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ON ORBITAL TOPOLOGICAL EQUIVALENCE OF CUBIC ODES IN TWO-DIMENSIONAL ALGEBRAS

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Abstract. Cubic differential systems in real commutative two-dimensional algebras are classified up to orbital topological equivalence via the solubility of polynomial equations in algebras. As a by-product, existence of bounded solutions in such systems is studied via complex structures in the algebras. Application to the existence of periodic solutions to n-dimensional differential systems "cubic at infinity" is given.

1. Introduction

1.1. Subject and goal of the paper. Given a real, two-dimensional, commutative, in general, non-associative algebra $A = (A, *)$, where "$*$" stands for the (binary) multiplication, consider an ODE

$$\frac{dx}{dt} = x(t) * x(t) * x(t) = x^3(t) \quad (x \in A,\ t \in \mathbb{R}). \tag{1.1}$$

The *main goal* of this paper is to describe qualitative profiles of (1.1) in terms of the solubility of several "basic" polynomial equations in A. Our interest in the above problem is motivated by several reasons.

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(a) *Classification of quadratic systems and two-dimensional algebras.* A lot of efforts has gone into classifying quadratic systems $\dot{x} = x(t) * x(t) = x^2(t)$, $x \in A$, up to several equivalences stronger than the topological one (see, for instance, [3], [33], [21], [10], [34]). Observe that, on the one hand, the right-hand side of (1.1) performs as an acceleration of the field $x \to x^2$, $x \in A$. On the other hand, for almost all $x \in A$, the multiplication table of A is completely determined by the vectors x^2, x^3 and $x^2 * x^2$. Therefore, the topological classification of cubic systems in A (together with the one of systems of the form $\dot{x} = x^2(t) * x^2(t)$) should give (i) a better understanding the hierarchy of the quadratic system equivalences mentioned above as well as (ii) a reasonable classification of real commutative two-dimensional algebras (cf. [31], [6]). This stream of ideas and applications goes beyond the scope of the present paper.

(b) *Bounded solutions to cubic systems in algebras.* Let $f = (f_1, \ldots, f_n)$: $\mathbb{R}^n \to \mathbb{R}^n$ be a k-homogeneous map meaning that all f_i are homogeneous polynomials of degree k. The corresponding differential system

$$\dot{x} = f(x) \quad (x \in \mathbb{R}^n) \tag{1.2}$$

is also called *k-homogeneous*. The problem of the existence of *bounded* solutions to (1.2) has attracted much attention for a long time (cf. [19], [27], [23], [8]).

Definition 1.1. Let $x = x(t)$ be a *non-constant* solution to (1.2). We say that $x = x(t)$ is *bounded* if there exists a real $\alpha > 0$ such that $\|x(t)\| < \alpha$ for all $t \in \mathbb{R}$.

As a consequence of the main topological classification result of this paper (see Theorems A1–A3) we obtain

Theorem B. *Let A be a real two-dimensional commutative algebra for which $x^3 \not\equiv 0$. System* (1.1) *has a bounded solution if and only if for some $x, u, v \in A$ at least one of the following four conditions is satisfied:*

(a) $x^3 = -x$ *and* $u^2v^2 = 0$ $(x, u^2, v^2 \neq 0)$,
(b) $x^3 = -x$ *and* $u^2u^2 = -u^2$ $(x, u^2 \neq 0)$,
(c) $x^3 = 0$ *and* $u^2u^2 = -u^2$ $(x, u^2 \neq 0)$,
(d) $x^3 = 0$ *and* $u^2v^2 = 0$ $(x, u^2, v^2 \neq 0)$,

provided A is not isomorphic to the algebra $\mathbb{N}_3$ with the multiplication table: $e_1^2 = e_2$, $e_2^2 = e_1$, $e_1e_2 = 0$.

Theorem B admits an immediate extension to the so-called *rank three algebras* of arbitrary dimension (i.e. the algebras for which any one-generated subalgebra has dimension ≤ 2 (see Definition 2.8, cf. [35]). The latter result is applied to study periodic solutions to n-dimensional systems being "cubic at infinity" (see Theorem C, cf. [27], [19], [23], [8]).

(c) *Occurence of* 3-*homogeneous maps in* (*binary*) *algebras.* It is easy to see that not every planar homogeneous polynomial map of degree three may be realized as a cubic map $x \to x^3$ in the corresponding (binary) algebra. As a by-product of Theorems A1–A3, we obtain explicit topological and algebraic obstructions for such a realization. For instance, if origin is isolated for the cubic map in a (binary) algebra A, then the topological index $\mathrm{ind}(0, x^3)$ cannot be equal to -3.

Observe, finally, that the present paper adjoins [6] (where important topological and algebraic equivalences on the set of real two-dimensional commutative algebras are considered) and [7] (where quadratic systems in non-associative algebras are studied). We also give complete proofs of several results from [6] which have been announced there without proofs.

Throughout the rest of this paper $\mathrm{Alg}_C(\mathbb{R}^n)$ *stands for the set of all real* n*-dimensional* (*binary*) *commutative algebras.*

1.2. Historical remarks. The problem of studying topological profiles of homogeneous systems was attacked using different techniques:

(a) direct analytical methods (see [13], [12], [33], [21], [28]),
(b) topological index methods (see [14], [32], [4], [19], [23]),
(c) Lyapunov-type function approach (see [19], [27], [23]),
(d) singularity theory methods (see [5]),
(e) invariant theory methods (see [10], [34], [9],
(f) algebraic approach (see [22], [17], [35], [36], [24]–[26], [29], [30]),

to mention a few. In particular, the authors of [4], [9], [29], [30] suggested effective methods to study topological profiles of (1.1). However, the importance of our approach rests on the following three observations:

(i) the techniques developed in [4], [9], [10], [34] are rather two-dimensional in nature with no meaning to be extended to higher dimensions,
(ii) the aprroaches used in (a)–(d) are not semi-algebraic in nature,
(iii) although the methods developed in (f) are algebraic, they, nevertheless, ignore the "complex structures" in algebras which, as we believe, are behind any *semi-algebraic* results on the existence of bounded solutions to polynomial systems (see Subsection 2.1 and [6], [7], [20]).

1.3. Overview. After the Introduction the paper is organized as follows. The first section contains algebraic preliminaries: complex structures (as well as their degenerate versions) are defined, some important properties of the fundamental forms associated to planar homogeneous maps are discussed and the notations frequently used throughout the paper are introduced. Section 3 is of a topological flavor: we discuss possible values of the Poincaré index of quadratic and cubic maps in connection with complex structures; in particular, we complete the proofs of several results annonced in [6] without proofs. In Section 4 we discuss some notions and facts related to the orbital topological equivalence (in short, OTE) of planar maps. Our main results are stated and proved in Sections 5–7: combining the methods and ideas described in Sections 2–4 with the results from [4], [9], we classify (up to OTE) phase portraits of system (1.1) occuring in a real (binary) two-dimensional algebra. The last Section contains applications to the existence of bounded solutions to cubic systems occuring in a rank three algebra (see Theorem B′). Observe that any 3-homogeneous map in $\mathbb{R}^n$ with isolated origin has topological index different from zero. Therefore, Theorem B′ has an immediate application to studying periodic solutions of systems being "cubic at infinity" (see Theorem C, cf. [19], [27], [23]).

2. Algebraic preliminaries

2.1. Idempotents, nilpotents, zero divisors and complex structures. For the non-associative algebras background we refer to [37].

Throughout this section $a \in A \in \mathrm{Alg}_C(\mathbb{R}^n)$ stands for a *non-zero element.* As usual, a is called an *idempotent* (resp. 2-*nilpotent*) if $a^2 = a$ (resp. $a^2 = 0$).

Proposition 2.1 (see [16]). *Let $A \in \mathrm{Alg}_C(\mathbb{R}^n)$ be an algebra without* 2-*nilpotents. Then A contains an idempotent.*

Obviously, if $a \in A$ is an idempotent (resp. 2-nilpotent), then $a^3 = a$ (resp. $a^3 = 0$). This gives rise to the following notions: a is called a *positive* 3-*idempotent* (resp. 3-*nilpotent*) if $a^3 = a$ and $a^2 \neq a$ (resp. $a^3 = 0$ and $a^2 \neq 0$).

Finally, a is called a *zero divisor* (resp. *square zero divisor*, *square nilpotent*) if there exists a non-zero $b \in A$ such that $ab = 0$ (resp. $a^2b^2 = 0$ $(a^2 \neq 0,\ b^2 \neq 0)$, $a^2a^2 = 0$ $(a^2 \neq 0)$).

The following notions generalizing complex unit in $\mathbb{C}$ were introduced in [6].

Definition 2.2. We say that $x \in A \in \mathrm{Alg}_C(\mathbb{R}^n)$ is a *negative square idempotent* if

$$x^2x^2 = -x^2 \qquad (x^2 \neq 0); \tag{2.1}$$

y is a *negative* 3-*idempotent* if

$$y^3 = -y, \tag{2.2}$$

z is a *negative square nilpotent* if

$$z^2n = -n \quad \text{for } n^2 = 0,\ n \neq 0. \tag{2.3}$$

As we will see later on, the algebraic notions introduced above completely determine the OTE classification of phase portraits of cubic systems occuring in algebras from $\mathrm{Alg}_C(\mathbb{R}^2)$. Therefore, the corresponding algebraic equations are sometimes refered to as the *basic* ones. Given an algebra $A \in \mathrm{Alg}_C(\mathbb{R}^2)$, the symbols $e(A)$ (resp. $n(A)$, $n_3(A)$) stand for the number of idempotents (resp. 2-nilpotents, 3-nilpotents (conted up to a non-zero scalar)). Also, we will denote by $e_3^+(A)$ (resp. $e_3^-(A)$, $e_{2,2}^-(A)$, $z(A)$, $z_{2,2}(A)$, $n_{2,2}(A)$) the set of positive 3-idempotents (resp. negative 3-idempotents, negative square idempotents, zero divisors, square zero divisors, square nilpotents); we will use "+" (resp. ∞) to indicate that the corresponding set is non-empty and finite (resp. infinite), and "$-$", otherwise.

2.2. Regular and singular algebras. Recall the following

Definition 2.3 (cf. [1], [31]). An algebra $A \in \mathrm{Alg}_C(\mathbb{R}^2)$ is called *regular* if there exists $u \in A$ such that the linear operator $L_u : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $x \to ux$ belongs to $\mathrm{GL}(2,\mathbb{R})$ (by the same token, $A^2 \supseteq A$). Otherwise, A is called *singular*.

