e_\varepsilon \to 0$ in $(E, \|\cdot\|_E)$ as $\varepsilon \to 0$. Consequently, we have $j(e_\varepsilon) \to 0$ in $(G, \mathcal{T})$ as $\varepsilon \to 0$.

Definition 2.11. Assume that $U = \mathrm{cl}(u_\varepsilon)_\varepsilon, V = \mathrm{cl}(v_\varepsilon)_\varepsilon \in \mathcal{A}(\mathcal{C}, E)$. We shall say that U and V are $(G, \mathcal{T})$-weakly equals if

$$(U - V) \overset{G,\mathcal{T}}{\sim} 0.$$

It will be denoted by $U \overset{G,\mathcal{T}}{\simeq} V$.

Proposition 2.12. *Assume that for every $T \in G$, there exists $(u_\varepsilon)_\varepsilon \in \mathcal{H}_A(E)$, so that*

$$j(u_\varepsilon) \to T \quad \text{in } (G, \mathcal{T}), \text{ as } \varepsilon \to 0.$$

Then, there exists, at least, an embedding i_G from $(G, \mathcal{T})$ into the N-generalized algebra $\mathcal{A}(\mathcal{C}, E)$. Furthermore, if for all $v \in E$ there exists $(u_\varepsilon)_\varepsilon \in \mathcal{H}_A(E)$ such that $(u_\varepsilon - v)_\varepsilon \in \mathcal{I}_{I_A}(E)$, then

$$(i_G \circ j)(u) \overset{G,\mathcal{T}}{\simeq} i_0(u) \quad \text{for all } u \in E. \tag{2.2}$$

Proof. For every $T \in G$, there exists $(u_\varepsilon)_\varepsilon \in \mathcal{H}_A(E)$, so that

$$j(u_\varepsilon) \to T \quad \text{in } (G, \mathcal{T}), \text{ as } \varepsilon \to 0.$$

Let us state $i_G(T) = \mathrm{cl}(u_\varepsilon)_\varepsilon$. The mapping i_G from G into $\mathcal{A}(\mathcal{C}, E)$ is obviously linear. Let us prove that i_G is one-to-one. If $i_G(T) = 0$ in $\mathcal{A}(\mathcal{C}, E)$ then

$$i_G(T) = \mathrm{cl}(e_\varepsilon)_\varepsilon \quad \text{for } (e_\varepsilon)_\varepsilon \in \mathcal{I}_{I_A}(E).$$

We have $e_\varepsilon \to 0$ in $(E, \|\cdot\|_E)$ which implies that $j(e_\varepsilon) \to 0$ in $(G, \mathcal{T})$, whenever $\varepsilon \to 0$. This leads to $T = 0$ in G, because $(G, \mathcal{T})$ is a Hausdorff space. The second property is obvious. □

Remark 2.13. If there exists another such embedding i'_G from $(G, \mathcal{T})$ into the N-generalized algebra $\mathcal{A}(\mathcal{C}, E)$ then

$$i_G(T) \overset{G,\mathcal{T}}{\simeq} i'_G(T) \quad \text{for all } T \in G.$$

Example 2.14. Let j be the canonical embedding of $(L^\infty(\Omega), \|\cdot\|_{L^\infty(\Omega)})$ in $(H^{-2}(\Omega), \sigma(H^{-2}(\Omega), H_0^2(\Omega)))$, where $\sigma(H^{-2}(\Omega), H_0^2(\Omega))$ denotes the weak topology on $H^{-2}(\Omega)$. We will say that $T \in H^{-2}(\Omega)$ and $\mathcal{U} = \mathrm{cl}(u_\varepsilon)_\varepsilon \in \mathcal{A}(\mathcal{C}, L^\infty(\Omega))$ are $H^{-2}(\Omega)$-associated if

$$j(u_\varepsilon) \to T \quad \text{in } (H^{-2}(\Omega), \sigma(H^{-2}(\Omega), H_0^2(\Omega))), \text{ as } \varepsilon \to 0,$$

and we will denote $\mathcal{U} \overset{2}{\sim} T$. Moreover, we will say that $\mathcal{U}, \mathcal{V} \in \mathcal{A}(\mathcal{C}, L^\infty(\Omega))$ are $H^{-2}(\Omega)$-weakly equals if $\mathcal{U} - \mathcal{V} \overset{2}{\sim} 0$ and we will denote $\mathcal{U} \overset{2}{\simeq} \mathcal{V}$.

2.3. Mapping on N-generalized algebra. The idea of extension of mapping has been introduced by A. Delcroix and D. Scarpalezos [4], in the framework of asymptotic algebras. But it is, in fact, a particular case of definition of mapping on $\mathcal{A}(\mathcal{C}, E)$-algebras.

If $\theta = (\theta_\varepsilon)_\varepsilon$ is a family of mappings from a normed algebra $(E, \|\cdot\|_E)$ into a normed algebra $(F, \|\cdot\|_F)$, one can view θ as a mapping from the N-generalized algebra $\mathcal{A}(\mathcal{C}, E)$ into the N-generalized algebra $\mathcal{A}(\mathcal{D}, F)$, where we have set $\mathcal{C} = A/I_A$ and $\mathcal{D} = B/I_B$ when A, I_A, B and I_B are as in Section 2.1. One remarks that the extension theorem of A. Delcroix and D. Scarpalezos [4] deals with the case where $\theta = (\theta)_\varepsilon$.

Theorem 2.15. *Let E and F be two normed algebras and $(\theta_\varepsilon)_\varepsilon$ a family of applications of E in F. We assume that*

(a) *$A \subset B$ and $I_A \subset I_B$,*

(b) *there exists a family of polynomial functions $(\Psi_\varepsilon)_\varepsilon$ of one variable with coefficients in A_+ so that*

$$\|\theta_\varepsilon(x)\|_F \leq \Psi_\varepsilon(\|x\|_E) \quad \textit{for all } \varepsilon > 0 \textit{ and all } x \in E,$$

(c) *there exists two families of polynomial functions* $(\Psi^1_\varepsilon)_\varepsilon$ *and* $(\Psi^2_\varepsilon)_\varepsilon$ *of one variable with coefficients in* A_+ *so that* $\Psi^2_\varepsilon(0) = 0$ *for all* $\varepsilon > 0$, *and*

$$\|\theta_\varepsilon(x+\xi) - \theta_\varepsilon(x)\|_F \le \Psi^1_\varepsilon(\|x\|_E)\Psi^2_\varepsilon(\|\xi\|_E). \quad \textit{for all } \varepsilon > 0 \textit{ and all } x, \xi \in E,$$

Then there exists an application $\Theta: \mathcal{A}(\mathcal{C}, E) \to \mathcal{A}(\mathcal{D}, F)$, *associating* $\mathrm{cl}(\theta_\varepsilon(x_\varepsilon))_\varepsilon$ *with* $\mathrm{cl}(x_\varepsilon)_\varepsilon$.

