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NON-COLLISION PERIODIC SOLUTIONS OF PRESCRIBED ENERGY PROBLEM FOR A CLASS OF SINGULAR HAMILTONIAN SYSTEMS

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Abstract. We study the existence of non-collision periodic solutions with prescribed energy for the following singular Hamiltonian systems:

$$\begin{cases} \ddot{q} + \nabla V(q) = 0, \\ \dfrac{1}{2}|\dot{q}|^2 + V(q) = H. \end{cases}$$

In particular for the potential $V(q) \sim -1/\mathrm{dist}\,(q, D)^\alpha$, where the singular set D is a non-empty compact subset of $\mathbb{R}^N$, we prove the existence of a non-collision periodic solution for all $H > 0$ and $\alpha \in (0, 2)$.

1. Introduction

In this paper we discuss the existence of non-collision periodic solutions for the following singular Hamiltonian systems with prescribed energy:

$$\text{(HS)} \qquad \begin{cases} \ddot{q} + \nabla V(q), = 0, \\ \dfrac{1}{2}|\dot{q}(t)|^2 + V(q(t)) = H \qquad \text{for all } t \in \mathbb{R}, \end{cases}$$

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where $q = (q_1, \dots, q_N) \in \mathbb{R}^N$, $N \ge 2$, $\dot{} = d/dt$, $H \in \mathbb{R}$, $V(q): \mathbb{R}^N \setminus D \to \mathbb{R}$ is a given potential and $D \subset \mathbb{R}^N$ is a set of singularities of $V(q)$. More precisely, we assume that $D \subset \mathbb{R}^N$ is a non-empty compact subset of $\mathbb{R}^N$ and

(V1) $V(q) \in C^1(\mathbb{R}^N \setminus D, \mathbb{R})$,
(V2) $V(q) < 0$ for all $q \in \mathbb{R}^N \setminus D$, $V(q)$, $\nabla V(q) \to 0$ as $|q| \to \infty$,
(V3) $-V(q) \to \infty$ as $\operatorname{dist}(q, D) \to 0$, where

$$\operatorname{dist}(x, D) = \inf_{y \in D} |x - y|.$$

Recently there exist many papers which deal with singular Hamiltonian systems in view of both prescribed energy problem and prescribed period problem. As to prescribed period problem, we refer to [1]–[3], [6], [9], [11]–[14], [17], [19]. See also a book by Ambrosetti–Coti Zelati [4] and references therein.

A typical example of potential satisfying (V1)–(V3) is

$$V(q) = -\frac{1}{\operatorname{dist}(q, D)^\alpha} \tag{1.1}$$

and the order α of the singularity plays an important role. Here we define *strong force condition* as follows:

(SF) There exists a neighbourhood Ω of D in $\mathbb{R}^N$ and $U \in C^1(\Omega \setminus D, \mathbb{R})$ such that

$$\begin{aligned} U(q) &\to \infty && \text{as } \operatorname{dist}(q, D) \to 0, \\ -V(q) &\ge |\nabla U(q)|^2 && \text{for all } q \in \Omega \setminus D. \end{aligned}$$

Condition (SF) is firstly introduced in Gordon [12] for $D = \{0\}$. We remark that (1.1) satisfies (SF) if and only if $\alpha \ge 2$. In fact, if $\alpha \ge 2$, then we can see that (SF) is satisfied with $U(q) = -\log|\operatorname{dist}(q, D)|$.

In this paper we consider the existence of non-collision periodic solutions of (HS) under weak force case ($\alpha \in (0, 2)$) and the general singular set D. Here we assume

(S) The boundary $S = \partial D$ of D is a compact C^3-manifold of $\mathbb{R}^N$.

Without loss of generality, we assume that $0 \in D$. We also consider the potentials which generalize (1.1). More precisely, we set

$$W(q) = V(q) + \frac{1}{\operatorname{dist}(q, S)^\alpha} \tag{1.2}$$

and assume

(W1) $W(q) \in C^2(\mathbb{R}^N \setminus D, \mathbb{R})$,
(W2) $\operatorname{dist}(q, S)^\alpha W(q)$, $\operatorname{dist}(q, S)^{\alpha+1} \nabla W(q)$, $\operatorname{dist}(q, S)^{\alpha+2} \nabla^2 W(q) \to 0$ as $\operatorname{dist}(q, S) \to 0$.

We remark that for the potential $V(q)$ of the form (1.2) satisfying (W1)–(W2), we can easily verify $V(q)$ satisfies (V1) and (V3).

It is well-known that the order α of the singularity has a close relation to the energy $H \in \mathbb{R}$ in the existence of periodic solutions of prescribed energy problem. Our main result is the following

Theorem 1.1. *Assume* $N \geq 2$, (S), (V2), (W1)–(W2) *and* $\alpha \in (0,2)$. *Then* (HS) *have at least one non-collision periodic solution for all* $H > 0$.

Theorem 1.1 claims that even if $V(q) = -1/\mathrm{dist}\,(q,D)^\alpha$ and $\alpha \in (0,2)$, we can obtain a non-collision periodic solution of (HS) for all $H > 0$. This case presents a great contrast to the case $D = \{0\}$ and $V(q) = -1/|q|^\alpha$. By simple calculation, we can see that for $D = \{0\}$ and $V(q) = -1/|q|^\alpha$, (HS) have a periodic solution if and only if

$$H > 0 \quad \text{for } \alpha > 2, \tag{1.3}$$

$$H = 0 \quad \text{for } \alpha = 2, \tag{1.4}$$

$$H < 0 \quad \text{for } \alpha \in (0,2). \tag{1.5}$$

Thus Theorem 1.1 is distinct from (1.5) with respect to the energy H. Indeed we also obtain the following non-existence result for $\alpha \in (0,2)$ and $H < 0$.

Theorem 1.2. *Assume* $N \geq 2$, $D = \overline{B_\rho(0)} = \{x \in \mathbb{R}^N : |x| \leq \rho\}$, $\alpha \in (0,2)$ *and*

$$V(q) = -\frac{1}{\mathrm{dist}\,(q,S)^\alpha} = -\frac{1}{(|q|-\rho)^\alpha}.$$

Then there exists a negative constant $H_-(\rho) \in (-\infty, 0)$ *such that* (HS) *have no non-constant periodic solutions for all* $H < H_-(\rho)$. *Moreover, we have*

$$H_-(\rho) \to -\infty \quad \textit{as } \rho \to 0.$$

Many authors generalized all cases (1.3)–(1.5) and showed the existence of periodic solutions for general potentials $V(q) \sim -1/|q|^\alpha$. See [15], [16] for the case (1.3), [22] for (1.4) and [8], [10], [18], [20], [21] for (1.5). See also [5] in which both (1.3) and (1.5) are studied. However, most works deal with the potentials which have only one point singular set, say, $D = \{0\}$ and it is natural that $H > 0$ under strong force condition ($\alpha > 2$) and $H < 0$ under weak force condition ($\alpha \in (0,2)$).

In the following sections, we give proofs of Theorems 1.1 and 1.2. We use variational methods to show Theorem 1.1. In Section 2, we introduce the modified potential

$$V_\varepsilon(q) = W(q) - \frac{1}{\mathrm{dist}\,(q,S)^\alpha} - \frac{\varepsilon\varphi(q)}{\mathrm{dist}\,(q,S)^4}$$

for $\varepsilon \in (0,1]$, where $\varphi(q)$ is a function whose support is contained in a small neighborhood of S and $\varphi(q) = 1$ near S. Then we set the following modified functional

$$I_\varepsilon(q) = \frac{1}{2}\int_0^1 |\dot{q}|^2\,dt \int_0^1 H - V_\varepsilon(q)\,dt.$$

Main purpose of Section 2 is to show the modified functional satisfies the Palais–Smale compactness condition and obtain the global existence of a deformation flow. In Section 3, we find a critical point $u_\varepsilon(t)$ through minimax methods for $N \geq 3$ and minimizing method for $N = 2$ due to Bahri–Rabinowitz [6]. We also obtain uniform bounds for critical values $I_\varepsilon(u_\varepsilon)$. In particular, we can obtain a positive lower bound for $I_\varepsilon(u_\varepsilon)$ by studying the orbits round singular set D precisely. A positive lower bound plays an important role in the proof of Theorem 1.1. In Section 4, we take a limit as $\varepsilon \to 0$ and show the existence of at least one non-collision periodic solution of (HS) for all $H > 0$. In the limit process we use re-scaling argument with respect to scale-change $q(\,\cdot\,) \to \delta^{-1}q(\delta^{(\alpha+2)/2}\,\cdot\,)$. See [1] and [19]. Lastly in Section 5, we prove Theorem 1.2.