Examples 2.4. All singular two-dimensional algebras (including the non-commutative ones over an arbitrary field) have been described in [31]. In particular, any element in such an algebra is a zero divisor. For completeness we present below the corresponding results for the real commutative case.

Take in $\mathbb{R}^2$ a basis (e_1, e_2) and set $e_1e_2 = 0$. Put $e_1^2 = e_1$ and define three algebras $\mathbb{N}_0$ (resp. $\mathbb{N}_1$ and $\mathbb{N}_2$) by setting $e_2^2 = e_1$ (resp. $e_2^2 = 0$ and $e_2^2 = -e_1$). Let $\mathbb{N}_\infty \in \mathrm{Alg}_C(\mathbb{R}^2)$ be the algebra with trivial multiplication. It is easy to see that $n(\mathbb{N}_i) = i$ ($i = 0, 1, 2, \infty$). Further, using the above zero divisor basis (e_1, e_2) define two additional algebras $\mathbb{N}_1^0$ (resp. $\mathbb{N}_2^0$) by setting $e_1^2 = e_2$, $e_2^2 = 0$ (resp. $e_1^2 = e_1 + e_2$, $e_2^2 = -e_1 - e_2$). In turn, $e(\mathbb{N}_1^0) = e(\mathbb{N}_2^0) = 0$ and $e(\mathbb{N}_0) = e(\mathbb{N}_1) = e(\mathbb{N}_2) = 1$.

Up to isomorphism the above six algebras exhaust singular algebras. Observe that none of them contains a negative 3-idempotent and only $\mathbb{N}_1^0, \mathbb{N}_0$ and $\mathbb{N}_2$ contain 3-nilpotents. Finally, $x^3 \equiv 0$ for any $x \in \mathbb{N}_1^0$; also, $\mathbb{N}_2$ is the only algebra containing a negative square idempotent.

2.3. Fundamental forms of homogeneous maps and trace vector. Let $g = (g_1, g_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a k-homogeneous map meaning that its coordinate functions g_i are homogeneous forms of degree $k = 2, 3$. Define a *fundamental* $(k+1)$-*form* F_{k+1} *associated to* g according to the formula:

$$F_{k+1}(x_1, x_2) = x_1 g_2(x_1, x_2) - x_2 g_1(x_1, x_2) \quad \text{for } (x_1, x_2) \in \mathbb{R}^2.$$

The importance of F_{k+1} rests on the following obvious observation: if $x \in \mathbb{R}^2$ satisfies the equation $g(x) = \lambda x$ for some $\lambda \in \mathbb{R}$, then $F_{k+1}(x) = 0$.

Take $A \in \mathrm{Alg}_C(\mathbb{R}^2)$ and denote by F_3^A (resp. F_4^A) the fundamental 3-form (resp. 4-form) associated to the map $x \to x^2$ (resp. $x \to x^3$), $x \in A$. It turns out that there is an intimate connection between F_3^A and F_4^A related to the solubility properties of the equation

$$x^3 = \lambda x \quad (x \in A,\ \lambda = -1, 0, 1). \tag{2.4}$$

To see this we need the following

Definition 2.5 (cf. [10], [6]). Let $A \in \mathrm{Alg}_C(\mathbb{R}^2)$ be an algebra with the multiplication table: $e_1^2 = a_1e_1 + b_1e_2$, $e_1e_2 = a_2e_2 + b_2e_2$, $e_2^2 = a_3e_1 + b_3e_2$. Set $p_1 = a_1 + b_2$, $p_2 = a_2 + b_3$. The vector $p = (p_1, p_2)$ is called a *trace vector* associated to A. Set $tr^{\perp}(A) = (-p_2, p_1)$.

Lemma 2.6. *Let $A \in \mathrm{Alg}_C(\mathbb{R}^2)$ and let F_3^A (resp. F_4^A) be a fundamental 3-form (resp. 4-form) associated to the map $x \to x^2$ (resp. $x \to x^3$), $x \in A$. Then*

$$F_4^A(x_1, x_2) = (p_1x_1 + p_2x_2)F_3^A(x_1, x_2). \tag{2.5}$$

Proof. By direct computation,

$$\begin{aligned} F_3(x_1, x_2) &= b_1x_1^3 + (2b_2 - a_1)x_1^2x_2 + (b_3 - 2a_2)x_1x_2^2 - a_3x_2^3, \\ F_4(x_1, x_2) &= (a_1b_1 + b_1b_2)x_1^4 + (a_2b_1 - a_1^2 + 2b_2^2 + a_1b_2 + b_1b_3)x_1^3x_2 \\ &\quad + (3b_2b_3 - 3a_1a_2)x_1^2x_2^2 \\ &\quad + (b_3^2 - a_3b_2 - a_3a_1 - b_3a_2 - 2a_2^2)x_1x_2^3 \\ &\quad + (-a_3a_2 - b_3a_3)x_2^4, \end{aligned} \tag{2.6}$$

and the result follows. □

Remark 2.7. Some comments related to formula (2.5) are in order:

(a) F_3^A divides F_4^A since $x^2 = x$ (resp. $x^2 = 0$) implies $x^3 = x$ (resp. $x^3 = 0$),

(b) if x is not a solution to

$$x^2 = \mu x \quad (x \in A,\ \mu = 0, 1) \tag{2.7}$$

and satisfies (2.4), then x is orthogonal to the trace vector, i.e. 3-nilpotents, positive and negative 3-idempotents are proportional to $\mathrm{tr}^{\perp}(A)$,

(c) if (2.4) admits infinitely many solutions, then $F_4^A \equiv 0$.

2.4. Rank of algebra. Pseudo-composition algebras. Lemma 2.6 together with Remark 2.7 give rise to the following two questions:

(1) Which algebras from $\mathrm{Alg}_C(\mathbb{R}^2)$ do admit infinitely many solutions to equation (2.4)?

(2) Which algebras from $\mathrm{Alg}_C(\mathbb{R}^2)$ are 3-nilpotent, negative 3-idempotent and positive 3-idempotent free?

To answer the first question we need

DEFINITION 2.8 (cf. [35], [6]). Let $A \in \mathrm{Alg}_C(\mathbb{R}^n)$.

(a) A is called a *rank three algebra* if there are a linear form γ_1 and a quadratic form γ_2 on A such that the identity

$$x^3 = \gamma_1(x)x^2 - \gamma_2(x)x \tag{2.8}$$

holds for all $x \in A$.

(b) The forms γ_1 and γ_2 are called *trace* and *norm*, respectively. To indicate a connection with the underlying algebra A we will sometimes write γ_2^A (resp. γ_1^A) instead γ_2 (resp. γ_1).

(c) If $\gamma_1 \equiv 0$, then A is called a *pseudo-composition* algebra.

(d) A is called a *rank two algebra* if there exists a linear form γ_1 on A such that $x^2 = \gamma_1(x)x$ for all $x \in A$.

EXAMPLES 2.9. (a) It is easy to see that any $A \in \mathrm{Alg}_C(\mathbb{R}^2)$ is of rank three. Moreover, $A \in \mathrm{Alg}_C(\mathbb{R}^n)$ is of rank three if and only if any one-generated subalgebra of A is of dimension ≤ 2.

(b) Obviously, the algebra $\mathbb{N}_\infty$ (see Examples 2.4) is of rank two. Also, the algebra $\mathbb{N}_{1/2} \in \mathrm{Alg}_C(\mathbb{R}^2)$ with the multiplication table $e_1^2 = e_1$, $e_1e_2 = e_2/2$, $e_2^2 = 0$ is of rank two. Obviously, $n(\mathbb{N}_{1/2}) = 1$ and $e(\mathbb{N}_{1/2}) = \infty$.

(c) Clearly, the algebras $\mathbb{N}_\infty$, $\mathbb{N}_1^0$ (see Examples 2.4) and $\mathbb{N}_3$ (see Theorem B) are pseudo-composition. Define two new algebras $\overline{\mathbb{C}}$ and $\overline{\mathbb{N}}$ with the multiplication tables $e_1^2 = e_1$, $e_1e_2 = -e_2$, $e_2^2 = -e_1$ and $e_1^2 = e_1$, $e_1e_2 = -e_2$, $e_2^2 = 0$, respectively. By direct computation, $\overline{\mathbb{C}}$ and $\overline{\mathbb{N}}$ are pseudo-composition as well. Also, $e_3^+(\overline{\mathbb{C}}) = \infty$, $e_3^-(\overline{\mathbb{C}}) = -$, $n(\overline{\mathbb{C}}) = n_3(\overline{\mathbb{C}}) = 0$, $e_3^-(\overline{\mathbb{N}}) = -$, $e_3^+(\overline{\mathbb{N}}) = \infty$, $n_3(\overline{\mathbb{N}}) = 0$, $n_2(\overline{\mathbb{N}}) = 1$, $e_3^+(\mathbb{N}_3) = e_3^-(\mathbb{N}_3) = \infty$, $n(\mathbb{N}_3) = 0$, $n_3(\mathbb{N}_3) = 2$.

The statement following below answers question (1).

PROPOSITION 2.10 (cf. [6]). *Take* $A \in \mathrm{Alg}_C(\mathbb{R}^2)$.

(a) *The following statements are equivalent:*
 - (a1) $F_3^A \equiv 0$.
 - (a2) *Equation* (2.7) *admits infinitely many solutions.*
 - (a3) *A is of rank two* (*in this case A is isomorphic to either* $\mathbb{N}_\infty$ *or* $\mathbb{N}_{1/2}$).

(b) *The following statements are equivalent:*
 - (b1) $F_4^A \equiv 0$.
 - (b2) *Equation* (2.4) *admits infinitely many solutions.*
 - (b3) *A is either of rank two or non-trivial pseudo-composition* (*in the latter case A is isomorphic to one of the four non-trivial algebras listed in Examples* 2.9(c)).

The following statement answers question (2).