Proof. First, let $(x_\varepsilon)_\varepsilon$ be in $\mathcal{H}_A(E)$ and let us show that $(\theta_\varepsilon(x_\varepsilon))_\varepsilon$ is in $\mathcal{H}_B(F)$. We have $(\|x_\varepsilon\|_E)_\varepsilon$ in A_+ so $(\Psi_\varepsilon(\|x_\varepsilon\|_E))_\varepsilon$ is also in A_+, since $(\Psi_\varepsilon)_\varepsilon$ has coefficients in A_+. Thus $(\|\theta_\varepsilon(x_\varepsilon)\|_F)_\varepsilon)$ belongs to $A_+ \subset B_+$, due to (a) and (b), which implies what we want. Then, let $(i_\varepsilon)_\varepsilon$ be in $\mathcal{I}_{I_A}(E)$ and let us show that $(\theta_\varepsilon(x_\varepsilon + i_\varepsilon) - \theta_\varepsilon(x_\varepsilon))_\varepsilon$ is in $\mathcal{I}_{I_B}(F)$. Since $(\|x_\varepsilon\|_E)_\varepsilon$ and $(\|i_\varepsilon\|_E)_\varepsilon$ are respectively in A_+ and I^+_A then $(\Psi^1_\varepsilon(\|x_\varepsilon\|_E))_\varepsilon$ and $(\Psi^2_\varepsilon(\|i_\varepsilon\|_E))_\varepsilon$ are respectively in A_+ and I^+_A, since, for $i \in \{1,2\}$, $(\Psi^i_\varepsilon)_\varepsilon$ has coefficients in A_+. Then, $(\Psi^1_\varepsilon(\|x_\varepsilon\|_E)\Psi^2_\varepsilon(\|i_\varepsilon\|_E))_\varepsilon$ is in I^+_A. Thus $(\|\theta_\varepsilon(x_\varepsilon + i_\varepsilon) - \theta_\varepsilon(x_\varepsilon)\|_F)_\varepsilon$ belongs to $I^+_A \subset I^+_B$, due to (a) and (c), which implies the required result. □

As a consequence, we obtain the following result.

Proposition 2.16. *Assume that* $A \subset B$ *and* $I_A \subset I_B$. *If* $(\theta_\varepsilon)_\varepsilon$ *is a family of continuous linear mappings from a normed algebra* E *into a normed algebra* F, *then* $(\theta_\varepsilon)_\varepsilon$ *also defines a mapping* Θ *from* $\mathcal{A}(\mathcal{C}, E)$ *into* $\mathcal{A}(\mathcal{D}, F)$.

Example 2.17. Let Ω be an open subset of $\mathbb{R}^d$ and $E = H^1(\Omega) \cap L^\infty(\Omega)$ with $\|u\|_E = \|u\|_{L^\infty(\Omega)} + \|u\|_{H^1(\Omega)}$. The canonical embedding $i: u \mapsto u$ is continuous as well as linear from the Banach algebra E into the Banach algebra $L^\infty(\Omega)$. Obviously, the mapping i verifies all the assumptions of the previous proposition; this is why we can define its extension $\mathcal{I}$ as a mapping from $\mathcal{A}(\mathcal{C}, E)$ into $\mathcal{A}(\mathcal{C}, L^\infty(\Omega))$.

In the same way, one can prove that:

Proposition 2.18. *Assume that* $(\theta_\varepsilon)_\varepsilon$ *is a family of mappings from a normed algebra* E *into the topological field* $(\mathbb{K}, |\cdot|)$, *so that*

(a) *there exists a family of polynomial functions* $(\Psi_\varepsilon)_\varepsilon$ *of one variable with coefficients in* A_+ *so that*

$$|\theta_\varepsilon(x)| \le \Psi_\varepsilon(\|x\|_E), \quad \textit{for all } \varepsilon > 0 \textit{ and all } x \in E,$$

(b) *there exists two families of polynomial functions* $(\Psi^1_\varepsilon)_\varepsilon$ *and* $(\Psi^2_\varepsilon)_\varepsilon$ *of one variable with coefficients in* A_+ *so that* $\Psi^2_\varepsilon(0) = 0$ *for all* $\varepsilon > 0$, *and*

$$|\theta_\varepsilon(x+\xi) - \theta_\varepsilon(x)| \le \Psi^1_\varepsilon(\|x\|_E)\Psi^2_\varepsilon(\|\xi\|_E), \quad \textit{for all } \varepsilon > 0 \textit{ and all } x, \xi \in E.$$

Then there exists an application $\Theta: \mathcal{A}(\mathcal{C}, E) \to \mathcal{C}$, *which associates* $\mathrm{cl}(\theta_\varepsilon(x_\varepsilon))_\varepsilon$ *with* $\mathrm{cl}(x_\varepsilon)_\varepsilon$.

Remark 2.19. If θ is a continuous linear mapping from a normed algebra $(E, \|\cdot\|_E)$ into the topological field $(\mathbb{K}, |\cdot|)$, then θ also defines a mapping, denoted by Θ, from $\mathcal{A}(\mathcal{C}, E)$ into the factor ring $\mathcal{C} = A/I_A$.

