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ON TRAJECTORIES OF ANALYTIC GRADIENT VECTOR FIELDS ON ANALYTIC MANIFOLDS

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To Professor Kazimierz Gęba on his 70th birthday

Abstract. Let $f: M \to \mathbb{R}$ be an analytic proper function defined in a neighbourhood of a closed "regular" (for instance semi-analytic or sub-analytic) set $P \subset f^{-1}(y)$. We show that the set of non-trivial trajectories of the equation $\dot{x} = \nabla f(x)$ attracted by P has the same Čech–Alexander cohomology groups as $\Omega \cap \{f < y\}$, where Ω is an appropriately choosen neighbourhood of P. There are also given necessary conditions for existence of a trajectory joining two closed "regular" subsets of M.

Let $f: M \to \mathbb{R}$ be a function of class C^1 on a riemannian manifold M. We can associate with f the gradient vector field ∇f on M. For any subset $P \subset M$ one may ask what is the topology of the family of non-trivial trajectories of ∇f that tend to P.

In the case where P is an isolated invariant set, one may apply the result obtained by R. Churchill [2] who used the Conley index [3] so as to describe the Čech–Alexander cohomologies of that family. Since the Conley index is defined only for isolated invariant sets, one cannot apply Churchill's results if P is not isolated in the set of critical points of f.

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In this paper we propose a different approach to the problem, which may be applied in the case where M is an analytic manifold, f is an analytic proper function and P is a compact "regular" (for instance semi-analytic or sub-analytic) subset of $f^{-1}(y)$ for some critical value y.

There exists such open $\Omega \supset P$, that for any open $T \supset P$ there is an open $\Omega' \supset P$ such that $\Omega' \subset T$ and the inclusion $\Omega' \cap \{f < y\} \subset \Omega \cap \{f < y\}$ is a homotopy equivalence.

The main result of the paper (Theorem 2.25) says that the Čech–Alexander cohomology groups of the family of non-trivial trajectories tending to P are isomorphic to the cohomologies of $\Omega \cap \{f < y\}$. In the case where P is a point this result has been already presented in [17].

We shall also give necessary conditions for existence of a trajectory joining two subsets $P_1 \subset f^{-1}(y_1)$ and $P_2 \subset f^{-1}(y_2)$ in the case where the interval (y_1, y_2) consists of regular values of f.

The proof requires advanced techniques of analytic geometry. The main tool is the Łojasiewicz inequality. In order to get result for a large class of "regular" subsets of M, in Section 1 we recall the notion of an analytic-geometric category, which was introduced by L. van den Dries and C. Miller [6]. (Semi-analytic and sub-analytic sets belong to any analytic-geometric category.)

In Section 2 we prove basic technical facts. Section 3 is devoted to the proof of the main Theorem 2.25. In Section 4 we give necessary conditions for existence of a trajectory joining P_1 and P_2.

References [1], [4], [5], [7], [9]–[11], [13], [15], [16], [18]–[20] present some important results on geometric and topological properties of gradient vector fields.

1. Analytic–geometric categories

An analytic–geometric category is an extension of the category of sub-analytic sets which has these sets among its object. The notion of analytic–geometric category has been introduced by L. van den Dries and C. Miller [6]. In this section we present some of their results. In exposition and notation we follow closely [6].

Throughout this paper, each manifold is assumed to be Haussdorff, with a countable basis for its topology and of the same (finite) dimension at all of its points. Also, "manifold" will mean "real analytic manifold" unless otherwise specified.

Definition 1.1. We say that an *analytic–geometric category* $\mathcal{C}$ is given if each manifold M is equipped with a collection $\mathcal{C}(M)$ of subsets of M such that the following conditions are satisfied for all manifolds M and N.

(AG1) $\mathcal{C}(M)$ is a boolean algebra of subsets of M, with $M \in \mathcal{C}(M)$.

(AG2) If $A \in \mathcal{C}(M)$, then $A \times \mathbb{R} \in \mathcal{C}(M \times \mathbb{R})$.

(AG3) If $f: M \to N$ is a proper analytic map and $A \in \mathcal{C}(M)$ then $f(A) \in \mathcal{C}(N)$.

(AG4) If $A \subset M$ and $(U_i)_{i \in I}$ is an open covering of M, then $A \in \mathcal{C}(M)$ if and only if $A \cap U_i \in \mathcal{C}(U_i)$ for all $i \in I$.

(AG5) Every bounded set in $\mathcal{C}(\mathbb{R})$ has finite boundary.

Definition 1.2. If $A \in \mathcal{C}(M)$, $B \in \mathcal{C}(N)$ then a continuous map $f: A \to B$ is called a *$\mathcal{C}$-map* if its graph belongs to $\mathcal{C}(M \times N)$.

Below we list some basic properties of a fixed analytic–geometric category $\mathcal{C}$. Let M, N be manifolds, and let $A \in \mathcal{C}(M)$, $B \in \mathcal{C}(N)$.

(A1) All sub-analytic subsets of a manifold are $\mathcal{C}$-sets in that manifold.

(A2) Every analytic map $f: M \to N$ is a $\mathcal{C}$-map.

(A3) Given an open covering $(U_i)_{i \in I}$ of M, a map $f: A \to N$ is a $\mathcal{C}$-map if and only if each restriction $f|_{U_i \cap A}: U_i \cap A \to N$ is a $\mathcal{C}$-map.

(A4) $A \times B \in \mathcal{C}(M \times N)$, and the projections $A \times B \to A$ and $A \times B \to B$ are $\mathcal{C}$-maps.

(A5) If $f: A \to N$ is a proper $\mathcal{C}$-map and $X \subset A$ is a $\mathcal{C}$-set, then $f(X) \in \mathcal{C}(N)$.

(A6) If A is closed in M and $f: A \to N$ is a $\mathcal{C}$-map, then $f^{-1}(B) \in \mathcal{C}(M)$.

(A7) If $B_1, \dots, B_k$ are $\mathcal{C}$-sets (in possibly different manifolds), then a map

$$f = (f_1, \dots, f_k): A \to B_1 \times \dots \times B_k$$

is a $\mathcal{C}$-map if and only if each $f_i: A \to B_i$ is a $\mathcal{C}$-map.

(A8) If $f, g: A \to \mathbb{R}$ are $\mathcal{C}$-maps, then $f \pm g$, $f \cdot g$ are $\mathcal{C}$-maps. If $g \neq 0$ on A then f/g is a $\mathcal{C}$-map.