2. Preliminaries

In this section we define modified functional $I_\varepsilon(u)$ and show some properties for $I_\varepsilon(u)$.

1.1. Functional setting. Firstly we recall some basic properties of distance function $\operatorname{dist}(x,S)$. Then we introduce the modified functional $I_\varepsilon(u)$.

For $z \in S$, we denote by $n(z)$ the unit outward normal vector of the surface S at z. We consider a map $\Phi: S \times [0,\infty) \to \mathbb{R}^N$ defined by

$$\Phi(z,s) = z + sn(z).$$

By the implicit function theorem, we have

Lemma 2.1. *Assume* (S). *Then there exists a constant $h_0 > 0$ such that*

$$\Phi|_{S\times[0,h_0)}: S \times [0,h_0) \to N_{h_0}(S)$$

is a diffeomorphism, where

$$N_{h_0}(S) = \{x \in \mathbb{R}^N \setminus D : \operatorname{dist}(x,S) < h_0\}.$$

Moreover, writing $(z(x), s(x)) = \Phi^{-1}(x)$, we have for $x \in N_{h_0}(S)$

$$\operatorname{dist}(x,S) = s(x), \quad \nabla\operatorname{dist}(x,S) = n(z(x)).$$

Let $\varphi \in C^\infty([0,\infty),\mathbb{R})$ satisfy $\varphi'(r) \leq 0$ for all $r \in [0,\infty)$ and

$$\varphi(r) = \begin{cases} 1 & \text{for } r \in [0,h_0/3], \\ 0 & \text{for } r \in [2h_0/3,\infty). \end{cases}$$

For a potential $V(q)$ satisfying (V1)–(V3), we define a modified potential $V_\varepsilon(q)$ by

$$V_\varepsilon(q) = V(q) - \varepsilon \frac{\varphi(\mathrm{dist}\,(q,S))}{\mathrm{dist}\,(q,S)^4} \quad \text{for } \varepsilon \in (0,1] \text{ and } q \in \mathbb{R}^N \setminus D.$$

Then we can easily see that $V_\varepsilon(q)$ satisfies (SF) for all $\varepsilon \in (0,1]$.

Next we use the following notation:

$$\begin{aligned} E &= \{u \in H^1(0,1;\mathbb{R}^N) : u(0) = u(1)\}, \\ \|u\|_E^2 &= \int_0^1 |\dot u(t)|^2\,dt + |[u]|^2, \quad \text{where } [u] = \int_0^1 u(t)\,dt, \\ \langle u, v\rangle &= \int_0^1 \dot u \dot v\,dt + [u][v], \\ \Lambda &= \{u \in E : u(t) \notin D \text{ for all } t \in [0,1]\}, \\ \partial\Lambda &= \{u \in E : u(t) \in S \text{ for some } t \in [0,1]\}. \end{aligned}$$

We also use the notation

$$\|u\|_p = \bigg(\int_0^1 |u|^p\,dt\bigg)^{1/p}$$

for $p \in [1,\infty)$. We define the following modified functional on Λ:

$$\begin{aligned} I_\varepsilon(u) &= \frac{1}{2}\int_0^1 |\dot u|^2\,dt \int_0^1 H - V_\varepsilon(u)\,dt \\ &= \frac{1}{2}\|\dot u\|_2^2 \int_0^1 H - V(u) + \frac{\varepsilon\varphi(\mathrm{dist}\,(u,S))}{\mathrm{dist}\,(u,S)^4}\,dt. \end{aligned} \tag{2.1}$$

We remark that Λ is open in E and $I_\varepsilon(u) \in C^2(\Lambda,\mathbb{R})$. If $u \in \Lambda$ is a critical point of $I_\varepsilon(u)$ with $I_\varepsilon(u) > 0$, then we have $\|\dot u\|_2^2 > 0$, that is, $u \not\equiv \mathrm{const}$. Moreover, setting

$$T = \bigg(\frac{(1/2)\int_0^1 |\dot u|^2\,dt}{\int_0^1 H - V_\varepsilon(u)\,dt}\bigg)^{1/2} > 0, \tag{2.2}$$

$$q(t) = u\Big(\frac{t}{T}\Big), \tag{2.3}$$

we see that $q(t)$ is a non-collision T-periodic solution of

$$\begin{cases} \ddot q + \nabla V_\varepsilon(q) = 0, \\ \dfrac{1}{2}|\dot q(t)|^2 + V_\varepsilon(q(t)) = H \quad \text{for all } t \in \mathbb{R}. \end{cases}$$

Thus in what follows, we study the existence of critical points of $I_\varepsilon(u)$ with positive functional levels and then pass to the limit as $\varepsilon \to 0$.

2.2. Palais–Smale condition for the modified functional. Firstly we remark that since $V_\varepsilon(u)$ satisfies (SF), the following lemma holds.

LEMMA 2.2. *Let $(u_j) \subset \Lambda$ be the sequence satisfying $u_j \rightharpoonup u_0 \in \partial\Lambda$ weakly in E. Then*

$$\int_0^1 -V_\varepsilon(u_j)\,dt \to \infty \quad \text{as } j \to \infty.$$

More precisely, we have

$$G(u_j) := \int_0^1 \frac{\varphi(\operatorname{dist}(u_j, S))}{\operatorname{dist}(u_j, S)^4}\,dt \to \infty \quad \text{as } j \to \infty.$$

We set $\mathcal{N} = \{u \in \Lambda : u(t) \in N_{h_0}(S) \text{ for all } t \in [0,1]\}$ and for $u \in \mathcal{N}$, we define

$$X(u) = n(z(u(1))) \in \mathbb{R}^N, \tag{2.4}$$

where we use the notation $u(t) = z(u(t)) + \operatorname{dist}(u(t), S)n(z(u(t)))$ for $u \in \mathcal{N}$ as in Lemma 2.1. Since $X(u)$ is a constant vector in $\mathbb{R}^N$, we identify $X(u)$ with the element of E. It is clear that $\|X(u)\|_E = 1$ for all $u \in \mathcal{N}$. We also define for $u \in \Lambda$,

$$d(u) = \inf_{\xi \in S} \|u - \xi\|_E.$$

We remark that if $d(u)$ small enough, then $u \in \mathcal{N}$. That is, there exists a constant $h_* > 0$ such that if $d(u) \le h_*$, then $u \in \mathcal{N}$. It is easily seen that $d: \Lambda \to \mathbb{R}$ is a locally Lipschitz continuous function.

