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# GEODESICS IN CONICAL MANIFOLDS

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Abstract. The aim of this paper is to extend the definition of geodesics to conical manifolds, defined as submanifolds of $\mathbb{R}^n$ with a finite number of singularities. We look for an approach suitable both for the local geodesic problem and for the calculus of variation in the large. We give a definition which links the local solutions of the Cauchy problem (1.1) with variational geodesics, i.e. critical points of the energy functional. We prove a deformation lemma (Theorem 2.2) which leads us to extend the Lusternik–Schnirelmann theory to conical manifolds, and to estimate the number of geodesics (Theorem 3.4 and Corollary 3.5). In Section 4, we provide some applications in which conical manifolds arise naturally: in particular, we focus on the brachistochrone problem for a frictionless particle moving in $S^n$ or in $\mathbb{R}^n$ in the presence of a potential $U(x)$ unbounded from below. We conclude with an appendix in which the main results are presented in a general framework.

## 1. Introduction and basic definition

The existence of geodesic is one of most studied problems in the calculus of variation. In this paper we want to study the presence of geodesics in a particular kind of manifolds, called conical manifolds, that appears in a natural way in some optimization problem (see Section 4.1).

We define the following type of topological manifolds.

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Definition 1.1. A conical manifold $M$ is a complete $n$-dimensional $C^0$ submanifold of $\mathbb{R}^m$ which is everywhere smooth, except for a finite set of points $V$. A point in $V$ is called vertex.

Usually there are two ways to introduce geodesics in a smooth manifold:

- (Local Cauchy problem) A geodesic is a solution of a suitable Cauchy problem, i.e. given $p \in M$, $v \in T_pM$, we look for a curve $\gamma\colon [0,\varepsilon] \to M$ such that

$$\begin{cases} D_s\gamma' = 0, \\ \gamma(0) = p, \\ \gamma'(0) = v. \end{cases} \tag{1.1}$$

- (Global Bolza problem) We consider the path space on $M$:

$$\begin{aligned} \Omega_{p,q} &:= \{\gamma \in H^1([0,1],M) : \gamma(0) = p,\ \gamma(1) = q\}, \\ \Omega_p &:= \{\gamma \in H^1([0,1],M) : \gamma(0) = \gamma(1) = p\}, \end{aligned}$$

  a geodesic is a critical point of the energy functional defined by[1]

$$E\colon \Omega \to \mathbb{R}, \quad E(\gamma) = \int_0^1 |\gamma'(s)|^2\, ds.$$

In conical manifolds the Cauchy problem (1.1) is not well posed, and the solution is neither unique, nor continuously dependent from the initial data. The functional approach gives us an easy result on minimal geodesics. However, this approach is not completely useful: we can not easily define a critical point of energy different from minimum, because the energy is not a $C^1$ functional.

Furthermore, the usual generalization of the derivative, the weak slope, cannot be applied to our case, because it requires some conditions on the manifolds $M$ which are not satisfied in the case of conical manifolds. The weak slope was introduced by Marco Degiovanni and Marco Marzocchi in [4] (see also [1]–[3]). Moreover, we refer to [5], [10] for a weak slope approach to geodesic problem and to [7] for a detailed comparison with our approach.

We give the following definition of geodesics, that appears to be the most suitable one for this kind of problem.

Definition 1.2. A path $\gamma \in \Omega$ is a geodesic if and only if

(a) the set $T = T_\gamma := \{s \in (0,1) : \gamma(s) = v\}$ is a closed set without internal part,
(b) $D_s\gamma' = 0$ for all $s \in [0,1] \setminus T$,
(c) $|\gamma'|^2$ is constant as a function in $L^1$.

[1] Hereafter we simply note $\Omega$ when we not need to specify the extremal points of paths.

We note that a geodesic may not be a local minimum for the length functional, for example, we consider a Euclidean cone and a broken geodesic passing through the vertex. However, this definition allows us to prove the main theorem of this paper (see Corollary 3.5).

Theorem 1.3. *Let $M$ be a conical manifold, $p \in M$. Then there are at least* $\operatorname{cat}\Omega$ *geodesics.*

We are relating Definition 1.2, which is local, with the topology of the path space, which is a tool of the calculus of variation in the large; furthermore, this approach allows us to find also non minimal geodesics.

Unfortunately, it's not easy to compute $\operatorname{cat}\Omega$ for conical manifolds. Set

$$\Omega^{\infty}_{p,q} := \{\gamma \in C^0([0,1], M) : \gamma(0) = p,\ \gamma(1) = q\}.$$

We know that, for a smooth manifold, there is an homotopy equivalence

$$\Omega^{\infty}_{p,q} \simeq \Omega_{p,q} \tag{1.2}$$

(for a proof see, for example [8, Theorem 1.2.10]). In general this result is false for conical manifolds; we show it by an example.

Example 1.4. Let $M = \{(x, x\sin(1/x)),\ x \in \mathbb{R}\} \subset \mathbb{R}^2$. This is an 1-dimensional conical manifold with vertex $O = (0,0)$. Let $p, q \in M$ be two opposite points with respect to $O$: we have that, while $\Omega^{\infty}_{p,q}$is connected, $\Omega_{p,q}$ is not, so the usual homotopy equivalence (1.2) does not hold.

Even if an explicit calculation of $\operatorname{cat}\Omega$ in general is very difficult, in Section 4, we will give a criterion for which (1.2) holds. Moreover, we show some applications in which conical manifolds appears naturally.

## 2. Deformation lemmas

We want to prove that our definition of geodesic is compatible with the energy functional, i.e. if there is no geodesic of energy $c$, then there is no change of the topology of functional $E$ at level $c$. To do that, we prove a deformation lemma (Theorem 2.2), that is the main result of this section.

Definition 2.1. Given $p \in M$ we set

$$\begin{aligned}\Omega^b &= \Omega^b_p := \{\gamma \in \Omega_p : E(\gamma) \le b\},\\ \Omega^b_a &= \Omega^b_{a,p} := \{\gamma \in \Omega_p : a \le E(\gamma) \le b\}.\end{aligned}$$

Theorem 2.2 (Deformation lemma). *Let $M$ be a conical manifold, $p \in M$. Suppose that there exists $c \in \mathbb{R}$ such that $\Omega^c$ contains only a finite number of geodesics. Then if $a, b \in \mathbb{R}$, and $a < b < c$ are such that the strip $[a,b]$ contains only regular values of $E$, $\Omega^a$ is a deformation retract of $\Omega^b$.*

In order to prove this theorem, we must study the structure of $\Omega^c$. For the moment, we consider a special case.

We suppose that $M$ has only a vertex $v$, and we study the special closed geodesic $\gamma_0$ for which there exists an unique $\sigma$ such that $\gamma_0(\sigma) = v$. We set $E(\gamma_0) = c_0$ and we suppose that there exists $a, b \in \mathbb{R}$, $c_0 < a < b$, such that $\Omega^b$ contains only the geodesics $\gamma_0$ (so $\Omega^b_a$ contains no geodesics).

At last let us set

$$L_1 = \int_0^\sigma |\gamma_0'|^2, \quad L_2 = \int_\sigma^1 |\gamma_0'|^2.$$

We identify now two special subsets of $\Omega^b_a$. Let

$$\Sigma = \{\gamma \in \Omega^b_a : v \in \operatorname{Im}\gamma\}, \tag{2.1}$$

for every $\gamma \in \Sigma$ it exists a set $T$ such that $\gamma(s) = v$ if and only if $s \in T$. Let

$$\Sigma_0 = \{\gamma \in \Sigma : D_s\gamma'(s) = 0 \text{ and } |\gamma'|^2 \text{ is constant on every connected component of } [0,1] \setminus T\}. \tag{2.2}$$

Indeed, we will see in the proof of the next lemma that, if $\gamma \in \Sigma_0$, then $\gamma([0,1]) = \gamma_0([0,1])$, so $\Sigma_0$ is the set of the piecewise geodesics that are equivalent to $\gamma_0$ up to affine reparametrization.

Lemma 2.3. *$\Sigma_0$ is compact.*

Proof. If $\gamma \in \Sigma_0$, only two situations occur: either there exists a unique $\tau$ such that $\gamma(\tau) = v$, or there exists $[\tau_1, \tau_2]$ such that $\gamma(t) = v$ if and only if $t \in [\tau_1, \tau_2]$. In fact, if it were two isolated consecutive points $s_1, s_2 \in T$ such that $\gamma(s_i) = v$, then,we can obtain by reparametrization a geodesic $\gamma_1 \neq \gamma_0$ in $\Omega^b$, that contradicts our assumptions (this proves also that $\gamma([0,1]) = \gamma_0([0,1])$).

Now take $(\gamma_n)_n \subset \Sigma_0$. For simplicity we can suppose that there exists a subsequence such that for all $n$ there exists a unique $\tau_n$ for which $\gamma_n(\tau_n) = v$ (else, definitely, there exists $[\tau_n^1, \tau_n^2]$ such that $\gamma_n(t) = v$ if and only if $s \in [\tau_n^1, \tau_n^2]$, but the proof follows in the same way).

If we consider $||\gamma||_{H^1} = E(\gamma)$, then we have $a \leq ||\gamma_n|| \leq b$, hence, up to subsequence, there exists $\overline{\gamma}$ such that $\gamma_n \to \overline{\gamma}$ in weak-$H^1$ norm and uniformly.