Proposition 2.11. *Take $A \in \mathrm{Alg}_C(\mathbb{R}^2)$ and assume A is 3-nilpotent, positive and negative 3-idempotent free.*

(a) *If A is a division algebra, then A admits the multiplication table $e_1^2 = e_2$, $e_1e_2 = \alpha e_1 + \beta e_2$, $e_2^2 = -e_2$ with either $\alpha = 1$ and $\beta \in \mathbb{R}$ or $\alpha > 1/4$ and $\beta = \pm(1-\alpha)(4\alpha - 1)^{-1/2}$.*

(b) *Assume A is a regular algebra with $n(A) = 0$ and let A admit a zero divisor basis (e_1, e_2). Then A admits a multiplication table*

$$e_1^2 = e_1 + \beta e_2, \quad e_2^2 = \alpha e_1 + e_2 \quad (\alpha + \beta = -2) \tag{2.9}$$

or the multiplication table

$$e_1^2 = \gamma e_1 + e_2, \quad e_2^2 = e_1 + \delta e_2 \quad (\gamma^3 + 2\gamma^2\delta^2 + \delta^3 = 0). \tag{2.10}$$

(c) *Let A be a regular algebra with a 2-nilpotent e_2. Assume A is neither pseudo-composition nor of rank two. Then either A is free from square nilpotents or A admits a multiplication table*

$$e_1^2 = e_2, \quad e_1e_2 = e_1 \pm \frac{1}{2}e_2. \tag{2.11}$$

(d) *Assume A is singular. Then A is isomorphic to either $\mathbb{N}_1$ or $\mathbb{N}_2^0$ (cf. Examples 2.4).*

Proof. "Canonical forms" of the multiplication tables appropriate for each of the cases (a)–(d) can be found in [6]. Take the vector $\mathrm{tr}^\perp(A)$ in each case and substitute into (2.6). □

3. Poincaré index and complex structures

3.1. Poincaré index. Given a smooth map $f: \mathbb{R}^2 \to \mathbb{R}^2$, assign to f its *Poincaré index* $\mathrm{ind}(0, f)$ according to the formula (cf. [18]):

$$\mathrm{ind}(0, f) := \frac{1}{2\pi} \int_\Gamma \frac{dt}{\|f\|^2}\left(f_1 \frac{df_2}{dt} - f_2 \frac{df_1}{dt}\right) \tag{3.1}$$

(here Γ stands for the unit circle and the integral is understood in the sense of Lebesgue; we assume the integral in (3.1) to exist and to be finite).

Observe that the standard construction of the topological index of a singular point of a planar vector field requires the point to be *isolated* (see, for instance, [18]). In such a case the topological index coincides with the Poincaré index. However, the algebraic maps we are dealing with may have non-isolated origin. Nevertheless, the right-hand side of (3.1) is correctly defined even in this case (i.e. the integrand has a removable singularity). The following statement is a direct consequence of formula (3.1).

Proposition 3.1. *Let $f = (f_1, f_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial k-homogeneous map, $f_1 = gh_1$, $f_2 = gh_2$ $(k = 2, 3)$, where g, h_1 and h_2 are (homogeneous) polynomials. Let $h = (h_1, h_2)$. Then*

$$\operatorname{ind}(0, f) = \operatorname{ind}(0, h).$$

3.2. Poincaré index of quadratic maps and complex structures. Take an algebra $A \in \mathrm{Alg}_C(\mathbb{R}^2)$ with non-trivial multiplication and consider the quadratic map $f\colon A \to A$ defined by $f(x) := x^2$.

Definition 3.2. We say that $A \in \mathrm{Alg}_C(\mathbb{R}^2)$ is an *algebra of index n* if $\operatorname{ind}(0, f) = n$. We will also use the symbol $\operatorname{ind}(A)$ for the index of A.

It was established in [6] that $\operatorname{ind}(A)$ coincides with the signature of γ_2, where γ_2 is the norm in A (see Definition 2.8(b)). In particular, (i) for (regular) algebras without 2-nilpotents $\operatorname{ind}(A) = 2, 0, -2$; (ii) for regular algebras with 2-nilpotents $\operatorname{ind}(A) = \pm 1$; (iii) for singular algebras $\operatorname{ind}(A) = 0$. More precisely,

Proposition 3.3 (cf. [6]). *Let $A \in \mathrm{Alg}_C(\mathbb{R}^2)$ be a regular algebra.*

(a) $\operatorname{ind}(A) = 2$ *if and only if A contains a negative square idempotent together with a negative* 3*-idempotent.*
(b) $\operatorname{ind}(A) = -2$ *if and only if A contains a negative square idempotent and does not contain a negative* 3*-idempotent.*
(d) $\operatorname{ind}(A) = 0$ *if and only if A does not contain a negative square idempotent and does not contain* 2*-nilpotents.*
(d) *If A contains a square nilpotent, then* $\operatorname{ind}(A) = 1$ *if and only if A contains a negative* 3*-idempotent.*
(e) *If A contains a* 2*-nilpotent and does not contain a square nilpotent, then* $\operatorname{ind}(A) = -1$ *if and only if A contains a negative square nilpotent.*

3.3. Poincaré index of cubic maps and complex structures. Some of the results considered in this subsection were presented in [6] without complete proofs or contained a mistake.

Take an algebra $A \in \mathrm{Alg}_C(\mathbb{R}^2)$ with $x^3 \not\equiv 0$ and denote by $\operatorname{ind}(0, x^3)$ the Poincaré index of the cubic map $x \to x^3$. Obviously, $\operatorname{ind}(0, x^3) = 0$ for A being singular. Also, if A contains neither 2-nilpotents, nor 3-nilpotents and nor negative 3-idempotents, then, by the standard Bole–Brouwer Theorem (see, for instance, [18, Theorems 4.1 and 4.3]), $\operatorname{ind}(0, x^3) = \operatorname{ind}(0, \mathrm{id}) = 1$, where id stands for the identity map (cf. [6]). Next we consider several cases.

Case 1. Algebras without both 2*- and* 3*-nilpotents.* Consider the algebras admitting a negative 3-idempotent (for the sake of convenience, including those with 2- or 3-nilpotents). Bearing in mind that the Poincaré index takes the same values for isomorphic algebras from $\mathrm{Alg}_C(\mathbb{R}^2)$, we choose for $A \in \mathrm{Alg}_C(\mathbb{R}^2)$

a basis in which the multiplication table takes the simplest form, namely: (e_1, e_1^2) with $e_1^3 = -e_1$ and $e_1^2 e_1^2 = 2\alpha e_1 + \beta e_1^2$ for some reals $\alpha \geq 0$ and β. Denote by $A_{\alpha,\beta}$ the corresponding algebra and let $\mathfrak{A}$ be the union of all $A_{\alpha,\beta}$.

LEMMA 3.4 ("apriori estimate"). *Take $A_{\alpha,\beta} \in \mathfrak{A}$. Then:*

(a) $n(A_{\alpha,\beta}) \neq 0$ *if and only if* $\alpha^2 + \beta = 0$,
(b) $\operatorname{ind}(0, A_{\alpha,\beta}) = 2$ *if and only if* $\alpha^2 + \beta < 0$,
(c) $\operatorname{ind}(0, A_{\alpha,\beta}) = 0$ *if and only if* $\alpha^2 + \beta > 0$,
(d) $n_3(A_{\alpha,\beta}) \neq 0$ *if and only if* $\beta = 1$.

PROOF. Statements (a)–(c) were proved in [6].

(d) Take $x = (x_1, x_2) \in A_{\alpha,\beta}$. Then

$$x^3 = (-x_1^3 + 2\alpha x_1^2 x_2 + (2-\beta)x_1 x_2^2 + 2\alpha(\beta - 1)x_2^3, \quad (-2+\beta)x_1^2 x_2 + 2\alpha x_1 x_2^2 + \beta^2 x_2^3). \tag{3.2}$$

Combining the resultant of the first and second components from (3.2) with statement (a) yields the result. □

REMARK 3.5. It is easy to see that all algebras from $\mathfrak{A}$ with $\beta = 1$ are isomorphic to $\mathbb{N}_3$.

Lemma 3.4 suggests a partition of the half-plane $(\alpha \geq 0, \beta)$ onto three mutually disjoint open blocks:

$(*)$ $\alpha^2 + \beta < 0$,
$(**)$ $-\alpha^2 < \beta < 1$,
$(***)$ $\beta > 1$,

(see Figure 3.1). The following statement was announced without a complete proof in [6].

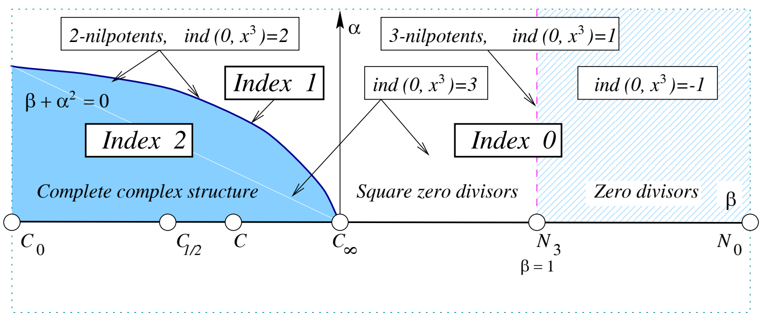

FIGURE 3.1. Algebras with negative 3-idempotents

LEMMA 3.6. *Take* $A_{\alpha,\beta} \in \mathfrak{A}$. *Then* $\operatorname{ind}(0, x^3) = 3$ *in the cases* $(*)$ *and* $(**)$; $\operatorname{ind}(0, x^3) = -1$ *in the case* $(***)$.

PROOF. By construction, for any algebra A belonging to some block, origin is an *isolated* singular point of the cubic map in A.

(a) Take an arbitrary algebra A_{α_0,β_0} satisfying $(*)$ and show that $\operatorname{ind}(0, x^3) = 3$ for this algebra.

To this end observe, first, that for the algebra $A_{0,-1}$ one has $0^2 - 1 < 0$. Also, it is easy to see that $A_{0,-1}$ is isomorphic to $\mathbb{C}$ – the algebra of complex numbers. Therefore, $\operatorname{ind}(0, x^3) = 3$ for $A_{-1,0}$. Put

$$\alpha = (1 - t)\alpha_0, \quad \beta = (1 - t)\beta_0 - t, \tag{3.3}$$

where $t \in [0, 1]$. According to $(*)$, formulae (3.3) determine a family of algebras $A_{\alpha(t),\beta(t)}$ satisfying

$$\alpha^2(t) + \beta(t) < 0 \tag{3.4}$$

for all $t \in [0, 1]$. Substitution of (3.4) into formula (3.2) determines the vector function $F\colon [0, 1] \times \mathbb{R}^2 \to \mathbb{R}^2$ continuos in both variables and such that $F(0, \cdot)$ (resp. $F(1, \cdot)$) coincides with the cubic map in A_{α_0,β_0} (resp. in $A_{0,-1}$). Moreover, bearing in mind (3.4) and Lemma 3.4, one obtains $F(t, x) \neq 0$ for all $t \in [0, 1]$ and $x \in \mathbb{R}^2 \setminus \{0\}$. Therefore, by the homotopy invariance of the index (cf. [18, Theorem 4.1]), $\operatorname{ind}(0, x^3) = 3$ for A_{α_0,β_0}.