LEMMA 2.3. *Suppose $(u_j) \subset \Lambda$ satisfies*

$$I_\varepsilon(u_j) \le M \quad \text{for some } M > 0. \tag{2.5}$$

Then

$$u_j \rightharpoonup u_0 \quad \text{for some } u_0 \in \partial\Lambda \text{ as } j \to \infty \tag{2.6}$$

if and only if

$$d(u_j) \to 0. \tag{2.7}$$

PROOF. The sufficiency is obvious. We prove only the necessity. We assume $(u_j) \subset \Lambda$ satisfies (2.5) and (2.6). Then it follows from (2.6) and Lemma 2.2 that

$$\int_0^1 H - V_\varepsilon(u_j)\,dt \to \infty \quad \text{as } j \to \infty.$$

Together with (2.5), we have $\|\dot{u}_j\|_2^2 \to 0$ as $j \to \infty$. Using (2.6) again, we can see that $\dot{u}_0 \equiv 0$, that is, $u_0 \equiv \xi$ for some $\xi \in S$ and $\|u_j - \xi\|_E \to 0$ as $j \to \infty$. Thus (2.7) holds. □

In what follows, we always assume $H > 0$ and identify E and E^* by the Reisz representation theorem. We prove the following

LEMMA 2.4. *For $\varepsilon \in (0,1]$ and $M > m > 0$, there exists a constant $h_1 = h_1(m,M) \in (0, \min\{h_0/3, h_*\})$ such that if $u \in \Lambda$ satisfies*

$$I_\varepsilon(u) \in [m, M], \tag{2.8}$$

$$d(u) \le h_1, \tag{2.9}$$

then we have

$$\langle I'_\varepsilon(u), X(u)\rangle \le -m, \tag{2.10}$$

$$\langle G'(u), X(u)\rangle \le 0. \tag{2.11}$$

PROOF. We can find a constant $h_1 \in (0, \min\{h_0/3, h_*\})$ such that (2.9) implies

$$\langle I'_\varepsilon(u), X(u)\rangle = \frac{1}{2}\|\dot u\|_2^2 \int_0^1 -\nabla V(u)X(u) - \frac{4\varepsilon\nabla \mathrm{dist}\,(u,S)X(u)}{\mathrm{dist}\,(u,S)^5}\,dt$$

and

$$\frac{1}{2} \le \nabla \mathrm{dist}\,(u,S)X(u) \le 1 \quad \text{for all } t \in [0,1].$$

Thus we have for $u \in \Lambda$ satisfying (2.9),

$$\langle I'_\varepsilon(u), X(u)\rangle \le \frac{1}{2}\|\dot u\|_2^2 \int_0^1 -\nabla V(u)X(u) - \frac{2\varepsilon}{\mathrm{dist}\,(u,S)^5}\,dt. \tag{2.12}$$

Moreover, choosing $h_1 > 0$ smaller if necessary, by (W1)–(W2), we obtain the following pointwise estimates:

$$-\nabla V(x)X(\xi) - \frac{2\varepsilon}{\mathrm{dist}\,(x,S)^5} \le -\frac{\varepsilon}{\mathrm{dist}\,(x,S)^5},$$

$$H - V(x) + \frac{\varepsilon}{\mathrm{dist}\,(x,S)^4} \le \frac{\varepsilon}{\mathrm{dist}\,(x,S)^5} \tag{2.14}$$

for all $x \in \mathbb{R}^N$ with $d(x) = \mathrm{dist}\,(x,S) \le h_1$ and $\xi \in S$. By (2.12) and (2.13), we have

$$\langle I'_\varepsilon(u), X(u)\rangle \le -\frac{1}{2}\|\dot u\|_2^2 \int_0^1 \frac{\varepsilon}{\mathrm{dist}\,(u,S)^5}\,dt \tag{2.15}$$

for all $u \in \Lambda$ satisfying (2.9). On the other hand, by (2.8) and (2.14), we have

$$\begin{aligned} m \le I_\varepsilon(u) &= \frac{1}{2}\|\dot u\|_2^2 \int_0^1 H - V(u) + \frac{\varepsilon}{\mathrm{dist}\,(u,S)^4}\,dt \\ &\le \frac{1}{2}\|\dot u\|_2^2 \int_0^1 \frac{\varepsilon}{\mathrm{dist}\,(u,S)^5}\,dt \end{aligned} \tag{2.16}$$

for all $u \in \Lambda$ satisfying (2.8) and (2.9). Thus we obtain (2.10) from (2.15) and (2.16). For $u \in \Lambda$ satisfying (2.9), we can easily obtain

$$\langle G'(u), X(u)\rangle = -\int_0^1 \frac{4\nabla \mathrm{dist}\,(u,S)X(u)}{\mathrm{dist}\,(u,S)^5}\,dt \le -\int_0^1 \frac{2}{\mathrm{dist}\,(u,S)^5}\,dt \le 0.$$

This completes the proof of Lemma 2.4. □

REMARK 2.5. Lemma 2.4, espcially (2.11) plays an important role in showing the global existence of a deformation flow. More precisely, near the singular set D, we define a deformation flow as a solution of $d/ds\eta = X(\eta)$. Since $X(u)$ is an unit outward normal vector of S, our deformation flow can not approach to D. See Lemma 2.8 for details. We also use Lemma 2.4 to show that $I_\varepsilon(u)$ satisfies the Palais–Smale condition. See below.

Now we prove the following Palais–Smale condition for $I_\varepsilon(u)$.

PROPOSITION 2.6. *Suppose that* $(u_j) \subset \Lambda$ *satisfies the following conditions:*

$$I_\varepsilon(u_j) \in [m, M] \quad \textit{for some } M > m > 0, \tag{2.17}$$

$$\|I'_\varepsilon(u_j)\|_{E^*} \to 0 \quad \textit{as } j \to \infty. \tag{2.18}$$

Then there exist a subsequence $(u_{j_k}) \subset \Lambda$ *and some* $u_0 \in \Lambda$ *such that*

$$u_{j_k} \to u_0 \quad \textit{strongly in } E.$$

PROOF. We devide the proof of Proposition 2.6 into several steps.

Step 1. Boundedness of (u_j).

Since $V_\varepsilon(u) < 0$, we have

$$I_\varepsilon(u) = \frac{1}{2}\|\dot{u}\|_2^2 \int_0^1 H - V_\varepsilon(u)\,dt \geq \frac{H}{2}\|\dot{u}\|_2^2.$$

Thus it follows from (2.17) that

$$\|\dot{u}_j\|_2^2 \leq \frac{2M}{H} =: C_1. \tag{2.19}$$

Next we show that there exists a constant $C_2 > 0$ such that

$$|[u_j]| \leq C_2. \tag{2.20}$$

Arguing indirectly, we assume that $|[u_j]| \to \infty$ as $j \to \infty$. Since

$$|[u_j]| \leq |u_j - [u_j]| + |u_j| \quad \text{for all } t \in [0,1]$$

and (2.19), we obtain

$$\inf_t |u_j(t)| \geq |[u_j]| - \max_t |u_j - [u_j]| \geq |[u_j]| - \|\dot{u}_j\|_2 \geq |[u_j]| - C_1^{1/2} \to \infty.$$

Hence

$$|u_j(t)| \to \infty \quad \text{as } j \to \infty. \tag{2.21}$$

Moreover, by (2.19) again, we have

$$\|u_j - [u_j]\|_E \leq \|\dot{u}_j\|_2 \leq C_1^{1/2}.$$

Thus we have from (2.18) that

$$\begin{aligned} o(1) &= I'_\varepsilon(u_j)(u_j - [u_j]) \\ &= \|\dot u_j\|_2^2 \int_0^1 H - V_\varepsilon(u_j)\,dt + \frac{1}{2}\|\dot u_j\|_2^2 \int_0^1 -\nabla V_\varepsilon(u_j)(u_j - [u_j])\,dt \\ &= 2I_\varepsilon(u_j) - \frac{1}{2}\|\dot u_j\|_2^2 \int_0^1 \nabla V_\varepsilon(u_j)(u_j - [u_j])\,dt. \end{aligned}$$

By (2.21) and (V2), we obtain $\nabla V_\varepsilon(u_j) \to 0$ as $j \to \infty$. Consequently we have $I_\varepsilon(u_j) \to 0$ and this contradicts (2.17). From (2.19) and (2.20), we see that (u_j) is bounded in E. As a consequence of Step 1, we can extract a subsequence — still denoted by (u_j) — such that

$$u_j \rightharpoonup u_0 \in E \quad \text{weakly in } E \text{ and strongly in } L^\infty. \tag{2.22}$$

Step 2. $u_0 \in \Lambda$.

Arguing indirectly, we assume that $u_0 \in \partial\Lambda$. From (2.17), (2.22) and Lemma 2.3, we have $d(u_j) \to 0$ as $j \to \infty$. Hence there exists a $j_0 \in \mathbb{N}$ such that $d(u_j) \le h_1$ for all $j \ge j_0$, where $h_1 > 0$ is a constant given in Lemma 2.4. By Lemma 2.4, we obtain

$$\langle I'_\varepsilon(u_j), X(u_j)\rangle \le -m \tag{2.23}$$

for all $j \ge j_0$. Since $\|X(u_j)\|_E = 1$ for $j \ge j_0$, (2.23) means $\|I'_\varepsilon(u_j)\|_{E^*} \ge m$ for all $j \ge j_0$ and this contradicts (2.18). Thus we have $u_0 \in \Lambda$.