Also, we know that for all $n$ there exists $\tau_n$ such that $\gamma_n(\tau_n) = v$ and

$$\gamma_n = \begin{cases} \gamma_0\left(\dfrac{\sigma}{\tau_n}s\right) & \text{for } s \in [0,\tau_n], \\ \gamma_0\left(\dfrac{1-\sigma}{1-\tau_n}s + \dfrac{\sigma-\tau_n}{1-\tau_n}\right) & \text{for } s \in (\tau_n,1]. \end{cases}$$

It exists $0 < p < 1$ such that $p \le \tau_n \le 1-p$, in fact

$$\begin{aligned} b &\ge \int |\gamma_n'|^2 = \int_0^{\tau_n} |\gamma_n'|^2 + \int_{\tau_n}^1 |\gamma_n'|^2 \\ &= \left[\frac{\sigma}{\tau_n}\right]^2 \int_0^{\tau_n} |\gamma_0'|^2\left(\frac{\sigma}{\tau_n}s\right) + \left[\frac{1-\sigma}{1-\tau_n}\right]^2 \int_{\tau_n}^1 |\gamma_0'|^2\left(\frac{1-\sigma}{1-\tau_n}s + \frac{\sigma-\tau_n}{1-\tau_n}\right) ds \\ &= \frac{\sigma}{\tau_n}\int_0^{\sigma} |\gamma_0'|^2(s')\,ds' + \frac{1-\sigma}{1-\tau_n}\int_0^{\sigma} |\gamma_0'|^2(s')\,ds' = \frac{L_1^2}{\sigma\tau_n} + \frac{L_2^2}{(1-\sigma)(1-\tau_n)}, \end{aligned}$$

so

$$b \ge \frac{L_1^2}{\sigma\tau_n} \ \Rightarrow\ \tau_n > \frac{L_1^2}{\sigma b},$$

and

$$b \ge \frac{L_2^2}{(1-\sigma)(1-\tau_n)} \ \Rightarrow\ \tau_n < 1 - \frac{L_2^2}{b(1-\sigma)}.$$

So a subsequence exists such that $\tau_n \to \tau$, $p \le \tau \le 1-p$. Obviously

$$\frac{\sigma}{\tau_n}s \to \frac{\sigma}{\tau}s,$$
$$\frac{1-\sigma}{1-\tau_n}s + \frac{\sigma-\tau_n}{1-\tau_n} \to \frac{1-\sigma}{1-\tau}s + \frac{\sigma-\tau}{1-\tau}.$$

So for almost all $s$ we have

$$\gamma_n \to \widetilde{\gamma}(s) = \begin{cases} \gamma_0\left(\dfrac{\sigma}{\tau}s\right) & \text{for } s \in [0,\tau], \\ \gamma_0\left(\dfrac{1-\sigma}{1-\tau}s + \dfrac{\sigma-\tau}{1-\tau}\right) & \text{for } s \in (\tau,1]. \end{cases} \tag{2.3}$$

Both $\gamma_n$ and $\widetilde{\gamma}$ are continuous, because $\gamma_0$ is continuous, so the convergence in (2.3) is uniform; furthermore, $\overline{\gamma} = \widetilde{\gamma}$ for the uniqueness of limit.

We have also that

$$||\gamma_n|| = \frac{L_1^2}{\sigma\tau_n} + \frac{L_2^2}{(1-\sigma)(1-\tau_n)} \to \frac{L_1^2}{\sigma\tau} + \frac{L_2^2}{(1-\sigma)(1-\tau)} = ||\overline{\gamma}||,$$

so $\gamma_n \xrightarrow{H^1} \overline{\gamma}$ and $a \le ||\overline{\gamma}|| \le b$, hence $\overline{\gamma} \in \Sigma_0$, that concludes the proof. □

Now we shall prove two technical lemmas which are crucial for this paper.

Lemma 2.4 (Existence of retraction in $\Sigma_0$). *There exist $R \supset \Sigma_0$, $\nu, \overline{t} \in \mathbb{R}^+$ and a continuous function $\eta_R\colon R \times [0,\overline{t}] \to \Omega$ such that, for all $t \in [0,\overline{t}]$, $\beta \in R$,*

(a) $\eta_R(\beta, 0) = \beta$,
(b) $E(\eta_R(\beta,t)) - E(\beta) < -\nu t$.

PROOF. We proceed by steps.

*Step 1.* At first we want to prove that, for any $\gamma \in \Sigma_0$, there are $\overline{t}, d, \nu \in \mathbb{R}^+$, and a local retraction $\mathcal{H}: B(\gamma, d) \times [0, \overline{t}] \to \Omega$ such that

- $\mathcal{H}(\beta, 0) = \beta$,
- $E(\mathcal{H}(\beta, t)) - E(\beta) < -\nu t$,

for all $t \in [0, \overline{t}]$, $\beta \in B(\gamma, d)$. Furthermore, we will see that $d$ is independent from $\gamma$.

By hypothesis there exists an unique $\sigma \in [0,1]$ such that $\gamma_0(\sigma) = v$. Furthermore, because $E(\gamma_0) = c_0$, we know also that $|\gamma_0'|^2 = c_0$ almost everywhere. Let $\gamma \in \Sigma_0$, then $\mathrm{Im}(\gamma) = \mathrm{Im}(\gamma_0)$. In analogy with Lemma 2.3 we suppose, without loss of generality, that there exists an unique $\tau \in [0,1]$ such that $\gamma(\tau) = v$, and both $\gamma'|_{(0,\tau)}$, $\gamma'|_{(\tau,1)}$ are constant, although we cannot say if they are equals. We can choose a suitable change of parameter $\varphi$ such that $\gamma(\varphi(s)) = \gamma_0(s)$. By this way we can construct a flow for $\gamma$ as follows:

$$\mathcal{H}(\gamma, t) = \gamma(\varphi_t(s)) = \begin{cases} \gamma\left(\dfrac{\tau}{a(t)} s\right) & \text{for } s \in [0, a(t)), \\ \gamma\left(\dfrac{\tau - 1}{a(t) - 1} s + \dfrac{a(t) - \tau}{a(t) - 1}\right) & \text{for } s \in [a(t), 1], \end{cases}$$

where $a(t) = (1-t)\tau + t\sigma$. Notice that

$$\gamma(\varphi_0(s)) = \gamma(s), \quad \gamma(\varphi_1(s)) = \gamma_0(s)$$

and

$$\gamma(\tau) = v = \gamma(\varphi_1(\sigma)) = \gamma(\varphi_t(a(t))).$$

We recall that $l(\gamma_0|_{(0,\sigma)}) = L_1$, $l(\gamma_0|_{(\sigma,1)}) = L_2$: obviously

$$\left[\frac{L_1}{\sigma}\right]^2 = \left[\frac{L_2}{1-\sigma}\right]^2 = c_0.$$

Furthermore

$$\left[\frac{\partial}{\partial s}\gamma(\varphi_t(s))\right]^2\bigg|_{(0,a(t))} = \left[\frac{L_1}{a(t)}\right]^2,$$

$$\left[\frac{\partial}{\partial s}\gamma(\varphi_t(s))\right]^2\bigg|_{(a(t),1)} = \left[\frac{L_2}{1-a(t)}\right]^2,$$

then

$$\begin{aligned} E(\mathcal{H}(\gamma, t)) &= \int_0^{a(t)} \frac{L_1^2}{a(t)^2}\, ds + \int_{a(t)}^1 \frac{L_2^2}{(1-a(t))^2}\, ds \\ &= \frac{L_1^2}{\sigma^2}\frac{\sigma^2}{a(t)} + \frac{L_2^2}{(1-\sigma)^2}\frac{(1-\sigma)^2}{1-a(t)} = c_0\left(\frac{\sigma^2}{a(t)} + \frac{(1-\sigma)^2}{1-a(t)}\right), \end{aligned}$$

so

$$\frac{\partial}{\partial t}E(\mathcal{H}(\gamma, t)) = c_0\left(\frac{(1-\sigma)^2}{(1-a(t))^2} - \frac{\sigma^2}{(a(t))^2}\right).$$

It is easy to see that, either for $\sigma < \tau$ as for $\sigma > \tau$, we have $\partial E(\mathcal{H}(\gamma, t))/\partial t < 0$, for all $t \in [0, 1)$, as expected. More over, because there is a $p > 0$ such that $p < \tau < 1 - p$ (as shown in the previous lemma), we can find $\bar{t}$, $\nu$ such that

$$\frac{\partial}{\partial t} E(\mathcal{H}(\gamma, t)) < 2\nu \quad \text{for all } t \in [0, \bar{t}].$$

Now we want to extend $\mathcal{H}$ in a neighbourhood of $\gamma$: it is useful, for finding that, to work on the whole space $H^1(I, \mathbb{R}^n)$. As above we consider $\gamma \in \Omega$.

Let $B_d = B^{H^1(I,\mathbb{R}^n)}(\gamma, d) \cap \Omega$. For all $\beta \in B_d$ we can say

$$\beta = \gamma + (\beta - \gamma) = \gamma + \delta, \quad ||\delta|| \leq d.$$

We can extend $\mathcal{H}$ as follows:

$$\mathcal{H}(\beta, t) = \mathcal{H}(\gamma + \delta, t) = \gamma(\varphi_t(s)) + \delta(\varphi_t(s)).$$

Obviously $\mathrm{Im}(\beta) = \mathrm{Im}(\mathcal{H}(\beta, t))$, so $\mathcal{H}(\beta, t) \in \Omega$.

We want to show that there exists a $d > 0$ such that

$$E(\mathcal{H}(\beta, t)) - E(\beta) < -\nu t \quad \text{for all } \beta \in B_d,$$

$$\begin{aligned} E(\mathcal{H}(\beta, t)) - E(\beta) &= \int |\gamma(\varphi_t(s))' + \delta(\varphi_t(s))'|^2 - \int |\gamma'(s) + \delta'(s)|^2 \\ &= \int |\gamma(\varphi_t(s))'|^2 - |\gamma'(s)|^2 + \int |\delta(\varphi_t(s))'|^2 - |\delta'(s)|^2 \\ &\quad + \int \langle \gamma(\varphi_t(s))', \delta(\varphi_t(s))' \rangle - \int \langle \gamma'(s), \delta'(s) \rangle. \end{aligned}$$

We have already shown that

$$\int |\gamma(\varphi_t(s))'|^2 - |\gamma'(s)|^2 < -2\nu t.$$

Let

$$\begin{aligned} A &= \int_0^1 |\delta(\varphi_t(s))'|^2 \, ds - \int_0^1 |\delta(s)'|^2 \, ds, \\ B &= \int_0^1 \langle \gamma(\varphi_t(s))', \delta(\varphi_t(s))' \rangle \, ds - \int_0^1 \langle \gamma'(s), \delta'(s) \rangle \, ds. \end{aligned}$$