(b) Take an arbitrary algebra A_{α_0,β_0} satisfying $(**)$. Consider the algebra $A_{0,1/2}$ and show that $\operatorname{ind}(0, x^3) = 3$ for it. Indeed, by direct computation utilizing (3.2), one obtains

$$x^3 = \Big(- x_1^3 + \frac{3}{2}x_1x_2^2, -\frac{3}{2}x_1^2x_2 + \frac{1}{4}x_2^3 \Big),$$

which is immediately homotopic to

$$\Big(- x_1^3 + \frac{3}{2}x_1x_2^2, -3x_1^2x_2 + \frac{1}{2}x_2^3 \Big). \tag{3.5}$$

By means of the formula

$$\Big(- x_1^3 + \frac{3}{2}(t+1)x_1x_2^2, -3x_1^2x_2 + \frac{1}{2}(t+1)x_2^3 \Big) \quad (t \in [0, 1]),$$

one deforms (3.5) into the cubic map in the algebra $\mathbb{C}$ of index 3.

Next, we can deform the multiplication table for A_{α_0,β_0} into the one for $A_{0,1/2}$ by means of the formulae:

$$\alpha = \alpha_0, \qquad \beta = (1 - 2t)\beta_0 + t \quad (t \in [0, 1/2]), \tag{3.6}$$

$$\alpha = (2 - 2t)\alpha_0, \quad \beta = 1/2, \qquad (t \in [1/2, 1]). \tag{3.7}$$

Bearing in mind $(**)$, (3.6), (3.7), (3.2), and using the same argument as in (a), one can prove that the cubic maps in A_{α_0,β_0} and $A_{0,1/2}$ are homotopic.

(c) Take an arbitrary algebra A_{α_0,β_0} satisfying $(***)$. Consider the algebra $A_{0,2}$ and show that $\operatorname{ind}(0,x^3)=-1$ for it. Indeed, by direct computation utilizing (3.2), $x^3=(-x_1^3,4x_2^3)$, i.e. $x^3=\operatorname{diag}\{-1,1\}\cdot(x_1^3,4x_2^3)^t$ from which it follows that $\operatorname{ind}(0,x^3)=-1\cdot 1=-1$ for $A_{0,2}$. Put

$$\alpha=(1-t)\alpha_0,\quad (1-t)\beta_0+2t, \tag{3.8}$$

where $t\in[0,1]$. Combining $(***)$, (3.8) with the same argument as in (a), one can easily prove that the cubic maps in A_{α_0,β_0} and $A_{0,2}$ are homotopic.

Lemma 3.6 is completely proved. □

It remains to recognize algebraically the above subsets $(*)$–$(***)$.

Proposition 3.7 (see [6]). *Take $A_{\alpha,\beta}\in\mathfrak{A}$.*

(a) *$A_{\alpha,\beta}$ is regular.*
(b) *$A_{\alpha,\beta}$ satisfies $(*)$ if and only if $A_{\alpha,\beta}$ contains a negative square idempotent.*
(c) *$A_{\alpha,\beta}$ satisfies $(**)$ or $(***)$ if and only if $A_{\alpha,\beta}$ is free from both* 2- *and* 3-*nilpotents and admits a zero divisor basis.*
(d) *$A_{\alpha,\beta}$ satisfies $(**)$ if and only if $A_{\alpha,\beta}$ is free from both* 2- *and* 3-*nilpotents and contains square zero divisors.*

Case 2. Regular algebras with 2-*nilpotents.* As was proved in [6], any (regular) algebra $A\in\operatorname{Alg}_C(\mathbb{R}^2)$ with a 2-nilpotent admits either (i) a square nilpotent, or (ii) a (unique) nilpotent ideal. In the case (i) A is isomorphic to an algebra with the multiplication table

$$e_1^2=\pm e_2,\quad e_1e_2=e_1+\mu e_2,\quad e_2^2=0 \tag*{(3.9)$_\pm$}$$

with $\mu\in\mathbb{R}$. In the case (ii) A admits the multiplication table of the form either

$$e_1^2=e_1,\quad e_1e_2=\lambda e_2,\quad e_2^2=0 \tag{3.10}$$

with $\lambda\in\mathbb{R}\setminus\{0\}$ (if $\lambda=0$, then A is singular), or

$$e_1^2=e_1+e_2,\quad e_1e_2=\frac{1}{2}e_2,\quad e_2^2=0. \tag{3.11}$$

We have

Proposition 3.8 (see [6]). *Let $A\in\operatorname{Alg}_C(\mathbb{R}^2)$ be a regular algebra with* 2-*nilpotents.*

(a) *If A admits a square nilpotent, then $\operatorname{ind}(0,x^3)=1+\operatorname{ind}(A)$; equivalently (cf. Proposition 3.3(d)), $\operatorname{ind}(0,x^3)=2$ if A contains a negative* 3-*idempotent, and $\operatorname{ind}(0,x^3)=0$, otherwise.*

(b) *If A does not admit a square nilpotent, then* $\operatorname{ind}(0, x^3) = 1$ *in the case* (3.11) *and* $\operatorname{ind}(0, x^3) = \operatorname{sign}(\lambda(1 + 2\lambda))$ *in the case* (3.10). *In particular* (*cf. Proposition* 3.3(e)), *if A does not admit a negative square nilpotent, then* $\operatorname{ind}(0, x^3) = 1$.

Case 3. Algebras with 3*-nilpotents.* Given an algebra $A \in \mathrm{Alg}_C(\mathbb{R}^2)$ with a 3-nilpotent e_1, choose in A the basis (e_1, e_1^2). Then the multiplication table for A takes the form:

$$e_1^2 = e_2, \quad e_1 e_2 = 0, \quad e_2^2 = \alpha e_1 + \beta e_2. \tag{3.12}$$

The following result presented in [6] with a minor mistake, describes possible values of $\operatorname{ind}(0, x^3)$ in algebras with 3-nilpotents.

PROPOSITION 3.9 (cf. [6]). *Let $A \in \mathrm{Alg}_C(\mathbb{R}^2)$ be an algebra containing* 3*-nilpotents. Then* $\operatorname{ind}(A) = 0$ *and one of the following holds:*

(a) $\operatorname{ind}(0, x^3)$ *is not defined for* $A \cong \mathbb{N}_1^0$,
(b) $\operatorname{ind}(0, x^3) = 2$ *if and only if A contains square zero divisors,*
(c) $\operatorname{ind}(0, x^3) = 1$ *if and only if A contains a negative* 3*-idempotent* ($A \cong \mathbb{N}_3$),
(d) $\operatorname{ind}(0, x^3) = 0$ *otherwise.*

PROOF. Statements (c), (d) and observation (a) were established in [6]. To show (d) (see [6]), one can assume that both $\gamma_1^A(x)$ and $\gamma_2^A(x)$ are not equal to zero identically (cf. Definition 2.8(b)).

In fact, the existence of a 3-nilpotent is equivalent to the existence of a linear form $\lambda(x)$ such that

$$\gamma_2^A(x) = -2\lambda(x)\gamma_1(x). \tag{3.13}$$

Bearing in mind (3.12), one easily obtains $\gamma_2^A(x) = -\alpha x_1 x_2$ and $\gamma_1^A(x) = \beta x_2$ from which it follows (see 3.13) that

$$\lambda(x) = \frac{\alpha}{2\beta} x_1. \tag{3.14}$$

Define a new algebra $B = (\mathbb{R}^2, \circ)$ by

$$x \circ y = x * y + \lambda(x) y + \lambda(y) x. \tag{3.15}$$

We have the following general statement due to S. Walcher (see [35]).

LEMMA 3.10. *Let $A = (\mathbb{R}^n, *)$ be a rank three algebra, λ a linear form on $\mathbb{R}^n$ and let $B = (\mathbb{R}^n, \circ)$ be the algebra with the multiplication defined by* (3.15). *Then B is a rank three algebra with the norm γ_2^B defined by*

$$\gamma_2^B(x) = \gamma_2^A(x) - \lambda(x^2) + 2\lambda(x)^2 + 2\lambda(x)\gamma_1^A \quad (x \in \mathbb{R}^n). \tag{3.16}$$

Continuation of the Proof of Proposition 3.9. Obviously,

$$\operatorname{ind}(0, x^3) = \operatorname{ind}(0, x^2 + 2\lambda(x)x) = \operatorname{ind}(B) \tag{3.17}$$

(here, of course, $\operatorname{ind}(0, x^3)$ stands for the index of the cubic map in A). According to (3.17), to complete the computation of $\operatorname{ind}(0, x^3)$, one should verify the signature of γ_2^B (cf. [6]). Due to (3.13) and (3.16) one has:

$$\gamma_2^B(x) = 2\lambda(x)^2 - \lambda(x^2). \tag{3.18}$$

Combining (3.18) with (3.14) yields:

$$\gamma_2^B(x) = \frac{\alpha^2}{2\beta^2}(x_1^2 - \beta x_2^2),$$

meaning that $\operatorname{ind}(B) = 1 - \operatorname{sign}(\beta)$. If $\beta > 0$, there are no square zero divisors in A; also, if $\beta < 0$, then $b = \sqrt{-\beta}e_1 + e_2$ is a square zero divisor, $e_1^2 b^2 = 0$.

The proof of Proposition 3.9 is complete. □

4. Orbital topological equivalence of k-homogeneous ODEs: background

4.1. Basic definition. All the differential systems we are dealing with in this section are supposed to be planar and k-homogeneous with $1 \leq k \leq 3$. Any such a system induces in a canonical way a system on the Poincaré 2-disc (see [15], [9], [34] for details).

Definition 4.1 (cf. [15], [9], [34], [5], [3]). Let (A) and (B) be two planar k-homogeneous systems of the form (1.2) with $1 \leq k \leq 3$ and let (A') and (B') be the corresponding projections on the Poincaré discs. (A) and (B) are called *orbitally topologically equivalent* (in short, OTE-equivalent) if there exists a homeomorphism of the Poincaré discs which carries singular points of (A') into singular points of (B') and which maps the phase curves of (A') into the phase curves of (B'), preserving the direction of the motion.

Consider a planar system of the form (1.2). Let $x_0 \in \mathbb{R}^2$ be a vector satisfying $f(x_0) = \lambda x_0$, $\lambda \in \mathbb{R}$. Following [10], we will call a straight line passing through origin and x_0 a *fixed direction*. Obviously, fixed directions are invariant with respect to the flow of (1.2). More specifically, if $\lambda \neq 0$ (resp. $\lambda = 0$), the fixed direction is called an *invariant line* (resp. *equilibrium line*) for (1.2). It follows immediately from Definition 4.1 that the homeomorphism realizing an orbital topological equivalence takes invariant lines (resp. equilibrium lines) onto the invariant lines (resp. equlibrium lines). Essentially, the orbital topological equivalence depends on the character of the sectors bounded by subsequent invariant lines and location of equilibrium lines (cf. [4], [10], [34], [9]).