Step 3. $u_j \to u_0$ strongly in E.

Since $I_\varepsilon(u_j) \ge m$, we have

$$\begin{aligned} \lim_{j\to\infty} \frac{1}{2}\|\dot u_j\|_2^2 \int_0^1 H - V_\varepsilon(u_0)\,dt &= \lim_{j\to\infty} \frac{1}{2}\|\dot u_j\|_2^2 \int_0^1 H - V_\varepsilon(u_j)\,dt \\ &= \lim_{j\to\infty} I_\varepsilon(u_j) \ge m > 0. \end{aligned}$$

Combined with $\|\dot u_j\|_2^2 \le C_1$, we obtain

$$\int_0^1 H - V_\varepsilon(u_0)\,dt > 0. \tag{2.24}$$

It follows from (2.18) that $I'_\varepsilon(u_j)u_0 \to 0$, that is,

$$\int_0^1 \dot u_j \dot u_0\,dt \int_0^1 H - V_\varepsilon(u_j)\,dt + \frac{1}{2}\|\dot u_j\|_2^2 \int_0^1 -\nabla V_\varepsilon(u_j)u_0\,dt \to 0.$$

Passing to the limit, we have

$$0 = \|\dot u_0\|_2^2 \int_0^1 H - V_\varepsilon(u_0)\,dt + \frac{1}{2}\lim_{j\to\infty} \|\dot u_j\|_2^2 \int_0^1 -\nabla V_\varepsilon(u_0)u_0\,dt. \tag{2.25}$$

Similarly it follows from $I'_\varepsilon(u_j)u_j \to 0$ that

$$0 = \lim_{j\to\infty} \|\dot u_j\|_2^2 \int_0^1 H - V_\varepsilon(u_0)\,dt + \frac{1}{2}\lim_{j\to\infty} \|\dot u_j\|_2^2 \int_0^1 -\nabla V_\varepsilon(u_0)u_0\,dt. \tag{2.26}$$

By (2.24)–(2.26), we have $\lim_{j\to\infty}\|\dot u_j\|_2^2 = \|\dot u_0\|_2^2$. Thus we obtain $u_j \to u_0$ strongly in E as $j\to\infty$. □

2.3. A deformation flow. Next we construct a deformation flow and prove the following proposition, which is so called Deformation Lemma.

PROPOSITION 2.7. *For $\varepsilon \in (0,1]$, we assume that $b > 0$ is not a critical value of $I_\varepsilon(u)$. Then for any $\overline{\delta} > 0$, there exists a constant $\delta \in (0,\overline{\delta})$ and $\eta(s,u) \in C([0,1]\times\Lambda,\Lambda)$ such that:*

(a) *$\eta(0,u) = u$ for all $u\in\Lambda$.*
(b) *$\eta(s,u) = u$ for all $s\in[0,1]$ if $I_\varepsilon(u) \notin [b-\overline{\delta}, b+\overline{\delta}]$.*
(c) *$\|\eta(s,u)-u\|_E \le 1$ for all $s\in[0,1]$ and $u\in\Lambda$.*
(d) *$I_\varepsilon(\eta(s,u)) \le I_\varepsilon(u)$ for all $s\in[0,1]$ and $u\in\Lambda$.*
(e) *If $I_\varepsilon(u) \le b+\delta$, then $I_\varepsilon(\eta(1,u)) \le b-\delta$.*

In the proof of Deformation Lemma, usually we can obtain a deformation flow $\eta(s,u)$ as a unique global solution of the negative gradient flow for $I_\varepsilon(u)$. However, in our case, it is not obvious that a deformation flow exists globally. That is, we need to show that $\eta(s,u)$ never enter the set $\partial\Lambda$. To prevent $\eta(s,u)$ from entering $\partial\Lambda$, we construct $\eta(s,u)$ in a different way from usual one. Near the singular set, we define $\eta(s,u)$ by using the unit outward normal vector of S instead of the negative gradient flow for $I_\varepsilon(u)$. Our construction is originated in Tanaka [21]. In [21], the construction of a deformation flow was studied in the case where the singular set D consists of finitely many points, say, $D = \{y_1,\dots,y_d\}$.

Suppose $b\in(m,M)$ is not a critical value of $I_\varepsilon(u)$. Let $\overline{\delta} > 0$ be a given number in Proposition 2.7. Since $I_\varepsilon(u)$ satisfies the Palais–Smale condition in the interval $[m,M]$, we see that there exist constants $\delta_1 \in (0,\overline{\delta}/3)$ and $a_0 > 0$ such that

$$\|I'_\varepsilon(u)\|_{E^*} \ge a_0 > 0 \quad \text{for all } u\in\Lambda \text{ with } I_\varepsilon(u)\in[b-2\delta_1, b+2\delta_1]. \tag{2.27}$$

We may assume without loss of generality that $[b-2\delta_1, b+2\delta_1] \subset [m,M]$. We introduce the following "cut-off" functions. $\chi(r)$, $\omega(r) \in C^\infty(\mathbb{R},[0,1])$ satisfy the following respectively:

$$\chi(r) = \begin{cases} 1 & \text{for } r\in(-\infty, h_1/2], \\ 0 & \text{for } r\in[h_1,\infty), \end{cases}$$

$$\omega(r) = \begin{cases} 1 & \text{for } r\in[b-\delta_1, b+\delta_1], \\ 0 & \text{for } r\notin[b-2\delta_1, b+2\delta_1]. \end{cases}$$

Then we set

$$Y(u) = \omega(I_\varepsilon(u))\bigg\{\chi(d(u))X(u) - (1-\chi(d(u)))\frac{I'_\varepsilon(u)}{\|I'_\varepsilon(u)\|_{E^*}}\bigg\},$$

where $X(u)$ is defined by (2.4). We remark that $Y:\Lambda\to E$ is a locally Lipschitz continuous function and

$$\|Y(u)\|_E \le 1 \quad \text{for all } u\in\Lambda. \tag{2.28}$$

We consider the following ordinary differential equation:

$$\frac{d}{ds}\eta = Y(\eta), \tag{2.29}$$

$$\eta(0,u) = u. \tag{2.30}$$

From Lemma 2.4, we have the following

LEMMA 2.8. *For any initial data* $u\in\Lambda$, (2.29)–(2.30) *have a unique solution* $\eta(s,u)$ *and*

$$\eta(s,u)\in C([0,\infty)\times\Lambda,\Lambda).$$

PROOF. By the definition of $Y(u)$, we can easily see that there exists a unique local solution $\eta(s,u)$ of (2.29)–(2.30) for all $u\in\Lambda$. We argue indirectly and assume that $\eta(s)=\eta(s,u_0)$ does not exist globally for some initial data $u_0\in\Lambda$ and we denote its maximal existence time by $[0,T)$. By (2.29) and (2.28), we see

$$\bigg\|\frac{d}{ds}\eta(s)\bigg\|_E \le 1 \quad \text{for all } s\in[0,T).$$

Thus we have

$$\|\eta(s)-\eta(t)\|_E \le |s-t| \quad \text{for all } s,\ t\in[0,T).$$

Let (s_j) be the sequence satisfying $s_j\nearrow T$. Since $\eta(s_j)$ is a Cauchy sequence, there exists $\eta_0\in E$ such that

$$\eta\to\eta_0 \quad \text{strongly in } E \text{ as } s\nearrow T. \tag{2.31}$$

Moreover, since T is the maximal existence time of $\eta(s)$, we see

$$\eta_0\in\partial\Lambda, \ \text{that is, } \eta_0(s_0)\in S \text{ for some } s_0\in[0,1]. \tag{2.32}$$

