

**ASYMPTOTIC BIFURCATION PROBLEMS  
FOR QUASILINEAR EQUATIONS  
EXISTENCE AND MULTIPLICITY RESULTS**

PAVEL DRÁBEK

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ABSTRACT. In this paper we address the existence and multiplicity results for

$$\begin{cases} -\Delta_p u - \lambda|u|^{p-2}u = h(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $p > 1$ ,  $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ ,  $h$  is a bounded function and the spectral parameter  $\lambda$  stays “near” the principal eigenvalue of the  $p$ -Laplacian.

We show how the bifurcation theory combined with certain asymptotic estimates yield desired results.

### 1. Introduction

This is a survey paper the results of which were presented at the conference “Topological and Variational Methods in Nonlinear Analysis” TVMNA 2003 which was held in Będlewo, Poland in June 2003. The results presented here concern the necessary and sufficient conditions for solvability of the quasilinear problem

$$(1.1) \quad \begin{cases} -\Delta_p u - \lambda|u|^{p-2}u = h(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

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where  $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$  is the  $p$ -Laplacian,  $p > 1$  is a real number,  $\Omega$  is a smooth domain in  $\mathbb{R}^N$ ,  $N \geq 1$ ,  $\lambda$  is a real spectral parameter and  $h = h(x, s)$  is a bounded function the properties of which are specified later.

Let us denote by  $\lambda_1$  the first eigenvalue of the  $p$ -Laplacian. It is well known that  $\lambda_1 > 0$  admits variational characterization

$$\lambda_1 = \inf \left\{ \int_{\Omega} |\nabla u|^p dx : \int_{\Omega} |u|^p dx = 1 \right\},$$

it is an isolated eigenvalue of  $-\Delta_p$  and simple in the sense that there is exactly one pair of normalized eigenfunctions  $\varphi_1$  and  $-\varphi_1$  chosen in such a way that  $\varphi_1(x) > 0$ ,  $x \in \Omega$  and  $\partial\varphi_1(x)/\partial\nu < 0$ ,  $x \in \Omega$  (see e.g. [1], [14], etc.).

In this paper we shall concentrate on the case when  $\lambda$  stays “near”  $\lambda_1$  and  $h(x, s)$  is either independent of  $s$  or it is a nonlinearity of the Landesman–Lazer type. To be more specific, we start with the following two motivations which deal with the semilinear problem ( $p = 2$ ) and special type of  $h$ .

As the *first motivation* we consider the following linear problem

$$(1.2) \quad \begin{cases} -\Delta u - \lambda_1 u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $f$  is a given function defined on  $\Omega$ . Note that  $h(x, s) = f(x)$  does not depend on  $s$  in this case. Without specifying function spaces (this depends on whether we deal with the weak or classical solutions to (1.2)) let us mention *necessary and sufficient condition* for the solvability of (1.2) provided by the linear *Fredholm alternative*. It states that (1.2) is solvable if and only if

$$(1.3) \quad \int_{\Omega} f \varphi_1 dx = 0$$

(see Figure 1.1).

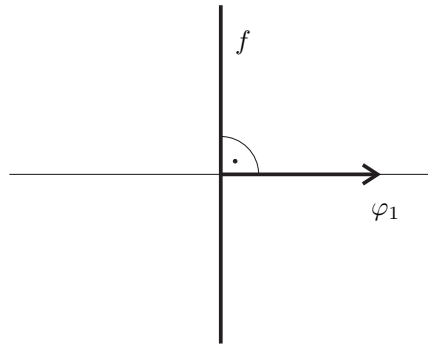


FIGURE 1.1. Geometrical interpretation of the necessary and sufficient condition provided by the linear Fredholm alternative.

