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THE INVARIANCE OF DOMAIN THEOREM FOR CONDENSING VECTOR FIELDS

In-Sook Kim

Abstract. Using degree theory for countably condensing maps due to Väth, we give an invariance of domain theorem for countably condensing vector fields. The key tool is Borsuk's theorem for odd countably condensing maps.

1. Introduction

Schauder in [6] showed that a profound topological theorem on the invariance of the n-dimensional domain first proved by Brouwer [1] can be carried over infinite dimensional spaces.

Let Ω be a domain in a Banach space E and $F:\Omega \to E$ a completely continuous map. If the vector field $f = I - F$ is injective, then its range $f(\Omega)$ is a domain in E. Here I is the identity map.

In other words, the existence of a solution of the equation

$$x - Fx = y, \quad x \in \Omega$$

for a fixed $y = y_0$ implies that the above equation has a solution for every y in some neighborhood of y_0. This theorem is related to the problem of solving nonlinear elliptic differential equations (see [6]). In this aspect, it is of practical

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importance that this problem is investigated for condensing maps. Using homotopy theory, Hahn [3] obtained the invariance of domain theorem for condensing vector fields (see also [2]).

The purpose of this paper is to extend the invariance of domain theorem to a larger class of countably condensing maps due to Väth [7], by using degree theory. To this end we first give some fundamental properties of countably condensing maps and then introduce a degree theory for countably condensing maps (see [7]–[9]). Next, we prove Borsuk's theorem for odd countably condensing maps. From this we obtain that an injective countably k-condensing vector field in the strong sense is an open map and therefore the invariance of domain theorem holds for these vector fields. This result has many consequences, such as surjectivity and homeomorphism. For the case of condensing or completely continuous vector fields, we refer to [4], [5], [10].

For arbitrary Fréchet spaces X and Y, a continuous map $F\colon X \to Y$ is said to be *compact* if its range $F(X)$ is contained in a compact subset of Y. F is said to be *completely continuous* if it is continuous and maps bounded subsets of X onto relatively compact sets.

For a nonempty subset K of a Fréchet space E, the closure, the boundary, the convex hull, and the closed convex hull of K in E are denoted by $\overline{K}$, ∂K, $\operatorname{co} K$, and $\overline{\operatorname{co}}\, K$, respectively.

2. Countably condensing maps

We introduce the concept of countably condensing maps due to Väth [7], [9].

Definition 2.1. Let E be a Fréchet space and $\mathcal{M}$ a collection of nonempty subsets of E containing all precompact subsets of E with the property that for any $M, N \in \mathcal{M}$, the sets $\overline{\operatorname{co}}\, M$, $M \cup N$, $M + N$, tM ($t \in \mathbb{R}$) and every subset of M belong to $\mathcal{M}$. A function $\gamma\colon \mathcal{M} \to [0, \infty]$ is called *a measure of noncompactness* on E provided that it satisfies the following properties:

(a) $\gamma(\overline{\operatorname{co}}\, M) = \gamma(M)$,
(b) $\gamma(M) = 0$ if and only if M is precompact,
(c) $\gamma(M \cup N) = \max\{\gamma(M), \gamma(N)\}$,
(d) $\gamma(M + N) \le \gamma(M) + \gamma(N)$,
(e) $\gamma(tM) = |t|\gamma(M)$ for any real number t.

Let X be a nonempty subset of a Fréchet space E and γ a measure of noncompactness on E. A continuous map $H\colon [0,1] \times X \to E$ is said to be *countably condensing* (with respect to γ) provided that $H([0,1] \times X) \in \mathcal{M}$ and for any countable set $C \subset X$ with $C \in \mathcal{M}$, the relation $\gamma(C) \le \gamma(H([0,1] \times C))$ implies that C is precompact. In particular, a continuous map $F\colon X \to E$ is said to be *countably condensing* (with respect to γ) if $F(X) \in \mathcal{M}$ and for any countable set $C \subset X$ with $C \in \mathcal{M}$, the relation $\gamma(C) \le \gamma(F(C))$ implies that C is precompact.