From (2.31), (2.32) and Lemma 2.2, we obtain

$$G(\eta(s))\to\infty \quad \text{as } s\nearrow T. \tag{2.33}$$

On the other hand, from Lemma 2.4 and (2.27), we see

$$(2.34)\quad \begin{aligned}\frac{d}{ds}I_\varepsilon(\eta(s)) &= \left\langle I'_\varepsilon(\eta(s)), \frac{d}{ds}\eta(s)\right\rangle = \langle I'_\varepsilon(\eta(s)), Y(\eta)\rangle \\ &= \omega(I_\varepsilon(\eta))\{\chi(d(\eta))\langle I'_\varepsilon(\eta), X(\eta)\rangle - (1-\chi(d(\eta)))\|I'_\varepsilon(\eta)\|_{E^*}\} \\ &\le -\omega(I_\varepsilon(\eta))(\chi(d(\eta))m + (1-\chi(d(\eta)))a_0) \le 0,\end{aligned}$$

that is, we have

$$I_\varepsilon(\eta(s)) \le I_\varepsilon(\eta(0)) = I_\varepsilon(u_0).$$

Hence it follows from (2.31), (2.32) and Lemma 2.3 that $d(\eta(s)) \to 0$ as $s \nearrow T$. Thus there exists a $T_0 \in (0,T)$ such that $d(\eta(s)) \le h_1$ for all $s \in [T_0, T)$. By the definition of $Y(u)$, (2.29) and Lemma 2.4, we see

$$\begin{aligned}\frac{d}{ds}G(\eta(s)) &= \left\langle G'(\eta(s)), \frac{d}{ds}\eta(s)\right\rangle \\ &= \langle G'(\eta(s)), Y(\eta(s))\rangle \le \langle G'(\eta(s)), X(\eta(s))\rangle \le 0\end{aligned}$$

for all $s \in [T_0, T)$. This is not compatible with (2.33). Therefore the unique solution $\eta(s,u)$ of (2.29)–(2.30) satisfies $\eta(s,u) \in C([0,\infty)\times\Lambda, \Lambda)$ for any initial data $u \in \Lambda$. □

Proof of Proposition 2.7. (a) follows from (2.30). By the definition of $\omega(r)$, we have

$$Y(u) = 0 \quad \text{if } I_\varepsilon(u) \notin [b-\overline{\delta}, b+\overline{\delta}].$$

Thus we obtain (b). Integrating (2.29) from 0 to 1 and using (2.28), we obtain (c). By (2.34), we see that $\eta(s,u)$ satisfies (d). Finally, if $I_\varepsilon(u) \in [b-\delta_1, b+\delta_1]$, then by (2.34) again, we have

$$\frac{d}{ds}I_\varepsilon(\eta(s,u)) \le -\min\{m, a_0\} =: -a_1.$$

Thus setting $\delta = \min\{\delta_1, a_1/2\}$, we obtain (e). □

3. Minimax methods for the modified functional

This section is devoted to showing the existence of a critical point of $I_\varepsilon(u)$. We use minimax methods for $N \ge 3$ and minimizing method for $N = 2$.

3.1. Definition of minimax values of $I_\varepsilon(u)$. In this subsection we set minimax values of the modified functional defined in (2.1). When $N \ge 3$, we set minimax values b_ε as follows. Identifying $[0,1]/\{0,1\} \simeq S^1$, we can associate each $\gamma \in C(S^{N-2}, \Lambda)$ with a mapping $\widetilde{\gamma}\colon S^{N-2}\times S^1 \to S^{N-1}$ by

$$\widetilde{\gamma}(x,t) = \frac{\gamma(x)(t)}{|\gamma(x)(t)|} \quad \text{for } x \in S^{N-2},\ t \in S^1 \simeq [0,1]/\{0,1\}.$$

Since $0 \in D$ and $\gamma(x)(t) \neq 0$ for all $x \in S^{N-2}$ and $t \in [0,1]$, $\widetilde{\gamma}(x,t)$ is well-defined. We denote the Brouwer degree of $\widetilde{\gamma}$ by $\deg\widetilde{\gamma}$ and define

$$\widetilde{\Gamma} = \{\gamma \in C(S^{N-2}, \Lambda) : \deg\widetilde{\gamma} \neq 0\}.$$

We can see $\widetilde{\Gamma} \neq \emptyset$ as in [6]. Then we set

$$b_\varepsilon = \inf_{\gamma \in \widetilde{\Gamma}} \max_{x \in S^{N-2}} I_\varepsilon(\gamma(x)), \qquad b_0 = \inf_{\gamma \in \widetilde{\Gamma}} \max_{x \in S^{N-2}} I(\gamma(x)),$$

where we define

$$I(u) = \frac{1}{2}\|\dot{u}\|_2^2 \int_0^1 H - V(u)\, dt.$$

When $N = 2$, we adopt the minimizing method. We associate each $u \in \Lambda$ a winding number $\operatorname{wind} u$ of $u(t)$ concerning $0 \in D$. Then we define

$$\widetilde{\Gamma} = \{u \in \Lambda : \operatorname{wind} u = 1\}$$

and set

$$b_\varepsilon = \inf_{\gamma \in \widetilde{\Gamma}} I_\varepsilon(u), \qquad b_0 = \inf_{\gamma \in \widetilde{\Gamma}} I(u).$$

Since $0 \leq I(u) \leq I_\varepsilon(u) \leq I_1(u)$ for all $u \in \Lambda$ and $\varepsilon \in (0,1]$, we have for $N \geq 2$,

$$0 \leq b_0 \leq b_\varepsilon \leq b_1 \quad \text{for } \varepsilon \in (0,1]. \tag{3.1}$$

3.2. Uniform bounds for b_ε and their consequences. Next we obtain uniform bounds for b_ε. In particular a positive lower bound for b_ε plays an important role.

Proposition 3.1. *There exist constants M, $m > 0$ independent of $\varepsilon \in (0,1]$ such that $0 < m \leq b_\varepsilon \leq M$.*

Existence of an uniform upper bound for b_ε follows from (3.1). To prove b_ε is bounded below away from 0, by (3.1), it suffices to show that $b_0 > 0$. We remark that we can not obtain $b_0 > 0$ if $D = \{0\}$. See Remark 3.4 below. We prove Proposition 3.1 for $N = 2$ and $N \geq 3$, respectively. Firstly we give a proof of Proposition 3.1 for $N = 2$.

Proof of Proposition 3.1 for $N = 2$. We choose a $\rho_0 > 0$ small enough so that $\overline{B_{\rho_0}(0)} \subset \operatorname{int} D$ and fix it. Then for all $u \in \Lambda$, we see that $\|\dot{u}\|_1 \geq 2\rho_0\pi$. Thus we have

$$I(u) = \frac{1}{2}\|\dot{u}\|_2^2 \int_0^1 H - V(u)\, dt \geq \frac{H}{2}\|\dot{u}\|_2^2 \geq \frac{H}{2}\|\dot{u}\|_1^2 = 2H\rho_0^2\pi^2 > 0$$

for all $u \in \widetilde{\Gamma}$. By the definition of b_0, we obtain $b_0 \geq 2H\rho_0^2\pi^2 > 0$. Therefore we have a desired lower bound. □

When $N \geq 3$, to show $b_0 > 0$, we need several lemmas. We set for $N \geq 3$,

$$A = \{u \in \Lambda : |[u]| \leq \|\dot{u}\|_2\}. \tag{3.2}$$

Then we have the following

LEMMA 3.2. *Assume $N \geq 3$. Then*

$$\gamma(S^{N-2}) \cap A \neq \emptyset \quad \text{for all } \gamma \in \widetilde{\Gamma}. \tag{3.3}$$

PROOF. We use the following notation:

$$\Lambda_0 = \{u \in E : u(t) \neq 0 \text{ for all } t \in [0,1]\},$$
$$\widetilde{\Gamma}_0 = \{\gamma \in C(S^{N-2}, \Lambda_0) : \deg\widetilde{\gamma} \neq 0\}.$$

We remark that $\Lambda \subset \Lambda_0$ and $\widetilde{\Gamma} \subset \widetilde{\Gamma}_0$. Thus it suffices to show (3.3) for all $\gamma \in \widetilde{\Gamma}_0$. We prove indirectly and assume that $\gamma(S^{N-2}) \cap A = \emptyset$ for all $\gamma \in \widetilde{\Gamma}_0$. Since $\gamma(x) \notin A$ for all $x \in S^{N-2}$, we have $\|\dot{\gamma}(x)\|_2 < |[\gamma(x)]|$. Thus we obtain

$$\max_{t\in[0,1]} |\gamma(x)(t) - [\gamma(x)]| \leq \|\dot{\gamma}(x)\|_2 < |[\gamma(x)]|.$$