(A9) If A is a C^1 submanifold of M and $f: A \to \mathbb{R}$ is a $\mathcal{C}$-map of class C^1 then $\mathrm{d}\, f: \mathrm{T}\, A \to \mathrm{T}\, \mathbb{R}$, where $\mathrm{T}\, A$ denotes the tangent bundle, is a $\mathcal{C}$-map.

(A10) There exists a locally finite Whithey stratification $\mathcal{S} \subset \mathcal{C}(M)$ with connected, relatively compact strata such that $A = \bigcup_{X \in \mathcal{S}} X$. Moreover, $\mathrm{cl}(A)$, $\mathrm{int}(A)$, $\mathrm{fr}(A) = \mathrm{cl}(A) \setminus A \in \mathcal{C}(M)$, and $\dim \mathrm{fr}(A) < \dim A$.

(A11) If $x \in \mathrm{fr}(A)$, then there is a $\mathcal{C}$-map $\gamma: [0, 1) \to M$ of class C^1 such that $\gamma((0, 1)) \subset A$ and $\gamma(0) = x$.

(A12) Assume that A is closed. Let $f: A \to N$ be a proper $\mathcal{C}$-map and $\mathcal{F}_M \subset \mathcal{C}(M)$, $\mathcal{F}_N \subset \mathcal{C}(N)$ be locally finite families. Then there is a locally finite C^1 Whitney stratification $(\mathcal{S}, \mathcal{T})$ of f with connected strata such that $\mathcal{S} \subset \mathcal{C}(M)$ is compatible with $\mathcal{F}_M$ (i.e. for all $X_1 \in \mathcal{S}$ and $X_2 \in \mathcal{F}_M$, either $X_1 \cap X_2 = \emptyset$ or $X_1 \subset X_2$) and $\mathcal{T} \subset \mathcal{C}(N)$ is compatible with $\mathcal{F}_N$. According to *the Thom Isotopy Lemma*, for any $X \in \mathcal{F}_M$ and $Y \in \mathcal{T}$ the map $f: X \cap f^{-1}(Y) \to Y$ is a trivial fibration. In particular, $f: f^{-1}(Y) \to Y$ is a trivial fibration too.

(A13) If A is closed, then there is a $\mathcal{C}$-map $p\colon M \to \mathbb{R}$ of class C^1 with $A = Z(p) := \{x \in M \mid p(x) = 0\}$. (This result generalizes the theorem proved by Bierstone, Milman and Pawłucki, in an unpublished paper, for the sub-analytic category.)

(A14) If $X \in \mathcal{C}(\mathbb{R}^n)$ is compact and $f, g\colon X \to \mathbb{R}$ are $\mathcal{C}$-maps with $Z(f) \subset Z(g)$, then there exists an odd, strictly increasing bijection $\psi\colon \mathbb{R} \to \mathbb{R}$ which is a $\mathcal{C}$-map of class C^1 such that $|\psi(g(x))| \leq |f(x)|$ for all $x \in X$.

(A15) There exists an o-minimal structure $\mathcal{G}$ on $\mathbb{R}$ such that all bounded $\mathcal{C}$-sets in $\mathbb{R}^n$ belong to $\mathcal{G}$.

(A16) Let $f\colon (a,b) \to \mathbb{R}$ be a $\mathcal{C}$-map. Then there are $a_0, a_1, \ldots, a_{k+1}$ with $a = a_0 < a_1 < \ldots < a_k < a_{k+1} = b$ such that $f|(a_i, a_{i+1})$ is C^1, and either constant or strictly monotone, for $i = 0, \ldots, k$.

2. Preliminaries

Assume that an analytic–geometric category $\mathcal{C}$ is fixed. Let M be a real analytic connected manifold. We shall assume that M is equipped with an analytic riemannian metric. If $v, w \in \mathrm{T}_x M$ then denote by $\langle v, w\rangle$ the scalar product of v and w, and put $\|v\| = \langle v, v\rangle^{1/2}$.

If $h\colon M \to \mathbb{R}$ is a differentiable function then we will denote by $\mathrm{d}\, h$ its differential and by ∇h the gradient with respect to the riemannian metric, i.e. for every $x \in M$ and $v \in \mathrm{T}_x M$, $\mathrm{d}\, h(x)(v) = \langle \nabla h(x), v\rangle$.

Let $f, p\colon M \to \mathbb{R}$ be $\mathcal{C}$-maps such that

(a) f is an analytic function,
(b) p is proper, non-negative of class C^1,
(c) $Z(p) \subset Z(f)$.

The last condition implies that $p > 0$ on $M \setminus Z(f)$.

Lemma 2.1. *The set of critical values of* p *is discrete.*

Proof. By (A12), there exists a locally finite C^1-Whitney stratification $(\mathcal{S}, T)$ of p with connected strata. $\mathcal{T}$ is a stratification of $\mathbb{R}$, so there exists a sequence $\ldots < y_i < y_{i+1} < \ldots$ $(-\infty < i < \infty)$ such that each element of $\mathcal{T}$ equals either $\{y_i\}$ or (y_i, y_{i+1}).

Assume that $x \in p^{-1}((y_i, y_{i+1}))$. One may find $X \in \mathcal{S}$ with $x \in X$. Since $p\colon X \to (y_i, y_{i+1})$ is a submersion, x is a regular point of $p|_X$, as well as p, which shows that every interval (y_i, y_{i+1}) consists of regular values of p. $\square$

If $r > 0$ is a regular value of p then $S(r) = p^{-1}(r) \in \mathcal{C}(M)$ is a compact C^1-manifold. Using (A9) and similar arguments as in the above lemma one may show that the set of critical values of $f|_{S(r)}$ is finite.

For a fixed $y_0 > 0$, denote $U_0 = f^{-1}((-y_0, 0))$. The set $\{x \in U_0 \mid \mathrm{d}\, p(x) \wedge \mathrm{d}\, f(x) = 0\}$ belongs to $\mathcal{C}(M)$. (Locally it is given by inequality $-y_0 < f < 0$ and equations $\mathrm{D_i}\, p\, \mathrm{D_j}\, f - \mathrm{D_j}\, p\, \mathrm{D_i}\, f = 0$. By (A9) $\mathrm{D_i}\, p$, $\mathrm{D_j}\, f$ are $\mathcal{C}$-maps.)