The above definition reduces to the usual definition of condensing maps if the condition holds for all sets in X, in place of all countable sets C.

We give the following result which enables us to define a degree for countably condensing maps. The basic idea of proof is given in [9, Corollary 3.1].

Proposition 2.2. *Let X be a closed nonempty subset of a Fréchet space E, V and K two compact subsets of E and γ a measure of noncompactness on E. If $H\colon [0,1]\times X \to E$ is a countably condensing map with respect to γ, there exists a compact convex subset S of E such that*

(a) *S contains K,*
(b) *$H([0,1]\times(X\cap S))+V$ is a subset of S,*
(c) *for each $(t,x)\in[0,1]\times X$, the relation $x\in \operatorname{co}[(\{H(t,x)\}+V)\cup S]$ implies $x\in S$.*

If we take $K=\{x_0\}$ for some $x_0\in X$, then $X\cap S$ is not empty. Such a closed convex nonempty set S satisfying conditions (b) *and* (c) *is called a fundamental set for H.*

Proof. Let Σ denote the collection of all subsets B of E satisfying the following conditions:

(a) B is closed and convex and $K\subset B$,
(b) $H([0,1]\times(X\cap B))+V\subset B$,
(c) for each $(t,x)\in[0,1]\times X$, the relation $x\in\operatorname{co}[(\{H(t,x)\}+V)\cup B]$ implies $x\in B$.

Then Σ is nonempty because $B_0=\overline{\operatorname{co}}\,[(H([0,1]\times X)+V)\cup K]\in\Sigma$. Set $S:=\bigcap_{B\in\Sigma}B$ and

$$S_1:=\overline{\operatorname{co}}\,[(H([0,1]\times(X\cap S))+V)\cup K].$$

Since $S\in\Sigma$, we have $S_1\subset S$ and so $H([0,1]\times(X\cap S_1))+V\subset H([0,1]\times(X\cap S))+V\subset S_1$. Moreover, if $x\in\operatorname{co}[(\{H(t,x)\}+V)\cup S_1]$, then $x\in\operatorname{co}[(\{H(t,x)\}+V)\cup S]$ implies $x\in S$ which yields $\operatorname{co}[(\{H(t,x)\}+V)\cup S_1]\subset S_1$ by definition of S_1 and hence $x\in S_1$. Consequently, $S_1\in\Sigma$. It follows from definition of S that $S\subset S_1$ and hence

$$S=\overline{\operatorname{co}}\,[(H([0,1]\times(X\cap S))+V)\cup K].$$

Thus, S is a closed, convex subset of E and $H([0,1]\times(X\cap S))+V\subset S$. It remains to show that S is compact. To prove this, we use Theorem 3.1 of [9]. Let C be any countable subset of $X\cap S$ such that

$$X\cap\operatorname{co}[(H([0,1]\times C)+V)\cup K]\subset\overline{C}\subset X\cap\overline{\operatorname{co}}\,[(H([0,1]\times C)+V)\cup K].$$

Observe that this set C belongs to $\mathcal{M}$ appearing in Definition 2.1 because $H([0,1]\times X)$ belongs to $\mathcal{M}$ and so does S. Using the properties of γ, we have

$$\begin{aligned}\gamma(C) &\le \gamma(\overline{\mathrm{co}}\,[(H([0,1]\times C)+V)\cup K]) = \max\{\gamma(H([0,1]\times C)+V),\gamma(K)\}\\ &\le \gamma(H([0,1]\times C))+\gamma(V) = \gamma(H([0,1]\times C)).\end{aligned}$$

Since H is countably condensing, C is precompact. Hence the compactness of S follows from Theorem 3.1 of [9]. This completes the proof. □

The following result will be needed for the degree of odd countably condensing maps.