The term $A$ can be estimate as follows, remembering the definition of $\varphi_t(s)$:

$$\begin{aligned} A &= \int_0^{a(t)} \left| \delta\left( \frac{\tau}{a(t)} s \right)' \right|^2 + \int_{a(t)}^1 \left| \delta\left( \frac{\tau - 1}{a(t) - 1} s + \frac{a(t) - \tau}{a(t) - 1} \right)' \right|^2 - \int_0^1 |\delta'(s)|^2 \\ &= \left[ \frac{\tau}{a(t)} \right]^2 \int_0^{a(t)} \left| \delta'\left( \frac{\tau}{a(t)} s \right) \right|^2 \\ &\quad + \left[ \frac{\tau - 1}{a(t) - 1} \right]^2 \int_{a(t)}^1 \left| \delta'\left( \frac{\tau - 1}{a(t) - 1} s + \frac{a(t) - \tau}{a(t) - 1} \right) \right|^2 - \int_0^1 |\delta'(s)|^2 \end{aligned}$$

$$
\begin{aligned}
&= \frac{\tau}{a(t)} \int_0^\tau |\delta'(s)|^2 + \frac{\tau - 1}{a(t)-1} \int_\tau^1 |\delta'(s)|^2 - \int_0^1 |\delta'(s)|^2 \\
&= \frac{\tau - a(t)}{a(t)} \int_0^\tau |\delta'(s)|^2 + \frac{\tau - a(t)}{a(t)-1} \int_\tau^1 |\delta'(s)|^2 \\
&\le \max\left[\left|\frac{\tau - a(t)}{a(t)}\right|, \left|\frac{\tau - a(t)}{a(t)-1}\right|\right] \int_0^1 |\delta'(s)|^2 \\
&\le |\tau - a(t)| \max\left[\frac{1}{a(t)}, \frac{1}{a(t)-1}\right] \int_0^1 |\delta'(s)|^2 \le K||\delta||^2_{H^1} t \le K d^2 \cdot t
\end{aligned}
$$

in fact $|\tau - a(t)| = |\tau - \sigma| t$. Furthermore, $K$ depends only on $\gamma_0$, because there exists $p > 0$ such that $\tau \in [p, 1-p]$ (as shown in Lemma 2.3).

In the same way we can estimate $B$:

$$
\begin{aligned}
B &= \frac{\tau - a(t)}{a(t)} \int_0^\tau \langle \gamma'(s), \delta'(s) \rangle + \frac{\tau - a(t)}{a(t)-1} \int_\tau^1 \langle \gamma'(s), \delta'(s) \rangle \\
&\le \left|\frac{\tau - a(t)}{a(t)}\right| \int_0^\tau \left|\langle \gamma'(s), \delta'(s) \rangle\right| + \left|\frac{\tau - a(t)}{a(t)-1}\right| \int_\tau^1 \left|\langle \gamma'(s), \delta'(s) \rangle\right| \\
&\le t|\tau - \sigma| \max\left[\frac{1}{a(t)}, \frac{1}{1-a(t)}\right] \int_0^1 \langle \gamma', \delta' \rangle \le K_1 d \cdot t,
\end{aligned}
$$

where, as above, $K_1$ is a constant depending only on $\gamma_0$.

Now, putting together all the pieces we have

$$
E(\mathcal{H}(\beta, t)) - E(\beta) \le -2\nu t + K d^2 t + K_1 d t < -\nu t
$$

if $d < \min(\nu/K_1, \sqrt{\nu/K})$.

*Step 2.* We want to prove that, for all $\varepsilon$ it exist a $0 < \widetilde{t} < \bar{t}$ such that

$$
\mathcal{H}(B(\beta, d'), t) \subset B(\beta, (1+\varepsilon)d')
$$

if $B(\beta, d') \subset B(\gamma, d)$, $t < \widetilde{t}$. We start proving that, for any $\beta, \beta_1 \in B(\gamma, d)$,

$$
\begin{aligned}
||\mathcal{H}(\beta, t) \; - \mathcal{H}(\beta_1, t)||^2_{H^1} &\le \left(\frac{\tau}{a(t)}\right)^2 \int_0^{a(t)} |\beta' - \beta_1'|^2 \left(\frac{\tau}{a(t)} s\right) ds \\
&\quad + \left(\frac{\tau - 1}{a(t)-1}\right)^2 \int_{a(t)}^1 |\beta' - \beta_1'|^2 \left(\frac{\tau - 1}{a-1} s + \frac{a - \tau}{a-1}\right) ds \\
&= \left(\frac{\tau}{a(t)}\right) \int_0^\tau |\beta' - \beta_1'|^2(r)\, dr + \left(\frac{\tau - 1}{a(t)-1}\right) \int_\tau^1 |\beta' - \beta_1'|^2(r)\, dr \\
&\le \max\left(\frac{\tau}{a(t)}, \frac{\tau - 1}{a(t)-1}\right) \int_0^1 |\beta' - \beta_1'|^2(r)\, dr \le M^2(t) ||\beta - \beta_1||^2_{H^1}
\end{aligned}
$$

for all $t$, where $M(t)$ is a continuous function such that $M(0) = 1$.

In particular for all $\varepsilon > 0$ there exists $\widetilde{t} > 0$ such that for $t \le \widetilde{t}$

$$
d(\mathcal{H}(\beta, t), \mathcal{H}(\beta_1, t)) < \left(1 + \frac{\varepsilon}{2}\right) ||\beta - \beta_1||_{H^1}.
$$

So, if $\beta_1 \in B(\beta, d')$, for all $\varepsilon > 0$ a $\widetilde{t}$ exists such that

$$\begin{aligned} d(\mathcal{H}(\beta_1, t), \beta) &\le d(\mathcal{H}(\beta_1, t), \mathcal{H}(\beta, t)) + d(\mathcal{H}(\beta, t), \beta) \\ &\le \left(1 + \frac{\varepsilon}{2}\right) d' + \frac{\varepsilon}{2} d' \le (1 + \varepsilon) d' \end{aligned}$$

for all $0 \le t \le \widetilde{t}$, because $\mathcal{H}(\beta, t)$ is continuous in $t$. Notice that $\widetilde{t}$ is independent from $\beta_1$, so we have that, chosen $\beta$ and $d'$ such that $B(\beta, d') \subset B(\gamma, d)$, then for every $\varepsilon > 0$ there exists $\widetilde{t} > 0$ such that

$$\mathcal{H}(B(\beta, d'), t) \subset B(\beta, (1 + \varepsilon) d') \quad \text{for all } 0 \le t \le \widetilde{t}. \tag{2.4}$$

*Step 3.* We have now to compound all these retraction. We follow an idea shown by Corvellec, Degiovanni and Marzocchi in [2, Theorem 2.8], and we combine it with the compactness of $\Sigma_0$.

Take $d$ as in the first step. Then $\bigcup_\gamma B(\gamma, d/4)$ covers $\Sigma_0$. By compactness we can choose

$$\gamma_1, \dots, \gamma_N \quad \text{such that } \bigcup_{i=1}^{N} B\left(\gamma_i, \frac{d}{4}\right) \supset \Sigma_0.$$

Set $B(\gamma_i, d/4) = B_i$, $R = \bigcup_i \overline{B}_i$ and $\nu = \min_i \nu_{\gamma_i}$ and $\mathcal{H}_i = \mathcal{H}_{\gamma_i}$. Let

$$\vartheta_i \colon H^1(I, M) \to [0, 1]$$

a partition of unity referred to $B_i$.

We want to define a sequence of continuous maps $\eta_h \colon R \times [0, \widetilde{t}_h] \to \Omega$, for $h = 1, \dots, N$, defined as follows:

$$\eta_1(\beta, t) = \begin{cases} \mathcal{H}_1(\beta, \vartheta_1 t) & \text{for } \beta \in \overline{B}_1, \\ \beta & \text{outside}, \end{cases}$$

$$\eta_h(\beta, t) = \begin{cases} \mathcal{H}_h(\eta_{h-1}(\beta, t), \vartheta_h t) & \text{for } \beta \in \overline{B}_h, \\ \eta_{h-1}(\beta, t) & \text{outside}. \end{cases}$$

We want that, for all $h$,

(1) $\eta_h(\beta, 0) = \beta$,
(2) $E(\eta_h(\beta, 0)) - E(\beta) \le -\nu t \sum_{i=1}^{h} \vartheta_i$,
(3) for all $i$ and all $\varepsilon$ there exists $\widetilde{t}_h$ such that

$$\eta_{h-1}(\overline{B}_i, t) \subset B(\gamma_i, (1 + \varepsilon)^{h-1} d/4) \quad \text{if } 0 \le t \le \widetilde{t}_h.$$

The proof of the first two conditions is obvious. The last condition, that assures the good definition of $\eta_h$, will be proved by induction on $h$.

(a) Case $h = 1$. If $B_i = B_1$ then $\eta_1(\beta, t) = \mathcal{H}_1(\beta, \vartheta t)$. Hence there exists $\widetilde{t}$ such that, if $0 \le t \le \widetilde{t}$

$$d(\gamma_1, \eta_1(\beta, t)) = d(\gamma_1, \mathcal{H}_1(\beta, \vartheta_1 t)) < (1+\varepsilon)\frac{d}{4}.$$

In fact we know that there exists $\widetilde{t}$ such that

$$d(\gamma_1, \mathcal{H}_1(\beta, t)) < (1+\varepsilon)\frac{d}{4},$$

for all $0 \le t \le \widetilde{t}$, and $\vartheta_1 \le 1$, so $\vartheta_1 t \le t \le \widetilde{t}$.

If $B_1 \cap B_i = \emptyset$, then $\eta_1(\overline{B}_i, t) = \overline{B}_i$ for all $t$, so the proof is obvious.

Finally, if $B_1 \cap B_i \neq \emptyset$, we know that $\overline{B}(\gamma_i, d/4) \subset B(\gamma_1, d)$, so we can say that $\eta_1(\overline{B}_i, t) = \mathcal{H}_1(\overline{B}_i, \vartheta_1 t)$, hence we can repeat the above deduction. Taking the minimum of $\widetilde{t}$ so found (they are a finite number) we can conclude.