2.4. An example of ordered generalized Sobolev algebra. Consider A and I_A as in Section 2.1, the Sobolev algebra $L^\infty(\Omega)$, endowed with its usual topology, with Ω an open bounded subset of $\mathbb{R}^d$. Thus, we can consider the algebra $\mathcal{A}(\mathcal{C}, L^\infty(\Omega))$. It is easy to prove, by means of Theorem 2.15, that the mapping

$$p: L^\infty(\Omega) \to L^\infty(\Omega), \quad u \mapsto u^+ = \sup\{u, 0\} = \frac{1}{2}(u + |u|)$$

can be extended as a mapping $\mathcal{P}$ from $\mathcal{A}(\mathcal{C}, L^\infty(\Omega))$ into itself, defined by:

$$\mathcal{P}(U) = \mathrm{cl}(p(u_\varepsilon))_\varepsilon \quad \text{for all } U = \mathrm{cl}(u_\varepsilon)_\varepsilon \in \mathcal{A}(\mathcal{C}, L^\infty(\Omega)),$$

due to the following relation:

$$|(r+s)^+ - r^+| \le |s| \quad \text{for all } r, s \in \mathbb{R}.$$

We are now able to state the following result:

Proposition 2.20. *The generalized Sobolev algebra* $\mathcal{A}(\mathcal{C}, L^\infty(\Omega))$ *is partially ordered by the following binary relation:*

$$U \le V \textit{ if and only if } \mathcal{P}(U - V) = 0 \textit{ for all } U, V \in \mathcal{A}(\mathcal{C}, L^\infty(\Omega)).$$

Proof. Obviously, the relation $\le$ is reflexive, then we have to prove, for $U, V, W \in \mathcal{A}(\mathcal{C}, L^\infty(\Omega))$, that:

$$\text{if } U \le V \text{ and } V \le U \text{ then } U = V, \tag{2.3}$$

$$\text{if } U \le V \text{ and } V \le W \text{ then } U \le W. \tag{2.4}$$

We state $U = \mathrm{cl}(u_\varepsilon)_\varepsilon$, $V = \mathrm{cl}(v_\varepsilon)_\varepsilon$ and $W = \mathrm{cl}(w_\varepsilon)_\varepsilon$.

Proof of (2.3). If $U \le V$ and $V \le U$ then, there exists $(\varphi_\varepsilon)_\varepsilon$ and $(\psi_\varepsilon)_\varepsilon$ in $\mathcal{I}_{I_A}(L^\infty(\Omega))$ so that $(u_\varepsilon - v_\varepsilon)^+ = \varphi_\varepsilon$ and $(v_\varepsilon - u_\varepsilon)^+ = \psi_\varepsilon$. As,

$$u_\varepsilon - v_\varepsilon = (u_\varepsilon - v_\varepsilon)^+ - (v_\varepsilon - u_\varepsilon)^+ = \varphi_\varepsilon - \psi_\varepsilon,$$

it follows that $(u_\varepsilon - v_\varepsilon)_\varepsilon = (\varphi_\varepsilon - \psi_\varepsilon)_\varepsilon \in \mathcal{I}_{I_A}(L^\infty(\Omega))$, whence $U = V$.

Proof of (2.4). If $U \le V$ and $V \le W$ then we have

$$(\|(u_\varepsilon - v_\varepsilon)^+\|_{L^\infty(\Omega)})_\varepsilon \in I_A, \quad (\|(v_\varepsilon - w_\varepsilon)^+\|_{L^\infty(\Omega)})_\varepsilon \in I_A.$$

By means of the solid property, we deduce, from the following inequality:

$$\|(u_\varepsilon - w_\varepsilon)^+\|_{L^\infty(\Omega)} \le \|(u_\varepsilon - v_\varepsilon)^+\|_{L^\infty(\Omega)} + \|(v_\varepsilon - w_\varepsilon)^+\|_{L^\infty(\Omega)},$$

that $((u_\varepsilon - w_\varepsilon)^+)_\varepsilon \in \mathcal{I}_{I_A}(L^\infty(\Omega))$, which yields $\mathcal{P}(U-W)=0$, that is to say $U \le W$. $\square$

Proposition 2.21. *For all $u, v \in L^\infty(\Omega)$, we have $i_0(u) \le i_0(v)$ if, and only if, $u \le v$ in $L^\infty(\Omega)$, that is $u \le v$ almost everywhere in Ω.*

Proof. If $i_0(u) \le i_0(v)$ then $\mathcal{P}(i_0(u) - i_0(v)) = 0$. Consequently, there exists $(\varphi_\varepsilon)_\varepsilon, (e_\varepsilon)_\varepsilon \in \mathcal{I}_{I_A}(L^\infty(\Omega))$ so that $(u - v + e_\varepsilon)^+ = \varphi_\varepsilon$, since we have $u_\varepsilon - v_\varepsilon = u - v + e_\varepsilon$ for all ε. Taking into account that

$$\varphi_\varepsilon \to 0 \quad \text{and} \quad e_\varepsilon \to 0 \quad \text{in } L^\infty(\Omega), \text{ as } \varepsilon \to 0,$$

it may be seen that $(u - v + e_\varepsilon)^+ = \varphi_\varepsilon \to 0$ a.e. in Ω, whence $(u-v)^+ = 0$ a.e. in Ω, since one can easily prove that

$$(u - v + e_\varepsilon)^+ \to (u-v)^+ \quad \text{in } L^\infty(\Omega), \text{ as } \varepsilon \to 0.$$

It means that $u \le v$ a.e. in Ω.

Conversely, if $u \le v$ a.e. in Ω then $(u-v)^+ = 0$ a.e. in Ω. By definition of $\mathcal{P}$ and i_0, this leads to $i_0(u) \le i_0(v)$. $\square$

Proposition 2.22. *Let $u \in L^\infty(\Omega)$ and $U \in \mathcal{A}(\mathcal{C}, L^\infty(\Omega))$. If $U \overset{L^1(\Omega)}{\sim} u$ (here $L^1(\Omega)$ is endowed with its usual topology) and $U \le 0$ then $u \le 0$ a.e. in Ω.*

Proof. We set $U = \mathrm{cl}(u_\varepsilon)_\varepsilon$. Since $U \overset{L^1(\Omega)}{\sim} u$ then, as ε goes to 0, $u_\varepsilon \to u$ in $L^1(\Omega)$, which gives $u_\varepsilon^+ \to u^+$ in $L^1(\Omega)$, by means of the Lebesgue dominated convergence theorem. Since $U \le 0$ then $\mathcal{P}(U) = 0$. Consequently, there exists a sequence of functions $(\varphi_\varepsilon)_\varepsilon \in \mathcal{I}_{I_A}(L^\infty(\Omega))$ so that $u_\varepsilon^+ = \varphi_\varepsilon$ for all ε. Taking into account that

$$\varphi_\varepsilon \to 0 \quad \text{in } L^\infty(\Omega), \text{ as } \varepsilon \to 0,$$

we find that $u_\varepsilon^+ = \varphi_\varepsilon \to 0$ a.e. in Ω, whence $u^+ = 0$ a.e. in Ω, which implies $u \le 0$ a.e. in Ω. $\square$

3. Solution of the nonlinear degenerate Dirichlet problem

After having solved the auxiliary problem by using an artificial viscosity regularization depending on a parameter ε, we solve our main problem (P) (see Section 1), in a generalized Sobolev algebra with the classical equality and with the weak one defined in Example 2.14. Then we perform a little qualitative study of the solution.