Proposition 2.3. *Let X be a closed, symmetric set in a Fréchet space E with $0\in X$. If $F\colon X\to E$ is an odd countably condensing map with respect to a measure γ of noncompactness, there exists a symmetric, convex and compact subset S of E with $0\in S$ such that $F(X\cap S)$ is a subset of S and the relation $x\in \mathrm{co}\,(\{Fx\}\cup S)$ implies $x\in S$.*

Proof. Let

$$\begin{aligned}\Sigma := \{B\subset E : B=\overline{\mathrm{co}}\,B,\ B=-B,\ 0\in B,\ F(X\cap B)\subset B,\\ x\in \mathrm{co}\,(\{Fx\}\cup B)\text{ implies } x\in B\}.\end{aligned}$$

Set $S:=\bigcap_{B\in\Sigma} B$ and $S_1:=\overline{\mathrm{co}}\,[F(X\cap S)\cup\{0\}]$. Then S_1 is symmetric because the sets X and S are symmetric and F is odd. It is obvous from $S\in\Sigma$ and definition of S_1 that the relation $x\in\mathrm{co}\,(\{Fx\}\cup S_1)$ implies $x\in S_1$. As in the proof of Proposition 2.2, a similar argument proves that

$$S=S_1=\overline{\mathrm{co}}\,[F(X\cap S)\cup\{0\}].$$

Thus, S is a closed, convex and symmetric subset of E and $F(X\cap S)\subset S$. To prove that S is compact, let C be any countable subset of $X\cap S$ such that

$$X\cap \mathrm{co}\,(F(C)\cup\{0\})\subset \overline{C}\subset X\cap\overline{\mathrm{co}}\,(F(C)\cup\{0\}).$$

Then the definition of γ implies

$$\gamma(C)\le\gamma(\overline{\mathrm{co}}\,(F(C)\cup\{0\}))=\max\{\gamma(F(C)),\gamma(\{0\})\}=\gamma(F(C)).$$

Since F is countably condensing, it follows that C is precompact. Applying Theorem 3.1 of [9], we obtain that S is compact. This completes the proof. □

3. Borsuk's theorem

Introducing a degree theory for countably condensing maps due to [7]–[9], we give an extension of Borsuk's theorem which is a key tool for the invariance of domain theorem.

Let Ω be an open bounded nonempty set in a Banach space E and y an arbitrary point of E. Let $F:\overline{\Omega} \to E$ be a countably condensing map such that $x - Fx \neq y$ for all $x \in \partial\Omega$. In view of Proposition 2.2, there exists a compact fundamental set S for F such that

$$F(\overline{\Omega} \cap S) + y \subset S.$$

Let $R: E \to S$ be any retraction onto S. Then it follows from $R = I$ on S and $y \notin (I - F)(\partial\Omega)$ that $F \circ R$ is compact on the closure of $R^{-1}(\Omega) \cap \Omega$ and $I - F \circ R \neq y$ on the boundary of $R^{-1}(\Omega) \cap \Omega$. Consequently, the Leray–Schauder degree for the compact map $F \circ R$, $d_{LS}(I - F \circ R, R^{-1}(\Omega) \cap \Omega, y)$, is defined.

Now we define a degree for the countably condensing map F as follows:

$$D(I - F, \Omega, y) := d_{LS}(I - F \circ R, R^{-1}(\Omega) \cap \Omega, y).$$

This definition does not depend on the choice of the compact fundamental set S and of the retraction R. To do this, we use the homotopy invariance property of the Leray–Schauder degree. The case of condensing maps can be found in [2].

We give some properties of the above degree which will be used later.

Lemma 3.1. *Let Ω be an open bounded nonempty set in a Banach space E and y a point of E. Let $F:\overline{\Omega} \to E$ be a countably condensing map such that $x - Fx \neq y$ for all $x \in \partial\Omega$. Then the degree has the following properties:*

- (a) *If $D(I - F, \Omega, y) \neq 0$, then the equation $x - Fx = y$ has a solution in Ω.*
- (b) (Homotopy invariance) *If $H:[0,1] \times \overline{\Omega} \to E$ is a countably condensing map and $y:[0,1] \to E$ is a continuous map such that $x - H(t,x) \neq y(t)$ for all $x \in \partial\Omega$ and $t \in [0,1]$, then $D(I - H(t,\,\cdot\,), \Omega, y(t))$ is independent of $t \in [0,1]$.*
- (c) *$D(I - F, \Omega, \,\cdot\,)$ is constant on $B_r(y)$, where*

  $$r = \rho(y, (I - F)(\partial\Omega)) = \inf\{\|x - Fx - y\| : x \in \partial\Omega\}.$$

  Here $B_r(y) = \{z \in E : \|z - y\| < r\}$ is the open ball with center y and radius r.