(b) Inductive step. Let $\eta_{h-1}$ be such that, given $\varepsilon > 0$, for all $i$ there exists a $\widetilde{t}_{h-1}$ for which

$$\eta_{h-1}(\overline{B}_i, t) \subset B\left(\gamma_i, (1+\varepsilon)^{h-1}\frac{d}{4}\right) \quad \text{for all } 0 \le t \le \widetilde{t}_h.$$

At first notice that we can choose $\varepsilon$ such that $\eta_{h-1}(\overline{B}_h, t) \subset B(\gamma_n, d)$, so $\eta_h$ is well defined.

Either if $B_i = B_h$ or if $B_h \cap B_i = \emptyset$ the proof is obvious.

Let $B_h \cap B_i \neq \emptyset$. If $\beta \in B_i \setminus \overline{B}_h$ then $\eta_h(\beta, t) = \eta_{h-1}(\beta, t)$, so

$$d(\gamma_h, \eta_h(\beta, t)) < (1+\varepsilon)^{h-1}\frac{d}{4} < (1+\varepsilon)^h\frac{d}{4}.$$

Otherwise, by inductive step

$$\eta_{h-1}(\overline{B}_h, t) \subset B\left(\gamma_h, (1+\varepsilon)^{h-1}\frac{d}{4}\right),$$

and, by (2.4) we have that there exists a $\widetilde{t}$ such that

$$\mathcal{H}_h(\eta_{h-1}(\beta, t)) \subset B\left(\gamma_h, (1+\varepsilon)^h\frac{\delta}{4}\right)$$

if $\beta \in \overline{B}_h \cap B_i$ and $0 \le t \le \widetilde{t}$, so the proof follows immediately. Because we have $N$ iterations, we choose $\overline{\varepsilon}$ such that $(1+\overline{\varepsilon})^N < 2$, and we define

$$\overline{t} = \min_h\{t_{\overline{\varepsilon}, h} \text{ previously found}\}.$$

By compactness $\overline{t} > 0$.

Set $\eta_R = \eta_N$, so we find a continuous map $\eta_R \colon R \times [0, \overline{t}] \to \Omega$ such that

- $\eta_R(\beta, 0) = \beta$,
- $E(\eta_t(\beta, t)) - E(\beta) \le -\nu t \sum_{i=1}^N \vartheta_i = -\nu t$,

for every $\beta \in R$, $0 \le t \le \overline{t}$. $\square$

Lemma 2.5. *For any $U \supset \Sigma_0$ there exist $\bar{t}$, $\nu \in \mathbb{R}^+$ and a continuous functional $\eta_U \colon \Omega_a^b \setminus U \times [0,\bar{t}] \to \Omega_a^b$ such that*

(a) $\eta_U(\,\cdot\,,0) = \mathrm{id}$,

(b) $E(\eta_U(\beta,t)) - E(\beta) \le -\nu t$,

*for all $t \in [0,\bar{t}]$, for all $\beta \in \Omega_a^b \setminus U$.*

Proof. We look for a pseudo gradient vector field $F$, such that, if $\eta_U$ is a solution of

$$\begin{cases} \dot{\eta}_U(t,\gamma) = F(\gamma), \\ \eta_U(0,\,\cdot\,) = \mathrm{id}, \end{cases} \tag{2.5}$$

then there exists $\nu > 0$ such that $E(\eta_U(t,\gamma)) - E(\gamma) < -\nu t$.

For every $S$ neighbourhood of $\Sigma$, we have that $-\nabla E$ is a good gradient field on $\Omega_a^b \setminus S$, in fact $E$ is smooth and satisfies the Palais Smale condition outside $S$, so, for $\Omega_a^b \setminus S$ does not contain critical points of $E$, we know that there exists a $\nu_0 \in \mathbb{R}^+$ such that $-||\nabla E||^2 < -\nu_0$. By integrating (2.5) with $F = -\nabla E$ we have that $E(\eta_U(\gamma,t)) - E(\gamma) < -\nu_0 t$ if $\eta_U(\gamma,t) \subset \Omega_a^b \setminus S$ for all $t$.

Now let $S_1$ be a neighbourhood of $S$ and let $U$ be a neighbourhood of $\Sigma_0$: we look for a pseudo gradient vector field on $S_1 \setminus U$. Although $E$ is non smooth, we can define $dE(\gamma)[w]$ for every $\gamma \in \Sigma \setminus U$, and for a suitable choice of $w$. It is sufficient to take $w$ vector field along $\gamma$ with

$$\operatorname{spt} w \subset \{s : \gamma(s) \neq v\}.$$

There exists $\nu_1$ such that for every $\gamma \in \Sigma \setminus U$ we can find $w_\gamma$ for which $dE(\gamma)[w_\gamma] < -2\nu_1$. This is possible because we can find a partition $0 = s_0 < \ldots < s_k = 1$ such that $\gamma(s_i) = v$ and $v \notin \operatorname{Im} \gamma|_{(s_i,s_{i+1})}$. Called $\gamma_i = \gamma|_{(s_i,s_{i+1})}$ we can shorten it by a vector field $w_i$ along $\gamma_i$, leaving its extremal point fixed, so we obtain a vector field $w_\gamma$ along $\gamma$ with

$$\operatorname{spt} w_\gamma \subset \{s : \gamma(s) \neq v\}, \quad \text{and} \quad dE(\gamma)[w_\gamma] < -2\nu_1,$$

in fact for these variations the (PS) condition for energy holds. Moreover, $\Sigma \setminus U$ does not contain any stationary point for these kind of variations.

Without loss of generality suppose now that exists a global chart $(V,\phi)$, $0 \in V \subset \mathbb{R}^n$ such that $\phi(0) = v$. The metric of $M$, read on $V$, lead us to consider a matrix $(g_{ij}(x))_{ij}$ whose coefficients are discontinuous at 0; if $\gamma$ is a path on $V$ we can compute is energy by taking

$$E(\gamma) = \int g_{ij}(\gamma)\gamma_i'\gamma_j'\,ds.$$

For the sake of simplicity we suppose also that $g_{ij}(x) = g(x)\delta_{ij}(x)$ where $\delta_{ij}$ are the coefficient of Euclidean metric. The general case does not present further difficulties.

Now we pass to coordinates $(V,\phi)$. Because $\gamma\in\Omega$, if $||\gamma-\gamma_1||_\Omega<\varepsilon$ then there exists $C\in\mathbb{R}^+$ such that $||\gamma-\gamma_1||_{L^\infty}<C\varepsilon$ by the Sobolev immersion, so also $||\phi(\gamma)-\phi(\gamma_1)||_{L^\infty}<C\varepsilon$.

In coordinates $dE(\gamma)[w]$ has the following form:

$$dE(\gamma)[w]=\int g(\gamma)\gamma' w'\,ds+\int\langle\nabla g,w\rangle|\gamma'|^2\,ds,$$

where $w\in H^1(I,V)$. Note that, even if $\nabla g$ does not exist everywhere, it is well defined on $\operatorname{spt} w$.

We have proved that for every $\gamma\in\Sigma\setminus U$ exists $w_\gamma$ such that $dE(\gamma)[w_\gamma]<-2\nu_1$; obviously we can prove the same for every $\gamma\in S_1\setminus U$. Given $\gamma\in S_1\setminus U$ and $w_\gamma$ as above, it exists a neighbourhood $V_\gamma$ of $\gamma$ such that

$$dE(\gamma_1)[w_\gamma]<-\nu_1\quad\text{for all }\gamma_1\in V_\gamma.$$

Let $||\gamma-\gamma_1||_{H^1}<\varepsilon$, then

$$\begin{aligned}\int g(\gamma)\gamma' w'_\gamma-\int g(\gamma_1)\gamma'_1 w'_\gamma&\le\int g(\gamma)(\gamma'-\gamma'_1)w'_\gamma+\int(g(\gamma)-g(\gamma_1))\gamma'_1 w'_\gamma\\&\le\sup_{t\in\operatorname{spt} w_\gamma}g(\gamma)||\gamma'-\gamma'_1||_{L^2}||w'||_{L^2}\\&\quad+\sup_{t\in\operatorname{spt} w_\gamma}[g(\gamma)-g(\gamma_1)]||\gamma'_1||_{L^2}||w'||_{L^2}\le\text{const.}\cdot\varepsilon,\end{aligned}$$

in fact $g(\gamma)\in C^\infty(\operatorname{spt} w_\gamma)$, so $\sup g(\gamma)$ is bounded. Furthermore, because

$$||\gamma-\gamma_1||_{L^\infty}<C\cdot\varepsilon,\quad\sup[g(\gamma)-g(\gamma_1)]\le C\cdot\varepsilon.$$

In the same way

$$\begin{aligned}&\int\langle\nabla g(\gamma),w_\gamma\rangle|\gamma'|^2-\int\langle\nabla g(\gamma_1),w_\gamma\rangle|\gamma'_1|^2\\&\quad\le\int\langle\nabla g(\gamma),w_\gamma\rangle(|\gamma'|^2-|\gamma'_1|^2)+\int\langle\nabla g(\gamma)-\nabla g(\gamma_1),w_\gamma\rangle|\gamma'_1|^2\le\text{const.}\cdot\varepsilon.\end{aligned}$$

So $dE(\gamma)[w_\gamma]-dE(\gamma_1)[w_\gamma]\le C\cdot\varepsilon$: we can choose a neighbourhood $V_\gamma$, for all $\gamma\in S_1\setminus U$, such that

$$dE(\gamma_1)[w_\gamma]<-\nu_1\quad\text{for all }\gamma_1\in V_\gamma.$$

The sets $V_\gamma$ covers the whole $S_1\setminus U$. Let $V_{\gamma_i}$ be a locally finite refinement of $V_\gamma$. Let $\beta_i$ be a partition of the unity associated to $V_{\gamma_i}$. Then

$$F_1=\sum\beta_i w_{\gamma_i}$$

is a pseudo-gradient vector field on $S_1\setminus U$ (for the details of such a construction see [15]). Now let $\alpha_j$ be a partition of the unity associated to $S_1\setminus U,\Omega_a^b\setminus S$, then

$$F=\alpha_1F_1-\alpha_2\nabla E$$

is the vector field we looked for, in fact we can find $\eta_U$ because $F$ is a Lipschitz vector field by definition. Even if $E$ isn't smooth, we can differentiate it along the direction of $F$, so

$$\begin{aligned} E(\eta_U(\gamma,t)) - E(\gamma) &= \int_0^t \frac{d}{d\tau} E(\eta_U(\gamma,\tau))\,d\tau \\ &= \int_0^t dE(\eta_U(\gamma,\tau))[\dot\eta_U(\gamma,\tau)]\,d\tau = \int_0^t dE(\eta_U(\gamma,\tau))[F]. \end{aligned}$$

Let $\nu = \min(\nu_0,\nu_1)$. Then

$$\begin{aligned} E(\eta_U(\gamma,t)) - E(\gamma) &= \int_0^t \alpha_1\, dE(\eta_U(\gamma,\tau))[F_1] - \alpha_2 ||\nabla E(\eta_U(\gamma,\tau))||^2 \\ &= \int_0^t \alpha_1 \sum \beta_i\, dE(\eta_U(\gamma,\tau))[w_{\gamma_i}] - \alpha_2 ||\nabla E(\eta_U(\gamma,\tau))||^2 \\ &\le \int_0^t -\alpha_1 \sum \beta_i \nu_1 - \alpha_2\nu_0 \le -\int_0^t \nu \le -\nu t. \qquad \square \end{aligned}$$

From Lemmas 2.4 and 2.5 we get the following result.