As the *second motivation* we consider semilinear Landesman–Lazer type problem

$$(1.4) \quad \begin{cases} -\Delta u - \lambda_1 u + \arctan u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that the function  $h = h(x, s)$  is again of a special form  $h(x, s) = f(x) - \arctan s$  in this case. This is a prototype of problems studied extensively in the last three decades of the 20th century (following the pioneering work of Landesman and Lazer [13]). Also in this case there is *necessary and sufficient condition* (the so-called *Landesman–Lazer-type condition*) for the solvability of (1.4) which generalizes condition (1.3) in a natural way. It states that (1.4) is solvable if and only if

$$(1.5) \quad -\frac{\pi}{2} < \int_{\Omega} f \varphi_1 dx < \frac{\pi}{2}$$

(we assume that  $\varphi_1$  is normalized here by  $\int_{\Omega} \varphi_1 dx = 1$ ) and it has also an instructive geometrical interpretation (see Figure 1.2).

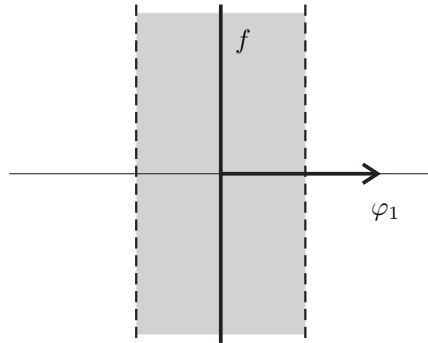


FIGURE 1.2. Geometrical interpretation of the necessary and sufficient condition provided by the Landesman–Lazer-type condition.

In this paper we discuss how previous classical results for semilinear (linear) problem extend to the general quasilinear case ( $p \neq 2$ ). In fact, we can give necessary and sufficient condition for solvability of

$$(1.6) \quad \begin{cases} -\Delta_p u - \lambda |u|^{p-2} u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

if  $\lambda = \lambda_1$  and  $p \neq 2$ . Moreover, we present some multiplicity results for (1.6) when  $\lambda$  stays “near”  $\lambda_1$  (including the case  $\lambda = \lambda_1$ ). On the other hand, results for the Landesman–Lazer-type problem *extend* to the quasilinear case as well but the Landesman–Lazer-type condition (1.5) is *only sufficient* in general.

Let us point out that the investigation of

$$(1.7) \quad \begin{cases} -\Delta_p u - \lambda|u|^{p-2}u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

is much more difficult than the study of

$$(1.8) \quad \begin{cases} -\Delta_p u - \lambda_1|u|^{p-2}u + g(u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

if  $p \neq 2$  (the situation is opposite if  $p = 2$ !). The reason consists in the fact that the special type of nonlinearity  $g$  in (1.8) helps to prove that the problem has “the right” structure. For instance, looking at (1.7) and (1.8) from the variational point of view then the nonlinearity of the Landesman–Lazer-type guarantees that the energy functional associated with (1.8) has either geometry of a global minimum or geometry of a saddle point and, moreover, it satisfies the Palais–Smale condition (see e.g. [2]–[4] and [12]). None of these properties are available for (1.7) and so “non standard” approaches must be looked for (see e.g. [6]–[11], [15]–[17]).

The results addressed in the following sections were obtained in papers Drábek, Girg, Takáč and Ulm [9] and Drábek, Girg and Takáč [8], and they rely on the *bifurcation theory*. In combination with the *method of lower and upper solutions* they lead to the following main results.

## 2. Main results

Note that by a solution we always mean a weak solution which is defined in a usual way.

THEOREM 2.1 (see [9]). *Let  $f \in L^\infty(\Omega)$  split as*

$$f = f^T + a\varphi_1, \quad \int_{\Omega} f^T \varphi_1 \, dx = 0, \quad a \in \mathbb{R}.$$

*Then for given  $f^T \neq 0$  there exist  $a_1(f^T) < 0 < a_2(f^T)$  such that*

$$(2.1) \quad \begin{cases} -\Delta_p u - \lambda_1|u|^{p-2}u = f^T + a\varphi_1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

*has at least*

- (a) *one solution provided  $a \in [a_1, a_2]$ ,*
- (b) *two distinct solutions provided  $a \in (a_1, 0) \cup (0, a_2)$ ,*
- (c) *no solution provided  $a \notin [a_1, a_2]$ .*

REMARK 2.2. The set of all  $f$  for which (2.1) has a solution forms a cone depicted in Figure 2.1. It can be actually proved that this cone has a nonempty interior with respect to the topology induced by  $L^\infty$  norm (see [7]). Let us also

note that the weak solution belongs to  $C^{1,\alpha}(\overline{\Omega})$  with some  $\alpha \in (0, 1)$  provided  $\partial\Omega$  is smooth enough ([8], [9]).