3.1. The regularized Dirichlet problem. Let us set

$$\mathcal{V}_A^+ = \left\{ (r_\varepsilon)_\varepsilon \in A^+ : \text{for all } \varepsilon > 0,\ r_\varepsilon \in \left]0,1\right],\ \lim_{\varepsilon \to 0} r_\varepsilon = 0,\ \left(\frac{1}{r_\varepsilon}\right)_\varepsilon \in A^+ \right\}.$$

Assume that $\mathcal{V}_A^+ \neq \emptyset$ and then, for all $(r_\varepsilon)_\varepsilon$ in $\mathcal{V}_A^+$, set $\Phi_\varepsilon = \Phi + r_\varepsilon \mathrm{id}$. This section consists in proving the following proposition:

PROPOSITION 3.1. *If $f \in L^\infty(\Omega)$ and $g \in L^\infty(\partial\Omega)$ then there exists one, and only one, function $u \in H^1(\Omega) \cap L^\infty(\Omega)$ solution of the regularized problem*

$$(\mathrm{P}_\varepsilon) \qquad \begin{cases} -\Delta\Phi_\varepsilon(u) + u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

PROOF. This proof goes in three steps.

Step 1. (Maximum's principle) We are going to prove that if $u \in H^1(\Omega)$ is a solution of this problem then

$$m \le u \le M \quad \text{a.e. in } \Omega,$$

with $m = \min\{\inf_\Omega f, \inf_{\partial\Omega} g\}$ and $M = \max\{\sup_\Omega f, \sup_{\partial\Omega} g\}$, which means that u belongs to $L^\infty(\Omega)$.

Indeed, for such a u, we have, for all v in $H_0^1(\Omega)$

$$\int_\Omega \nabla\Phi_\varepsilon(u)\nabla v\,dx + \int_\Omega uv\,dx = \int_\Omega fv\,dx,$$

where dx denotes the Lebesgue measure on Ω. Let us consider the function $v = (\Phi_\varepsilon(u) - \Phi_\varepsilon(M))^+$ then v is in $H_0^1(\Omega)$, so

$$\begin{aligned} \int_\Omega (\nabla(\Phi_\varepsilon(u) - \Phi_\varepsilon(M))^+)^2\,dx + \int_\Omega u(\Phi_\varepsilon(u) - \Phi_\varepsilon(M))^+\,dx \\ = \int_\Omega f(\Phi_\varepsilon(u) - \Phi_\varepsilon(M))^+\,dx, \end{aligned}$$

since $\Phi_\varepsilon(M)$ is a constant. Consequently,

$$\begin{aligned} \|(\Phi_\varepsilon(u) - \Phi_\varepsilon(M))^+\|^2_{H_0^1(\Omega)} = \int_\Omega (f - M)(\Phi_\varepsilon(u) - \Phi_\varepsilon(M))^+\,dx \\ - \int_\Omega (u - M)(\Phi_\varepsilon(u) - \Phi_\varepsilon(M))^+\,dx. \end{aligned}$$

By definition of M, the first integral is negative and, since the functions id and Φ_ε are increasing, the second one is non negative. Then

$$\|(\Phi_\varepsilon(u) - \Phi_\varepsilon(M))^+\|^2_{H_0^1(\Omega)} \le 0,$$

that is $\Phi_\varepsilon(u) \le \Phi_\varepsilon(M)$ a.e. in Ω, which implies the first part of the required result, since Φ_ε is an increasing function. For the second part, we use a similar method by taking $v = (\Phi_\varepsilon(u) - \Phi_\varepsilon(m))^-$.

Step 2. (Existence of a solution in $H^1(\Omega)$) This result is obtained by using the Schauder's fixed point theorem related to a weakly sequentially continuous mapping from a reflexive and separable Banach space into itself. Let us consider $w_0 \in H^1(\Omega)$ the unique solution of the following linear Dirichlet problem:

$$\begin{cases} -\Delta w_0 = 0 & \text{in } \Omega, \\ w_0 = g & \text{on } \partial\Omega. \end{cases}$$

Then a solution of the regularized problem is of the form w_0+w, with $w \in H^1_0(\Omega)$ and for all v in $H^1_0(\Omega)$, one has

$$\int_\Omega \Phi'_\varepsilon(w_0+w)\nabla w_0 \nabla v\,dx + \int_\Omega \Phi'_\varepsilon(w_0+w)\nabla w \nabla v\,dx + \int_\Omega (w_0+w)v\,dx = \int_\Omega fv\,dx.$$

Consequently, for all $h \in H^1_0(\Omega)$, let us look for w_h in $H^1_0(\Omega)$ so that, for all v in $H^1_0(\Omega)$,

$$\int_\Omega \{\widetilde{\Phi}'_\varepsilon(w_0+h)\nabla w_h \nabla v + w_h v\}\,dx = \int_\Omega \{fv - w_0 v - \widetilde{\Phi}'_\varepsilon(w_0+h)\nabla w_0 \nabla v\}\,dx,$$

where $\widetilde{\Phi}_\varepsilon$ is defined by

$$\widetilde{\Phi}_\varepsilon(x) = \begin{cases} \Phi(m) + r_\varepsilon x & \text{if } x \le m, \\ \Phi(x) + r_\varepsilon x & \text{if } x \in]m, M[, \\ \Phi(M) + r_\varepsilon x & \text{if } x \ge M. \end{cases}$$