4.2. Systems with isolated origin. We refer to [5], [3] for the definition of elliptic, hyperbolic and parabolic sectors for a planar dynamical system. The following result (which is well-known as the Bendixon Theorem (see, for instance, [3])) links the quantity and quality of the sectors with the Poincaré index.

PROPOSITION 4.2. *Assume system* (1.2) *is planar, has isolated origin and admits finitely many invariant lines. Then*

$$\operatorname{ind}(0, f) = 1 + \frac{e-h}{2},$$

where e (*resp.* h) *stands for the number of elliptic* (*resp. hyperbolic*) *sectors for the system.*

In what follows we will essentially use the following

PROPOSITION 4.3 (cf. [4], [5], [10], [34], [9]). *Let* (A) *and* (B) *be two planar systems of the form* (1.2) *having isolated origin.*

(a) *Assume* (A) *and* (B) *do not contain invariant lines. Then* (A) *and* (B) *are orbitally topologically equivalent if and onlu if origin is either a focus or center for* (A) *and* (B) *simultaneously.*

(b) *Assume* (A) *and* (B) *do contain* (*finitely many*) *invariant lines. Then* (A) *and* (B) *are orbitally topologically equivalent if and only if they have the same number of elliptic, hyperbolic and parabolic sectors.*

4.3. Systems with non-isolated origin: secant and tangent equlibrium lines. Assume now that origin is not an isolated singular point for a planar system (A) of the form (1.2). Then the components of $f = (f_1, f_2)$ have a common (homogeneous) factor: $f_1 = \mu\overline{f}_1$, $f_2 = \mu\overline{f}_2$. Changing time $ds = \mu\, dt$ yields the system

$$\begin{cases} \dot{x}_1 = \overline{f}_1(x_1, x_2), \\ \dot{x}_2 = \overline{f}_2(x_1, x_2), \end{cases} \tag{4.1}$$

for which $\overline{f}_1$ and $\overline{f}_2$ do not have comon factor (cf. [10], [34], [9]). Also, the zeros of μ coincide with equlibrium lines of (A). In addition, the orbits of (4.1) are the same as the orbits of (1.2) in the region $\mu > 0$ while in the region $\mu < 0$ the orientation is reversed. Therefore, to obtain the orbital topological classification of the (cubic) systems with non-isolated origin, one should consider quadratic and linear systems (see [10], [34], [7]) with (homogeneous) zeros of μ "laid over" (cf. [9]). As it is important to know the location of zeros of μ, we arrive at the following

Definition 4.4. Given a planar system (1.2) with $\mu(\,\cdot\,)$ being a common factor of f_1 and f_2, denote by l a straight line on which $\mu(\,\cdot\,)$ vanishes. Take the system (4.1).

(a) l is called *e-secant* (resp. *h-secant, p-secant)* for system (1.2) if there exists an elliptic (resp. hyperbolic, parabolic) sector s for (4.1) such that l intersects the interior of s.

(b) l is called *tangent* if it covers some invariant ray for (4.1).

5. Orbital topological equivalence of cubic systems in algebras without both 2- and 3-nilpotents

5.1. Poincaré index, invariant lines and multiple roots of F_4^A. Clearly, invariant lines for system (1.1) can be generated by idempotents, positive 3-idempotents, or negative 3-idempotents. We have

Proposition 5.1. *Take $A \in \mathrm{Alg}_C(\mathbb{R}^2)$.*

(a) *Assume A is free from 2-nilpotents. Then system* (1.1) *admits at least one invariant line.*

(b) *Assume A is free from both 2- and 3-nilpotents. Then system* (1.1) *admits infinitely many line solutions if and only if A is isomorphic to a pseudo-composition algebra $\overline{\mathbb{C}}$* (*see Definition* 2.8(c) *and Examples* 2.9(c)).

Proof. Statement (a) is a direct consequence of Proposition 2.1. Statement (b) follows from Proposition 2.10 and the fact that $\overline{\mathbb{C}}$ is the only pseudo-composition algebra without both 2- and 3-nilpotents. □

Proposition 5.2 (cf. [6, Proposition 6.1]). *Let $A \in \mathrm{Alg}_C(\mathbb{R}^2)$ be free from both 2- and 3-nilpotents. Assume also that A contains a negative 3-idempotent.*

(a) *A may contain one, two or three idempotents provided A admits either a negative square idempotent or square zero divisor* (*in both cases* $\mathrm{ind}(0, x^3) = 3$).

(b) *If A admits a zero divisor basis and does not admit square zero divisors* (*i.e.* $\mathrm{ind}(0, x^3) = -1$), *then A has precisely one idempotent.*

(c) *F_4^A admits a* (*homogeneous*) *root of multiplicity three if and only if A is isomorphic to the algebra $\mathbb{C}_{1/2}$ with the multiplication table $e_1^2 = e_2$, $e_1e_2 = -e_1/2$, $e_2^2 = -e_2$.*

Proof. Since A contains a negative 3-idempotent, it admits the multiplication table $e_1^2 = e_2$, $e_1e_2 = -e_1$, $e_2^2 = 2\alpha e_1 + \beta e_2$ (see Subsection 3.3). Hence (see (2.6)), $F_3^A(x_1, x_2) = x_1^3 + (\beta + 2)x_1x_2^2 - 2\alpha x_2^3$ with the discriminant $D = -4((\beta + 2)^3 + 27\alpha^2)$. Combining this with Lemma 3.6 yields statements (a) and (b). Statement (c) was established in [11]. □

Proposition 5.3. *Let* $A \in \mathrm{Alg}_C(\mathbb{R}^2)$ *be free from both* 2- *and* 3-*nilpotents. Assume also* A *contains a positive* 3-*idempotent* (*in particular,* $\mathrm{ind}(0, x^3) = 1$). *Assume, in addition,* A *is not isomorphic to* $\overline{\mathbb{C}}$.

(a) *If* A *admits a negative square idempotent then* A, *contains precisely three idempotents.*

(b) *If* A *does not admit a negative square idempotent, then* A *may contain one, two or three idempotents.*

(c) F_4^A *admits a* (*homogeneous*) *root of multiplicity three if and only if* A *is isomorphic to the algebra* $\mathbb{D}_{1/2}$ *with the multiplication table* $e_1^2 = e_2$, $e_1e_2 = e_1/2$, $e_2^2 = e_2$.

Proof. By assumption, A admits the multiplication table $e_1^2 = e_2$, $e_1e_2 = e_1$, $e_2^2 = 2\alpha e_1 + \beta e_2$. By direct computation, A contains 2-nilpotents if and only if $\alpha^2 + \beta = 0$. Moreover, A admits a negative square idempotent if and only if $\alpha^2 + \beta < 0$. To complete the proof of statements (a) and (b) one can use the same argument as in the proof of Proposition 5.2. For the proof of statement (c) we refer to [11]. □

Proposition 5.4. *Let* $A \in \mathrm{Alg}_C(\mathbb{R}^2)$ *be free from both* 2- *and* 3-*nilpotents. Assume also that* A *is free from both negative and positive* 3-*idempotents. Then:*

(a) F_4^A *always contains a* (*homogeneous*) *root of multiplicity two.*

(b) A *may contain one, two or three idempotents.*

Proof. (a) By assumption, F_4^A should contain a (homogeneous) root of multiplicity ≥ 2. Therefore, we have to show that the multiplicity of the root cannot be greater than two. Indeed, if A contains a negative square idempotent (i.e. (see Proposition 3.3) $\mathrm{ind}(A) = -2$), then A contains three idempotents (see Proposition 5.3) and the result follows. So that assume A contains a zero divisor basis and we must consider two cases (cf. Proposition 2.11(b)).

Assume A satisfies (2.9). By direct computation, $F_3^A = -(2+\alpha)x_1^3 + x_1x_2^2 - x_1^2x_2 - \alpha x_2^3$ and, since the trace vector associated to A has the form $(1,1)$, we obtain

$$F_3^A = (x_1 + x_2)(-(2+\alpha)x_1^2 + (1+\alpha)x_1x_2 - \alpha x_2^3). \tag{5.1}$$

Assume

$$F_3^A = (x_1 + x_2)^2(tx_1 + qx_2) \quad (t, q \in \mathbb{R}). \tag{5.2}$$

Combining (5.1) and (5.2) yields $-2-\alpha = t$, $1+\alpha = t+q$, $-\alpha = q$ that is impossible.

Suppose A satisfies (2.10). Since $\gamma = 0$ yields $\delta = 0$ and vice versa, we can assume that γ and δ are different from zero (recall, A is free from 3-nilpotents).

By direct computation, $F_3^A = x_1^3 + \delta x_1 x_2^2 - \gamma x_1^2 x_2 - x_2^2$. Since the trace vector associated to A has the form (γ, δ), we obtain

$$F_3^A = (\gamma x_1 + \delta x_2)\bigg(\frac{1}{\gamma} x_1^2 + \bigg(1 + \frac{\gamma}{\delta^2}\bigg) x_1 x_2 - \frac{1}{\delta} x_2^2\bigg). \tag{5.3}$$

Assume

$$F_3^A = (\gamma x_1 + \delta x_2)^2 (t x_1 + q x_2). \tag{5.4}$$

Combining (5.3), (5.4), (2.10) and $\gamma^2 + \delta^2 \neq 0$ yields $\gamma = \delta = -1$ that is impossible, since, by assumption, A is regular. Statement (a) is proved.

(b) To show statement (b), take the second factor in the right-hand side of (5.1) and observe that its discriminant may be positive, negative or equal to zero. □

5.2. Classification. Our study of orbitally topologically equivalent classes is essentially based on the following simple observation.

Lemma 5.5. *Let $A \in \mathrm{Alg}_C(\mathbb{R}^2)$ be free from both* 2- *and* 3-*nilpotents. Assume also A does not have a negative* 3-*idempotent. Assume, finally, A is not isomorphic to $\overline{\mathbb{C}}$. Then the phase portrait to* (1.1) *has finitely many sectors and each of them is parabolic.*

Proof. By Proposition 5.1(b), system (1.1) admits finitely many sectors. By assumption, all the invariant lines pass through either idempotents or positive 3-idempotents. Let x_0 be either an idempotent or positive 3-idempotent, and let $s(t, x_0)$ (resp. $s(t, -x_0)$) be a phase curve passsing through x_0 (resp. $-x_0$).

