**T**opological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 25, 2005, 101–133

# BOUNDARY VALUE PROBLEMS FOR FIRST ORDER SYSTEMS ON THE HALF-LINE

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Abstract. We prove existence theorems for first order boundary value problems on $(0,\infty)$, of the form $\dot{u}+F(\,\cdot\,,u)=f$, $Pu(0)=\xi$, where the function $F=F(t,u)$ has a $t$-independent limit $F^{\infty}(u)$ at infinity and $P$ is a given projection. The right-hand side $f$ is in $L^p((0,\infty),\mathbb{R}^N)$ and the solutions $u$ are sought in $W^{1,p}((0,\infty),\mathbb{R}^N)$, so that they tend to 0 at infinity. By using a degree for Fredholm mappings of index zero, we reduce the existence question to finding *a priori* bounds for the solutions. Nevertheless, when the right-hand side has exponential decay, our existence results are valid even when the governing operator is not Fredholm.

## 1. Introduction

Let $F=F(t,u)\colon[0,\infty)\times\mathbb{R}^N\to\mathbb{R}^N$ be a given function and $P\colon\mathbb{R}^N\to X_1$ the projection associated with some splitting $\mathbb{R}^N=X_1\oplus X_2$. We discuss the existence of solutions of problems of the type

$$\begin{cases}\dot{u}(t)+F(t,u(t))=f(t) & \text{for a.e. } t>0,\\ Pu(0)=\xi,\end{cases}\tag{1.1}$$

$$\lim_{t\to\infty}u(t)=0,\tag{1.2}$$

where $f\in L^p((0,\infty),\mathbb{R}^N)$ and $\xi\in X_1$ are given.

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2000 *Mathematics Subject Classification.* 43A12, 34B15, 47A53, 47H11.

*Key words and phrases.* Ordinary differential equation, half-line, Sobolev space, boundary and initial value problem, Fredholm operator, a-priori bounds.

This paper is a continuation of our work [12], to which we refer for an exposition of the status of the problem and various references to the literature. While $p = 2$ in [12] and the emphasis is on the "autonomous" case when $F = F(u)$ is independent of $t$, we consider here the general case. We also discuss the existence of exponentially decaying solutions when the right-hand side $f$ has exponential decay in a suitable sense.

To motivate our investigations, we begin with three examples.

Example 1.1. If $X_1 = \mathbb{R}^N$, we are seeking solutions of the usual initial value problem

$$\begin{cases} \dot{u}(t) + F(t, u(t)) = f(t) & \text{for a.e. } t > 0, \\ u(0) = \xi, \end{cases}$$

which vanish at infinity. This is far more specific than mere local existence and our results provide conditions on $F$ ensuring that such solutions exist for all or for some $f \in L^p((0,\infty), \mathbb{R}^N)$ and $\xi \in \mathbb{R}^N$. Naturally, for this problem, the uniqueness of the solution is true as well.

Example 1.2. If $X_1 = \{0\}$, we are seeking solutions of

$$\dot{u}(t) + F(t, u(t)) = f(t) \quad \text{for a.e. } t > 0,$$

which vanish at infinity, without any requirement when $t = 0$. It is noteworthy that the solution of this problem, if any, may be *unique* (and hence the problem in Example 1.1 has no solution in general). For instance, it is so if $-F(t, \cdot)$ is monotone in some neighbourhood of 0 for all $t > 0$ large enough (easy verification). Our results provide conditions on $F$ ensuring that solutions exist, which are compatible with the possible monotonicity of $-F(t, \cdot)$.

Example 1.3. Consider the second order system

$$\begin{cases} \ddot{v}(t) + G(t, v(t), \dot{v}(t)) = g(t), \\ v(0) = \xi, \end{cases} \tag{1.3}$$

where $G \in C^1([0,\infty) \times \mathbb{R}^{2M}, \mathbb{R}^M)$, $g \in L^p((0,\infty), \mathbb{R}^M))$ and $\xi \in \mathbb{R}^M$ are given. This has the form (1.1) with $N = 2M$ after setting

$$u = \begin{pmatrix} v \\ w \end{pmatrix}, \quad F(t, u) = \begin{pmatrix} -w \\ G(t, v, w) \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

and choosing $X_1 = \mathbb{R}^M \times \{0\}$ and $X_2 = \{0\} \times \mathbb{R}^M$. The condition (1.2) then means that

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \dot{v}(t) = 0. \tag{1.4}$$

Note however that $\dot{v}(0)$, and hence $u(0)$, is not prescribed. Note also that only the right-hand sides of the form $f = (0, g)$ are relevant in this problem. There is an analogous Neumann problem in which $v(0) = \xi$ is replaced by $\dot{v}(0) = \xi$ and $v(0)$ is not prescribed, so that $X_1 = \{0\} \times \mathbb{R}^M$ and $X_2 = \mathbb{R}^M \times \{0\}$.