Theorem 2.6. *Let $M$ be a conical manifold with only a vertex $v$, and consider the special closed geodesic $\gamma_0$ for which there exists an unique $\sigma$ such that $\gamma_0(\sigma) = v$. Set $E(\gamma_0) = c_0$. Suppose that there exist $a, b \in \mathbb{R}$, $c_0 < a < b$ such that $\Omega^b$ contains only the geodesics $\gamma_0$. Then $\Omega^b \simeq \Omega^a$.*

Proof. Given $R$ as in Lemma 2.4, we choose $U$ and $V$ neighbourhoods of $\Sigma_0$ such that

$$\Sigma_0 \subsetneq U \subsetneq V \subsetneq R.$$

We know that, for such an $U$, there exists a retraction $\eta_U$ defined as in Lemma 2.5. For the sake of simplicity we will suppose that $\eta_U$ and $\eta_R$ (see Lemma 2.4) are defined for $0 \le t \le 1$ and that $\nu$ is the same for both of them.

Let $\theta_1 \colon \Omega^b \to [0,1]$ a continuous map such that

$$\theta_1|_U \equiv 0, \quad \theta_1|_{\Omega^b \setminus V} \equiv 1.$$

Then we define a continuous map

$$\mu_1 \colon \Omega^b \times [0,1] \to \Omega^b, \quad \mu_1(\beta,t) = \eta_U(\beta,\theta_1(\beta)t).$$

We know that $E(\mu_1(\beta,t)) - E(\beta) \le -\nu t\theta_1(\beta)$, so $\mu_1(\Omega^b,1) \subset V \cup \Omega^{b-\nu}$. In fact, if $\mu_1(\beta,t) \notin V$ for all $t$, then $E(\mu_1(\beta,t)) - E(\beta) \le -\nu t$, so $\mu_1(\beta,1) \in \Omega^{b-\nu}$.

By $\mu_1$ we have retracted $\Omega^b$ on $\Omega^{b-\nu} \cup V$. Now we define a continuous map $\theta_2 \colon \Omega^b \to [0,1]$ such that

$$\theta_1|_{\Omega^b_{b-\nu/2}} \equiv 1, \quad \theta_1|_{\Omega^{b-\nu}} \equiv 0.$$

Then set

$$\mu_2 : V \cup \Omega^{b-\nu} \times [0,1] \to \Omega^b, \quad \mu_2(\beta, t) = \eta_R(\beta, \theta_2(\beta)t);$$

$\mu_2$ is a continuous map that retracts $V \cup \Omega^{b-\nu}$ on $\Omega^{b-\nu/2}$. By iterating this algorithm we can retract continuously $\Omega^b$ on $\Omega^a$. $\square$

Now we can prove the deformation lemma.

Proof of Theorem 2.2. Let $\{\gamma_i\}_{i=1,\dots,N}$ be the set of geodesics in $\Omega^b$. We start defining some special subset of $\Omega_a^b$, as in (2.1) and (2.2); let

$$\Sigma = \{\gamma \in \Omega_a^b : \operatorname{Im} \gamma \cap V \neq \emptyset\}$$

(we recall that $V$ is the set of vertexes); for $i = 1, \dots N$, set

$$\Sigma_i = \{\gamma \in \Sigma : \gamma = \gamma_i \text{ up to affine reparametrization}\}.$$

We note that for $i \neq j$ then $\Sigma_i \cap \Sigma_j = \emptyset$, because the geodesics are different. For these $\Sigma_i$ we can find a retraction $\eta_\Sigma$ as in Lemma 2.4: indeed, for every $U \supset \bigcup_{i=0}^N \Sigma_i$ there exists a retraction $\eta_U$ on $\Omega_a^b \setminus U$ in analogy with Lemma 2.5. Finally, we compound these two maps $\eta_\Sigma$ and $\eta_U$ following the proof of Theorem 2.6 and we conclude. $\square$

Theorem 2.7 (Second deformation lemma). *Let $M$ be a conical manifold, $p \in M$. Suppose that there exists $c \in \mathbb{R}$ such that $\Omega^c$ contains only a finite number of geodesics and that there exists $a, b \in R$, $a < b < c$ such that the strip $[a, b)$ contains only regular values of $E$. Set $Z$ the set of geodesics and $Z_b = Z \cap E^{-1}(b)$, then there exists a neighbourhood $U$ of $Z_b$ such that*

$$\Omega^b \setminus U \simeq \Omega^a.$$

Proof. We can prove this corollary following the lines of Theorem 2.2. $\square$

As previously said, Lemma 2.4, which is crucial for this work, is based on a generalization of [2, Theorem 2.8]. Indeed, using a slight modification of the weak slope tool, this result and the deformation lemmas can be reformulated in a more general context. This theoretic frame is briefly discussed in the Appendix.

## 3. Category theory

First, we recall some well known results relative to the Lusternik and Schnirelmann category. This theory was presented in [9] in a finite dimensional framework, then generalized to Banach manifold by R. Palais [14].

DEFINITION 3.1. Let $X$ be a topological space, $A \subset X$. If $A \neq \emptyset$ we say that

$$\operatorname{cat} A = \operatorname{cat}_X A = k$$

if and only if $k$ is the least integer for which there are $F_1, \dots, F_k$ closed contractible subsets of $X$ such that $\bigcup_k F_k$ covers $A$. We define also

$$\operatorname{cat} \emptyset = \operatorname{cat}_X \emptyset = 0.$$

THEOREM 3.2. *Let $X$ be a topological space. Then:*

(a) *if $A \subset B \subset X$ then $\operatorname{cat}_X A \leq \operatorname{cat}_X B$,*
(b) *if $A, B \subset X$ then $\operatorname{cat}_X A \cup B \leq \operatorname{cat}_X A + \operatorname{cat}_X B$,*
(c) *if $A, B \subset X$, $A$ closed, and there is $\eta \in C([0,1] \times A, X)$ such that*

$$B = \eta(1, A), \quad \eta(0, u) = u \quad \text{for all } u \in A,$$

*then $\operatorname{cat}_X A \leq \operatorname{cat}_X B$,*
(d) *if $Y$ is a topological space, $y \in Y$, then $\operatorname{cat}_{X+Y}(A \times \{y\}) + \operatorname{cat}_X A$.*

PROOF. The points (a), (b) and (d) are trivial. We have only to prove (c). By hypothesis, we can find $F_1, \dots, F_k$ such that $B \subset F_1 \cup \ldots \cup F_k$. Set

$$C_i = \{u \in A : \eta(1, u) \in F_i\}.$$

Obviously, $C_i$ are closed and contractible. Since $C_1 \cup \ldots \cup C_k$ covers $A$ we obtain the thesis. $\square$

By Theorem 2.7 we are able to reconstruct the category theory for the energy functional defined on a conical manifold.

LEMMA 3.3. *Let $M$ be a conical manifold, $p \in M$. Suppose that there exists $\overline{c} \in \mathbb{R}$ such that $\Omega^{\overline{c}}$ contains only a finite number of geodesics. Let $c < \overline{c}$ a critical level for $E$. Then, set $U$ a neighbourhood of $Z_c$ there exists $\varepsilon > 0$ such that*

$$\operatorname{cat} \Omega^{c+\varepsilon} \leq \operatorname{cat} \Omega^{c-\varepsilon} + \operatorname{cat} U.$$

PROOF. We know, by the second deformation lemma, that $\Omega^{c-\varepsilon}$ is a deformation retract of $\Omega^{c+\varepsilon} \setminus U$: applying Theorem 3.2 we obtain

$$\operatorname{cat} \Omega^{c+\varepsilon} \leq \operatorname{cat} \Omega^{c+\varepsilon} \setminus U + \operatorname{cat} U \leq \operatorname{cat} \Omega^{c-\varepsilon} + \operatorname{cat} U. \qquad \square$$

THEOREM 3.4. *Let $M$ be a conical manifold, let $p \in M$ and let $a < b \in \mathbb{R}$. Then $\Omega_a^b$ contains at least $\operatorname{cat} \Omega^b - \operatorname{cat} \Omega^a$ geodesics.*

PROOF. We suppose that there is a finite number of critical levels in $[a, b]$ (otherwise there is nothing to prove). Set $a \leq c_0 < c_1 < \ldots < c_k \leq b$ these critical levels, and set, for all $i$, $U_i$ a neighbourhood of $Z_{c_i}$. We know that there exists an $\varepsilon$ such that for all $i$

$$\operatorname{cat}\Omega^{c_i+\varepsilon} \leq \operatorname{cat}\Omega^{c_i-\varepsilon} + \operatorname{cat} U_i. \tag{3.1}$$

By iterating (3.1), and using the deformation lemma, we obtain

$$\begin{aligned}\operatorname{cat}\Omega^{c_k+\varepsilon} &\leq \operatorname{cat}\Omega^{c_k-\varepsilon} + \operatorname{cat} U_k \leq \operatorname{cat}\Omega^{c_{k-1}+\varepsilon} + \operatorname{cat} U_k \\ &\leq \operatorname{cat}\Omega^{c_{k-2}+\varepsilon} + \operatorname{cat} U_{k-1} + \operatorname{cat} U_k \leq \ldots \leq \operatorname{cat}\Omega^{c_0-\varepsilon} + \sum_{i=0}^{k} \operatorname{cat} U_i.\end{aligned}$$

Because $\operatorname{cat}\Omega^b \leq \operatorname{cat}\Omega^{c_k+\varepsilon}$ and $\operatorname{cat}\Omega^{c_0-\varepsilon} \leq \operatorname{cat}\Omega^a$ we have that

$$\operatorname{cat}\Omega^b - \operatorname{cat}\Omega^a \leq \sum_{i=0}^{k} \operatorname{cat} U_i.$$

Suppose now that there are a finite number of geodesics for any critical level. Because every point has a contractible neighbourhood, we can choose $U_i$ such that

$$\operatorname{cat} U_i \leq \# Z_{c_i},$$

thus

$$\operatorname{cat}\Omega^b - \operatorname{cat}\Omega^a \leq \sum_{i} \# Z_{c_i}. \qquad \square$$

From Theorem 3.4 the main result of this paper follows.