Proof. (a) Let S and R be as in the above definition. Since $d_{LS}(I - F \circ R, R^{-1}(\Omega) \cap \Omega, y) \neq 0$, there exists a point x in $R^{-1}(\Omega) \cap \Omega$ such that $x - (F \circ R)x = y$, where R is the retraction onto S. From $Rx \in \Omega \cap S$ and $F(\overline{\Omega} \cap S) + y \subset S$ it follows that $x = F(Rx) + y \in S$ and so $Rx = x$. We conclude that the equation $x - Fx = y$ has a solution in Ω.

(b) Since H is countably condensing, by Proposition 2.2, there exists a compact fundamental set S for H such that

$$H([0,1]\times(\overline{\Omega}\cap S))+y([0,1])\subset S.$$

Let $R\colon E\to S$ be any retraction onto S. By the compactness of S, the set $H([0,1]\times R(\overline{R^{-1}(\Omega)\cap\Omega}))$ is relatively compact. Now fix $t\in[0,1]$. Assume to the contrary that $x-H(t,Rx)=y(t)$ for some $x\in\partial(R^{-1}(\Omega)\cap\Omega)$. From $Rx\in\overline{\Omega}\cap S$ it follows that $x=H(t,Rx)+y(t)\in S$. $Rx=x$ implies $x-H(t,x)=y(t)$ with $x\in\partial\Omega$, which is a contradiction to the hypothesis. Thus, $x-H(t,Rx)\neq y(t)$ for all $x\in\partial(R^{-1}(\Omega)\cap\Omega)$.

For each $t\in[0,1]$, as in the above definition, we have

$$D(I-H(t,\cdot\,),\Omega,y(t))=d_{LS}(I-H(t,\cdot\,)\circ R,\quad R^{-1}(\Omega)\cap\Omega,y(t)).$$

The homotopy invariance property of the Leray–Schauder degree says that, for all $t\in[0,1]$, $d_{LS}(I-H(t,\cdot\,)\circ R,R^{-1}(\Omega)\cap\Omega,y(t))$ is constant. Therefore, $D(I-H(t,\cdot\,),\Omega,y(t))$ is also independent of t.

(c) We will claim that $D(I-F,\Omega,y')=D(I-F,\Omega,y)$ for every $y'\in B_r(y)$. Fix $y'\in B_r(y)$. Consider $H(t,\cdot\,)\equiv F$ for all $t\in[0,1]$ and $y\colon[0,1]\to E$ given by $y(t)=(1-t)y+ty'$ for $t\in[0,1]$. For every $x\in\partial\Omega$ and $t\in[0,1]$, we have

$$\|x-Fx-y(t)\|\geq\|x-Fx-y\|-\|y(t)-y\|>r-tr\geq 0$$

and hence $x-H(t,x)\neq y(t)$. Statement (b) implies that

$$\begin{aligned}D(I-F,\Omega,y')&=D(I-H(1,\cdot\,),\Omega,y(1))\\&=D(I-H(0,\cdot\,),\Omega,y(0))=D(I-F,\Omega,y).\end{aligned}$$

This completes the proof. □

Now we can prove Borsuk's theorem for countably condensing maps. The method is to use Borsuk's theorem for odd compact maps, as in the condensing case (see [2, Theorem 9.4]).