REMARK 2.3. Theorem 2.1 actually states that (1.3) is sufficient for solvability of (1.7) for arbitrary  $p > 1$  but it is not necessary if  $p \neq 2$ .

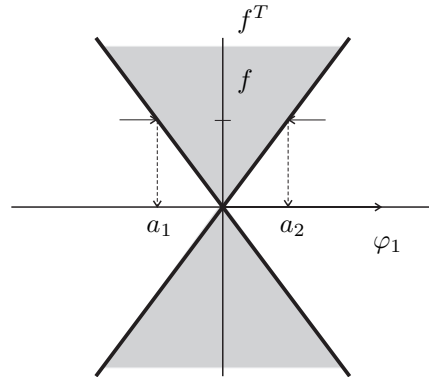


FIGURE 2.1. The set of all  $f$  for which (2.1) has at least one (two) solution(s).

THEOREM 2.4 (see [9]). *Let us consider*

$$(2.2) \quad \begin{cases} -\Delta_p u - \lambda |u|^{p-2} u = f^T + a\varphi_1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $f^T$  and  $a$  are as above. Then there exists  $\varepsilon > 0$  such that

- (a) for every  $\varepsilon' \in (0, \varepsilon)$  there is  $\eta = \eta(f^T, \varepsilon, \varepsilon') > 0$  such that  $\varepsilon' < |a| < \varepsilon$  and  $\lambda \in (\lambda_1 - \eta, \lambda_1) \cup (\lambda_1, \lambda_1 + \eta)$  imply that (2.2) has at least three distinct solutions, of which at least one is positive and at least one is negative,
- (b)  $\lambda = \lambda_1$  and  $0 < |a| < \varepsilon$  imply that problem (2.2) has at least two solutions, of which at least one is negative if  $(p-2)a < 0$ , and at least one is positive if  $(p-2)a > 0$ .

Moreover, there exists  $\widehat{\eta} = \widehat{\eta}(f^T) > 0$  such that for  $a = 0$  problem (2.2) has at least three distinct solutions (among them at least one positive and one negative) provided either  $1 < p < 2$  and  $\lambda \in (\lambda_1 - \widehat{\eta}, \lambda_1)$ , or  $p > 2$  and  $\lambda \in (\lambda_1, \lambda_1 + \widehat{\eta})$ .

THEOREM 2.5 (see [9]). *There exist  $a_0 = a_0(f^T) > 0$  and  $\delta = \delta(f^T) > 0$  such that*

- (a) if either  $\lambda \in (\lambda_1 - \delta, \lambda_1)$  and  $a \geq a_0$ , or else  $\lambda \in (\lambda_1, \lambda_1 + \delta)$  and  $a \leq -a_0$ , then (2.2) can have only positive solutions,
- (b) if either  $\lambda \in (\lambda_1 - \delta, \lambda_1)$  and  $a \leq -a_0$ , or else  $\lambda \in (\lambda_1, \lambda_1 + \delta)$  and  $a \geq a_0$ , then (2.2) can have only negative solutions.

Concerning the perturbed problem

$$(2.3) \quad \begin{cases} -\Delta_p u - \lambda|u|^{p-2}u + g(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

we only state two results for special choice of  $g$  which illustrate the striking difference between the semilinear ( $p = 2$ ) and quasilinear ( $p \neq 2$ ) case.