We deal with the general problem (1.1)–(1.2) by writing it as an equation

$$\Phi(u) = (f, \xi) \quad \text{for } u \in W^{1,p}, \tag{1.5}$$

where $1 \leq p < \infty$ and, from now on,

$$W^{1,p} = W^{1,p}((0,\infty), \mathbb{R}^N), \quad L^p = L^p((0,\infty), \mathbb{R}^N) \tag{1.6}$$

and

$$\Phi: u \in W^{1,p} \to \Phi(u) = (\dot{u} + F(\,\cdot\,, u), Pu(0)) \in L^p \times X_1. \tag{1.7}$$

Since $W^{1,p} \subset \{u \in C^0([0,\infty), \mathbb{R}^N) : \lim_{t\to\infty} u(t) = 0\}$ the solutions of $\Phi(u) = (f, \xi)$ satisfy both (1.1) and (1.2).

We establish the existence of solutions of (1.5) by a degree theory argument. Since we are dealing with a boundary value problem on an *infinite* interval, the Leray–Schauder degree is not adequate. Instead, we use a degree for proper Fredholm mappings of index zero. This approach does not require an approximation by problems on bounded intervals, a procedure unlikely to produce solutions in $W^{1,p}$ since it usually yields no control of their behavior at infinity.

We need to impose conditions on $F$ and $P$ ensuring that

(1) $\Phi \in C^1(W^{1,p}, L^p \times X_1)$,
(2) $D\Phi(u): W^{1,p} \to L^p \times X_1$ is Fredholm of index 0 for all $u \in W^{1,p}$,
(3) $\Phi: W^{1,p} \to L^p \times X_1$ is proper on the closed bounded subsets of $W^{1,p}$.

The assumptions that $F(t,0) = 0$ and that $F(t,u)$ has a $t$-independent limit $F^\infty(u)$ as $t \to \infty$ are helpful in verifying (2) and (3) above. Naturally, we are led to consider the linearization of (1.1), namely

$$\begin{cases} \dot{w}(t) + D_u F(t, u(t)) w(t) = f(t) & \text{for a.e. } t > 0, \\ Pw(0) = \xi. \end{cases} \tag{1.8}$$

For this reason, in Section 2, we begin by discussing the Fredholm properties of the simplest linear case of (1.1), where $F(t,u) = Au$ for some $A \in \mathcal{L}(\mathbb{R}^N)$. We show that Fredholmness is equivalent to $A$ having no imaginary eigenvalues and that the index depends only upon the dimensions of $X_1$ and of the sum of the generalized eigenspaces of $A$ corresponding to the eigenvalues with positive real part (but not on $1 \leq p < \infty$).

While the results of Section 2 are close in spirit to those in Massera and Shäffer [6] or Palmer [8], [9] (see also [4]), there are differences regarding the emphasis, the functional setting or the wording of some results. The case $A = A(t)$ is also discussed in these references, but here the autonomous linear problem will provide all the information we require for the non-autonomous linear equations like (1.8). When $p = 2$, a partial treatment can also be found in of [12, Section 2].

The short Sections 3 and 4 address nonlinear issues, namely the smoothness, Fredholm and properness properties of the operator $\Phi$. They complement Sections 3 and 4 of [12] by providing some technical proofs left out in that paper. Properties fully established in [12] are quoted without proof.

The general existence result based on degree theory is given in Theorem 5.2. Corollary 5.3 is a variant in which the main remaining issue is to find *a priori* bounds in $W^{1,p}$ for the solutions. In Section 6, further conditions are shown to ensure the existence of such bounds. Detailed proofs are given only when $p = 2$ and the hypotheses and calculations are simpler, but the procedure is general.

The first "concrete" existence theorem is Theorem 7.1, which follows at once from the material developed earlier. It requires, among other things, that $DF^\infty(0)$ have no imaginary eigenvalues, a property equivalent to the Fredholmness of $\Phi$ above. The remainder of Section 7 is devoted to the discussion of examples.

In Section 8, we investigate the exponential decay of the solutions when the right-hand side itself exhibits exponential decay, elaborating upon the abstract results in [11]. In turn, this is used in Section 9 to prove another existence theorem for exponentially decaying solutions (Theorem 9.1). This theorem is valid even when $\Phi$ above is not Fredholm and markedly different from Theorem 7.1 or its variants in several other respects.