The existence and uniqueness of w_0 and w_h are ensured by the Lax–Milgram's theorem. Moreover, for the test-function $v = w_h$, we get

$$\begin{aligned} \int_\Omega &\{\widetilde{\Phi}'_\varepsilon(w_0+h)|\nabla w_h|^2 + |w_h|^2\}\,dx \\ &= \int_\Omega f w_h\,dx - \int_\Omega \{\widetilde{\Phi}'_\varepsilon(w_0+h)\nabla w_0 \nabla w_h - w_0 w_h\}\,dx. \end{aligned}$$

Meanwhile,

$$\int_\Omega \{\widetilde{\Phi}'_\varepsilon(w_0+h)|\nabla w_h|^2 + |w_h|^2\}\,dx \ge r_\varepsilon \int_\Omega \{|\nabla w_h|^2 + |w_h|^2\}\,dx,$$
$$\int_\Omega f w_h\,dx \le C(\Omega)\|f\|_{L^\infty(\Omega)}\|w_h\|_{H^1_0(\Omega)}$$

and

$$\begin{aligned} -\int_\Omega &\{\widetilde{\Phi}'_\varepsilon(w_0+h)\nabla w_0 \nabla w_h + w_0 w_h\}\,dx \\ &\le C(\Omega)(1 + r_\varepsilon + \|\Phi'\|_{L^\infty(\mathbb{R})})\|w_0\|_{H^1(\Omega)}\|w_h\|_{H^1_0(\Omega)} \\ &\le C(\Omega)(2 + \|\Phi'\|_{L^\infty(\mathbb{R})})\|w_0\|_{H^1(\Omega)}\|w_h\|_{H^1_0(\Omega)}, \end{aligned}$$

where $C(\Omega)$ denotes a constant depending on Ω. Thus,

$$\|w_h\|_{H^1_0(\Omega)} \le \frac{1}{r_\varepsilon} C(\Omega)[\|f\|_{L^\infty(\Omega)} + (2 + \|\Phi'\|_{L^\infty(\mathbb{R})})\|w_0\|_{H^1(\Omega)}].$$

Noticing that $\|w_0\|_{H^1(\Omega)}$ depends only on g and Ω and not on ε, we obtain that $\|w_h\|_{H^1_0(\Omega)} \le C(\Omega, f, g)/r_\varepsilon$, which implies that the closed ball $B(0, R_\varepsilon)$ of center 0 and radius $R_\varepsilon = C(\Omega, f, g)/r_\varepsilon$ of the separable Hilbert space $H^1_0(\Omega)$ is stable by the application

$$\Pi\colon H^1_0(\Omega) \to H^1_0(\Omega), \quad h \mapsto w_h.$$

Now we have to prove that for all sequence $(h_n)_n$ of $B(0,R_\varepsilon)$ converging weakly to h, when n tends to ∞, the sequence $(\Pi(h_n))_n$ converges weakly to $\Pi(h)$. Let us consider such a sequence $(h_n)_n$. Since $(\Pi(h_n))_n$ is bounded, we can extract a subsequence, still denoted by $(\Pi(h_n))_n$, so that

$$\Pi(h_n) \rightharpoonup \chi \quad \text{in } H_0^1(\Omega).$$

As the imbedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact, after another extraction, we have

$$\begin{cases} \Pi(h_n) \to \chi & \text{in } L^2(\Omega), \\ h_n \to h & \text{in } L^2(\Omega) \text{ and a.e. in } \Omega. \end{cases}$$

Since $\widetilde{\Phi}'_\varepsilon$ is a bounded and piecewise continuous function and, using the Lebesgue dominated convergence theorem, we have also

$$\widetilde{\Phi}'_\varepsilon(w_0+h_n) \to \widetilde{\Phi}'_\varepsilon(w_0+h) \quad \text{in } L^2(\Omega).$$

Moreover, for all n in $\mathbb{N}$ and all v in $H_0^1(\Omega)$, we have

$$\int_\Omega \{\widetilde{\Phi}'_\varepsilon(w_0+h_n)\nabla w_{h_n}\nabla v + w_{h_n} v\}\,dx = \int_\Omega \{fv - w_0 v - \widetilde{\Phi}'_\varepsilon(w_0+h_n)\nabla w_0 \nabla v\}\,dx.$$

Passing to the limit, as n tends to the infinity, in this previous equality, we obtain that, for all v in $H_0^1(\Omega)$,

$$\int_\Omega \{\widetilde{\Phi}'_\varepsilon(w_0+h)\nabla \chi \nabla v + \chi v\}\,dx = \int_\Omega \{fv - w_0 v - \widetilde{\Phi}'_\varepsilon(w_0+h)\nabla w_0 \nabla v\}\,dx.$$

Meanwhile, for all h in $H_0^1(\Omega)$, there is one and only one $w_h = \Pi(h)$, so $\Pi(h) = \chi$ and the whole sequence $(\Pi(h_n))_n$ converges weakly to $\Pi(h)$ in $H_0^1(\Omega)$. We can now apply the fixed point theorem and conclude that there is w in $H_0^1(\Omega)$ so that $\Pi(w) = w$. Setting $u = w_0 + w$, we have u in $H^1(\Omega)$ and, for all v in $H_0^1(\Omega)$,

$$\int_\Omega \widetilde{\Phi}'_\varepsilon(u)\nabla u \nabla v\,dx + \int_\Omega uv\,dx = \int_\Omega fv\,dx,$$

that is to say that u is solution of

$$\begin{cases} -\Delta \widetilde{\Phi}_\varepsilon(u) + u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Using a method similar to the first step, for this problem, we can prove that

$$m \le u \le M \quad \text{a.e. in } \Omega,$$

which shows, in fact, that u is solution of the regularized problem and u belongs to $H^1(\Omega) \cap L^\infty(\Omega)$. Moreover,

$$\|u\|_{H^1(\Omega)} \le \|w_0\|_{H^1(\Omega)} + \|w\|_{H_0^1(\Omega)}.$$

But, by definition of w_0, we have $\|w_0\|_{H^1(\Omega)} \leq C(\Omega)\|g\|_{L^\infty(\partial\Omega)}$ and we prove that

$$\|w\|_{H_0^1(\Omega)} \leq \frac{1}{r_\varepsilon} C(\Omega)[\|f\|_{L^\infty(\Omega)} + (2 + \|\Phi'\|_{L^\infty(\mathbb{R})})\|w_0\|_{H^1(\Omega)}],$$

so

$$\|u\|_{H^1(\Omega)} \leq \frac{C(\Omega)}{r_\varepsilon}[\|f\|_{L^\infty(\Omega)} + (2 + \|\Phi'\|_{L^\infty(\mathbb{R})})\|g\|_{L^\infty(\partial\Omega)}]. \tag{3.1}$$