That is, we see that

$$\gamma(x) \subset B_{|[\gamma(x)]|}([\gamma(x)]). \tag{3.4}$$

Next we set

$$\gamma_s(x) = s[\gamma(x)] + (1-s)\gamma(x)(t).$$

By (3.4), we see that $\gamma_s(x) \in C([0,1] \times S^{N-2}, \Lambda_0)$. Moreover, since $\gamma_0(x) = \gamma(x) \in \widetilde{\Gamma}_0$, it follows from the homotopy invariance of Brouwer degree that $\gamma_1(x) \in \widetilde{\Gamma}_0$. Thus $\gamma_1(x)\colon S^{N-2} \times S^1 \to S^{N-1}$ is an onto mapping. On the other hand, $\gamma_1(x) = [\gamma(x)]$ is independent of t. Consequently $\gamma_1\colon S^{N-2} \to S^{N-1}$ is onto. This is a contradiction. □

LEMMA 3.3. *There exists a constant $m > 0$ such that*

$$\inf_{u\in A} I(u) \geq m > 0.$$

PROOF. We choose a $\rho_0 > 0$ small enough so that $\overline{B_{\rho_0}(0)} \subset \operatorname{int} D$ and fix it. If $[u] \in \overline{B_{\rho_0/2}(0)}$, then we have $\operatorname{dist}([u], S) \geq \rho_0/2$. Taking into account of $u \in \Lambda$, that is, u goes around of D, we see that $\|\dot{u}\|_1 \geq \rho_0/2$. Thus we have

$$\|\dot{u}\|_2 \geq \frac{\rho_0}{2} \quad \text{for all } u \in \Lambda \text{ with } [u] \in \overline{B_{\rho_0/2}(0)}.$$

On the other hand, if $[u] \notin \overline{B_{\rho_0/2}(0)}$, then we have

$$\|\dot{u}\|_2 \geq |[u]| \geq \frac{\rho_0}{2} \quad \text{for all } u \in A \text{ with } [u] \notin \overline{B_{\rho_0/2}(0)}.$$

Hence we obtain

$$\|\dot{u}\|_2 \geq \frac{\rho_0}{2} > 0 \quad \text{for all } u \in A.$$

Therefore

$$I(u) \geq \frac{H}{2}\|\dot{u}\|_2^2 \geq \frac{H}{8}\rho_0^2 > 0 \quad \text{for all } u \in A$$

and this completes the proof of Lemma 3.3. □

PROOF OF PROPOSITION 3.1 FOR $N \geq 3$. From Lemmas 3.2 and 3.3, we have

$$\max_{x \in S^{N-2}} I(\gamma(x)) \geq \inf_{u \in A} I(u) \geq m > 0 \quad \text{for all } \gamma \in \widetilde{\Gamma}.$$

Thus

$$b_0 = \inf_{\gamma \in \widetilde{\Gamma}} \max_{x \in S^{N-2}} I(\gamma(x)) \geq m > 0.$$

By (3.1), we have a desired lower bound. □

REMARK 3.4. $b_0 > 0$ is a key of our proof. In general, we can not obtain $b_0 > 0$ if $D = \{0\}$. For example, if $D = \{0\}$ and $V(u) = -1/|u|^\alpha$, then we have $b_0 = 0$. Indeed for $N \geq 3$ and $\gamma(x) \in \widetilde{\Gamma}_0$, we see that $\ell\gamma(x) \in \widetilde{\Gamma}_0$ for all $\ell > 0$. Moreover, we have

$$\begin{aligned} I(\ell\gamma(x)) &= \frac{1}{2}\|\ell\dot{\gamma}(x)\|_2^2 \int_0^1 H + \frac{1}{|\ell\gamma(x)|^\alpha}\, dt \\ &= \frac{H}{2}\ell^2\|\dot{\gamma}(x)\|_2^2 + \frac{1}{2}\ell^{2-\alpha}\|\dot{\gamma}(x)\|_2^2 \int_0^1 \frac{1}{|\gamma(x)|^\alpha}\, dt. \end{aligned}$$

Thus we obtain

$$\max_{x \in S^{N-2}} I(\ell\gamma(x)) \to 0 \quad \text{as } \ell \to 0.$$

Therefore $b_0 = 0$. When $N = 2$, we also obtain $b_0 = 0$ in the same way as $N \geq 3$.

From Propositions 2.6, 2.7 and 3.1, we see that each $b_\varepsilon > 0$ is a critical value of $I_\varepsilon(u)$ and we obtain the following

PROPOSITION 3.5. *For $\varepsilon \in (0,1]$, there is a critical point $u_\varepsilon(t) \in \Lambda$ of $I_\varepsilon(u)$ such that*

$$I_\varepsilon(u_\varepsilon) = b_\varepsilon, \qquad I'_\varepsilon(u_\varepsilon) = 0.$$

Moreover, there exist constants m, M, $C > 0$ independent of $\varepsilon \in (0,1]$ such that, for $\varepsilon \in (0,1]$,

$$\begin{gathered} m \leq I_\varepsilon(u_\varepsilon) \leq M, \qquad \|u_\varepsilon\|_E \leq C, \\ \frac{1}{2}|\dot{u}_\varepsilon(t)|^2 + T_\varepsilon^2 V_\varepsilon(u_\varepsilon(t)) = T_\varepsilon^2 H \quad \text{for all } t \in \mathbb{R}, \end{gathered}$$

where

$$T_\varepsilon = \left(\frac{\|\dot{u}_\varepsilon\|_2^2/2}{\int_0^1 H - V_\varepsilon(u_\varepsilon)\, dt}\right)^{1/2}.$$

PROOF. One can easily obtain $\|u_\varepsilon\|_E \leq C$ by repeating Step 1 of Proposition 2.6 with u_j replaced by u_ε. □

As to the period T_ε, we have the following

LEMMA 3.6. *There exist constants T_1, $T_2 > 0$ independent of $\varepsilon \in (0,1]$ such that*

$$0 < T_1 \le T_\varepsilon \le T_2 \quad \textit{for all } \varepsilon \in (0,1].$$

PROOF. Since $I_\varepsilon(u_\varepsilon) \in [m, M]$ and $V_\varepsilon(u) < 0$, we have

$$M \ge I_\varepsilon(u_\varepsilon) = \frac{1}{2}\|\dot{u}_\varepsilon\|_2^2 \int_0^1 H - V_\varepsilon(u_\varepsilon)\,dt \ge \frac{H}{2}\|\dot{u}_\varepsilon\|_2^2.$$

Thus we have

$$T_\varepsilon = \left(\frac{\|\dot{u}_\varepsilon\|_2^2/2}{\int_0^1 H - V_\varepsilon(u_\varepsilon)\,dt}\right)^{1/2} \le \frac{M^{1/2}}{H} =: T_2.$$

Arguing indirectly, we assume, for some $\varepsilon_j \to 0$, $T_{\varepsilon_j} \to 0$ as $j \to \infty$. Then we have

$$\|\dot{u}_{\varepsilon_j}\|_2^2 \to 0 \quad \text{as } j \to \infty \tag{3.5}$$

or

$$\int_0^1 H - V_{\varepsilon_j}(u_{\varepsilon_j})\,dt \to \infty \quad \text{as } j \to \infty. \tag{3.6}$$

Since $I_\varepsilon(u_\varepsilon) \in [m, M]$, both (3.5) and (3.6) hold. Thus we can easily see that, for some $\xi \in S$,

$$\|u_{\varepsilon_j} - \xi\|_E \to 0 \quad \text{as } j \to \infty. \tag{3.7}$$

It follows from (3.7) that there exists a $j_0 \in \mathbb{N}$ such that $d(u_{\varepsilon_j}) \le h_1$ for all $j \ge j_0$. Thus we have

$$\frac{1}{2} \le \nabla \text{dist}\,(u_{\varepsilon_j}, S) X(u_{\varepsilon_j}) \le 1 \quad \text{for all } j \ge j_0.$$