Corollary 3.5. *Let $M$ be a conical manifold, $p \in M$. Then there are at least* $\operatorname{cat}\Omega$ *geodesics.*

Proof. If there is an infinite number of geodesics, there is nothing to prove.

Otherwise, we can apply the previous theorem and we conclude by a limiting process. (consider that $\Omega^{-1} = \emptyset$ and that $\Omega^b \simeq \Omega$ for $b \gg 1$). $\square$

## 4. An application

We show a topological lemma necessary to provide some applications.

Let $X$ a smooth submanifold of $\mathbb{R}^n$. Given $g \in L^\infty(X, \mathbb{R}^+)$, set

$$E(\gamma) = \int_0^1 g(\gamma(s))|\gamma'|^2\, ds.$$

We set

$$\begin{aligned} G(I,X) &= \{\gamma \in C^0(I,X) : E(\gamma) \text{ is well defined and finite}\}, \\ G(S^1,X) &= \{\gamma \in C^0(S^1,X) : E(\gamma) \text{ is well defined and finite}\}.\end{aligned}$$

Obviously we have that

$$H^1(I,X) \subset G(I,X) \subset C^0(I,X), \quad H^1(S^1,X) \subset G(S^1,X) \subset C^0(S^1,X).$$

We recall that

$$\begin{aligned}\Omega &= \Omega_p X = \{\gamma \in H^1([0,1],X) : \gamma(0)=\gamma(1)=p\},\\ \Omega^\infty &= \Omega_p^\infty X = \{\gamma \in C^0(S^1,X) : \gamma(0)=\gamma(1)=p\},\end{aligned}$$

as previously defined. We define also the free loop space on $X$ as

$$\begin{aligned}\Lambda &= \Lambda X = \{\gamma \in H^1(S^1,X)\},\\ \Lambda^\infty &= \Lambda^\infty X = \{\gamma \in C^0(S^1,X)\}.\end{aligned}$$

In analogous way we set $G$ (resp. $G_p$) the subspace of $\Lambda^\infty$ (resp. $\Omega^\infty$) in which $E$ is well defined and finite, according with previous definitions. These definitions allow us to formulate the following lemma.

Lemma 4.1. *Let $X, g$ and $E(\,\cdot\,)$ be as above. Then*

$$\operatorname{cat}_G G \geq \operatorname{cat}_{\Lambda^\infty}\Lambda^\infty; \tag{4.1}$$

$$\operatorname{cat}_{G_p} G_p \geq \operatorname{cat}_{\Omega^\infty}\Omega^\infty. \tag{4.2}$$

*In particular, if $X$ is a connected and non contractible manifold then*

$$\operatorname{cat} G_p = \operatorname{cat}\Omega^\infty = \infty. \tag{4.3}$$

Proof. We show only (4.1), then (4.2) follows in the same way. Because $X$ is a smooth manifold, it is well known that there is an homotopic equivalence between $\Lambda^\infty$ and $\Lambda$ (see e.g. [8, Theorem 1.2.10]). Then

$$\operatorname{cat}_{\Lambda^\infty}\Lambda = \operatorname{cat}_{\Lambda^\infty}\Lambda^\infty.$$

Now, because $\Lambda \subset G \subset \Lambda^\infty$, we have $\operatorname{cat}_G G \geq \operatorname{cat}_{\Lambda^\infty}\Lambda = \operatorname{cat}_{\Lambda^\infty}\Lambda^\infty$, that proves (4.2).

Formula (4.3) is a standard result and can be found, for example in [6, Corollary 1.2] $\square$

By this result we can compute $\operatorname{cat}\Omega$ in some concrete case, as shown in the next example.

Example 4.2. Let $M \subset \mathbb{R}^n$ a compact conical manifold, $V$ the set of its vertexes. Suppose that there exists a compact smooth manifold $X \subset \mathbb{R}^k$ and an homeomorphism $\psi: M \to X$ such that $\psi_{|_{M\setminus V}} \in C^\infty(M \setminus V, X)$, then there exists $g^*$ an induced metric on $X$ defined by

$$g_p^*(v,w) := \begin{cases} g_{\psi^{-1}(p)}(d\psi^{-1}(v),(d\psi^{-1}(w)) & \text{on } X \setminus \psi(V),\\ 0 & \text{otherwise.}\end{cases}$$

If $|d\psi^{-1}| \in L^\infty(X)$, we have that $g^*$ is bounded with respect to the Euclidean metric of $X$ and that

$$\Omega_{\psi^{-1}(p)}(M) = G_p(X) := \left\{\gamma \in C^0([0,1],X) : \int_0^1 g^*_\gamma|\gamma'|^2 < \infty\right\}.$$

In this case, we can apply Lemma 4.1 to compute the category of based (or free) loop space of $M$.

We also state an immersion theorem that is, in some sense, the converse of previous example.

Theorem 4.3 (Nash immersion for conical manifolds). *Let $X$ a smooth manifold and let $g$ a continuous non negative and bounded bilinear tensor such that there exist $V$ a finite set of points and $g$ is smooth and positive defined on $X \setminus V$. Then:*

(a) *If $V = \{x\}$, then, for $N$ sufficiently large, there exists $M \subset \mathbb{R}^N$ a conical manifold and a continuous map $\psi: X \to M$ such that $\psi_{|X\setminus V}$ is a $C^\infty$ isometry.*
(b) *If $V = \{x_1, \ldots, x_k\}$, for every $x_i$ it exists $\rho_i > 0$ such that $B(x_i,\rho_i)$ is isometric (in the sense above specified) to some conical manifold $M_i \subset \mathbb{R}^N$.*

Proof. We start proving (a). By hypothesis, $(X \setminus V, g)$ is a Riemannian manifold, so, by Nash theorem [12], it can be embedded in $\mathbb{R}^N$, for $N$ sufficiently large. Let $\psi: X \setminus V \to M$ be this embedding.

We can continuously extend $\psi$ to the whole $X$. In fact, let $\{x_n\}_n$ be a Cauchy sequence converging to $x$; because $g$ is bounded, then $\{\psi(x_n)\}_n$ is a Cauchy sequence in $\mathbb{R}^N$, so there exists $y \in \mathbb{R}^N$ such that $\lim \psi(x_n) = y$. Set $\psi(x) := y$: obviously we have that

$$\psi(B_X(x,\rho)) \subset B_{\mathbb{R}^N}(y,r),$$

and $r \xrightarrow{\rho\to 0} 0$, so $\psi$ is continuous at $x$.

Then, set $M := \psi(X)$, we have that $M$ is a conical manifold with vertex $y$, isometric to $X$.

To proof (b), it is sufficient to choose $\rho_i$ such that $B(x_i,\rho_i)$ are all disjoint. Then we apply the previous result with $X = B(x_i,\rho_i)$. □

By this result, we formulate a result which will be useful in the next of this paper.

Theorem 4.4. *In the above hypothesis, we have that*

$$\textit{number of geodesics in } X \geq \operatorname{cat} G$$

Proof. If $X$ has an unique vertex, it is isometric to a conical manifold $M$. Then, by applying Lemma 4.1, we obtain the proof. If the manifold $X$ has several vertexes, we are in the case (b) of previous theorem.

Anyway, by the local isometries, we can prove an analogous of deformation lemma for geodesics in $X$. Also an analogous of Theorem 3.4 follows. This, paired with Lemma 4.1 gives us the proof. $\square$

**4.1. Brachistochrones.** In this section we want to study the brachistocrones problem. A brachistochrone is a curve $\gamma$ which minimizes the time of transit for a particle moving from a point $p$ towards a point $q$. We study this problem on $(S^n, \langle\,\cdot\,,\,\cdot\,\rangle)$ an Euclidean sphere embedded in $\mathbb{R}^{n+1}$. We suppose that the particle moves in the presence of a potential $U\colon S^n \to \mathbb{R}$ without friction. Also, we are interested to any curve stationary for the time of transit functional.

Be $p, q \in S^n$, $E \in \mathbb{R}^+$ the energy of the particle, $U \in C^\infty(S^n, \mathbb{R})$ the given potential. It is well known that, if there exist $c_1, c_2 \in \mathbb{R}$ such that

$$-\infty < c_1 < U(\,\cdot\,) < c_2 < E,$$

then this problem is equivalent to the geodesic problem for the Riemannian manifold

$$\left(S^n, g := g_x = \frac{\langle\,\cdot\,,\,\cdot\,\rangle}{E - U(x)}\right),$$

and that the metric $g$ is equivalent to the Euclidean metric on $S^n$, so the problem has always a solution. Furthermore, it is also well known that a solution exists even if the upper bound on $U(x)$ does not exists.

In this section we want to study the problem for a given potential

$$U \in C^\infty(S^n \setminus V, \mathbb{R})$$

where $V = \{x_1, \dots x_k\}$ a finite set of points on the sphere, and

$$U(x) \stackrel{x \to x_i}{\longrightarrow} -\infty.$$

As we will see in the next section, potential in $S^n$ with these kind of singularities may appear from non singular potential defined in $\mathbb{R}^n$.