Since p is proper, by (A5) the set

$$\Sigma = \{(p(x), f(x)) \mid x \in U_0 \text{ and } \mathrm{d}\, p(x) \wedge \mathrm{d}\, f(x) = 0\} \subset \mathbb{R}^2$$

belongs to $\mathcal{C}(\mathbb{R}^2)$. If $(r, y) \in \Sigma$ and $r > 0$ is sufficiently small then r is a regular value of p and y is a critical value of $f|_{S(r)}$, and so $\Sigma \cap \{r\} \times \mathbb{R}$ is finite. Since p is positive on U_0, Σ is a closed 1-dimensional $\mathcal{C}(\mathbb{R}^2)$-set near the origin. Put $\Delta = \mathrm{cl}(\Sigma)$.

Since $f < 0$ on U_0, $\Sigma \cap \mathbb{R} \times \{0\} = \emptyset$ and then the origin is isolated in $\Delta \cap \mathbb{R} \times \{0\}$. Since $Z(p) \subset Z(f)$, $\Sigma \cap \{0\} \times \mathbb{R} = \emptyset$ and then the origin is isolated in $\Delta \cap \{0\} \times \mathbb{R}$. By (A10), in a neighbourhood of the origin, $\Delta = \Sigma \cup \{(0, 0)\}$ is an union of a finite family of curves belonging to $\mathcal{C}(M)$, and $\Sigma \subset \mathbb{R} \times \mathbb{R}_-$. From (A15), we may assume that Δ belongs to $\mathcal{G}$, and then it admits a cell decomposition [6, p. 509]. In particular, there is a $\mathcal{C}$-map $y_1\colon [0, \varepsilon] \to \mathbb{R}$ such that for every $(r, y) \in \Delta$, $0 \le r \le \varepsilon$, we have $y \le y_1(r)$, $y_1(0) = 0$, and $y_1(r) < 0$ on $(0, \varepsilon]$. According to (A16), we may assume that y_1 is strictly decreasing. The function $y_2(r) = r$, $0 \le r \le \varepsilon$, is a $\mathcal{C}$-map, and $Z(y_1) = Z(y_2) = \{0\}$. From (A14), there exists an odd, strictly increasing bijection $\psi\colon \mathbb{R} \to \mathbb{R}$ which is a $\mathcal{C}$-map of class C^1 such that $2|\psi(y_2(r))| = 2\psi(r) \le |y_1(r)|$ for $0 \le r \le \varepsilon$. Then

$$2\psi(r) \le |y|$$

for all $(r, y) \in \Delta$ sufficiently close to the origin. (Notice that $r \ge 0$ for every $(r, y) \in \Delta$.) In other words, for any $x \in U_0$ with $p(x)$ close to zero, if $\mathrm{d}\, p(x) \wedge \mathrm{d}\, f(x) = 0$ then

$$f(x) \le -2\psi(p(x)). \tag{2.1}$$

Lemma 2.2. *Let N be a positive integer. Put $g(x) = f(x) + \psi(p(x))^N$. Then $\mathrm{d}\, g(x)$ does not vanish at any $x \in U_0$ with $p(x)$ sufficiently close to zero.*

Proof. Suppose, contrary to our claim, that

$$Z(p) \cap \mathrm{cl}(\{x \in U_0 \mid \mathrm{d}\, g(x) = 0\})$$

is not empty. It is an $\mathcal{C}(M)$-set, and then there exists an injective $\mathcal{C}$-map $\gamma\colon [0, 1) \to M$ of class C^1 such that $\gamma(0) \in Z(p)$ and $p(\gamma(t)) \ne 0$, $\mathrm{d}\, g(\gamma(t)) = 0$ for $t \in (0, 1)$. Since $Z(p) \subset Z(f)$ and $\psi(0) = 0$, $g(\gamma(0)) = 0$ and then $g \circ \gamma = 0$.

Hence $f = -(\psi \circ p)^N \ge -\psi \circ p$ and $0 = \mathrm{d}\, g \wedge \mathrm{d}\, p = \mathrm{d}\, f \wedge \mathrm{d}\, p$ along γ, which contradicts (2.1). □

For a fixed positive integer N denote $V = Z(g) = \{x \in M \mid f(x) = -\psi(p(x))^N\}$. Since $p \ge 0$, $V \subset \{f \le 0\}$.

Lemma 2.3. *Suppose that there are constants $c > 0$, $0 < \rho < 1$, such that $\|\nabla f(x)\| \geq c|f(x)|^\rho$ for $x \in U_0$ with $p(x)$ small enough. Take a positive integer N such that $N\rho < N - 1$. If $x \in V \setminus Z(f)$ and $p(x)$ is sufficiently close to zero then the scalar product $\langle \nabla f(x), \nabla g(x)\rangle$ is positive. In particular $\nabla g(x) \neq 0$, which implies that $V \setminus Z(f)$ is a C^1-hypersurface in a neighbourhood of $Z(p)$.*

Proof. For all x with $p(x)$ close to zero we have

$$\|\nabla((\psi \circ p)^N)(x)\| = N|\psi(p(x))|^{N-1}|\psi'(p(x))| \cdot \|\nabla p(x)\| \leq C_N|\psi(p(x))|^{N-1}$$

($C_N > 0$). If $x \in V \cap U_0$ then $p(x) > 0$ and $\psi(p(x)) > 0$. If $p(x)$ is small enough then

$$\|\nabla f(x)\| \geq c|f(x)|^\rho = c|\psi(p(x))|^{N\rho} > C_N|\psi(p(x))|^{N-1} \geq \|\nabla((\psi \circ p)^N)(x)\|.$$

Hence

$$\langle \nabla f(x), \nabla g(x)\rangle \geq \|\nabla f(x)\|^2 - \|\nabla f(x)\| \cdot \|\nabla((\psi \circ p)^N)(x)\| > 0. \qquad \square$$

From now on we shall suppose that f and N satisfy the assumptions of Lemma 2.3. Denote

$$H = \{g \leq 0\} = \{x \in M \mid f(x) \leq -\psi(p(x))^N\}.$$

Corollary 2.4. *If $x \in V \setminus Z(f)$ and $p(x)$ is sufficiently close to zero then $\nabla f(x)$ is transversal to V and points outside H.*

Lemma 2.5. *Assume that $A, B \in \mathcal{C}(M)$, A is closed, and $h_1, h_2 : M \to \mathbb{R}$ are $\mathcal{C}$-maps such that each $h_i|_A$ is non-negative, proper and $Z(h_1) \cap A = Z(h_2) \cap A$. If $r > 0$ is small enough then sets*

$$\begin{aligned} L(r, h_i, A \cap B) &= \{x \in A \cap B \mid h_i(x) = r\}, \\ M(r, h_i, A \cap B) &= \{x \in A \cap B \mid 0 < h_i(x) \leq r\}, \end{aligned}$$

where $i = 1, 2$, are homotopy equivalent. Moreover, for any $r > 0$ there is $y > 0$ with $M(y, h_2, A \cap B) \subset M(r, h_1, A \cap B)$, and for any $y > 0$ there is $r > 0$ with $M(r, h_1, A \cap B) \subset M(y, h_2, A \cap B)$.