Hence we have for $j \ge j_0$

$$\begin{aligned} 0 = I'_{\varepsilon_j}(u_{\varepsilon_j})X(u_{\varepsilon_j}) &= \frac{1}{2}\|\dot{u}_{\varepsilon_j}\|_2^2 \int_0^1 -\nabla W(u_{\varepsilon_j})X(u_{\varepsilon_j}) \\ &\quad - \frac{\alpha \nabla \text{dist}\,(u_{\varepsilon_j}, S)X(u_{\varepsilon_j})}{\text{dist}\,(u_{\varepsilon_j}, S)^{\alpha+1}} - \frac{4\varepsilon_j \nabla \text{dist}\,(u_{\varepsilon_j}, S)X(u_{\varepsilon_j})}{\text{dist}\,(u_{\varepsilon_j}, S)^5}\,dt \\ &\le \frac{1}{2}\|\dot{u}_{\varepsilon_j}\|_2^2 \int_0^1 -\nabla W(u_{\varepsilon_j})X(u_{\varepsilon_j}) \\ &\quad - \frac{\alpha}{2\text{dist}\,(u_{\varepsilon_j}, S)^{\alpha+1}} - \frac{2\varepsilon_j}{\text{dist}\,(u_{\varepsilon_j}, S)^5}\,dt \\ &\le \frac{1}{2}\|\dot{u}_{\varepsilon_j}\|_2^2 \int_0^1 -\nabla W(u_{\varepsilon_j})X(u_{\varepsilon_j}) - \frac{\alpha}{2\text{dist}\,(u_{\varepsilon_j}, S)^{\alpha+1}}\,dt. \end{aligned} \tag{3.8}$$

Moreover, choosing h_1 smaller if necessary, we see

$$-\nabla W(x)X(\xi) - \frac{\alpha}{2\text{dist}\,(x, S)^{\alpha+1}} \le -\frac{\alpha}{4\text{dist}\,(x, S)^{\alpha+1}} \tag{3.9}$$

for all $x \in \mathbb{R}^N$ with $\operatorname{dist}(x, S) \le h_1$ and $\xi \in S$. By (3.8) and (3.9), we have for $j \ge j_0$

$$0 = I'_{\varepsilon_j}(u_{\varepsilon_j})X(u_{\varepsilon_j}) \le \frac{1}{2}\|\dot{u}_{\varepsilon_j}\|_2^2 \int_0^1 -\frac{\alpha}{4\operatorname{dist}(u_{\varepsilon_j}, S)^{\alpha+1}}\, dt < 0.$$

This is a contradiction. □

By Proposition 3.5 and Lemma 3.6, we can choose a sequence $\varepsilon_j \to 0$ such that for some $u_0 \in E$ and $T \in [T_1, T_2]$

$$u_{\varepsilon_j} \rightharpoonup u_0 \quad \text{weakly in } E, \tag{3.10}$$

$$T_{\varepsilon_j} \to T \quad \text{as } j \to \infty. \tag{3.11}$$

There is a possibility that the limit function $u_0 \in \partial\Lambda$, that is, u_0 may enter the singular set D. $q_0(t) = u_0(t/T)$ is called a *generalized solution* in [6]. If we can show

$$u_0 \notin D \quad \text{for all } t \in [0, 1], \tag{3.12}$$

then the proof of Theorem 1.1 is established. In the following section, we show (3.12).

4. Limit process of the sequence of critical points and proof of Theorem 1.1

In this section we study the regularity of u_0 and give a proof of Theorem 1.1. The argument in this section is similar to [1], but we give a proof for reader's convenience. Let $u_{\varepsilon_j} \in \Lambda$ be a critical point of $I_{\varepsilon_j}(u)$ obtained in Proposition 3.5, which satisfies (3.10) and (3.11). We show (3.12) indirectly and we assume that $u_0(t_\infty) \in D$ for some $t_\infty \in [0, 1]$.

Since $u_{\varepsilon_j}(t) \to u_0(t)$ in $L^\infty(0, 1)$, we can find a sequence $(t_j) \subset [0, 1]$ such that

$$\delta_j = \operatorname{dist}(u_{\varepsilon_j}(t_j), S) \equiv \min_{t\in[0,1]} \operatorname{dist}(u_{\varepsilon_j}(t), S) \to 0. \tag{4.1}$$

After extracting a subsequence, we can assume

$$t_j \to t_\infty \quad \text{and} \quad u_{\varepsilon_j}(t_j) \to u_0(t_\infty) \in S.$$

For notational convenience, we assume $0 \in S$ and $u_0(t_\infty) = 0$, that is, $u_{\varepsilon_j}(t_j) \to 0$. We also choose an orthonormal basis $\{e_1, \dots, e_N\}$ of $\mathbb{R}^N$ such that $n(0) = e_1$.

Setting $z_j = z(u_{\varepsilon_j}(t_j))$, we introduce a re-scaling function $x_j(s)$ by

$$x_j(s) = \frac{1}{\delta_j}(u_{\varepsilon_j}(\delta_j^{(\alpha+2)/2}s + t_j)) - z_j) \quad \text{for } s \in \mathbb{R},$$

where $\delta_j > 0$ is defined by (4.1). We obtain the following properties as to the behavior of x_j.

LEMMA 4.1. *$x_j(s)$, z_j and $\delta_j > 0$ satisfy*

$$\delta_j \to 0,\ z_j \to 0 \quad \textit{as } n \to \infty, \tag{4.2}$$

$$|x_j(s)| \textit{ takes its minimum at } s = 0, \tag{4.3}$$

$$|x_j(0)| = 1,\ x_j(0) \perp \dot{x}_j(0), \tag{4.4}$$
$$x_j(0) \to e_1 \quad \textit{as } n \to \infty,$$

$$\ddot{x}_j(s) + \delta_j^{\alpha+1} T_{\varepsilon_j}^2 \nabla V_{\varepsilon_j}(\delta_j^{(\alpha+2)/2} s + t_j, \delta_j x_j + z_j) = 0 \quad \textit{in } \mathbb{R}, \tag{4.5}$$

$$\frac{1}{2}|\dot{x}_j(s)|^2 + \delta_j^{\alpha} T_{\varepsilon_j}^2 V_{\varepsilon_j}(\delta_j^{(\alpha+2)/2} s + t_j, \delta_j x_j + z_j) = \delta_j^{\alpha} T_{\varepsilon_j}^2 H \quad \textit{in } \mathbb{R}. \tag{4.6}$$

Moreover, if $\delta_j x_j(s) + z_j \in N_{h_0/2}(S)$, then we have

$$\begin{aligned}
\delta_j^{\alpha} T_{\varepsilon_j}^2 V_{\varepsilon_j}(\delta_j^{(\alpha+2)/2} s + t_j, \delta_j x_j + z_j) &= -\frac{\delta_j^{\alpha} T_{\varepsilon_j}^2}{\operatorname{dist}(\delta_j x_j + z_j, S)^{\alpha}} \\
&\quad + \delta_j^{\alpha} T_{\varepsilon_j}^2 W(\delta_j^{(\alpha+2)/2} s + t_j, \delta_j x_j + z_j) - \frac{\varepsilon_j \delta_j^{\alpha} T_{\varepsilon_j}^2}{\operatorname{dist}(\delta_j x_j + z_j, S)^4} \\
&= -\frac{T_{\varepsilon_j}^2}{\operatorname{dist}(x_j, \delta_j^{-1}(S - z_j))^{\alpha}} + \delta_j^{\alpha} T_{\varepsilon_j}^2 W(\delta_j^{(\alpha+2)/2} s + t_j, \delta_j x_j + z_j) \\
&\quad - \frac{\varepsilon_j}{\delta_j^{4-\alpha}} \frac{T_{\varepsilon_j}^2}{\operatorname{dist}(x_j, \delta_j^{-1}(S - z_j))^4}
\end{aligned}$$

and we can rewrite (4.5)–(4.6) as

$$\begin{aligned}
\ddot{x}_j(s) &+ \frac{\alpha T_{\varepsilon_j}^2 n(z(\delta_j x_j + z_j))}{\operatorname{dist}(x_j, \delta_j^{-1}(S - z_j))^{\alpha+1}} \\
&- \delta_j^{\alpha+1} T_{\varepsilon_j}^2 \nabla W(\delta_j^{(\alpha+2)/2} s + t_j, \delta_j x_j + z_j) \\
&+ \frac{4\varepsilon_j}{\delta_j^{4-\alpha}} \frac{T_{\varepsilon_j}^2 n(z(\delta_j x_j + z_j))}{\operatorname{dist}(x_j, \delta^{-1}(S - z_j))^5} = 0 \quad \text{in } \mathbb{R},
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
\frac{1}{2}|\dot{x}_j(s)|^2 &- \frac{T_{\varepsilon_j}^2}{\operatorname{dist}(x_j, \delta_j^{-1}(S - z_j))^{\alpha}} \\
&+ \delta_j^{\alpha} T_{\varepsilon_j}^2 \nabla W(\delta_j^{(\alpha+2)/2} s + t_j, \delta_j x_j + z_j) \\
&- \frac{\varepsilon_j}{\delta_j^{4-\alpha}} \frac{T_{\varepsilon_j}^2}{\operatorname{dist}(x_j, \delta_j^{-1}(S - z_j))^4} = T_{\varepsilon_j}^2 H \delta_j^{\alpha}.
\end{aligned} \tag{4.8}$$