Step 3. (Uniqueness of the solution in $H^1(\Omega)$) Let u_1 and u_2 in $H^1(\Omega)$ be two solutions of the regularized problem, then for all v belonging to $H_0^1(\Omega)$, one has

$$\int_\Omega \nabla(\Phi_\varepsilon(u_1) - \Phi_\varepsilon(u_2))\nabla v \, dx + \int_\Omega (u_1 - u_2)v \, dx = 0.$$

Taking $v = \Phi_\varepsilon(u_1) - \Phi_\varepsilon(u_2)$, we can write that

$$\|\Phi_\varepsilon(u_1) - \Phi_\varepsilon(u_2)\|^2_{H_0^1(\Omega)} + \int_\Omega (u_1 - u_2)(\Phi_\varepsilon(u_1) - \Phi_\varepsilon(u_2)) \, dx = 0.$$

But it is the sum of two non negative terms, so both are equal to zero. In particular, $\|\Phi_\varepsilon(u_1) - \Phi_\varepsilon(u_2)\|^2_{H_0^1(\Omega)} = 0$, that is $u_1 = u_2$, since Φ_ε is an injective function. □

3.2. Strong solution of the generalized Dirichlet problem. We are going to apply Theorem 2.15 with $E = L^\infty(\Omega) \times L^\infty(\partial\Omega)$, $F = H^1(\Omega) \cap L^\infty(\Omega)$ and

$$\theta_\varepsilon \colon E \to F, \quad (f, g) \mapsto \theta_\varepsilon(f, g) = u_\varepsilon,$$

where u_ε is the solution of problem (P_ε). Before, we are going to show the two following lemmas.

Lemma 3.2. *For all (f, g) in E and $u_\varepsilon = \theta_\varepsilon(f, g)$ in F, we have*

$$\|u_\varepsilon\|_F \leq \frac{C(\Omega, \|\Phi'\|_{L^\infty(\mathbb{R})})}{r_\varepsilon}\|(f, g)\|_E.$$

Proof. This result is an immediate consequence of inequality (3.1) since $\max\{|m|, |M|\}$ is less than $\|(f, g)\|_E = \|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\partial\Omega)}$. □

Lemma 3.3. *For all (f, g), (δ, η) in E, $u_\varepsilon = \theta_\varepsilon(f, g)$ in F and $u_\varepsilon + \nu_\varepsilon = \theta_\varepsilon(f + \delta, g + \eta)$ in F, we have*

$$\|\nu_\varepsilon\|_F \leq \frac{C(\Omega, \|\Phi'\|_{L^\infty(\mathbb{R})})}{r_\varepsilon}\|(\delta, \eta)\|_E.$$

Proof. By definition of θ_ε, we have

$$\begin{cases} -\Delta\Phi_\varepsilon(u_\varepsilon) + u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon = g & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta\Phi_\varepsilon(u_\varepsilon+\nu_\varepsilon)+u_\varepsilon+\nu_\varepsilon=f+\delta & \text{in } \Omega,\\ u_\varepsilon+\nu_\varepsilon=g+\eta & \text{on } \partial\Omega,\end{cases}$$

so

$$\begin{cases} -\Delta\chi_\varepsilon(\nu_\varepsilon)+\nu_\varepsilon=\delta & \text{in } \Omega,\\ \nu_\varepsilon=\eta & \text{on } \partial\Omega,\end{cases}$$

with $\chi_\varepsilon=\Phi_\varepsilon(u_\varepsilon+\,\cdot\,)-\Phi_\varepsilon(u_\varepsilon)$ which satisfies the same hypothesis as Φ_ε of Section 3.1. Consequently, ν_ε is the solution of a similar problem as (P_ε) and satisfies an inequality of the same type as (3.1), that is

$$\|\nu_\varepsilon\|_F\le\frac{C(\Omega,\|\chi'\|_{L^\infty(\mathbb{R})})}{r_\varepsilon}\|(\delta,\eta)\|_E,$$

where $\chi=\chi_\varepsilon-r_\varepsilon\mathrm{id}$. And inequality $\|\chi'\|_{L^\infty(\mathbb{R})}\le\|\Phi'\|_{L^\infty(\mathbb{R})}$ implies the required result. □

Theorem 3.4. *If $(\mathcal{F},\mathcal{G})$ belongs to $\mathcal{A}(\mathcal{C},L^\infty(\Omega)\times L^\infty(\partial\Omega))$ then there is one, and only one, generalized function $\mathcal{U}=\mathrm{cl}(u_\varepsilon)_\varepsilon$, belonging to $\mathcal{A}(\mathcal{C},H^1(\Omega)\cap L^\infty(\Omega))$, so that*

$$\begin{cases}\mathrm{cl}[-\Delta\Phi_\varepsilon(u_\varepsilon)]_\varepsilon+\mathcal{U}=\mathcal{F} & \text{in } \mathcal{A}(\mathcal{C},L^\infty(\Omega)),\\ \Gamma(\mathcal{U})=\mathcal{G} & \text{in } \mathcal{A}(\mathcal{C},L^\infty(\partial\Omega)),\end{cases}\tag{3.2}$$

where, by definition, $\Gamma(\mathcal{U})=\mathrm{cl}(u_{\varepsilon|\partial\Omega})_\varepsilon=\mathrm{cl}(g_\varepsilon)_\varepsilon$, when $\mathcal{G}=\mathrm{cl}(g_\varepsilon)_\varepsilon$.