Proof. Denote $M_i(r) = M(r, h_i, A \cap B)$. Proper $\mathcal{C}$-maps on a closed set A admit a Whitney stratification compatible with any locally finite family of subsets of $\mathcal{C}(M)$, in particular with $A \cap B \cap Z(h_i)$, $A \cap B \cap \{h_i > 0\}$, $i = 1, 2$. Hence, from (A12), there is $r_0 > 0$ such that $h_i : M_i(r_0) \to (0, r_0]$ is a trivial fibration. Since $Z(h_1) \cap A = Z(h_2) \cap A$, there are $0 < r_3 < r_2 < r_1 < r_0$ such that

$$M_2(r_3) \subset M_1(r_2) \subset M_2(r_1) \subset M_1(r_0).$$

Inclusions $M_2(r_3) \subset M_2(r_1)$ and $M_1(r_2) \subset M_1(r_0)$ are homotopy equivalences, and so the inclusion $M_1(r_2) \subset M_2(r_1)$ is a homotopy equivalence.

Now it is enough to observe that inclusions $L(r, h_i, A \cap B) \subset M_i(r)$ are homotopy equivalences for $0 < r \le r_0$. $\square$

Denote

$$\begin{aligned} L(r) &= S(r) \cap \{x \in M \mid f(x) < 0\}, \\ L'(r) &= S(r) \cap H = S(r) \cap \{x \in M \mid f(x) \le -\psi(p(x))^N\}, \\ M(r) &= \{x \in M \mid 0 < p(x) \le r,\ f(x) < 0\}, \\ M'(r) &= \{x \in M \mid 0 < p(x) \le r,\ f(x) \le -\psi(p(x))^N\}. \end{aligned}$$

By (A6), the sets defined above belong to $\mathcal{C}(M)$ for every r. From Lemma 2.5, if $r > 0$ is small enough then the inclusion $L(r) \subset M(r)$ is a homotopy equivalence. Applying the same arguments one may prove that for each positive integer N, if $r > 0$ is small enough then $L'(r) \subset M'(r)$ is a homotopy equivalence.

Lemma 2.6. *If $r > 0$ is small enough then $L(r)$, $L'(r)$, $M(r)$, and $M'(r)$ are homotopy equivalent.*

Proof. Since ψ is continuous and $\psi(0) = 0$, $2\psi(r) > \psi(r)^N$ if $r > 0$ is small enough. By (2.1), for any negative critical value y of $f|_{S(r)}$, $y \le -2\psi(r) < -\psi(r)^N$. Hence f has no critical points in $\{x \in S(r) \mid -\psi(r)^N \le f(x) < 0\}$, and then $L'(r) = \{x \in S(r) \mid f(x) \le -\psi(r)^N\}$ is a deformation retract of $L(r)$. $\square$

For $y > 0$ denote

$$F(y) = H \cap f^{-1}(-y) = \{x \in M \mid f(x) \le -\psi(p(x))^N,\ f(x) = -y\}.$$

Lemma 2.7. $Z(p) \cap H = Z(f) \cap H = Z(p)$. *If $r, y > 0$ are small enough then $F(y)$, $L'(r)$, $L(r)$, and $M(r)$ are homotopy equivalent.*

Proof. H belongs to $\mathcal{C}(M)$. Since p is proper and non-negative, $\psi(p(x))^N|_H$ is proper non-negative. Of course, $f \le 0$ on H. For any $a > 0$,

$$\{x \in H \mid -a \le f(x)\} \subset \{x \in H \mid -a \le -\psi(p(x))^N \le 0\}.$$

The last set is compact, and then $\{x \in H \mid -a \le f(x) \le a\}$ is compact too. Hence $(-f)|_H$ is proper, non-negative.

Since $Z(p) \subset Z(f)$, $Z(p) \cap H = Z(p) = Z(-f) \cap H$. Using notation introduced in Lemma 2.5,

$$L'(r) = L(r, p, H), \quad F(y) = L(y, -f, H).$$

Hence, if $r, y > 0$ are small enough then $L'(r)$ and $F(y)$ are homotopy equivalent. Now it is enough to apply Lemma 2.6. $\square$

Denote $H(y) = \{x \in H \mid f(x) \ge -y\}$ nd $W(r) = \{x \in H \mid p(x) \le r\}$. Then

$$F(y) = H(y) \cap f^{-1}(-y), \quad L'(r) = W(r) \cap p^{-1}(r) = W(r) \cap S(r).$$

Applying the same arguments as in the proof of Lemma 2.5, one may show

Corollry 2.8. *For any* $r > 0$ *there exists* $y > 0$ *such that* $F(y) \subset H(y) \subset W(r)$. *For any* $y > 0$ *there exits* $r > 0$ *such that* $L'(r) \subset W(r) \subset H(y)$.

Corollary 2.9. *For any* $r > 0$ *there exists* $y > 0$ *and* $0 < r_1 < r$ *such that* $f < -y$ *on* $L'(r)$ *and* $f > -y$ *on* $L'(r_1)$.

Lemma 2.10. *Let* Ω *be an open neighbourhood of* $Z(p)$ *such that for any neighbourhood* $T \subset \Omega$ *there is a neighbourhood* $\Omega' \subset T$ *such that the inclusion* $\Omega' \cap \{f < 0\} \subset \Omega \cap \{f < 0\}$ *is a homotopy equivalence. Denote*

$$D(r) = \{x \in M \mid p(x) \leq r\}.$$

If $r > 0$ *is small enough then the inclusion*

$$M(r) = D(r) \cap \{f < 0\} \subset \Omega \cap \{f < 0\}$$

is a homotopy equivalence.

Proof. Since p is a proper $\mathcal{C}$-map and $Z(p) \cap \{f < 0\} = \emptyset$, there is $r_0 > 0$ such that $D(r_0) \subset \Omega$ and $p: D(r_0) \cap \{f < 0\} \to (0, r_0]$ is a topologically trivial fibration.