As to the behavior of $\varepsilon_j / \delta_j^{4-\alpha}$, we have

LEMMA 4.4.
$$\limsup_{j \to \infty} \frac{\varepsilon_j}{\delta_j^{4-\alpha}} \le \frac{2 - \alpha}{2}.$$

Proof. By (4.3), we have

$$0 \le \frac{1}{2}\frac{d^2}{ds^2}\bigg|_{s=0} |x_j(s)|^2 = (\ddot{x}_j(0), x_j(0)) + |\dot{x}_j(0)|^2.$$

Since $x_j(0) \to e_1$, $n(\delta_j x_j(0) + z_j) \to e_1$ and $\operatorname{dist}(x_j(0), \delta_j^{-1}(S - z_j)) = 1$, it follows from (3.11), (4.2), (4.7), (4.8) and (W2) that

$$0 \le 2 - \alpha - \limsup_{j\to\infty} \frac{2\varepsilon_j}{\delta_j^{4-\alpha}}. \qquad \square$$

Extracting a subsequence, still denoted by j, we may assume there exists a constant $d \in [0, (2-\alpha)/2]$ such that

$$\frac{\varepsilon_j}{\delta_j^{4-\alpha}} \to d \quad \text{as } n \to \infty. \tag{4.9}$$

Using (3.11), (4.4), (4.8) and (4.9) again, we may assume, without loss of generality, that

$$\dot{x}_j(0) \to (2(1+d))^{1/2} T e_2 \quad \text{as } n \to \infty.$$

Since

$$\operatorname{dist}(\delta_j x, \delta_j^{-1}(S - z_j)) \to |(x, e_1)|, \qquad n(\delta_j x + z_j) \to e_1,$$

the continuous dependence of solutions on initial data and equation implies the following

Lemma 4.3. *For any $\ell > 0$, $x_j(s)$ converges in $C^2([-\ell,\ell], \mathbb{R})$ to a function $x(s)$, which satisfies*

$$\ddot{x} + \frac{\alpha T^2 e_1}{|(x, e_1)|^{\alpha+1}} + \frac{4dT^2 e_1}{|(x, e_1)|^5} = 0 \quad \textit{in } \mathbb{R},$$
$$x(0) = e_1, \quad \dot{x}(0) = (2(1+d))^{1/2} T e_2.$$

Moreover, $|(x(s), e_1)|$ takes its local minimum at $s = 0$.

End of the Proof of Theorem 1.1. Writing $x(s) = (x_1(s), \ldots, x_N(s))$, we have

$$\begin{aligned} \ddot{x}_1 + \frac{\alpha T^2}{x_1^\alpha} + \frac{4dT^2}{x_1^5} &= 0, \quad x_1(0) = 1, \quad \dot{x}_1(0) = 0, \\ \ddot{x}_2 &= 0, \quad x_2(0) = 0, \quad \dot{x}_2(0) = (2(1+d))^{1/2} T, \\ \ddot{x}_i &= 0, \quad x_i(0) = 0, \quad \dot{x}_i(0) = 0 \quad \text{for } i = 3, \ldots, N. \end{aligned} \tag{4.10}$$

It follows from (4.10) that

$$\ddot{x}_1(0) = -\alpha T^2 - 4dT^2 < 0.$$

But this contradicts the fact that $|x_1(s)| = |(x(s), e_1)|$ takes its local minimum at $s = 0$. Thus we see that $u_0(t_\infty) \notin D$ and this completes the proof of Theorem 1.1. $\square$

5. Proof of Theorem 1.2

In this section we give a proof of Theorem 1.2. We assume $D = \{x \in \mathbb{R}^N : |x| \le \rho\}$, $\alpha \in (0,2)$ and

$$V(q) = -\frac{1}{\mathrm{dist}\,(q,S)^\alpha} = -\frac{1}{(|q|-\rho)^\alpha}$$

and consider the following Hamiltonian system with prescribed energy:

$$\ddot{q} + \frac{\alpha q}{(|q|-\rho)^{\alpha+1}|q|} = 0, \tag{5.1}$$

$$\frac{1}{2}|\dot{q}|^2 - \frac{1}{(|q|-\rho)^\alpha} = H. \tag{5.2 }$$

The corresponding functional to (5.1)–(5.2) is

$$I(u) = \frac{1}{2}\|\dot{u}\|_2^2 \int_0^1 H + \frac{1}{(|u|-\rho)^\alpha}\,dt. \tag{5.3}$$

We claim that there exists a constant $H_- = H_-(\rho) \in (-\infty, 0)$ such that if (5.1)–(5.2) have a non-constant periodic solution, then $H \ge H_-(\rho)$. Indeed if $u \in \Lambda$ is a non-constant critical point of (5.3), then we have

$$0 = I'(u)u = \|\dot{u}\|_2^2 \int_0^1 H - V(u) - \frac{1}{2}\nabla V(u)u\,dt. \tag{5.4}$$

Since u is a non-constant critical point of $I(u)$, we obtain $\|\dot{u}\|_2^2 > 0$. Thus we have from (5.4)

$$H = \int_0^1 V(u) + \frac{1}{2}\nabla V(u)u\,dt \tag{5.5}$$

for any non-constant critical point $u \in \Lambda$. We study the behavior of $V(u) + (1/2)\nabla V(u)u$ precisely. Setting $|u| = R$ for $u \in \Lambda$, we define $f\colon (\rho,\infty) \to \mathbb{R}$ by

$$\begin{aligned} f(R) := V(u) + \frac{1}{2}\nabla V(u)u &= -\frac{1}{(R-\rho)^\alpha} + \frac{\alpha}{2}\frac{1}{(R-\rho)^{\alpha+1}}R \\ &= \frac{1}{(R-\rho)^{\alpha+1}}\Big(\rho - \frac{2-\alpha}{2}R\Big). \end{aligned} \tag{5.6}$$

Since $\alpha \in (0,2)$, direct calculation yields

$$f'(R) = \frac{\alpha}{(R-\rho)^{\alpha+2}}\left(\frac{2-\alpha}{2}R - \frac{3}{2}\rho\right),$$

that is,

$$f'\left(\frac{3}{2-\alpha}\rho\right) = 0. \tag{5.7}$$

By (5.6) and (5.7), we define $H_-(\rho) \in (-\infty, 0)$ by

$$H_-(\rho) := \inf_{R>\rho} f(R) = f\left(\frac{3}{2-\alpha}\rho\right) = -\frac{1}{2}\left(\frac{2-\alpha}{1+\alpha}\right)^{\alpha+1}\frac{1}{\rho^\alpha}. \tag{5.8}$$

It follows from (5.5)–(5.8) that if there exists a non-constant periodic solution of (5.1)–(5.2), then $H \geq H_-(\rho)$. Therefore (5.1)–(5.2) have no non-constant periodic solutions for all $H < H_-(\rho)$. Moreover it follows from (5.8) that we can easily see

$$H_-(\rho) \to -\infty \quad \text{as } \rho \to 0. \qquad \square$$

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