By definition of direction on a phase curve (see [3, p. 13]), the direction on a $s(t, x_0)$ (resp. $s(t, -x_0)$) coincides with the direction of x_0 (resp. $-x_0$). In particular, any two phase rays determining a sector for (1.1) are directed from origin to infinity, i.e. the sector is parabolic. □

Combining Lemma 5.5 with Lemma 3.6, Proposition 3.7 and Bole–Brouwer Theorem (see [18], Theorems 4.1 and 4.3) yields

Corollary 5.6. *Assume $A \in \mathrm{Alg}_C(\mathbb{R}^2)$ is free from both* 2- *and* 3-*nilpotents. Assume A is not isomorphic to $\overline{\mathbb{C}}$. Then* $\mathrm{ind}(0, x^3) = 1$ *if and only if the phase portrait to* (1.1) *admits only parabolic sectors.*

We are now in a position to formulate the main result of this section.

Theorem A1. *Let $A \in \mathrm{Alg}_C(\mathbb{R}^2)$ be an algebra free from both* 2- *and* 3-*nilpotents. Up to orbital topological equivalence, there exist precisely ten different phase portraits for system* (1.1) *listed in the first colomn of Table* 5.1 (*the corresponding numbers coincide with the ones presented in* [9, pp. 444–445]*, where the geometric types are drawn as well*). *Every orbital topological equivalence class* *is completely characterized by algebraic conditions given in Table* 5.1 (*for the notations used in Table* 5.1 *we refer to Subsection* 2.1).

Type	$e(A)$	$e_3^+(A)$	$e_3^-(A)$	$e_{2,2}^-(A)$	$z_{2,2}(A)$
(4)	3	$+$	$-$	$+$ or $-$	$-$
(5)	3	$-$	$+$	$+\ (-)$	$-\ (+)$
(8)	2 (3)	$+\ (-)$	$-$	$-$ ($+$ or $-$)	$-$
(9)	2	$-$	$+$	$+\ (-)$	$-\ (+)$
(10)	1	$-$	$+$	$-$	$-$
(11)	1	$+$	$-$	$-$	$-$
(12)	1	$-$	$+$	$+\ (-)$	$-\ (+)$
(13)	2	$-$	$-$	$-$	$-$
(15)	1	$-$	$-$	$-$	$-$
(64)	3	∞	$-$	$+$	$-$

Table 5.1. Cubic systems with isolated origin

Proof. As it follows from [4] and [9] (cf. [29], [30]), a planar 3-homogeneous system (1.2) having isolated origin, admits eighteen orbitally topologically non-equivalent phase portraits (see [9, pp. 444–445, Figures (1)–(17) and (64)]). Below we (i) clarify which of them can be realized for system (1.1) and (ii) classify them in purely algebraic terms.

By Proposition 5.1(a), system (1.1) has at least one invariant line, hence origin cannot be a center or focus, therefore, Figure (16) and (17) cannot appear for system (1.1). Further, by Lemma 3.6 and Bole–Brouwer Theorem, $\mathrm{ind}(0, x^3)$ may take three values: -1, 1 and 3. Consider Figure (1) from [9]. Obviously (cf. Proposition 4.2), $\mathrm{ind}(0, f) = -3$, hence Figure (1) cannont appear for system (1.1). Next, combining Corollary 5.6 with Proposition 4.2 shows that Figures (3), (7) and (14) also cannot appear for system (1.1). Finally, combining Propositions 3.6, 3.7, 5.2(b) and 4.2 yields that Figures (2) and (6) also cannot appear for system (1.1).

It follows immediately from Proposition 5.1(b) and [9] that system (1.1) admits the phase portrait of type (64) if and only if A is isomorphic to $\overline{\mathbb{C}}$. Also (see [4], [9]), if $\mathrm{ind}(0, x^3) \neq 1$, then the value of the index together with the number of invariant lines completely determine the phase portrait (up to OTE). Combining this observation with Propositions 5.2(a) and (b), 3.6 and 3.7 yields the algebraic characterization of phase portraits of types (5), (9), (12) and (10) given in Table 5.1.

It remains to consider algebras with $\mathrm{ind}(0,x^3)=1$. It follows from [4], [9] that for any $i=1,2,3,4$ there exists a planar 3-homogeneous system (1.2) with $\mathrm{ind}(0,f)=1$ and precisely i invariant lines. Moreover, a phase portrait to (1.2) is unique (up to OTE) in the case $i=1,3,4$. This observation together with Propositions 5.3 and 5.4 justify the algebraic characterization of Figures (4), (8) and (15) in Table 5.1 (as well as the fact that Figures (11) and (13) can be realized for system (1.1)). To differ algebraically Figure (11) from Figure (13) observe, first, that according to [4], [9], F_4^A has two (homogeneous) roots both of multiplicity two only in the case of Figure (13). This situation can be realized if and only if A contains precisely two idempotents and does not contain a positive 3-idempotent (cf. Proposition 5.4(a), (b)). Theorem A1 is completely proved.□

As an immediate consequence we have

Corollary 5.7. *Assume $A\in\mathrm{Alg}_C(\mathbb{R}^2)$ satisfies conditions of Theorem* A1. *Then system* (1.1) *admits a bounded solution if and only if A contains a negative* 3*-idempotent together with either a negative square idempotent or square zero divisor.*

6. Orbital topological equivalence of qubic systems in algebras with 3-nilpotents

6.1. Fundamental forms and location of 3-nilpotents. Throughout this section $A\in\mathrm{Alg}_C(\mathbb{R}^2)$ stands for an algebra with a 3-nilpotent e_1. Then in the basis (e_1,e_1^2) the multiplication table for A takes the form (3.12). Obviously, A is singular if and only if $\alpha=0$; in this case A is isomorphic to one of the following three algebras (cf. Examples 2.4): $\mathbb{N}_0$ (for $\beta=1$), $\mathbb{N}_1^0$ (for $\beta=0$) and $\mathbb{N}_2$ (for $\beta=-1$). If A is regular, then without loss of generality one can assume that $\alpha=1$. Then (cf. Proposition 3.9 and [6]), $\mathrm{ind}(0,x^3)=0$ (resp. 2) if and only if $\beta>0$ (resp. $\beta<0$); also $\mathrm{ind}(0,x^3)=1$ for $\beta=0$ (in this case $A=\mathbb{N}_3$ (cf. Theorem B)).

To recognize possible phase portraits for (1.1) in $A=(\mathbb{R}^2,*)$, assign to A an algebra $B=(\mathbb{R}^2,\circ)$ according to formulae (3.14) and (3.15). Then (see (3.17)) $\mathrm{ind}(0,x^3)$ in A coincides with $\mathrm{ind}(B)$. Moreover, $x*x*x=l(x)\cdot x\circ x$, where $l(\,\cdot\,)$ is a linear form vanishing on the 3-nilpotent e_1. Therefore, the OTE classification of systems (1.1) in algebras with 3-nilpotents is essentially based on the two statements following below.

Proposition 6.1. *Let $A=(\mathbb{R}^2,*)\in\mathrm{Alg}_C(\mathbb{R}^2)$ be a regular algebra containing a* 3*-nilpotent with the multiplication table* (3.12)*, where $\alpha=1$, and let $B=(\mathbb{R}^2,\circ)\in\mathrm{Alg}_C(\mathbb{R}^2)$ be the algebra defined according to* (3.14) *and* (3.15)*. Then:*

(a) $F_3^B = \beta \cdot F_3^A$ *(in particular, if* $\beta \neq 0$ *(i.e.* A *is not isomorphic to* $\mathbb{N}_3$*), then* A *and* B *have the same idempotents).*

(b) *If* $\operatorname{ind}(A) = 0$ *(i.e.* $\beta > 0$*), then* A *contains precisely one idempotent.*

(c) *If* $\operatorname{ind}(A) = 1$ *(i.e.* $\beta = 0$*), then* A *contains precisely one idempotent; in this case* B *is of rank two and, therefore, contains infinitely many idempotents.*

(d) *If* $\operatorname{ind}(A) = 2$ *(i.e.* $\beta < 0$*), then* A *may contain one, two or three idempotents.*

PROOF. (a) By direct computation, $F_3^B = \beta x_1^3 + \beta^2 x_1 x_2^2 - \beta x_2^3 = \beta \cdot F_3^A$. A careful analysis of the discriminant of F_3^A yields (b)–(d). □

PROPOSITION 6.2. *Let* $A, B \in \operatorname{Alg}_C(\mathbb{R}^2)$ *be as in Proposition* 6.1 *and let* l *be an equilibrium line for* (1.1) *passing through a* 3*-nilpotent. Then:*

(a) l *is not tangent for system* (1.1) *in* A.

(b) *if* $\operatorname{ind}(A) = 2$ *(i.e.* $\beta < 0$*), then* l *is e-secant for system* (1.1) *in* A.

PROOF. Statement (a) follows from Proposition 6.1(a) and Lemma 2.6 (see also Remark 2.7).

To show (b) assume $\operatorname{ind}(A) = 2$. If A contains precisely one idempotent, then statement (b) follows from (a) and Proposition 6.1(a). Assume A contains two idempotents (i.e. $\beta = -3/2^{2/3}$). In this case the partition of the phase plane of the system

$$\dot{x} = x \circ x \quad (x \in B) \tag{6.1}$$

onto sectors is determined by the straight lines $x_2 = 2^{-2/3} x_1$ and $x_2 = -2^{1/3} x_1$. Obviously, the vector $e_1 = (1, 0)$ which is a 3-nilpotent in A, and the vector $(1, -2^{1/3}/3)$ which is proportinal to a negative 3-idempotent in B, belong to the same (elliptic) sector. It remains to consider the case when A contains three idempotents.

We start with the particular case when $\beta = -2$. Then the partition onto sectors of the phase plane of (6.1) is given by the straight lines $x_2 = -x_1$ and $x_2 = 2x_1/(1 \pm \sqrt{5})$. Straightforward computations show that in this case e_1 and the vector $(1, -3/8)$ which is proportinal to a negative 3-idempotent in B, belong to the same (elliptic) sector. Take now an arbitrary algebra $A = A_\beta$ containing three idempotents (i.e. $\beta < -3/2^{2/3}$) along with the corresponding $B = B_\beta$ and assume that e_1 is p-secant. Replacing in (3.12) β with $t\beta - 2(1-t)$, $t \in [0, 1]$, one deforms continuously A_{-2} to A_β via the algebras containg 3-nilpotents and three idempotents. Obviously, this deformation takes idempotents to idempotents, therefore, the standard continuity argument implies the existence of $t_0 \in (0, 1)$ such that l is tangent for system (1.1) in $A_{t_0\beta - 2(1-t_0)}$. Contradiction with statement (a) completes the proof. □

6.2. Classification. The main result of this section is

Theorem A2. *Let $A \in \mathrm{Alg}_C(\mathbb{R}^2)$ be an algebra containig a* 3*-nilpotent. Assume $x^3 \not\equiv 0$. Up to orbital topological equivalence, there exist precisely six different phase portraits for system* (1.1) *listed in the first colomn of Table* 6.1 (*the corresponding numbers coincide with the ones presented in* [9, *pp.* 444–445], *where the geometric types are drawn as well*)*. Every orbital topological equivalence class is completely characterized by algebraic conditions given in Table* 6.1 (*for the notations used in Table* 6.1 *we refer to Subsection* 2.1).