Take $0 < r \leq r_0$. There exists an open neighbourhood Ω' of $Z(p)$ such that $\Omega' \subset D(r)$ and $\Omega' \cap \{f < 0\} \subset \Omega \cap \{f < 0\}$ is a homotopy equivalence. There exists $r' < r$ such that $D(r') \subset \Omega'$. Since $D(r') \cap \{f < 0\} \subset D(r) \cap \{f < 0\}$ is a homotopy equivalence and

$$D(r') \cap \{f < 0\} \subset \Omega' \cap \{f < 0\} \subset D(r) \cap \{f < 0\} \subset \Omega \cap \{f < 0\},$$

the inclusion $M(r) = D(r) \cap \{f < 0\} \subset \Omega \cap \{f < 0\}$ is a homotopy equivalence. $\square$

Note that there exists Ω satisfying the above lemma. It is enough to take $\Omega = \mathrm{int}(D(r))$, for $r > 0$ small enough. From Lemmas 2.7 and 2.10 we get

Corollary 2.11. *Let* Ω *be an open neighbourhood of* $Z(p)$ *such that for any neighbourhood* $T \subset \Omega$ *there is a neighbourhood* $\Omega' \subset T$ *such that the inclusion* $\Omega' \cap \{f < 0\} \subset \Omega \cap \{f < 0\}$ *is a homotopy equivalence. Then if* $y > 0$ *is small enough then the inclusion* $F(y) \subset \Omega \cap \{f < 0\}$ *is a homotopy equivalence.*

Denote $V(y) = \{x \in V \mid f(x) \geq -y\}$.

Proposition 2.12. *For any* $y > 0$ *there exists* $r > 0$ *such that* $V \cap D(r) \subset V(y)$, *and for any* $r > 0$ *there exists* $y > 0$ *such that* $V(y) \subset V \cap D(r)$. *In particular,* $V \cap Z(f) = V \cap Z(p)$. *If* $y > 0$ *is small enough then* $f: V(y) \setminus Z(p) \to$

$[-y, 0)$ *is a topologically trivial fibration, and* $V(y) \setminus Z(p)$ *is either void or a* C^1-*hypersurface.*

Proof. V is a closed $\mathcal{C}(M)$-set. Since $\psi: \mathbb{R} \to \mathbb{R}$ is an odd continuous bijection and p is proper, $-f = (\psi \circ p)^N: V \to \mathbb{R}$ is a proper $\mathcal{C}$-map, which implies the first part of the proposition.

By (A12), there is a sequence $\ldots < y_i < y_{i+1} < \ldots$ $(-\infty < i < \infty)$ such that $f: V \cap f^{-1}((y_i, y_{i+1})) \to (y_i, y_{i+1})$ is a topologically trivial fibration. Then it is enough to take $y > 0$ and i such that $(-2y, 0) \subset (y_i, y_{i+1})$. The last part of the Proposition is a consequence of Lemma 2.3. □

Let $f: M \to \mathbb{R}$ be a proper analytic function. There are positive constants c, y_0 and $0 < \rho < 1$ such that the Łojasiewicz inequality holds on $U_0 = f^{-1}((-y_0, 0))$, i.e. $\|\nabla f(x)\| \geq c|f(x)|^\rho$ for $x \in U_0$ [14], [15]. (Kurdyka and Parusiński [12] proved that the inequality also holds if f is a continuous subanalytic function.)

Corollary 2.13. $\nabla f \neq 0$ *on* U_0.

If $x \in U_0$ then denote by $\phi(t, x)$, where $a(x) < t < b(x)$, the maximal solution in U_0 of the differential equation $\dot{x} = \nabla f / \|\nabla f\|$ with $\phi(0, x) = x$. Then $f \circ \phi(t, x)$ is increasing, $\mathrm{d}/\mathrm{dt}[f \circ \phi] = 1$, and

$$\lim_{t \to a(x)} f \circ \phi(t, x) = -y_0, \qquad \lim_{t \to b(x)} f \circ \phi(t, x) = 0.$$

Of course, trajectories of $\dot{x} = \nabla f / \|\nabla f\|$ are the same as trajectories of $\dot{x} = \nabla f$ in U_0.

If $x_1, x_2 \in U_0$ belong to the same trajectory, let $\ell(x_1, x_2)$ denote the lenght of the trajectory between x_1 and x_2. Using the same arguments as in [11, p. 765] one may prove

Lemma 2.14. $\ell(x_1, x_2) \leq c_0 \big||f(x_1)|^{1-\rho} - |f(x_2)|^{1-\rho}\big|$, *where* $c_0 = [c(1-\rho)]^{-1}$.

Hence the lenght of each trajectory is bounded by $c_0|y_0|^{1-\rho}$, and then the limit $\omega(x) = \lim_{t \to b(x)} \phi(t, x)$ does exist, $\omega(x) \in Z(f)$, and

$$\ell(x, \omega(x)) = \lim_{t \to b(x)} \ell(x, \phi(t, x)) \leq c_0 |f(x)|^{1-\rho}.$$

For $x, y \in M$ let $d(x, y)$ denote the distance in M defined as the infimum of the lenght of all continuous picewise C^1-curves in M from x to y. For any $A \subset M$, let $d(x, A) = \inf_{y \in A} d(x, y)$.

Of course, $d(x, \omega(x)) \leq \ell(x, \omega(x))$. For $x \in Z(f)$ put $\omega(x) = x$. Then $\omega: U_0 \cup Z(f) = f^{-1}((-y_0, 0]) \to Z(f)$ is a continuous retraction by a strong deformation.

Let $P \in \mathcal{C}(M)$ be a closed subset of $Z(f)$. The Bierstone–Milman–Pawłucki theorem (A13) implies that there exists a non-negative $\mathcal{C}$-map $p: M \to \mathbb{R}$ of class

C^1 with $P = Z(p)$. In particular, $Z(p) \subset Z(f)$. Since P is compact, one may easily modify the proof of (A13) presented in [6] so as to show that there exists a proper p. U_0 is relatively compact, so the norm of $\mathrm{d}\,p$ is bounded on U_0 and, after multiplying p by an appriopriate constant, we may assume that

$$p(x) \leq d(x, Z(p)) \quad \text{on } U_0. \tag{2.2}$$

Let $\psi: \mathbb{R} \to \mathbb{R}$ be an odd, strictly increasing bijection which is a $\mathcal{C}$-map of class C^1, defined as in Section 2. Let $D(r) = \{x \in M \mid p(x) \leq r\}$. Clearly, $D(r) \subset W(r)$.

Lemma 2.15. *Let N be a positive integer such that $N(1-\rho) > 1$. Then there exists $r_0 > 0$ such that if $x \in U_0 \cap D(r_0)$ and $-\psi(p(x))^N \leq f(x)$ then*

$$\ell(x, \omega(x)) < p(x)/2,$$

so that $\omega(x) \notin Z(p)$.