**THEOREM 2.6** (see [8]). *Let  $f \in L^\infty(\Omega)$  and  $g(s) = e^{-s^2}$ ,  $s \in \mathbb{R}$ . Then*

- (a) *for  $p = 2$ , condition  $\int_\Omega f \varphi_1 dx > 0$  is necessary for solvability of (2.3),*
- (b) *for  $1 < p < 2, 2 < p < 3$ , condition  $\int_\Omega f \varphi_1 dx = 0$  is sufficient for solvability of (2.3).*

**THEOREM 2.7** (see [8]). *Let  $f \in L^\infty(\Omega)$  and  $g(s) = \operatorname{arccotan}|s|^{q-1}s$ ,  $s \in \mathbb{R}$ . Then*

- (a) *for  $p = 2$ ,  $q > 0$ , condition*

$$0 < \int_\Omega f \varphi_1 dx < \pi$$

*is necessary and sufficient for solvability of (2.3),*

- (b) *for  $1 < p < 2$ ,  $q > p - 1$ , condition*

$$0 \leq \int_\Omega f \varphi_1 dx < \pi$$

*is sufficient for solvability of (2.3).*

### 3. Bifurcation approach

Set

$$(J(u), w) = \int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla w \rangle dx,$$

$$(S(u), w) = \int_\Omega |u|^{p-2} u w dx,$$

$$(H(u), w) = \int_\Omega h(\cdot, u) w dx,$$

$u, w \in W_0^{1,p}(\Omega)$ . Here  $(\cdot, \cdot)$  is the duality between  $W_0^{1,p}(\Omega)$  and its dual space,  $\langle \cdot, \cdot \rangle$  is the scalar product in  $\mathbb{R}^N$  and  $\|\cdot\|$  will further denote the norm on  $W_0^{1,p}(\Omega)$ :

$$\|u\| = \left( \int_\Omega |\nabla u|^p dx \right)^{1/p}.$$

Then operator equation

$$(3.1) \quad J(u) - \lambda S(u) = H(u)$$

is a weak formulation of (1.1).

For  $u \neq 0$  let  $w := u/\|u\|^2$  and

$$G(v) := \begin{cases} \|v\|^{2(p-1)}H(v\|v\|^{-2}) & \text{for } v \neq 0, \\ 0 & \text{for } v = 0. \end{cases}$$

Then (3.1) is equivalent to

$$(3.2) \quad J(v) - \lambda S(v) = G(v).$$

Since

$$\lim_{\|v\| \rightarrow 0} \frac{G(v)}{\|v\|^{p-1}} = 0,$$

the global bifurcation result can be applied to (3.2) (see [6]). It is proved in [6] that  $(\lambda_1, 0)$  is a global bifurcation point for (3.2). Going back to the original equation (3.1) we obtain that  $(\lambda_1, \infty)$  is an *asymptotic bifurcation point* for (3.1) (i.e. there exists a sequence  $\{(\lambda_n, u_n)\}_{n=1}^\infty \subset \mathbb{R} \times W_0^{1,p}(\Omega)$  such that (3.1) holds with  $\lambda = \lambda_n$ ,  $u = u_n$  and  $(\lambda_n, \|u_n\|) \rightarrow (\lambda_1, \infty)$ ) and, moreover, there are maximal closed sets  $\mathcal{C}^+, \mathcal{C}^- \subset \mathbb{R} \times W_0^{1,p}(\Omega)$  of solutions of (3.1) such that there exist sequences of pairs  $(\mu_n, u_n) \in \mathcal{C}^+$  and  $(\hat{\mu}_n, \hat{u}_n) \in \mathcal{C}^-$  such that  $\mu_n \rightarrow \lambda_1$ ,  $\hat{\mu}_n \rightarrow \lambda_1$ ,  $\|u_n\| \rightarrow \infty$  and  $\|\hat{u}_n\| \rightarrow \infty$ , together with  $u_n/\|u_n\| \rightarrow \varphi_1/\|\varphi_1\|$  and  $\hat{u}_n/\|\hat{u}_n\| \rightarrow -\varphi_1/\|\varphi_1\|$ . This is very useful information, however, this information alone does not tell us anything about the solvability of (3.1) for  $\lambda = \lambda_1$ !