Type	$e(A)$	$n(A)$	$n_3(A)$	$e^-_{2,2}(A)$	$z_{2,2}(A)$
(36)	1	0	1	$-$	$-$
(65)	1	2	1	$+$	$-$
(62)	1	0	2	$-$	$+$
(38)	1	0	1	$-$	$+$
(32)	2	0	1	$-$	$+$
(24)	3	0	1	$-$	$+$

Table 6.1. Cubic systems in algebras with 3-nilpotents

Proof. If A is isomorphic to $\mathbb{N}_1^0$ (cf. Examples 2.4), then $x^3 \equiv 0$.

Further, up to isomorphism, $\mathbb{N}_2$ is the only non-division algebra containing a negative square idempotent; also, $\mathbb{N}_2$ is free from square zero divisors, $n_2(\mathbb{N}_2) = 2$ and $e(\mathbb{N}_2) = n_3(\mathbb{N}_2) = 1$ (cf. Examples 2.4 and [6]). Since e_1 is not colinear to the idempotent $-e_2$, we obtain Figure (65) from [9].

Next, if A is isomorphic to $\mathbb{N}_3$, then $e(A) = 1$, $n_2(A) = 0$ and $n_3(A) = 2$. Obviously, e_1 and e_2 are square zero divisors in $\mathbb{N}_3$. Since A is pseudo-compoistion, we obtain Figure (62) from [9] (cf. Remark 2.7(d) and Propositions 2.10(b) and 6.1(c)).

Assume $A = (\mathbb{R}^2, *)$ either is isomorphic to the (singular) algebra $\mathbb{N}_0$ or admits a regular $B = (\mathbb{R}^2, \circ)$ with $\mathrm{ind}(B) = 0$. Then (cf. Proposition 6.1(b)) $e(a) = n_3(A) = 1$, $n(A) = 0$ and A does not admit negative square idempotents or square zero divisors. Proposition 6.2(a) provides Figure (36).

Finally, if $\mathrm{ind}(A) = 2$, then possible values of $e(A)$ in compilance with Propositions 6.1(d), 6.2(b) and 3.9 provide Figures (38), (32) and (24), respectively.□

Corollary 6.3. *Assume $A \in \mathrm{Alg}_C(\mathbb{R}^2)$ satisfies conditions of Theorem* A2*. Assume A is not isomorphic to $\mathbb{N}_3$. Then system* (1.1) *admits a bounded solution if and only if A contains either a square zero divisor or a negative square idempotent.*

7. Orbital topological equivalence of cubic systems in algebras with 2-nilpotents

7.1. Fundamental forms and location of 2-nilpotents. Throughout this section $A = (\mathbb{R}^2, *)$ stands for a regular algebra with a 2-nilpotent e_2.

Take a basis $(e_1, e_2) \in \mathbb{R}^2$. Then (cf. [6]) $x * x * x = x_1 \cdot f(x)$, $x = x_1 e_1 + x_2 e_2$, where $f: \mathbb{R}^2 \to \mathbb{R}^2$ is a quadratic map. Let $B = (\mathbb{R}^2, \circ) \in \mathrm{Alg}_C(\mathbb{R}^2)$ be the algebra with the multiplication defined by

$$x \circ y = \frac{1}{2}(f(x+y) - f(x) - f(y)) \quad (x, y \in \mathbb{R}^2). \tag{7.1}$$

Obviously, $\mathrm{ind}(0, x^3)$ in A coincides with $\mathrm{ind}(B)$, therefore, the OTE classification of systems (1.1) in (regular) algebras with 2-nilpotents is essentially based on the following two statements.

PROPOSITION 7.1. *Let $A \in \mathrm{Alg}_C(\mathbb{R}^2)$ be an algebra containing a 2-nilpotent e_2 and let $B \in \mathrm{Alg}_C(\mathbb{R}^2)$ be the algebra with the multiplication defined by* (7.1). *Suppose the image of the field $x \to x^3$, $x \in A$, is two-dimensional. Then $F_4^A = x_1 F_3^B$ (in particular, any idempotent, positive 3-idempotent or negative 3-idempotent of A is an idempotent of B as well). In addition,*

(a) *If $\mathrm{ind}(A) = 2$, then B may contain one, two or three idempotents.*
(b) *If $\mathrm{ind}(A) = 0$, then B may contain two or three idempotents.*
(c) *If $\mathrm{ind}(A) = \pm 1$ and A is neither pseudo-composition nor of rank two, then B may contain one* (*cf.* (3.10)) *or zero* (*cf.* (3.11)) *idempotents.*

PROOF. (a) Assume $\mathrm{ind}(A) = 2$. Then A (resp. B) admits the multiplication table $(3.9)_-$ (resp.

$$e_1^2 = -e_1 - \mu e_2, \quad e_1 e_2 = \mu e_1 + (\mu^2 - 1) e_2, \quad 2e_1 + 2\mu e_2). \tag{7.2}$$

Combining $(3.9)_-$ with (7.2) and Lemma 2.6 yields $F_4^A = x_1 F_3^B$. In particular, the negative 3-idempotent in A is always an idempotent in B. In addition, if $\mu^2 > 2$ (resp. $\mu^2 = 2$), then A has two (resp. one) idempotent(s) being also idempotent(s) in B.

(b) Assume $\mathrm{ind}(A) = 0$. Then A (resp. B) admits the multiplication table $(3.9)_+$ (resp.

$$e_1^2 = e_1 + \mu e_2, \quad e_1 e_2 = \mu e_1 + (\mu^2 + 1) e_2, \quad 2e_1 + 2\mu e_2). \tag{7.3}$$

Combining $(3.9)_+$ with (7.3) and Lemma 2.6 yields $F_4^A = x_1 F_3^B$. In particular, A always contains two idempotents being also idempotents in B. Finally (cf. Proposition 2.11(c)), if A contains a positive 3-idempotent, then B contains an additional idempotent.

(c) Assume $\mathrm{ind}(A) = \pm 1$. If A admits the multiplication table (3.10), then one has for B:

$$e_1^2 = 0, \quad e_1e_2 = \left(\frac{\lambda}{2} + \lambda^2\right)e_2, \quad e_2^2 = 0. \tag{7.4}$$

If A admits (3.11), then one has for B:

$$e_1^2 = e_1 + \frac{3}{2}e_2, \quad e_1e_2 = \frac{1}{2}e_2, \quad e_2^2 = 0. \tag{7.5}$$

Obviously, in both cases $F_4^A = x_1F_3^B$. In particular (see Proposition 2.11(c)), B contains one idempotent in the case (7.4) and does not contain an idempotent in the case (7.5). □

Proposition 7.2. *Let A be as in Proposition* 7.1 *and let l be an equilibrium line for* (1.1) *passing through a* 2*-nilpotent e_2. Then:*

(a) *If* $\mathrm{ind}(A) = 2$ *and B contains precisely one idempotent, then l is e-secant.*
(b) *If* $\mathrm{ind}(A) = 2$ *and B contains precisely two idempotents, then l is p-secant.*
(c) *If* $\mathrm{ind}(A) = 2$ *and B contains three idempotents, then l is p-secant.*
(d) *If* $\mathrm{ind}(A) = 0$*, then l is h-secant.*
(e) *If* $\mathrm{ind}(A) = \pm 1$ *and A is neither pseudo-composition nor of rank two, then l is tangent.*

Proof. If A satisfies (a), then (see $(3.9)_-$) the only idempotent in B is proportional to $\mathrm{tr}^\perp(A) = (-1, \mu)$. Since $\mathrm{ind}(B) = 2$ and $\mathrm{tr}^\perp(A)$ is not proportional to e_2, l must be e-secant.

Assume A satisfies (b). Then (see $(3.9)_-$ and Proposition 7.1) B contains an idempotent proportional to $\mathrm{tr}^\perp(A) = (-1, \mu)$ and one more $d = (1/2)(\mu, 1)$ (here $\mu = \pm\sqrt{2}$). On the other hand (see (7.1)), the vector $\mathrm{tr}^\perp(B) = (-3\mu, \mu^2 - 2)$ is proportional to a negative 3-idempotent in B. By direct computation, e_2 and $\mathrm{tr}^\perp(B)$ belong to the sectors lying on different sides of the straight line passing through $\mathrm{tr}^\perp(A)$, i.e. l is p-secant.

Assume A satisfies (c). Then (see $(3.9)_-$ and Proposition 7.1) B contains an idempotent proportional to $\mathrm{tr}^\perp(A) = (-1, \mu)$ and two more: $d_\pm = (1/2)(\mu \pm \sqrt{\mu^2 - 2}, 1)$. Since none of them is proportional to e_2, l is not tangent.

To show that l is p-secant, consider the mutual location of e_2 and the vector $\mathrm{tr}^\perp(B) = (-3\mu, \mu^2 - 2)$. Assume for a moment, that $\mu = -4$ (resp. $\mu = 4$). Then, by direct computation, the vectors e_2 and $\mathrm{tr}^\perp(B) = (12, 14)$ (resp. $(-12, 14)$) belong to the sectors lying on different sides of the straight line passing through $\mathrm{tr}^\perp(A)$, i.e. l is p-secant. Take now an arbitary algebra $A = A_\mu$ containing three idempotents (i.e. $\mu^2 > 2$). Then combining the same homotopy argument as in the proof of Proposition 6.2 with fact that l is not tangent, one can deform A_μ with $\mu > \sqrt{2}$ to the algebra A_4 (resp. with $\mu < -\sqrt{2}$ to the algebra A_{-4}).

Suppose A satisfies (d). Then (see $(3.9)_+$ and Proposition 7.1), the vectors $d_\pm = (1/2)(-\mu \pm \sqrt{\mu^2+2}, 1)$ are idempotents in B. Also, if $\mu \neq \pm 1/2$ (cf. (2.11)), then the vector $\mathrm{tr}^\perp(A) = (-1, \mu)$ is proportional to an additional idempotent in B. Obviously, e_2 is not proportional to $\mathrm{tr}^\perp(A)$ or $d_\pm$ for any μ. Therefore l is not tangent.