Proof. Since ψ is odd of class C^1, there is c_1 such that $\psi(r) \leq c_1 r$, for $r > 0$ small enough. If $-\psi(p(x))^N \leq f(x)$ then

$$\ell(x, \omega(x)) \leq c_0 |f(x)|^{1-\rho} \leq c_0 \psi(p(x))^{N(1-\rho)} \leq c_0 c_1^{N(1-\rho)} p(x)^{N(1-\rho)}.$$

Since $N(1-\rho) > 1$, it is enough to take $r_0 > 0$ with $c_0 c_1^{N(1-\rho)} r_0^{N(1-\rho)} < r_0/2$. If that is the case, $\ell(x, \omega(x)) < d(x, Z(p))/2$, and then $\omega(x) \notin Z(p)$. □

Corollary 2.16. *If $x \in U_0$ and $\omega(x) \in Z(p)$, put*

$$c(x) = \sup\{t \mid p(\phi(t, x)) = r_0\}.$$

Then $\phi(c(x), x) \in L'(r_0)$ and $\phi(t, x) \in W(r_0)$ for any $t \geq c(x)$.

Proposition 2.17. *Let N be a positive integer such that $N(1-\rho) > 1$. Then there exists $y_1 > 0$ such that*

(a) *for any positive $y < y_1$, $F(y)$ is either void or a compact C^1-manifold with boundary*

$$\begin{aligned} \partial F(y) &= \{x \in M \mid \psi(p(x))^N = y,\ f(x) = -y\} \\ &= \{x \in M \mid \psi(p(x)) = y^{1/N},\ f(x) = -y\}, \end{aligned}$$

(b) *if $x \in U_0$ and $\omega(x) \in Z(p)$ then $\phi(t, x)$ cuts $F(y)$ transversally at exactly one point.*

Proof. f and $(\psi \circ p)^N$ are proper $\mathcal{C}$-maps of class C^1. Hence the set of critical values of these maps is discrete. Then there exists $y_1 > 0$ such that $y_0 > y_1$ and the interval $(0, y_1)$ consists of regular values of $-f$ and $(\psi \circ p)^N$. We may assume that $y_1^{1/N} > 2y_1$.

(a) If $0 < y < y_1$ then $F(y)$ is the intersection of a C^1-hypersurface $f^{-1}(-y)$ and $H = \{f \leq -(\psi \circ p)^N\}$. H is "bounded" by $V = \{f = -(\psi \circ p)^N\}$, so it is enough to show that both the sets cut transversally.

From Lemma 2.3 and Proposition 2.12, $V \setminus Z(f)$ is a C^1-hypersurface, and using the well-known curve selection lemma argument one may prove that f restricted to $V \setminus Z(f)$ has no critical points for $|f(x)|$ small enough. Hence, if $y_1 > 0$ is small enough then $f^{-1}(-y)$ is transversal to V.

(b) From Corollaries 2.8, 2.9 we may assume that $H(y_1) \subset W(r_0) \subset D(r_0)$ and $f < -y_1$, on $L'(r_0)$. Take any $0 < y < y_1$. From Corollary 2.9, there exists $r_1 < r_0$ such that $f > -y$ on $L'(r_1)$.

Take $x \in U_0$ such that $\omega(x) \in Z(p)$. From Corollary 2.16, $x_1 = \phi(c(x), x) \in L'(r_0)$ and $\phi(t, x) \in W(r_0)$ for any $t \geq c(x)$.

Since $p(x_1) = r_0$ and $\lim_{t \to b(x)} p(\phi(t, x)) = 0$, there exists x_2 on the trajectory such that $p(x_2) = r_1$, and then $x_2 \in L'(r_1)$. Hence $f(x_1) < -y_1 < -y < f(x_2)$, so that the trajectory must cut transversally $f^{-1}(-y)$ at a point belonging to $W(r_0) \cap f^{-1}(-y) \subset F(y)$. □

The set $H(y)$ is compact, its boundary is the union of $F(y)$ and $V(y)$, where

$$F(y) \cap V(y) = \{x \in M \mid -y = f(x) = -\psi(p(x))^N\} = \partial F(y).$$

If $0 < y \ll 1$ then $H(y) \setminus Z(p) \subset U_0$, so that $\nabla f \neq 0$ on $H(y) \setminus Z(p)$.

If $x \in F(y) \setminus \partial F(y)$ then $f(x) = -y < -\psi(p(x))^N$. Then there is $d > 0$ such that $f(\phi(t, x)) \geq -y$ and $f(\phi(t, x)) \leq -\psi(\phi(t, x))^N$, i.e. $\phi(t, x) \in H(y)$, for $t \in [0, d]$.

Denote $d(x) = \sup\{t \mid \phi(t, x) \in H(y)\}$ and $\gamma(x) = \lim_{t \to d(x)} \phi(t, x)$. Clearly, $\gamma(x) \in V(y)$.

If $x \in V(y) \setminus Z(p)$ then $g(x) = f(x) + \psi(p(x))^N = 0$. By Lemma 2.3, $\phi(t, x)$ is transversal to $V(y)$ and there is $e > 0$ such that $g(\phi(t, x)) > 0$ for $t \in (0, e)$, and then $\phi(t, x) \notin H(y)$.

That means that a trajectory enters $H(y)$ on $F(y) \setminus \partial F(y)$, and either leaves it on $V(y) \setminus Z(p)$ at $\gamma(x) \neq 0$ or is attracted by $Z(p)$. In the second case, $\gamma(x) = \omega(x) = 0$. From Proposition 2.17, if a trajectory is attracted by $Z(p)$ then it must first cut $F(y)$ and enter $H(y)$. From Lemma 2.15, if a trajectory leaves $H(y)$ on $V(y) \setminus Z(p)$ then it is not attracted by $Z(p)$. Hence we have

Theorem 2.18. *If $N(1 - \rho) > 1$ and $y > 0$ is small enough then there is one-to-one correspondence between the set of non-trivial trajectories attracted by $Z(p)$ and*

$$\Gamma(y) = \{x \in F(y) \mid \gamma(x) \in Z(p)\}.$$

The above theorem allows us to equip the set of non-trivial trajectories attracted by the origin with the topology induced from $\Gamma(y)$. In the remainder of this section we shall show that this space has the same Čech–Alexander cohomology groups as $F(y)$.