On the other hand, if we knew that e.g.  $(\mu, u) \in \mathcal{C}^\pm$ ,  $\|u\|$  “large”, imply that  $\mu < \lambda_1$  then we immediately get a solution of (3.1) for  $\lambda = \lambda_1$ . Indeed, the isolatedness of  $\lambda_1$ , together with this property would imply the existence of  $(\lambda_n, u_n) \in \mathbb{R} \times W_0^{1,p}(\Omega)$  which solve (3.1) and  $\lambda_n \rightarrow \lambda_1+$ ,  $\|u_n\| \leq \text{const}$ . The existence of a solution to (3.1) for  $\lambda = \lambda_1$  now follows from the standard compactness argument and passage to the limit for  $n \rightarrow \infty$ . So, we need an asymptotic estimate for solutions  $(\lambda, u)$  of (3.1), where  $\lambda$  is “close” to  $\lambda_1$  and  $\|u\|$  is “large”.

#### 4. Asymptotic estimate

In order to formulate an asymptotic estimate we need the linearization of (3.1) around the first eigenfunction  $\varphi_1$ . Namely, we define the matrix

$$\mathbb{A} = |\nabla\varphi_1|^{p-2} \left( \mathbb{I} + (p-2) \frac{\nabla\varphi_1 \otimes \nabla\varphi_1}{|\nabla\varphi_1|^2} \right)$$

(here  $\mathbb{I}$  is the identity matrix  $N \times N$ ), the quadratic form

$$Q(v, v) = \frac{1}{2} \int_\Omega \langle \mathbb{A} \nabla v, \nabla v \rangle dx - \frac{\lambda_1}{2} (p-1) \int_\Omega \varphi_1^{p-2} v^2 dx$$

and the weighted space  $\mathcal{D}_{\varphi_1}$  endowed with the norm

$$\|v\|_{\mathcal{D}_{\varphi_1}} = \left( \int_\Omega |\nabla\varphi_1|^{p-2} |\nabla v|^2 dx \right)^{1/2}.$$

Without discussing the technical details we shall present the key asymptotic estimate due to which the bifurcation diagrams can be made more precise.

**THEOREM 4.1** (see [8], [9]). *Let  $\{\mu_n\}_{n=1}^\infty \subset \mathbb{R}$  and  $\{u_n\}_{n=1}^\infty \subset W_0^{1,p}(\Omega)$  be sequences, and let  $\delta > 0$  be such that*

- (a)  $\lambda_1 + \mu_n < \lambda_2 - \delta$  for all  $n \in \mathbb{N}$  ( $\lambda_2$  is the second eigenvalue of  $-\Delta_p$ ),
- (b)  $h(\cdot, u_n(\cdot)) \rightharpoonup^* \tilde{h}(\cdot)$  weakly-star in  $L^\infty(\Omega)$ ,
- (c)  $\|u_n\| \rightarrow \infty$  as  $n \rightarrow \infty$ ,
- (d) in addition assume that for all  $n \in \mathbb{N}$  and  $v \in W_0^{1,p}(\Omega)$ ,

$$(4.1) \quad \int_{\Omega} |\nabla u_n|^{p-2} \langle \nabla u_n, \nabla v \rangle dx = (\lambda_1 + \mu_n) \int_{\Omega} |u_n|^{p-2} u_n v dx + \int_{\Omega} h(x, u_n) v dx.$$