Proof. We are going to apply theorem 1 with $E=L^\infty(\Omega)\times L^\infty(\partial\Omega)$, $F=H^1(\Omega)\cap L^\infty(\Omega)$ and

$$\theta_\varepsilon\colon E\to F\quad (f,g)\mapsto\theta_\varepsilon(f,g)=u_\varepsilon,$$

where u_ε is the solution of problem (P_ε). In order to obtain the required result, it suffices to use the two previous lemmas and apply Theorem 2.15 with

$$\Psi_\varepsilon(x)=\frac{C(\Omega,\|\Phi'\|_{L^\infty(\mathbb{R})})}{r_\varepsilon}x=\Psi_\varepsilon^2(x)$$

and $\Psi_\varepsilon^1(x)=1$, for all x in $\mathbb{R}$. The fact that Φ' is bounded, ensures that $(C(\Omega,\|\Phi'\|_{L^\infty(\mathbb{R})})/r_\varepsilon)_\varepsilon$ is in A^+. We set then

$$\mathcal{U}=\Theta(\mathcal{F},\mathcal{G})=\mathrm{cl}(u_\varepsilon)_\varepsilon=\mathrm{cl}(\theta_\varepsilon(f_\varepsilon,g_\varepsilon))_\varepsilon$$

when $\mathcal{F}=\mathrm{cl}(f_\varepsilon)_\varepsilon$ and $\mathcal{G}=\mathrm{cl}(g_\varepsilon)_\varepsilon$. □

3.3. Weak solution of the generalized Dirichlet problem. In this section, we define the notion of weak solution by using the weak equality defined in Example 2.14.

Theorem 3.5. *With the assumptions of Theorem* 3.4, *if* $\mathcal{F} = \mathrm{cl}(f_\varepsilon)_\varepsilon$ *and* $\mathcal{G} = \mathrm{cl}(g_\varepsilon)_\varepsilon$ *are such that*

$$\exists (r_\varepsilon)_\varepsilon \in \mathcal{V}_A^+,\ \lim_{\varepsilon\to 0^+} r_\varepsilon \max\{\|g_\varepsilon\|_{L^\infty(\partial\Omega)}, \|f_\varepsilon\|_{L^\infty(\Omega)}\} = 0, \tag{3.3}$$

then there is one, and only one, generalized function $\mathcal{U}$ *belonging to* $\mathcal{A}(\mathcal{C}, H^1(\Omega)\cap L^\infty(\Omega))$ *and such that*

$$\begin{cases} -\Delta\Phi(\mathcal{U}) + \mathcal{U} \overset{2}{\simeq} \mathcal{F} & \text{in } \mathcal{A}(\mathcal{C}, L^\infty(\Omega)), \\ \Gamma(\mathcal{U}) = \mathcal{G} & \text{in } \mathcal{A}(\mathcal{C}, L^\infty(\partial\Omega)), \end{cases} \tag{3.4}$$

with $\Delta\Phi(\mathcal{U}) = \mathrm{cl}(\Delta\Phi(u_\varepsilon))_\varepsilon = \mathrm{cl}(u_\varepsilon - r_\varepsilon\Delta u_\varepsilon - f_\varepsilon)_\varepsilon$.

Proof. Since $\mathrm{cl}[-\Delta\Phi_\varepsilon(u_\varepsilon)]_\varepsilon + \mathcal{U} = \mathcal{F}$ in $\mathcal{A}(\mathcal{C}, L^\infty(\Omega))$, so $H^{-2}(\Omega)$-weakly equal and $\Phi_\varepsilon = \Phi + r_\varepsilon\mathrm{id}$, it is sufficient to prove that

$$Cl(-r_\varepsilon\Delta u_\varepsilon)_\varepsilon \overset{2}{\simeq} 0.$$

Let φ be in $H_0^2(\Omega)$, using Green's formula, one has

$$\begin{aligned}\int_\Omega -r_\varepsilon\Delta u_\varepsilon\varphi\,dx &= r_\varepsilon\bigg(\int_\Omega \nabla u_\varepsilon\nabla\varphi\,dx - \int_{\partial\Omega} g_\varepsilon\varphi\,d\nu\bigg) \\ &= r_\varepsilon\bigg(\int_{\partial\Omega} u_\varepsilon\frac{\partial\varphi}{\partial\nu}\,d\nu - \int_\Omega u_\varepsilon\Delta\varphi\,dx - \int_{\partial\Omega} g_\varepsilon\varphi\,d\nu\bigg) \\ &= -\,r_\varepsilon\int_\Omega u_\varepsilon\Delta\varphi dx.\end{aligned}$$

Consequently, using Cauchy–Schwartz's inequality, one has

$$\bigg|\int_\Omega -r_\varepsilon\Delta u_\varepsilon\varphi\,dx\bigg| \le r_\varepsilon\max(\|g_\varepsilon\|_{L^\infty(\partial\Omega)}, \|f_\varepsilon\|_{L^\infty(\Omega)})C(\Omega)\|\Delta\varphi\|_{L^2(\Omega)}.$$

The assumption (3.3) implies that

$$\lim_{\varepsilon\to 0}\int_\Omega -r_\varepsilon\Delta u_\varepsilon\varphi\,dx = 0. \qquad \square$$

Remark 3.6. This theorem leads us to notice that we can have a Dirac generalized function in the second member of the problem. Indeed, a representative of a Dirac generalized function can be: $\delta_\varepsilon(x) = \varepsilon^{-d}\varphi(\varepsilon^{-1}x)$ for all $x \in \mathbb{R}^d$ where φ is a compactly supported function defined on $\mathbb{R}^d$. The hypothesis (3.3) is satisfied with $r_\varepsilon = \varepsilon^{d+q}$ for all $q \in \mathbb{N}$, and, for example, we take A and I_A as in Example 2.4.

3.3. Non positive solutions. In this section, we prove that the solution is non positive, in a sense to be defined, when the data is. We start by defining what non positive means here.

Definition 3.7. An element $\mathcal{U} \in \mathcal{A}(\mathcal{C}, H^1(\Omega) \cap L^\infty(\Omega))$ is said to be non positive if and only if the corresponding element $\mathcal{I}(\mathcal{U})$, of the generalized Sobolev algebra $\mathcal{A}(\mathcal{C}, L^\infty(\Omega))$, is non positive.

In this definition, $\mathcal{I}$ denotes the extension of the canonical embedding of $H^1(\Omega) \cap L^\infty(\Omega))$ into $L^\infty(\Omega)$, introduced in Example 2.17. This mapping is an embedding of $\mathcal{A}(\mathcal{C}, H^1(\Omega) \cap L^\infty(\Omega))$ into $\mathcal{A}(\mathcal{C}, L^\infty(\Omega))$.