Lemma 2.19. *$\gamma: F(y) \to V(y)$ is a continuous function, and $\gamma: F(y) \setminus \Gamma(y) \to V(y) \setminus Z(p)$ is a homeomorphism.*

Proof. Let $x \in F(y) \setminus \Gamma(y)$. Its trajectory is transversal to both $F(y)$ and $V(y)$, so γ is the Poincaré mapping in some neighbourhood of x, and then $\gamma: F(y) \setminus \Gamma(y) \to V(y) \setminus Z(p)$ is a homeomorphism.

If $x \in \Gamma(y)$ then $\gamma(x) = \omega(x) \in Z(p)$. Take a sequence $F(y) \ni x_n \to x$. Since $\gamma(x_n) \in V(y)$, by Lemma 2.15 and (2.2) we have

$$d(\gamma(x_n), \omega(x_n)) \leq \ell(\gamma(x_n), \omega(x_n)) \leq p(\gamma(x_n))/2 \leq d(\gamma(x_n), Z(p))/2.$$

If there is $\delta > 0$ such that $\delta < p(\gamma(x_n))$ then $\delta < d(\gamma(x_n), Z(p))$, and so $\delta/2 \leq d(\omega(x_n), Z(p))$. Then $\lim \omega(x_n) \neq \omega(x)$, which contradicts the continuity of ω.

Hence $\lim p(\gamma(x_n)) = 0$, and then $\lim d(\gamma(x_n), \omega(x_n)) = 0$. Since $\lim \omega(x_n) = \omega(x)$, $\lim \gamma(x_n) = \gamma(x)$. Then γ is continuous at each point in $\Gamma(y)$. □

Lemma 2.20. $\gamma(F(y)) = V(y) \cap \mathrm{cl}(U_0)$.

Proof. The inclusion "$\subset$" is obvious.

From the previous lemma, $V(y) \setminus Z(p) \subset \gamma(F(y))$. It is easy to see that $\omega(f^{-1}(-y)) = Z(f) \cap \mathrm{cl}(U_0)$. In particular, for any $z \in Z(p) \cap \mathrm{cl}(U_0)$ there exists $x \in f^{-1}(-y)$ with $\omega(x) = z$. From Theorem 2.18, $Z(p) \cap \mathrm{cl}(U_0) \subset \gamma(F(y))$. Now it is enough to observe that $V(y) \cap \mathrm{cl}(U_0)$ is the disjoint union of $V(y) \setminus Z(p)$ and $Z(p) \cap \mathrm{cl}(U_0)$. □

Corollary 2.21. *If $Z(p) = Z(p) \cap \mathrm{cl}(U_0)$ then $\gamma(F(y)) = V(y)$.*

From now on we shall assume that $Z(p) = Z(p) \cap \mathrm{cl}(U_0)$.

Lemma 2.22. *For any open neighbourhood T of $\Gamma(y)$ in $F(y)$ the image $\gamma(T)$ is an open neighbourhood of $Z(p)$ in $V(y)$.*

Proof. T is open, so $F(y) \setminus T$ is compact. Then $\gamma(F(y) \setminus T)$ is compact in $V(y)$, so $V(y) \setminus \gamma(F(y) \setminus T)$ is open in $V(y)$. It is enough to show that it equals $\gamma(T)$.

We have $V(y) \setminus \gamma(F(y) \setminus T)) \subset \gamma(T)$ because $\gamma(F(y)) = V(y)$. Let $z \in \gamma(T)$. If $z \in Z(p)$ then $z \notin \gamma(F(y) \setminus T)$ because $\Gamma(y) \subset T$. If $z \notin Z(p)$ then $z = \gamma(x)$ for some $x \in T \setminus \Gamma(y)$. But $\gamma|_{F(y) \setminus \Gamma(y)}$ is a homeomorphism, so $z \notin \gamma(F(y) \setminus T)$. □

Lemma 2.23. *If $y > 0$ is small enough then there exists a descending family $\Gamma(y) = T_1 \supset T_2 \supset \dots$ of open neighbourhoods of $\Gamma(y)$ in $F(y)$ such that*

(a) *every inclusion $T_{n+1} \subset T_n$ is a homotopy equivalence, so that the induced homomorphism of cohomology groups $H^*(T_n) \to H^*(T_{n+1})$ and $H^*(F(y)) \to H^*(T_n)$ are isomorphisms,*

(b) *for every open neighburhood T of $\Gamma(y)$ in $F(y)$ there is n such that $T_n \subset T$.*

Proof. As in the proof of Proposition 2.12, if $y > 0$ is small enough then $f: V(y) \to [-y, 0]$ is proper and $f: V(y) \setminus V(f) \to [-y, 0)$ is a topologically trivial fibration. $V(y) \cap Z(f) = V(y) \cap Z(p)$, hence there exists a descending family $V(y) = T_1' \supset T_2' \supset \dots$ of open neighbourhoods of $Z(p)$ such that every inclusion $T_{n+1}' \subset T_n'$ is a homotopy equivalence, and for every neighbourhood T' of $Z(p)$ in $V(y)$ there is n such that $T_n' \subset T'$. Set $T_n = \gamma^{-1}(T_n')$. Each T_n is an open neighbourhood of $\Gamma(y)$.

$\gamma: F(y) \setminus \Gamma(y) \to V(y) \setminus Z(p)$ is a homeomorphism, hence (a) holds. Let T be an open neighbourhood of $\Gamma(y)$ in $F(y)$. From Lemma 2.22, $\gamma(T)$ is an open neighbourhood of $Z(p)$ in $V(y)$, and then there is n with $T_n' \subset \gamma(T)$. Hence $T_n \subset T$. □

Theorem 2.24. *If $y > 0$ is small enough then the Čech–Alexander cohomology groups $\overline{H}^*(F(y)) = H^*(F(y))$ and $\overline{H}^*(\Gamma(y))$ are isomorphic.*

Proof. The family $T_1 \supset T_2 \supset \dots$ described in the previous Lemma is cofinal in the family of all open neighbourhoods of $\Gamma(y)$ in $F(y)$. Then we have an isomorphism of direct limits

$$\overline{H}^*(\Gamma(y)) = \lim_{\overrightarrow{T_n}} H^*(T_n) \cong H^*(F(y)). \qquad \square$$