Theorem 3.2. *Let Ω be an open bounded set in a Banach space E that is symmetric with respect to $0\in\Omega$. Let $F\colon\overline{\Omega}\to E$ be a countably condensing map such that F is odd on $\partial\Omega$ and $0\notin(I-F)(\partial\Omega)$. Then $D(I-F,\Omega,0)$ is odd.*

Proof. Consider a continuous homotopy $H\colon[0,1]\times\overline{\Omega}\to E$ given by

$$H(t,x):=\frac{1}{1+t}Fx-\frac{t}{1+t}F(-x)\quad\text{for } x\in\overline{\Omega}\text{ and } t\in[0,1].$$

Then H is clearly countably condensing because F is countably condensing. Since F is odd on $\partial\Omega$, we have $H(t,x)=Fx\neq x$ for all $x\in\partial\Omega$ and $t\in[0,1]$. Define $G\colon\overline{\Omega}\to E$ by $Gx:=1/2(Fx-F(-x))$ for $x\in\overline{\Omega}$. Lemma 3.1 implies that

$$D(I-F,\Omega,0)=D(I-H(0,\cdot\,),\Omega,0)=D(I-H(1,\cdot\,),\Omega,0)=D(I-G,\Omega,0).$$

Since G is an odd countably condensing map on $\overline{\Omega}$, by Proposition 2.3, there exists a symmetric compact fundamental set S for G with $0 \in S$ such that $G(X \cap S)$ is a subset of S. Let $R_0: E \to S$ be any retraction onto S and $R: E \to S$ a map defined by

$$Rx = \frac{1}{2}\,[R_0 x - R_0(-x)] \quad \text{for } x \in E.$$

Then R is a retraction onto S. Since $0 \notin (I - G)(\partial\Omega)$, as in the above definition, we have

$$D(I - G, \Omega, 0) = d_{LS}(I - G \circ R, R^{-1}(\Omega) \cap \Omega, 0).$$

Note that the set $R^{-1}(\Omega) \cap \Omega$ is open, bounded, and symmetric with respect to $0 \in R^{-1}(\Omega) \cap \Omega$. The last follows from $0 \in S$ and $R = I$ on S; the symmetry is possible because R is odd and Ω is symmetric. Since $G \circ R$ is odd on the closure of $R^{-1}(\Omega) \cap \Omega$, Borsuk's theorem for odd compact maps implies that $d_{LS}(I - G \circ R, R^{-1}(\Omega) \cap \Omega, 0)$ is odd (see [2, Theorem 8.3]). Therefore, we conclude that $D(I - F, \Omega, 0)$ is odd. This completes the proof. $\square$

4. The invariance of domain theorem

For our purpose, we need a notion of countably k-condensing maps in the strong sense.

Definition 4.1. Let X be a nonempty subset of a Fréchet space E, $\gamma: \mathcal{M} \to [0, \infty]$ a measure of noncompactness on E, and k a nonnegative real number. A continuous map $H: [0,1] \times X \to E$ is said to be *countably k-condensing* (with respect to γ) if $H([0,1] \times X) \in \mathcal{M}$ and $\gamma(H([0,1] \times C)) \le k\gamma(C)$ for each countable set C in X with $C \in \mathcal{M}$. Moreover, a continuous map $F: X \to E$ is said to be *countably k-condensing in the strong sense* (with respect to γ) if $F(X) \in \mathcal{M}$ and $\gamma(F(\overline{\mathrm{co}}\, C)) \le k\gamma(C)$ for each countable set C in X with $\overline{\mathrm{co}}\, C \subset X$ and $C \in \mathcal{M}$.

An additional condition $\overline{\mathrm{co}}\, C \subset X$ may be automatically dropped whenever X is a closed convex subset of E. In this case, every countably k-condensing map $F: X \to E$ in the strong sense is countably k-condensing.

The above definition includes the usual definition of k-condensing maps whenever the condition is satisfied for all sets in X, instead of all countable sets C.

A map $f: X \to E$ is called a (*countably*) *k-condensing vector field* if $f = I - F$ and F is (countably) k-condensing.

The following shows that a similar result of Hahn in the condensing case can be extended to countably condensing maps in the strong sense. For the proof we follow the basic line of proof in [3, Hilfssatz 2].

Lemma 4.2. *Let X be a closed, convex and symmetric neighborhood of 0 in a Fréchet space E. Let γ be a measure of noncompactness on E and k a nonnegative real number. If $F\colon X \to E$ is a countably k-condensing map in the strong sense with respect to γ, then the map $H\colon [0,1] \times X \to E$ defined by*

$$H(t,x) := F\bigg(\frac{1}{1+t}x\bigg) - F\bigg(-\frac{t}{1+t}x\bigg) \quad \textit{for } x \in X \textit{ and } t \in [0,1]$$

is countably k-condensing.