Throughout the paper, $\langle \cdot\,, \cdot \rangle$ is a given inner product on $\mathbb{R}^N$ with induced norm $|\cdot|$. For $1 \le p \le \infty$, the associated norm on $L^p = L^p((0,\infty),\mathbb{R}^N)$ will be denoted by $|\cdot|_{0,p}$ and the norm on $W^{1,p}$ by $\|\cdot\|_{1,p}$, so that, if $p < \infty$,

$$\|u\|_{1,p} = \{|u|_{0,p}^p + |\dot{u}|_{0,p}^p\}^{1/p}.$$

Several of our arguments will also implicitly use the fact that if $u \in W^{1,p}$, then $u$ is absolutely continuous and its derivative in the sense of distributions is its a.e. derivative (see for instance [2]).

## 2. Linear systems with constant coefficients

We begin by recalling some basic facts from linear algebra (see [1, Chapter III, Section 13]).

Given $A \in \mathcal{L}(\mathbb{R}^N)$, we consider $A$ acting on $\mathbb{C}^N$ through $A(u+iv) = Au+iAv$ where $u, v \in \mathbb{R}^N$. The spectrum of $A$ is denoted by $\sigma(A)$ and we define

$$\sigma_0(A) = \{\lambda \in \sigma(A) : \operatorname{Re}\lambda = 0\} \quad \text{and} \quad \sigma_\pm(A) = \{\lambda \in \sigma(A) : \operatorname{Re}\lambda \gtrless 0\}.$$

Setting

$$\widetilde{X}_0 = \bigoplus_{\lambda\in\sigma_0(A)} \bigcup_{k\in\mathbb{N}} \ker(A-\lambda I)^k \quad \text{and} \quad \widetilde{X}_\pm = \bigoplus_{\lambda\in\sigma_\pm(A)} \bigcup_{k\in\mathbb{N}} \ker(A-\lambda I)^k,$$

we have that $\mathbb{C}^N = \widetilde{X}_0 \oplus \widetilde{X}_+ \oplus \widetilde{X}_-$, that $A\widetilde{X}_0 \subset \widetilde{X}_0$, $A\widetilde{X}_\pm \subset \widetilde{X}_\pm$ and that $\sigma(A_0) = \sigma_0(A)$ and $\sigma(A_\pm) = \sigma_\pm(A)$, where $A_0 = A_{|\widetilde{X}_0}$ and $A_\pm = A_{|\widetilde{X}_\pm}$. Then, setting

$$X_0 = \{z \in \widetilde{X}_0 : \operatorname{Im} z = 0\} \quad \text{and} \quad X_\pm = \{z \in \widetilde{X}_\pm : \operatorname{Im} z = 0\},$$

we call $X_+$ the positive (generalized) eigenspace of $A$ and it follows that

$$\mathbb{R}^N = X_0 \oplus X_+ \oplus X_- \quad \text{and} \quad AX_0 \subset X_0,\ AX_\pm \subset X_\pm. \tag{2.1}$$

Furthermore, there exist constants $K, \alpha > 0$ such that ([1, Chapter III, Theorem 13.3])

$$\|e^{tA_-}\| \le Ke^{-\alpha t} \quad \text{and} \quad \|e^{-tA_+}\| \le Ke^{-\alpha t} \quad \text{for all } t \ge 0. \tag{2.2}$$

The projections onto $X_0, X_\pm$ associated with this decomposition are denoted by $P_-$, $P_\pm$, respectively. It follows that, if $f \in L^p$, then $e^{tA}P_- f = e^{tA_-}P_- f \in L^1$. This will be used repeatedly.

Lemma 2.1. *Suppose that $\sigma_0(A) = \emptyset$. If $1 \le p < \infty$, the initial value problem*

$$\begin{cases} \dot{u} + Au = f, \\ u(0) = \xi, \end{cases} \tag{2.3}$$

*has a solution $u \in W^{1,p}$ if and only if $f \in L^p$ and $P_-\xi = -\int_0^\infty e^{\tau A}P_- f(\tau)\,d\tau$. In this case,*

$$u(t) = e^{-tA}P_+\xi + \int_0^t e^{-(t-\tau)A}P_+ f(\tau)\,d\tau - \int_t^\infty e^{(\tau-t)A}P_- f(\tau)\,d\tau. \tag{2.4}$$

Proof. Suppose first that $u \in W^{1,p}$ satisfies (2.3). Then, $f = \dot{u} + Au \in L^p$ and, by a simple integration by parts,

$$\int_0^\infty e^{\tau A}P_- f(\tau)\,d\tau = \lim_{t\to\infty} \int_0^t e^{\tau A_-}P_-(\dot{u} + Au)\,d\tau = \lim_{t\to\infty} e^{\tau A_-}P_- u(\tau)|_0^t = -P_-\xi,$$

since $(de^{tA_-}/dt)P_- = e^{tA_-}A_-P_- = e^{tA}P_-A$.