Since $\mathrm{ind}(A) = 0$ and (see Proposition 7.1(b)) A contains at least two idempotents, the phase portrait of (1.1) should contain both hyperbolic and parabolic sectors. Assume l is p-secant. Then both rays constituting a hyperbolic sector cannot be directed to infinity, but this contradicts the fact that A is negative 3-idempotent free. Therefore, l is h-secant.

Assume, finally, A satisfies (e). By direct computation, $F_4^A = (2\lambda^2 + \lambda - 1) \cdot x_1^3 x_2$ for the case (3.10) and $F_4^A = 3x_1^4/2$ for the case (3.11) and the result follows. $\square$

7.2. Classification: algebras without both nilpotent ideals and 3-nilpotents. The main result of this Section is contained in

Theorem A3. *Let $A \in \mathrm{Alg}_C(\mathbb{R}^2)$ be an algebra containig a* 2*-nilpotent. Assume A is nilpotent ideal free. Assume A does not contain* 3*-nilpotents. Up to orbital topological equivalence, there exist precisely six different phase portraits for system* (1.1) *listed in the first colomn of Table* 7.1 (*the corresponding numbers coincide with the ones presented in* [9, pp. 444–445], *where the geometric types are drawn as well*). *Every orbital topological equivalence class is completely characterized by algebraic conditions given in Table* 7.1 (*for the notations used in Table* 7.1 *we refer to Subsection* 2.1).

Type	$e(A)$	$n(A)$	$n_{2,2}(A)$	$e_3^+(A)$	$e_3^-(A)$
(68)	0	2	$-$	$-$	$-$
(38)	0	1	$+$	$-$	$+$
(33)	1	1	$+$	$-$	$+$
(25)	2	1	$+$	$-$	$+$
(28)	2	1	$+$	$-$	$-$
(20)	2	1	$+$	$+$	$-$

Table 7.1. Cubic systems in algebras without both nilpotent ideals and 3-nilpotents

Proof. Obviously (cf. Examples 2.4), $\mathbb{N}_2^0$ is the only singular algebra being free from both nilpotent ideals and 3-nilpotents. By direct computation,

$$x^3 = (x_1 - x_2)^2(x_1 + x_2)(e_1 + e_2).$$

Since $n(A) = 2$ and $e(A) = 0$, one obtains Figure (68).

The OTE classification of phase portraits of system (1.1) occuring in regular algebras containing square nilpotents can be obtained directly from Propositions 7.1(a), (b) and 7.2(a)–(d) in compliance with [9]. □

Corollary 7.3. *Assume $A \in \mathrm{Alg}_C(\mathbb{R}^2)$ satisfies conditions of Theorem* A3. *Then system* (1.1) *admits a bounded solution if and only if A contains a square nilpotent together with negative* 3*-idempotent.*

7.3. Classification: 3-nilpotent free algebras with nilpotent ideals. Although for the algebras containing nilpotent ideals, possible values of $\mathrm{ind}(A) = \pm 1$ agree with negative square nilpotent dichotomy (cf. Proposition 3.3(e)), a complete OTE classification of the *quadratic* systems cannot be done by means of reasonable polynomial equations in algebras. From this point of view, passing to the *cubic* systems only aggravates the situation. Therefore, to obtain the OTE classification for cubic systems, we prefer a direct analysis of spectral properties of the matrix naturally associated to the cubic map rather than looking for sophisticated polynomial equations in algebras. Another reason justifying this approach is the statement following below.

Proposition 7.4. *Let $A \in \mathrm{Alg}_C(\mathbb{R}^2)$ be a* 3*-nilpotent free algebra containing a nilpotent ideal. Then:*

(a) *A is negative and positive* 3*-idempotent free.*
(b) *System* (1.1) *does not admit a bounded solution.*

Proof. Set

$$\overline{M} = \begin{pmatrix} 1 & 0 \\ 3/2 & 1 \end{pmatrix}, \quad \text{and} \quad M = \begin{pmatrix} 1 & 0 \\ 0 & 2\lambda^2 + \lambda \end{pmatrix}.$$

Obviously (cf. (7.4) and (7.5)), $x^3 = x_1^2 \cdot \overline{M}x$ in the case (3.11) and $x^3 = x_1^2 \cdot Mx$ in the case (3.10), $x = (x_1, x_2)$, from which Statement (a) follows immediately.

Take A satisfying (3.11). Since $e(A) = 0$ and $n(A) = 1$, one obtains Figure (56) from [9] for system (1.1) ("Jordan node").

Assume A satisfies (3.10) and consider several cases. If $\lambda = 0$, then $A = \mathbb{N}_1$ is a singular algebra (cf. Examples 2.4). Also, if $\lambda = -1/2$, then the field $x \to x^3$ has one-dimensional image. In both cases $e(A) = n(A) = 1$, the matrix M is degenerate ($\mathrm{ind}(0, x^3) = 0$) and one obtains for system (1.1) Figure (36) from [9].

Suppose now the eigenvalues of M are equal, i.e. $2\lambda^2 + \lambda = 1$. Then A is either of rank two ($\lambda = 1/2$) or pseudo-composition ($\lambda = -1$); cf. Examples 2.9.

In both cases $e_3^+(A) = \infty$ and M determines a "bicritical node", so that one obtains Figure (63) from [9].

The above four algebras split the family (3.10) onto five subsets. If $\operatorname{ind}(0, x^3) = -1$ (i.e. $-1/2 < \lambda < 0$), then M determines a "saddle" and one obtains Figure (45) from [9]. Further, in the remaining cases $\operatorname{ind}(0, x^3) = 1$ and still $e(A) = n(A) = 1$, therefore M determines a "node". If the absolute value of the eigenvalue of M corresponding to the 2-nilpotent is less than the absolute value of the eigenvalue corresponding to the idempotent (i.e. $-1 < \lambda < -1/2$ or $0 < \lambda < 1/2$), then the phase curves adjoin at infinity the invariant line and one obtains Figure (53); otherwise (i.e. $\lambda < -1$ or $\lambda > 1/2$), phase curves adjoin the equilibrium line and one obtains Figure (52).

The proof of Proposition 7.4 is complete. □

Remarks 7.5. (a) In fact, the proof of Proposition 7.4 contains a complete OTE classification of system (1.1) in the considered case.

(b) It should be pointed out that passing from binary algebra invariants (solubility of polynomial equations) to unary algebra invariants (spectral properties of matrices) is not just ad hoc trick used in the proof of Proposition 7.4. First, it resembles the following fact (cf. [2]): if $\mathrm{GL}(2, \mathbb{C})$ acts via isomorphisms on the set of all *complex* two-dimensional (binary) algebras, then the orbits of algebras containing nilpotent ideals are *semi-stable*, meaning that all polynomial invariants vanish on these orbits; in particular, these orbits require special moduli spaces. On the other hand, it appeals to a better understanding of a connection between adjoining phase curves of system (1.1) and (topological) properties of the corresponding algebras. However, both aspects go beyond the scope of our paper.

8. Application: periodic solutions to systems "cubic at infinity" in rank three algebras

8.1. Bounded solutions to cubic systems in rank three algebras. In this section we apply our classification results to systems occuring in rank three algebras (in general, of dimension $n \geq 2$; cf. Definition 2.8). To this end we need

Definition 8.1. We say that an algebra $A \in \mathrm{Alg}_C(\mathbb{R}^n)$ admits a $*$-*complex structure* if there exists a two-dimensional subalgebra D satisfying at least one of the conditions (a)–(d) from Theorem B provided D is not isomorphic to $\mathbb{N}_3$ (see Introduction).

Combining Example 2.9(a), Corollaries 5.7, 6.3, 7.3 and Proposition 7.4(b) with the fact that any phase curve for a cubic system occuring in a rank three algebra is planar, one obtains

Theorem B′. *Let $A \in \mathrm{Alg}_C(\mathbb{R}^n)$ be a rank three algebra. Then system*

$$\dot{x} = x^3 \quad (x \in A) \tag{8.1}$$

admits a bounded solution if and only if A admits a $*$-complex structure.

It should be noticed that being of completely semi-algebrac nature, Theorem B′ is in a sharp contrast to the corresponding results on bounded solutions to homogeneous systems given in [27], [19], [23] and based on the guiding function approach. At the same time, the problem of the existence of bounded solutions to polynomial systems is not "semi-algebraic", in general (see [5]), therefore one cannot expect to obtain finitely many "basic" algebraic equations responsible for the existence of bounded solutions (cf. [27], [19], [23]).

8.2. Systems "cubic at infinity". Observe that from the viewpoint of possible applications (see [27], [19], [23]), any result on the *non-existence* of bounded solutions to homogeneous systems is also of great interest. As a consequence of Theorem B′ we have

Corollary 8.2. *Let $A \in \mathrm{Alg}_C(\mathbb{R}^n)$ be a rank three algebra. Assume A is* 2*-nilpotent,* 3*-nilpotent and negative* 3*-idempotent free. Then system* (8.1) *does not admit a bounded solution.*

Combining this result with the main result from [27] one obatins

Corollary 8.3. *Let $A \in \mathrm{Alg}_C(\mathbb{R}^n)$ be a rank three algebra. Assume A is* 2*-nilpotent,* 3*-nilpotent and negative* 3*-idempotent free. Then for any continuos map $h: A \times \mathbb{R} \to A$ being ω-periodic in $t \in \mathbb{R}$ and small at infinity* (*i.e.*

$$\lim_{\|x\| \to \infty} \sup_t \|x\|^{-3} \|h(t,x)\| = 0)$$

system

$$\dot{x} = x^3 + h(x,t) \quad (x \in A,\ t \in \mathbb{R})$$

admits at least one ω-periodic solution.

Example 8.4. Here we give the simplest example illustrating Corollary 8.3 (for more involved examples we refer to [35]). Take a quadratic map $f: \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$f(x_1, \dots, x_n) = (x_1^2 - x_2^2 - x_3^2 - \dots - x_n^2, -2x_1x_2, -2x_1x_3, \dots, -2x_1x_n). \tag{8.2}$$

Using (8.2) one can equip $\mathbb{R}^n$ with the multiplication structure $(\mathbb{R}^n, *)$ (cf. (7.1)). By direct computation, $x * x * x = \|x\|^2 x$, from which it follows that $(\mathbb{R}^n, *)$ is a pseudo-composition algebra (in particular, it is of rank three) being also 2-nilpotent, 3-nilpotent and negative 3-idempotent free. Therefore, $(\mathbb{R}^n, *)$ satisfies all the conditions of Corollary 8.3.

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