Then  $\mu_n \rightarrow 0$  and, writing  $u_n = t_n^{-1}(\varphi_1 + v_n^T)$  with  $t_n \in \mathbb{R}$ ,  $t_n \neq 0$ , and  $v_n^T \in \{w \in W_0^{1,p}(\Omega) : \int_{\Omega} w \varphi_1 dx = 0\}$ , we have  $t_n \rightarrow 0$ ,  $|t_n|^{-p} t_n v_n^T \rightarrow V^T$  in  $\mathcal{D}_{\varphi_1}$ , if  $p > 2$  and in  $W_0^{1,2}(\Omega)$  if  $1 < p < 2$ , and

$$(4.2) \quad \mu_n = -|t_n|^{p-2} t_n \int_{\Omega} h(\cdot, u_n) \varphi_1 dx + (p-2)|t_n|^{2(p-1)} Q(V^T, V^T) + \\ + (p-1)|t_n|^{2(p-1)} \left( \int_{\Omega} \tilde{h} \varphi_1 dx \right) \left( \int_{\Omega} \varphi_1^{p-1} V^T dx \right) + o(|t_n|^{2(p-1)}).$$

Moreover,  $V^T \in \{u \in \mathcal{D}_{\varphi_1} : \int_{\Omega} u \varphi_1 dx = 0\}$  is the unique solution to

$$2Q(V^T, \Phi) = \int_{\Omega} h^* \Phi dx \quad \text{for all } \Phi \in \mathcal{D}_{\varphi_1},$$

where we have denoted

$$h^* = \tilde{h} - \left( \int_{\Omega} \tilde{h} \varphi_1 dx \right) \varphi_1^{p-1}.$$

**REMARK 4.2.** In particular,  $V^T \neq 0$  (and hence  $Q(V^T, V^T) > 0$ ) if and only if  $\tilde{h}$  is not a real multiple of  $\varphi_1^{p-1}$ . This is the case if e.g.  $\int_{\Omega} \tilde{h} \varphi_1 dx = 0$  and  $\tilde{h} \neq 0$ .

Now, we shall investigate some special choices of  $h = h(x, s)$  and show how the asymptotics above applies to get a priori estimates. Let us start with the case  $h(x, s) = f(x)$ , i.e. we shall deal with the problem

$$(4.3) \quad \begin{cases} -\Delta_p u - \lambda |u|^{p-2} u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Assume that  $\int_{\Omega} f \varphi_1 dx = 0$  and  $f \neq 0$ . Then (4.2) is of the form

$$(4.4) \quad \mu_n = (p-2)|t_n|^{2(p-1)} Q(V^T, V^T) + o(|t_n|^{2(p-1)}).$$

Since  $Q(V^T, V^T) > 0$  (cf. Remark 4.1), we have that  $\mu_n < 0$  for  $1 < p < 2$  and  $\mu_n > 0$  for  $p > 2$ , respectively. This means that every solution  $(\lambda, u) \in \mathbb{R} \times W_0^{1,p}(\Omega)$  of (4.3) with “large” norm  $\|u\|$  must satisfy  $\lambda < \lambda_1$  if  $1 < p < 2$



and  $\lambda > \lambda_1$  for  $p > 2$ , respectively (see Figure 5.1). In particular, we get at least one solution of (4.3) with  $\lambda = \lambda_1$  (cf. the end of Section 3) and, moreover, the fact that all such solutions are a priori bounded for  $p \neq 2$ . The last statement is very different from one which holds in the linear case  $p = 2$ . The reader should also notice that there is no information provided by asymptotics (4.4) if  $p = 2$ .

Assume that  $f = f^T + a\varphi_1$ ,  $\int_{\Omega} f^T \varphi_1 dx = 0$ ,  $a \in \mathbb{R}$ ,  $a \neq 0$ . Then (4.2) reduces to

$$\mu_n = -|t_n|^{p-2} t_n a \int_{\Omega} \varphi_1^2 dx + o(|t_n|^{p-1}).$$

This means that every solution  $(\lambda, u) \in \mathbb{R} \times W_0^{1,p}(\Omega)$  of (4.3) with “large” norm  $\|u\|$  satisfies:  $(\lambda, u) \in \mathcal{C}^+$  implies  $\lambda < \lambda_1$  for  $a > 0$  and  $\lambda > \lambda_1$  for  $a < 0$ , and  $(\lambda, u) \in \mathcal{C}^-$  implies  $\lambda > \lambda_1$  for  $a > 0$  and  $\lambda < \lambda_1$  for  $a < 0$ . These facts are true for all  $p > 1$ . More careful analysis which takes into account also terms of order  $|t_n|^{2(p-1)}$  yields to the situations depicted in Figures 5.2–5.4. These results combined with the method of lower and upper solutions lead to the statements of Theorems 2.1, 2.4 and 2.5 (see [9] for details).