Proof. Let C be any countable subset of X such that $C \in \mathcal{M}$. Let ε be an arbitrary positive real number. Choose a positive integer n such that $1/n < \varepsilon$. Then we have

$$\gamma(H([0,1] \times C)) = \max\bigg\{\gamma\bigg(H\bigg(\bigg[\frac{j}{n}, \frac{j+1}{n}\bigg] \times C\bigg)\bigg) : j = 0, \dots, n-1\bigg\}$$

and

$$H\bigg(\bigg[\frac{j}{n}, \frac{j+1}{n}\bigg] \times C\bigg) \subset F\bigg(\overline{\mathrm{co}}\bigg(\frac{n}{n+j} C \cup \{0\}\bigg)\bigg) - F\bigg(-\overline{\mathrm{co}}\bigg(\frac{j+1}{n+j+1} C \cup \{0\}\bigg)\bigg).$$

For $j = 0, \dots, n-1$, since F is countably k-condensing in the strong sense, the definition of γ implies that

$$\begin{aligned}\gamma\bigg(H\bigg(\bigg[\frac{j}{n}, \frac{j+1}{n}\bigg] \times C\bigg)\bigg) &\le k\bigg(\frac{n}{n+j} + \frac{j+1}{n+j+1}\bigg)\gamma(C) \\ &\le k\bigg(1 + \frac{1}{n+j}\bigg)\gamma(C) < k(1+\varepsilon)\gamma(C).\end{aligned}$$

As the inequality $\gamma(H([0,1] \times C)) < (1+\varepsilon)k\gamma(C)$ holds for all $\varepsilon > 0$, we obtain that $\gamma(H([0,1] \times C)) \le k\gamma(C)$. Therefore, H is countably k-condensing. This completes the proof. $\square$

Applying degree theory, we show when each countably k-condensing vector field is an open map. One of essential conditions is local injectivity. Recall that a map $f\colon \Omega \to E$ is said to be *locally injective* if for every $x \in \Omega$ there exists a neighborhood U of x such that the restriction of f to U, $f|_U$, is injective.

Theorem 4.3. *Let Ω be an open nonempty subset of a Banach space E and $k \in [0,1)$. Let $F\colon \Omega \to E$ be a countably k-condensing map in the strong sense. If the vector field $f = I - F$ is locally injective, then f is an open map.*

Proof. It suffices to show that for every $x_0 \in \Omega$ there exists an open ball $B_r(x_0)$ such that $f(B_r(x_0))$ contains an open ball with center $f(x_0)$. Passing to $\Omega - x_0$ and $\tilde{f}(x) = f(x + x_0) - f(x_0)$ for $x \in \Omega - x_0$, if necessary, we may suppose that $x_0 = 0$ and $f(0) = 0$. We can choose an open ball $B_r(0)$ with $r > 0$ such that $\overline{B}_r(0) \subset \Omega$ and the restriction $f|_{\overline{B}_r(0)}$ is injective.

Consider a continuous homotopy $H\colon[0,1]\times\overline{B}_r(0)\to E$ given by

$$H(t,x):=F\left(\frac{1}{1+t}x\right)-F\left(-\frac{t}{1+t}x\right)\quad\text{for } x\in\overline{B}_r(0)\text{ and } t\in[0,1].$$

Then H is countably k-condensing by Lemma 4.2 and $H(0,\,\cdot\,)=F$ and $H(1,x)=F(x/2)-F(-x/2)$ for $x\in\overline{B}_r(0)$. Moreover, $x\neq H(t,x)$ for all $x\in\partial B_r(0)$ and $t\in[0,1]$. Indeed, if $x=H(t,x)$ for some $x\in\partial B_r(0)$ and $t\in[0,1]$, then $f(x/(1+t))=f(-tx/(1+t))$ and by the injectivity of f, $x/(1+t)=-tx/(1+t)$, that is, $x=0$, which is a contradiction.