Proposition 3.8. *With the assumptions of Theorem* 3.5*, if the generalized functions* $\mathcal{F}= \mathrm{cl}(f_\varepsilon)_\varepsilon \in \mathcal{A}(\mathcal{C}, L^\infty(\Omega))$ *and* $\mathcal{G}= \mathrm{cl}(g_\varepsilon)_\varepsilon \in \mathcal{A}(\mathcal{C}, L^\infty(\partial\Omega))$ *are non positive, then* $\mathcal{U} = \Theta(\mathcal{F}, \mathcal{G}) = \mathrm{cl}(\theta_\varepsilon(f_\varepsilon, g_\varepsilon))_\varepsilon \in \mathcal{A}(\mathcal{C}, H^1(\Omega) \cap L^\infty(\Omega))$*, the solution to our main problem, is non positive.*

Proof. Using the hypothesis on $\mathcal{F}$, $\mathcal{G}$ and the results of Section 2.4, one can claim that each data admits a non positive representative. And then it suffices to show that $\mathcal{U} = \Theta(\mathcal{F}, \mathcal{G}) = \mathrm{cl}(\theta_\varepsilon(f_\varepsilon, g_\varepsilon))_\varepsilon$ admits a non positive representative, since a non positive representative of $\mathcal{U}$ is also one for $\mathcal{I}(\mathcal{U})$. Let f_ε and g_ε be the non positive representatives of $\mathcal{F}$ and $\mathcal{G}$, and $u_\varepsilon = \theta_\varepsilon(f_\varepsilon, g_\varepsilon)$. Using the maximum's principle as in the proof of Proposition 3.1 with

$$M_\varepsilon = \max\{\sup_{\Omega} f_\varepsilon \ , \ \sup_{\partial\Omega} g_\varepsilon\} = 0,$$

we obtain that $u_\varepsilon \leq 0$ a.e. Ω, and for all ε. $\square$

Remark 3.9. In fact, we solved the following obstacle problem:

- For $\mathcal{F} = \mathrm{cl}(f_\varepsilon)_\varepsilon \in \mathcal{A}(\mathcal{C}, L^\infty(\Omega))$ and $\mathcal{G}= \mathrm{cl}(g_\varepsilon)_\varepsilon \in \mathcal{A}(\mathcal{C}, L^\infty(\partial\Omega))$ non positive, find $\mathcal{U} \in \mathcal{A}(\mathcal{C}, H^1(\Omega) \cap L^\infty(\Omega))$ so that

$$\begin{cases} -\Delta\Phi(\mathcal{U}) + \mathcal{U} \overset{2}{\simeq} \mathcal{F} & \text{in } \mathcal{A}(\mathcal{C}, L^\infty(\Omega)), \\ \Gamma(\mathcal{U}) = \mathcal{G} & \text{in } \mathcal{A}(\mathcal{C}, L^\infty(\partial\Omega)), \\ \mathcal{U} \leq 0 & \text{in } \mathcal{A}(\mathcal{C}, L^\infty(\Omega)), \end{cases} \tag{3.5}$$

which is a generalized version of this one:

- Find $u: \Omega \mapsto \mathbb{R}$ so that

$$\begin{cases} -\Delta\Phi(u) + u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \\ u \leq 0 & \text{on } \Omega, \end{cases}$$

  where f and g are non positive given functions.

Acknowledgements. The authors are pleased to acknowledge A. Delcroix for helpful advice and suggestions.

References

[1] H. Brezis, *Analyse fonctionnelle — Théorie et applications*, Dunod, Paris, 1999.

[2] J.-F. Colombeau, *New Generalized Functions and Multiplications of Distributions*, North Holland, Amsterdam–Oxford–New-York, 1984.

[3] J.-F. Colombeau, *Multiplication of distributions: a tool in mathematics numerical engineering and theorical physics*, Lectures Notes in Math., vol. 1532, Springer–Verlag, 1992.

[4] A. Delcroix and D. Scarpalezos, *Topology on asymptotic algebras of generalized functions and applications*, Monats. Math. **129** (2000), 1–14.

[5] Y. V. Egorov, *A contribution to the theory of generalized functions*, Russian Math. Surveys **45** (1990), 1–49.

[6] G. Gagneux and M. Madaune-Tort, *Analyse mathématique de modèles non linéaires de l'ingénierie pétrolière*, Math. Appl. **22** (1996).

[7] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equation of Second Order*, Springer–Verlag, 1977.

[8] M. Grosser, M. Kunzinger, M. Oberguggenberger and R. Steinbauer, *Geometric theory of generalized functions with applications to general relativity*, Math. Appl. (2002).

[9] J.-L. Lions, *Quelques méthodes de rèsolution des problèmes aux limites non linéaires*, Gauthier-Villars, Paris, 1969.

[10] J.-A. Marti, *Fundamental structures and asymptotic microlocalization in sheaves of generalized functions*, Integral Transform. Spec. Funct. **6** (1998), 223–228.

[11] J.-A. Marti and S. P. Nuiro, *Génération d'algèbres différentielles et applications*, Proceedings of the Third Catalan Days on Applied Mathematics, November 27–29, Lleida (Spain), 1996.

[12] ———, *Analyse algébrique d'un problème de Dirichlet non linéaire et singulier*, Topol. Methods Nonlinear Anal. **13** (1999), 301–311.

[13] J.-A. Marti, S. P. Nuiro and V. S. Valmorin, *A non linear Goursat problem with irregular data*, Integral Transform. Spec. Funct. **6** (1998), 229–246.

[14] M. Oberguggenberger, *Nonlinear theories of generalized functions*, Proceedings of the International Conference on Applications of Nonstandard Analysis, Functional Analysis and Probability Theory, Blaubeuren, 1992.

[15] F. Treves, *Topological Vector Spaces — Distributions and Kernels*, Academic Press, 1967.

Manuscript received September 29, 2004

Séverine Bernard
Université des Antilles et de la Guyane
Laboratoire AOC
Campus de Fouillole
97159 Pointe-à-Pitre cedex, GUADELOUPE

E-mail address: severine.bernard@univ-ag.fr

Silvère P. Nuiro
Université des Antilles et de la Guyane
Laboratoire GRIMAAG
Campus de Fouillole
97159 Pointe-à-Pitre cedex, GUADELOUPE

E-mail address: paul.nuiro@univ-ag.fr