Now, let  $h$  be of a special form  $h(x, s) = f(x) - g(s)$ , where  $\lim_{s \rightarrow \pm\infty} g(s) = 0$ . This corresponds to the problem

$$(4.5) \quad \begin{cases} -\Delta_p u - \lambda |u|^{p-2} u + g(u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and (4.2) is of the form

$$(4.6) \quad \begin{aligned} \mu_n = & -|t_n|^{p-2} t_n \int_{\Omega} [f - g(u_n)] \varphi_1 dx + (p-2) |t_n|^{2(p-1)} Q(V^T, V^T) \\ & + (p-1) |t_n|^{2(p-1)} \left( \int_{\Omega} f \varphi_1 dx \right) \left( \int_{\Omega} \varphi_1^{p-1} V^T dx \right) + o(|t_n|^{2(p-1)}). \end{aligned}$$

This asymptotics for  $\mu_n$  and  $t_n$  offers variety of possibilities depending on the asymptotic behaviour of  $g$ . In particular, if  $g(s) = e^{-s^2}$  and  $\int_{\Omega} f \varphi_1 dx = 0$  then (4.6) reduces to

$$(4.7) \quad \begin{aligned} \mu_n = & |t_n|^{p-2} t_n \int_{\Omega} e^{u_n} \varphi_1 dx \\ & + (p-2) |t_n|^{2(p-1)} Q(V^T, V^T) + o(|t_n|^{2(p-1)}). \end{aligned}$$

It can be proved (see [8]) that  $\mu_n < 0$  for  $n$  large if  $1 < p < 2$ , and  $\mu_n > 0$  for  $n$  large if  $2 < p < 3$ . Hence the bifurcation diagram for (4.5) with  $g(s) = e^{-s^2}$  and  $\int_{\Omega} f \varphi_1 dx = 0$  behaves like in Figure 5.1 and the assertion of Theorem 2.6(b) follows. The assertion of Theorem 2.6(a) follows easily multiplying (2.3) (with  $g(s) = e^{-s^2}$ ) by  $\varphi_1$  and integrating by parts. Similarly, (4.6) is used to prove Theorem 2.7(b) if  $g(s) = \operatorname{arccotan}|s|^{q-1}s$ ,  $1 < p < 2$ ,  $q > p - 1$ . On the other hand the assertion of Theorem 2.7(a) is the classical result of Landesman and Lazer [13].

### 5. Bifurcation diagrams

In this section we present some bifurcation diagrams which correspond to various situations considered in Section 4.

The reader should have in mind that these pictures have to be viewed “asymptotically” where  $(\lambda, u) \in \mathcal{C}^\pm$  always means that  $\|u\|$  is “large”.

The reader is invited to make the following three *experiments*:

- (a) Take  $a = 0$  and let  $p$  vary from 1 to  $\infty$ . Observe how the left part of Figure 5.1 is deformed into its right part assuming the “linear” shape for  $p = 2$ .