Notice that any countably k-condensing map with $k<1$ is countably condensing. Since H is countably condensing, the homotopy invariance property stated in Lemma 3.1 implies that

$$D(I-F,B_r(0),0)=D(I-H(1,\,\cdot\,),B_r(0),0).$$

In view of Theorem 3.2, since $H(1,\,\cdot\,)$ is an odd countably condensing map on $\overline{B}_r(0)$, we have

$$D(I-H(1,\,\cdot\,),B_r(0),0)\neq 0.$$

Set $s:=\rho\,(0,(I-F)(\partial B_r(0)))$. Since F is countably condensing on $\overline{B}_r(0)$, Lemma 3.1 implies that

$$D(I-F,B_r(0),y)=D(I-F,\,B_r(0),0)\quad\text{for every } y\in B_s(0).$$

Consequently, for every $y\in B_s(0)$, we have $D(f,B_r(0),y)\neq 0$; hence there exists an $x\in B_r(0)$ such that $f(x)=y$. Thus, $B_s(0)\subset f(B_r(0))$. This completes the proof. □

In [3, Satz 5], Hahn obtained the following condensing case of Theorem 4.3, by using homotopy theory; see [2, Theorem 9.5].

COROLLARY 4.4. *Let Ω be an open nonempty set in a Banach space E and $k\in[0,1)$. Let $F\colon\Omega\to E$ be a k-condensing map. If $f=I-F$ is locally injective, then f is an open map.*

PROOF. Let C be any countable subset of Ω such that $\overline{\mathrm{co}}\,C\subset\Omega$ and $C\in\mathcal{M}$, as in Definition 4.1. Since F is a k-condensing map, it follows that

$$\gamma(F(\overline{\mathrm{co}}\,C))\leq k\gamma(\overline{\mathrm{co}}\,C)=k\gamma(C).$$

Thus, F is countably k-condensing in the strong sense. Apply Theorem 4.3. □

Now we present that Theorem 4.3 implies a number of important results, such as surjectivity and homeomorphism. For the condensing or completely continuous case, we refer to [4] and [10].

COROLLARY 4.5. *Let Ω be an open nonempty set in a Banach space E, $k \in [0,1)$, and $F\colon \Omega \to E$ a countably k-condensing map in the strong sense. If the vector field $f = I - F$ is locally injective and $f(\Omega)$ is closed in E, then $f(\Omega) = E$.*

PROOF. By Theorem 4.3 and the hypothesis, $f(\Omega)$ is an open and closed subset of E. From the connectedness of E it follows that $f(\Omega) = E$. □

COROLLARY 4.6. *Let E be a Banach space, $k \in [0,1)$, and $F\colon E \to E$ a countably k-condensing map in the strong sense. If the vector field $f = I - F$ is injective and $f(E)$ is closed in E, then f is a homeomorphism.*

PROOF. In view of Theorem 4.3, f is an open map. Since f is bijective by Corollary 4.5, we conclude that f^{-1} is continuous and so f is a homeomorphism. □

Finally we give an invariance of domain theorem for countably k-condensing maps in the strong sense. Here a set Ω is called a *domain* if it is open and connected.

THEOREM 4.7. *Let Ω be a domain in a Banach space E, $k \in [0,1)$, and $F\colon \Omega \to E$ a countably k-condensing map in the strong sense. If the vector field $f = I - F$ is locally injective, then its range $f(\Omega)$ is a domain in E.*

PROOF. This is an immediate consequence of Theorem 4.3 and the fact that the continuous image of a connected set is connected. □

COROLLARY 4.8. *Let Ω be a domain in a Banach space E and $F\colon \Omega \to E$ a completely continuous map. If $f = id - F$ is locally injective, then $f(\Omega)$ is a domain in E.*

PROOF. Since F is a completely continuous map, it is obviously countably k-condensing in the strong sense with respect to α, where k is any nonnegative real number and α is the Kuratowski measure of noncompactness on E (see [2]). Now Theorem 4.7 is applicable. □

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IN-SOOK KIM
Department of Mathematics
Sungkyunkwan University
Suwon 440–746, KOREA

E-mail address: iskim@math.skku.ac.kr