Theorem 2.25. *Let $f: M \to \mathbb{R}$ be a proper analytic function. Take a positive constant y_0 such that f has no critical points in $f^{-1}((-y_0, 0))$. Let $\omega: f^{-1}((-y_0, 0]) \to Z(f)$ be the continuous deformation defined by the flow associated with ∇f. Let $P \subset Z(f)$ be a closed $\mathcal{C}$-set and $\Gamma(y) = \{x \in f^{-1}(-y) \mid \omega(x) \in P\}$ for $0 < y < y_0$. Then there is one-to-one correspondence between $\Gamma(y)$ and the set of non-trivial trajectories of the equation $\dot{x} = \nabla f$ attracted by P. Let Ω be an open neighbourhood of P such that for any neighbourhood $T \subset \Omega$ there is a neighbourhood $\Omega' \subset T$ such that the inclusion $\Omega' \cap \{f < 0\} \subset \Omega \cap \{f < 0\}$ is a homotopy equivalence. Then $\Omega \cap \{f < 0\}$ has the homotopy type of a compact manifold with boundary, and the Čech–Alexander cohomology groups $\overline{H}^*(\Gamma(y))$ and $\overline{H}^*(\Omega \cap \{f < 0\}) = H^*(\Omega \cap \{f < 0\})$ are isomorphic.*

Proof. Take positive constants c and $0 < \rho < 1$ such that the Łojasiewicz inequality holds on $U_0 = f^{-1}((-y_0, 0))$, i.e. $\|\nabla f(x)\| \geq c|f(x)|^\rho$ for $x \in U_0$.

The set of non-trivial trajectories attracted by P is equal to the set of trajectories attracted by $P \cap \mathrm{cl}(U_0)$. Hence we may assume that $P = P \cap \mathrm{cl}(U_0)$.

There is a proper non-negative $\mathcal{C}$-map $p: M \to \mathbb{R}$ of class C^1 such that $P = Z(p)$. Let N be a positive integer with $N(1 - \rho) > 1$. From Theorem 2.18, $\Gamma(y)$ is homeomorphic to $\Gamma(y')$ for y' small enough. From Theorem 2.24, $\overline{H}^*(\Gamma(y))$

is isomorphic to $H^*(F(y'))$. Corollary 2.11 implies that $\Omega \cap \{f < 0\}$ has the homotopy type of a compact manifold $F(y')$, and then the groups $\overline{H}^*(\Gamma(y))$ and $\overline{H}^*(\Omega \cap \{f < 0\}) = H^*(\Omega \cap \{f < 0\})$ are isomorphic. □

Remark 2.26. If P is a compact component of $\{x \in Z(f) \mid \mathrm{d} f(x) = 0\}$, then one may also apply Churchill's results [2] and get the same result as above.

Example 2.27. Suppose that M is a 3-dimensional manifold, and there exists a diffeomorphism $h: \mathbb{R}^3 \to h(\mathbb{R}^3) \subset M$ such that

$$\operatorname{sign} f \circ h(x,y,z) = \operatorname{sign}((1 - x^2 - y^2)y^2 - z^2).$$

Take $P = h([-1,1] \times \{0\} \times \{0\})$. P is not isolated in in $\{x \in Z(f) \mid \mathrm{d} f(x) = 0\} \supset h(\mathbb{R} \times \{0\} \times \{0\})$, so one cannot apply the Churchill result. One may check that $\Omega = h((-3/2, 3/2) \times (-1/2, 1/2) \times (-1/2, 1/2))$ satisfies assumptions of the above theorem, and $\Omega \cap \{f < 0\}$ has the homotopy type of the bouquet $S^1 \vee S^1 \vee S^1$. Hence $\overline{H}^0(\Gamma(y)) = \mathbb{Z}$, $\overline{H}^1(\Gamma(y)) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, and $\overline{H}^k(\Gamma(y)) = 0$ for $k > 1$.

3. Trajectories joining two $\mathcal{C}$-sets

Let $f: M \to \mathbb{R}$ be a proper analytic function. Assume that f has no critical points in $U = f^{-1}((y_1, y_2))$. Then there are constants $c > 0$, $0 < \rho < 1$ such that $\|\nabla f(x)\| \geq c|f(x) - y_i|^\rho$ for all x in U which are sufficiently close to $Z_i = f^{-1}(y_i)$ $(i = 1, 2)$.

Let $P_i \subset Z_i$ be a closed $\mathcal{C}$-set. In this section we shall give necessary conditions for existence of a trajectory of the gradient field emanating from P_1 and attracted by P_2.

Take $x \in U$. Let $\phi(t, x)$, where $a(x) < t < b(x)$, denote the maximal solution in U of the equation $\dot{x} = \nabla f / \|\nabla f\|$ with $\phi(0, x) = x$. Put

$$\alpha(x) = \lim_{t \to a(x)} \phi(t, x), \quad \omega(x) = \lim_{t \to b(x)} \phi(t, x).$$

For $x \in Z_1$ (resp. $x \in Z_2$) put $\alpha(x) = x$ (resp. $\omega(x) = x$). Then $\alpha: f^{-1}([y_1, y_2)) \to Z_1$, as well as $\omega: f^{-1}((y_1, y_2]) \to Z_2$, is a continuous retraction by a strong deformation.

Theorem 3.1. *Let $\sigma: f^{-1}([y_1, y_2)) \to Z_1$ be a continuous retraction by a strong deformation. Let $\Omega \subset \{f > y_1\}$ be an open neighbourhood of P_2 such that for any open neighbourhood $T \subset \Omega$ of P_2 there is an open neighbourhood $\Omega' \subset T$ such that the inclusion $\Omega' \cap U \subset \Omega \cap U$ is a homotopy equivalence. Denote by $\sigma^*: H^*(Z_1, Z_1 \setminus P_1) \to H^*(\Omega \cap U)$ the homomorphism of cohomology groups induced by $\Omega \cap U \ni x \mapsto \sigma(x) \in (Z_1, Z_1 \setminus P_1)$. If $\sigma^* \neq 0$ then there is $x \in U$ such that $\alpha(x) \in P_1$ and $\omega(x) \in P_2$.*

Proof. Suppose that $\sigma^* \neq 0$. Since α is homotopy equivalent to σ, then $\alpha^*\colon H^*(Z_1, Z_1 \setminus P_1) \to H^*(\Omega \cap U)$ is a non-trivial homomorphism.

Take $y \in (y_1, y_2)$ which lies very close to y_2, so that $\Gamma(y) = f^{-1}(y) \cap \omega^{-1}(P_2) \subset \Omega \cap U$ induces an isomorphism of the Čech–Alexander cohomology groups. Hence, the homomorphism $H^*(Z_1, Z_1 \setminus P_1) \to \overline{H}^*(\Gamma(y))$, induced by $\Gamma(y) \ni x \mapsto \alpha(x) \in (Z_1, Z_1 \setminus P_1)$, is non-trivial. Thus there exists $x \in \Gamma(y)$ such that $\alpha(x) \in P_1$, and clearly $\omega(x) \in P_2$. □

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