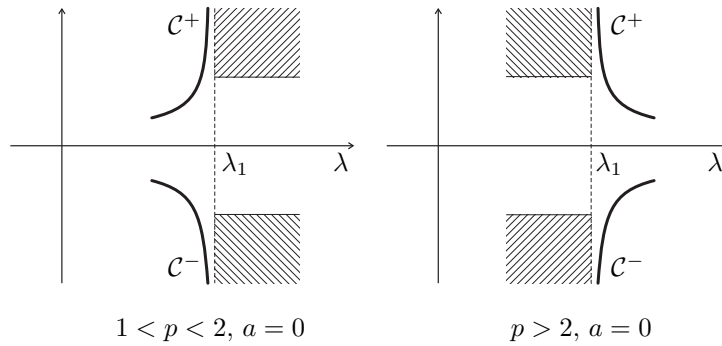


FIGURE 5.1

- (b) Take  $1 < p < 2$  and let  $a$  vary from  $-\infty$  to  $\infty$ . Observe how the right part of Figure 5.2 is deformed into its left part assuming intermediate shapes depicted in the left parts of Figures 5.4, 5.1 and 5.3.

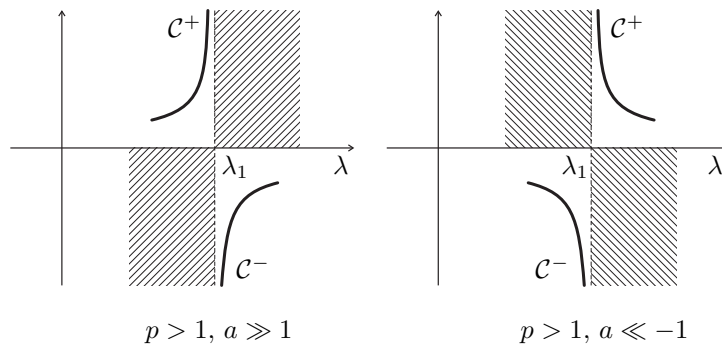


FIGURE 5.2

- (c) Take  $p > 2$  and let  $a$  vary from  $-\infty$  to  $\infty$ . Observe again how the right part of Figure 5.2 is deformed into its left part assuming intermediate shapes depicted in the right parts of Figures 5.4, 5.1 and 5.3.

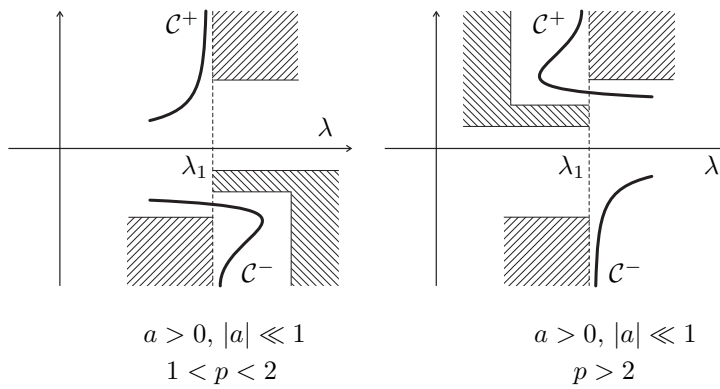


FIGURE 5.3

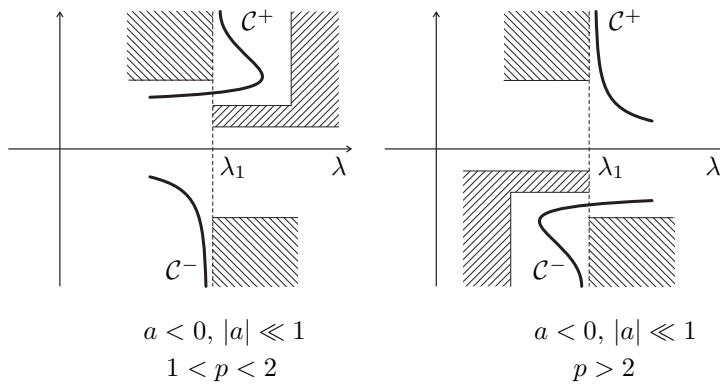


FIGURE 5.4

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PAVEL DRÁBEK  
Centre of Applied Mathematics  
University of West Bohemia  
Univerzitní 22  
306 14 Plzeň, CZECH REPUBLIC  
*E-mail address:* pdrabek@kma.zcu.cz