Now suppose that $f \in L^p$ and $P_-\xi = -\int_0^\infty e^{\tau A}P_- f(\tau)\,d\tau$. The unique continuous function satisfying the initial value problem (2.3) in the sense of distributions is

$$u(t) = e^{-tA}\xi + \int_0^t e^{(\tau-t)A} f(\tau)\,d\tau.$$

Since $\sigma_0(A) = \emptyset$ we have that $P_+ + P_- = I$ and so $u(t) = P_+u(t) + P_-u(t)$ where

$$\begin{aligned} P_+u(t) &= e^{-tA}P_+\xi + \int_0^t e^{-(t-\tau)A}P_+f(\tau)\,d\tau, \\ P_-u(t) &= e^{-tA}\bigg\{P_-\xi + \int_0^t e^{\tau A}P_-f(\tau)\,d\tau\bigg\} = -\int_t^\infty e^{(\tau-t)A}P_-f(\tau)\,d\tau. \end{aligned}$$

This proves (2.4).

Next, by (2.2), we find $|e^{-tA}P_+\xi| \le Ke^{-\alpha t}|P_+\xi|$ and

$$\begin{aligned} \bigg|\int_0^t e^{-(t-\tau)A}P_+f(\tau)\,d\tau\bigg| &\le K\int_0^t e^{-\alpha(t-\tau)}|P_+f(\tau)|\,d\tau, \\ \bigg|\int_t^\infty e^{(\tau-t)A}P_-f(\tau)\,d\tau\bigg| &\le K\int_t^\infty e^{-\alpha(\tau-t)}|P_-f(\tau)|\,d\tau. \end{aligned}$$

Clearly $e^{-tA}P_+\xi = e^{-tA_+}P_+\xi \in L^p$. Also, the functions $\int_0^t e^{-\alpha(t-\tau)}|P_+f(\tau)|\,d\tau$ and $\int_t^\infty e^{-\alpha(\tau-t)}|P_-f(\tau)|\,d\tau$ are, respectively, the convolution of $|P_+f|$ extended by 0 for $t < 0$ with the function $e^{-\alpha t}\chi_{(0,\infty)} \in L^1$ and the convolution of $|P_-f|$ extended by 0 for $t < 0$ with the function $e^{\alpha t}\chi_{(-\infty,0)} \in L^1$. Young's inequality implies that both these functions are in $L^p$, which shows that $u \in L^p$. Since $\dot u = f - Au$, this implies that $u \in W^{1,p}$, completing the proof. $\square$

Remark 2.2. If $\sigma_0(A) = \emptyset$, it follows from Lemma 2.1 and its obvious analog on $(-\infty, 0)$ that, given $f \in L^p(\mathbb{R}, \mathbb{R}^N)$, there is one and only one solution of $\dot u + Au = f$ in $W^{1,p}(\mathbb{R}, \mathbb{R}^N)$ ($f$ uniquely determines $P_\pm u(0)$, and hence $u(0)$, and formula (2.4) and its analog on $(-\infty, 0)$ yield the restriction of $u$ to $\mathbb{R}\setminus\{0\}$).

Theorem 2.3. *If $1 \le p < \infty$, the bounded linear differential operator $D_A\colon W^{1,p} \to L^p$ defined by $D_Au = \dot u + Au$ is a Fredholm operator if and only if $\sigma_0(A) = \emptyset$. If so, then* $\ker D_A = \{e^{-tA}\xi : \xi \in X_+\}$ *and* $\operatorname{rge} D_A = L^p$, *so that* $\operatorname{ind} D_A = \dim X_+$.

Proof. Let $\widetilde D_A$ denote the extension of $D_A$ to the complex spaces

$$\widetilde{W}^{1,p} = W^{1,p}((0,\infty), \mathbb{C}^N) = W^{1,p} + iW^{1,p}, \quad \widetilde{L}^p = L^p((0,\infty), \mathbb{C}^N) = L^p + iL^p.$$

It is easily seen that $D_A\colon W^{1,p} \to L^p$ is a Fredholm operator if and only if $\widetilde D_A\colon \widetilde W^{1,p} \to \widetilde L^p$ is a Fredholm operator.