For the sake of simplicity, we suppose that there exist $c > 0$ for which $E > c > U(\,\cdot\,)$. We define a metric on $S^n$ by

$$g := g_x = \begin{cases} \dfrac{\langle\,\cdot\,,\,\cdot\,\rangle}{E - U(x)} & \text{on } S^n \setminus V, \\ 0 & \text{otherwise,} \end{cases}$$

and we look for $g_x$-geodesics between two given points $p, q \in S^n$. Set, as usual

$$
\begin{aligned}
G(p) &= \left\{\gamma \in C^0([0,1], S^n) : \gamma(0) = \gamma(1) = p,\ \frac{1}{2}\int_0^1 g(\gamma', \gamma') < \infty\right\},\\
G(p,q) &= \left\{\gamma \in C^0([0,1], S^n) : \gamma(0) = p,\ \gamma(1) = q,\ \frac{1}{2}\int_0^1 g(\gamma', \gamma') < \infty\right\},
\end{aligned}
$$

we know that $g$ satisfies the hypothesis of Lemma 4.1, so $\operatorname{cat} G(p) = \infty$.

It's easy to prove that there is an homotopy equivalence between $G(p)$ and $G(p,q)$, in fact, for any given couple of points $p$, $q$, there exists a continuous curve $\gamma$ which joins them, with $E(\gamma) < \infty$ (because $g$ is bounded). Then there is a map

$$i\colon G(p) \to G(p,q), \quad \beta \mapsto \beta + \gamma,$$

where $\beta + \gamma$ is the usual composition of paths.

Of course there exists the inverse map

$$i^{-1}\colon G(p,q) \to G(p), \quad \beta \mapsto \beta + (-\gamma)$$

and $i^{-1} \circ i$ is homotopic equivalent to $1_{G(p)}$.

By the above consideration and by Nash theorem we have that

$$\infty = \operatorname{cat} G(p) = \operatorname{cat} G(p,q) = \text{number of geodesics between } p \text{ and } q,$$

thus we can count the number of brachistochrones on the sphere in presence of our potential $U$.

**4.2. Brachistocrones in $\mathbb{R}^n$.** A more interesting application is the study of the same brachistochrone problem in $\mathbb{R}^n$ (indeed this was the very beginning of our research). Let $U \in C^\infty(\mathbb{R}^n, \mathbb{R})$ and $E > 0$ such that

- $E > U(x)$,
- $-U(x) = O(|x|^\alpha)$ when $|x| \gg 1$, for some $\alpha > 0$.

We are looking for brachistocrones joining two given points $p, q \in \mathbb{R}^n$ in presence of potential $U(x)$. As above we look for geodesics in

$$\left(\mathbb{R}^n, g_x := \frac{\langle\,\cdot\,,\,\cdot\,\rangle}{E - U(x)}\right)$$

where $1/(E - U(x)) \in C^\infty(\mathbb{R}^n, \mathbb{R} \setminus \{0\}) \cap L^\infty(\mathbb{R}^n)$.

We can map $\mathbb{R}^n$ in $S^n \subset \mathbb{R}^n + 1$ by the stereographic map $\pi$. The inverse map is

$$
\pi^{-1} : \begin{cases} S^n \setminus N \subset \mathbb{R}^{n+1} \to \mathbb{R}^n, \\ \begin{pmatrix} y_1 \\ \vdots \\ y_{n+1} \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1/(1 - y_{n+1}) \\ \vdots \\ y_n/(1 - y_{n+1}) \end{pmatrix}, \end{cases}
$$

where $N$ is the north pole of $S^n$. As usual we can induce a metric $g^*$ on $S^n$ defined by

$$g^*(y)(v,w) = \begin{cases} g_{\pi^{-1}(y)}(d\pi^{-1}v, d\pi^{-1}(w)) & \text{on } S^n \setminus N, \\ 0 & \text{for } y = N. \end{cases}$$

It is easy to see that

$$|d\pi^{-1}| = O\left(\frac{1}{\sqrt{1-y_{n+1}}}\right),$$

that, read on $\mathbb{R}^n$, becomes

$$|d\pi^{-1}| = O\left(\frac{1}{|x|}\right).$$

So, if $\alpha > 2$, then $g^*$ is bounded with respect to the Euclidean metric on $S^n$, and we can apply Lemma 4.1.

Furthermore, by Nash embedding, there is an isometry with a compact conical manifold, so we can easily state that there is an infinite number of brachistocrones joining $p$ and $q$, although we cannot say if they are bounded in $\mathbb{R}^n$, and so physical meaningful.

## 5. Appendix. The theoretic frame

As said, our deformation lemmas (Lemma 2.2 and Theorem 2.7) are obtained modifying a weak slope theory resutlt. In this section we present the *k-slope*, a generalization of the weak slope, which allows us to reformulate the main results of this paper in a more general framework.

We start recalling the definition of weak slope.

Definition 5.1. Let $(X,d)$ be a metric space and let $f\colon X \to \mathbb{R}$ be a continuous functional. The *weak slope* of $f$ at $u \in X$ (noted $|df|(u)$) is the supremum of $\sigma$'s in $[0,\infty)$ such that therer exists $\delta > 0$ and $\mathcal{H}\colon B(u,\delta) \times [0,\delta] \to X$ continuous with

$$d(\mathcal{H}(v,t), v) \le t, \tag{5.1}$$

$$f(\mathcal{H}(v,t)) - f(v) \le -\sigma t, \tag{5.2}$$

for every $v \in B(u,\delta)$, $t \in [0,\delta]$.

Due to (5.1) we can prove a deformation property for continuous functionals ([2, Theorem 2.8]): this inequality allows us to compound the local maps $\mathcal{H}$ finding a global retraction.

Unhappily, these tools are not completely useful for our purposes. In particular we was not able to prove an estimate like (5.1). In our work we override these difficulties using the compactness of sets $\Sigma_i$ and compounding explicitly all the local retractions. This method has a generalization that we present here.

**5.1 The $k$-slope.** We define an extension of weak slope which will be called *$k$-slope.*

Definition 5.2. Let $(X,d)$ be a metric space. Let $f\colon X\to\mathbb{R}$ be a continuous functional and let $u\in X$. We define the *$k$-slope* of $f$ at $u\in X$ (noted $|d_k f|(u)$) as the supremum of $\sigma\in[0,\infty)$ such that exist $\delta>0$, $k_u\colon[0,\delta]\to\mathbb{R}^+$ continuous, $k_u(0)=0$, and a continuous map $\mathcal{H}\colon B(u,\delta)\times[0,\delta]\to X$ which satisfies

$$d(\mathcal{H}(v,t),v)\le k_u(t), \tag{5.3}$$

$$f(\mathcal{H}(v,t))-f(v)\le-\sigma t, \tag{5.4}$$

for all $v\in B(u,\delta)$, for all $t\in[0,\delta]$.

In analogy with the weak slope theory we can prove the following property.

Proposition 5.3. *If $f$ is continuous, $|d_k f|$ is lower semi-continuous.*

Proof. If $|d_k f|(u)=0$ the proof is obvious. Otherwise, for any $0<\sigma<|d_k f|(u)$ there exist $\delta$ and $\mathcal{H}\colon B(u,\delta)\times[0,\delta]\to X$ as in Definition 5.2. Let $u_h\to u$. Definitively we have $u_h\in B(u,\delta/2)$, so we can take the restriction of $\mathcal{H}$ to $B(u_h,\delta/2)\times[0,\delta/2]$, to have $|d_k f(u_h)|\ge\sigma$. This completes the proof. $\square$

Obviously we say that $u\in X$ is a *critical point* if $|d_k f|(u)=0$.

**5.2. The deformation lemma.** We are able now to formulate the wanted deformation property.

Theorem 5.4. *Let $(X,d)$ be a metric space, and $f\colon X\to\mathbb{R}$ a continuous functional. Suppose that $\sigma\in\mathbb{R}^+$ exists such that $|d_k f|(u)\ge\sigma$ for all $u\in X$. Let $C\subset X$ be a compact subspace such that*

$$k_u(t)\le t\quad\text{for all } u\in X\setminus C. \tag{5.5}$$

*Then there exists a $\tau\in\mathbb{R}^+$ and a continuous function $\mu\colon X\times[0,\tau]\to X$ such that*

$$\begin{aligned}\mu(u,0)&=u &&\text{for all } u\in X,\\ f(\mu(u,t))-f(u)&\le-\sigma t &&\text{for all } u\in X,\ t\in[0,\tau].\end{aligned}$$

Before proving Theorem 5.4, we prove two deformation lemmas for $C$ and $X\setminus C$ analogues to Lemmas 2 4 and 2.5. To conclude the proof we will attach the retractions found.

We recall a topological lemma by John Milnor useful for the next results.

LEMMA 5.5 (Milnor's lemma). *Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of a paracompact space $X$. There is a locally finite open cover $V_{j,\lambda}$ refining $\{U_\alpha\}$ such that $V_{j,\lambda}\cap V_{j,\mu}=\emptyset$ if $\lambda\neq\mu$.*

PROOF. For the proof we refer to [13, Lemma 2.4]. Here we report only how to construct the open cover $\{V_{i,\lambda}\}_{i,\lambda}$.

By an initial refinement we can take $\{U_\alpha\}$ locally finite. Then, let $\Lambda_j$ be the set of $(j+1)$-ples $\lambda=\{\alpha_0,\dots,\alpha_j\}$ of elements in $A$. Let $\{\varphi_\alpha\}_\alpha$ be a partition of unity with $\operatorname{spt}\varphi_\alpha\subset U_\alpha$. For $\lambda\in\Lambda_j$ let

$$V_{j,\lambda}=\{x\in X:\varphi_\alpha>0 \text{ if } \alpha\in\lambda \text{ and } \varphi_\gamma<\varphi_\alpha \text{ if } \alpha\in\lambda,\ \gamma\notin\lambda\},$$

so we have found our locally finite open cover $V_{j,\lambda}$. □

With this lemma, we prove the deformation results.

LEMMA 5.6 (Deformation lemma for $C$). *Let $(X,d)$ be a metric space, and $C\subset X$ be a compact set. Let $\sigma\in\mathbb{R}^+$ and let $f\colon X\to\mathbb{R}$ be a continuous function such that*

$$|d_k f|(u)>\sigma \quad \text{for all } u\in C.$$

*Then there exist $\widetilde{C}\supset C$, $\tau\in\mathbb{R}^+$ and $\eta\colon\widetilde{C}\times[0,\tau]\to X$ a continuous functional such that:*

(a) $\eta(u,0)=u$ *for all* $u\in\widetilde{C}$,
(b) $f(\eta(u,t))-f(u)\leq-\sigma t$ *for all* $u\in\widetilde{C}$, $t\in[0,\tau]$.

PROOF. We know by hypothesis that $|d_k f|(u)\geq\sigma$, so for every $u\in C$ there exist a $\delta_u>0$, a continuous map $k_u\colon[0,\delta_u]\to\mathbb{R}^+$, $k_u(0)=0$, and a continuous function

$$\mathcal{H}_u\colon B(u,\delta_u)\times[0,\delta_u]\to X$$

satisfying (5.3) and (5.4). By Milnor's Lemma we know that the open cover $\{B(u,\delta_u/2),\ u\in C\}$ admits a locally finite refinement $\{V_{j,\lambda}:j\in\mathbb{N},\ \lambda\in\Lambda_j\}$ such that

$$\lambda\neq\mu\Rightarrow V_{j,\lambda}\cap V_{j,\mu}=\emptyset.$$

By compactness of $C$ we can suppose that $\{V_{j,\lambda}\}$ be a finite family. In particular there will be an $h_0$ and a finite number of elements in $\Lambda_j$ such that the family $\{V_{j,\lambda}:j=1,\dots,h_0,\ \lambda\in\Lambda_j\}$ covers the whole $C$.

Let $\vartheta_{j,\lambda}\colon X\to[0,1]$ be a family of continuous functionals with

$$\operatorname{spt}\vartheta_{j,\lambda}\subset V_{j,\lambda},\quad \sum_{j=1}^{h_0}\sum_{\lambda\in\Lambda_j}\vartheta_{j,\lambda}(u)=1.$$

For every $(j,\lambda)$ let $V_{j,\lambda}\subset B(u_{j,\lambda},\delta_{u_{j,\lambda}})$. To simplify the notations set $\delta_{j,\lambda}=\delta_{u_{j,\lambda}}$, $k_{j,\lambda}=k_{u_{j,\lambda}}$ and $\mathcal{H}_{j,\lambda}=\mathcal{H}_{u_{j,\lambda}}$. Let $\tau_0$ be a positive real number such that

$0 < \tau_0 < \min \delta_{j,\lambda}$, so every $k_{j,\lambda}$ is well defined on $[0,\tau_0]$. Let

$$k(t) = \bigvee_{j,\lambda} k_{j,\lambda}(t);$$

let $\tau_1$ be a positive real number such that

$$\max_{t\in[0,\tau_1]} k(t) \le \frac{1}{2}\,\frac{\min \delta_{j,\lambda}}{h_0 \sum_j \#\Lambda_j}.$$

Set $\tau = \min\{\tau_0,\tau_1\}$.

Now, called $\widetilde{C} = \bigcup_{j,\lambda} \overline{V}_{j,\lambda}$, we want to define a sequence of continuous map $\eta_h \colon \widetilde{C} \times [0,\tau] \to X$ such that

$$d(\eta_h(v,t),v) \le \sum_{j=1}^{h} \sum_{\lambda\in\Lambda_j} k(\vartheta_{j,\lambda}(v)t), \tag{5.6}$$

$$f(\eta_h(v,t)) - f(v) \le -\sigma\bigg(\sum_{j=1}^{h} \sum_{\lambda\in\Lambda_j} \vartheta_{j,\lambda}(v)\bigg)t. \tag{5.7}$$

First of all we set

$$\eta_1(v,t) = \begin{cases} \mathcal{H}_{1,\lambda}(v,\vartheta_{1,\lambda}(v)t) & \text{if } v \in \overline{V}_{1,\lambda}, \\ v & \text{if } v \notin \bigcup\limits_{\lambda\in\Lambda_1} V_{1,\lambda}. \end{cases}$$

Obviously $\eta_1$ satisfies (5.6) and (5.7); now we proceed by induction: assume that we have defined $\eta_{h-1}$ satisfying (5.6) and (5.7). For every $v \in \overline{V}_{h,\lambda}$ we have

$$d(\eta_{h-1}(v,t),v) \le \sum_{j=1}^{h-1} \sum_{\lambda\in\Lambda_j} k(\vartheta_{j,\lambda}t) \le (h-1)\sum_j \#\Lambda_j \max_{t\in[0,\tau]} k(t) \le \frac{1}{2}\,\delta_{h,\lambda},$$

hence $\eta_{h-1}(v,t) \in B(u_{h,\lambda},\delta_{h,\lambda})$, so the map

$$\eta_h(v,t) = \begin{cases} \mathcal{H}_{h,\lambda}(\eta_{h-1}(v,t),\vartheta_{h,\lambda}(v)t) & \text{if } v \in \overline{V}_{h,\lambda}, \\ \eta_{h-1}(v,t) & \text{if } v \notin \bigcup\limits_{\lambda\in\Lambda_h} V_{h,\lambda}, \end{cases}$$

is well defined and satisfies (5.6) and (5.7). Now we set

$$\eta(u,t) = \eta_{h_0}(u,t),$$

so we have that $\eta \colon \widetilde{C} \times [0,\tau] \to X$ is continuous. Furthermore,

$$d(\eta(v,t),v) \le \sum_{j=1}^{h} \sum_{\lambda\in\Lambda_j} k(\vartheta_{j,\lambda}(v)t) \Rightarrow \eta(0,v) = v,$$

$$f(\eta(v,t)) - f(v) \le -\sigma\bigg(\sum_{j=1}^{h-1} \sum_{\lambda\in\Lambda_j} \vartheta_{j,\lambda}(v)\bigg)t = -\sigma t,$$

that concludes the proof $\square$

In this lemma we have used the compactness of $C$ to compound the local retractions without using the property (5.1) of the weak slope. To find a retraction on $X \setminus C$ we must suppose that $k_u(t) \le t$ and proceed as in Degiovanni, Marzocchi and Corvellec work [2].

Lemma 5.7 (deformation lemma for $X \setminus C$). *Let $(X,d)$ be a metric space, $\sigma \in \mathbb{R}^+$ and $f: X \to \mathbb{R}$ be a continuous function such that*

$$|d_k f|(u) > \sigma \quad \text{for all } u \in X.$$

*Suppose also that there exists a compact set $C \subset X$ such that*

$$k_u(t) \le t \quad \text{for all } u \in X \setminus C, \tag{5.8}$$

*where $k_u$ is defined as in* (5.3). *Then exist $\tau \in \mathbb{R}^+$ and $\eta: X \setminus C \times [0,\tau] \to X$ a continuous functional such that*

(a) *$\eta(u,0) = u$ for all $u \in X \setminus C$,*
(b) *$f(\eta(u,t)) - f(u) \le -\sigma t$ for all $u \in X \setminus C$, $t \in [0,\tau]$.*

Proof. For all details see [2, Theorem 2.8]. We note only that the proof is quite similar to Lemma 5.6, but for proving that $\eta_h$ is well defined we must use the inequality (5.8) to obtain a good estimate of $d(\eta_{h-1}(v,t),v)$. □

By Lemmas 5.6 and 5.7 the proof of main theorem follows as usual.

Proof of Theorem 5.4. Let $V \subset X$ be such that $C \subset V \subset \widetilde{C}$. We can also choose $V$ such that $B(V,\rho) \subset \widetilde{C}$ for some $\rho > 0$. Set $\eta_C$ and $\eta_{X\setminus C}$ the retraction found respectively in Lemmas 5.6 and 5.7. For the sake of simplicity we suppose that they are defined for all $t \in [0,1]$. Let $\theta: X \to [0,1]$ be a continuous map such that

$$\theta_1|_C \equiv 0, \quad \theta_1|_{X\setminus V} \equiv 1,$$

and let $\theta_2 = 1 - \theta_1$. Then we define a continuous map $\mu_1: X \times [0,1]: \to X$ by

$$\mu_1(u,t) = \begin{cases} \eta_{X\setminus C}(u, \theta_1(u)t) & \text{for } u \in X \setminus C, \\ u & \text{otherwise.} \end{cases}$$

We know that

$$f(\mu_1(u,t)) - f(u) \le -\sigma t \theta_1(u) \quad \text{and} \quad d(\mu_1(u,t),u) \le t\theta_1(u).$$

Now let

$$\mu_2 = \begin{cases} \eta_C(\mu_1(u,t), \theta_2(u)t) & \text{for } u \in V, \\ \mu_1(u,t) & \text{otherwise.} \end{cases}$$

We found that

$$d(\mu_1(u,t),u) \le \theta_1(u)t \le \rho \quad \text{if } t \le \rho,$$

so $\mu_2$ is well defined on $X \times [0,\rho]$.

Obviously we have that $\mu_2(u,0)=0$ for all $u\in X$. Furthermore, if $u\in X\setminus V$ we have that

$$f(\mu_2(u,t))-f(u)=f(\mu_1(u,t))-f(u)\leq -\sigma t,$$

and, if $u\in V$, then

$$\begin{aligned} f(\mu_2(u,t))-f(u) &= f(\eta_C(\mu_1(u,t),\theta_2(u)t))-f(u)\\ &= f(\eta_C(\mu_1(u,t),\theta_2(u)t))-f(\mu_1(u,t))+f(\mu_1(u,t))-f(u)\\ &\leq -\sigma\theta_2(u)t-\sigma\theta_1(u)t=-\sigma t. \end{aligned}$$

So, we set $\tau=\rho$ and $\mu=\mu_2$ and we conclude the proof. □

We provide a final remark: we observe that if there exist a compact set $C\subset X$, then we are allowed to weaken the standard definition of weak slope. This improvement is useful only in $C$, because the compactness allows us to compound the local retraction explicitly.

In $X\setminus C$ we must recover condition (5.1) adding the hypothesis (5.5) of Theorem 5.4: in a non compact set this estimate makes possible a continuous composition of local retractions.

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