Suppose first that $D_A$ is Fredholm and, by contradiction, that $\sigma_0(A) \ne \emptyset$, so that $i\xi \in \sigma(A)$ for some $\xi \in \mathbb{R}$. Let then $z \in \mathbb{C}^N$ with $|z| = 1$ be such that $Az = i\xi z$ and set

$$u(t) = e^{-i\xi t}z \quad \text{and} \quad u_n(t) = \varphi_n(t)u(t) \quad \text{where } \varphi_n(t) = \bigg(\frac{p}{n}\bigg)^{1/p}e^{-t/n}.$$

Then $\int_0^\infty \varphi_n(t)^p\,dt = 1$ for all $n \in \mathbb{N}$, whence $u_n \in \widetilde W^{1,p}$ with $|u_n|_{0,\infty} = (p/n)^{1/p}$, $|u_n|_{0,p} = 1$ and $|\dot u_n|_{0,p} = \sqrt{1/n^2 + \xi^2}$ since $\dot u_n = \dot\varphi_n u + \varphi_n \dot u = -(1/n + i\xi)u_n$.

Thus $(u_n)$ is a bounded sequence in $\widetilde{W}^{1,p}$ and $\widetilde{D}_A u_n = \dot{u}_n + Au_n = \dot{u}_n + i\xi u_n = -u_n/n$ and so $|\widetilde{D}_A u_n|_{0,p} = 1/n$.

Since $\widetilde{D}_A : \widetilde{W}^{1,p} \to \widetilde{L}^p$ is a Fredholm operator, it is an isomorphism of any chosen closed complement $\widetilde{Y}$ of its null-space $\widetilde{Z}$ onto its (closed) range. By writing $u_n = z_n + y_n$ with $z_n \in \widetilde{Z}$ and $y_n \in \widetilde{Y}$, it follows from $|\widetilde{D}_A u_n|_{0,p} = 1/n$ that $(y_n)$ tends strongly to 0 in $\widetilde{W}^{1,p}$.

Now, since $\widetilde{Z}$ is finite dimensional, the bounded sequence $(z_n) = (u_n - y_n)$ has a norm-convergent subsequence $(z_{n_k})$ in $\widetilde{W}^{1,p}$, with limit $z$. From the above, $(u_{n_k})$ is also strongly convergent to $z$ in $\widetilde{W}^{1,p}$. But $z = 0$ since $|u_{n_k}|_{0,\infty} = (p/n_k)^{1/p}$ and the embedding of $\widetilde{W}^{1,p}$ in $\widetilde{L}^\infty$ is continuous. Therefore, $(u_{n_k})$ tends strongly to 0 in $\widetilde{L}^p$, which contradicts $|u_{n_k}|_{0,p} = 1$. This contradiction proves that $\sigma_0(A) = \emptyset$ if $D_A : W^{1,p} \to L^p$ is a Fredholm operator.

Suppose now that $\sigma_0(A) = \emptyset$. Then $\mathbb{R}^N = X_+ \oplus X_-$ and it follows from Lemma 2.1 that $u \in \ker D_A$ if and only if $\dot{u} + Au = 0$ and $P_- u(0) = 0$. Thus $\ker D_A = \{e^{-tA}\xi : \xi \in X_+\}$.

Given any $f \in L^p$, we can define $\eta \in X_-$ by setting $\eta = -\int_0^\infty e^{\tau A} P_- f(\tau) d\tau$. By Lemma 2.1, there is $u \in W^{1,p}$ such that $D_A u = f$ and $u(0) = \eta$. Hence $\operatorname{rge} D_A = L^p$. It follows that $D_A : W^{1,p} \to L^p$ is Fredholm with *ind* $D_A = \dim \ker D_A = \dim X_+$. $\square$

We now come to the main result of this section.

Theorem 2.4. *If $1 \le p < \infty$, the bounded linear operator $\Lambda : W^{1,p} \to L^p \times X_1$ defined by*

$$\Lambda u = (D_A u, Pu(0)) \tag{2.5}$$

*is a Fredholm operator if and only if $\sigma_0(A) = \emptyset$. If so,*

$$\begin{aligned} \ker \Lambda &= \{e^{-tA}\xi : \xi \in X_+ \cap X_2\}, \\ \operatorname{rge} \Lambda &= \left\{ (f,\eta) \in L^p \times X_1 : \eta + \int_0^\infty e^{\tau A} P_- f(\tau)\, d\tau \in X_+ + X_2 \right\} \end{aligned}$$

*and* $\operatorname{ind} \Lambda = \dim X_+ - \dim X_1$.

Proof. Consider the operators $\Lambda_1, \Lambda_2 : W^{1,p} \to L^p \times X_1$ defined by

$$\Lambda_1 u = (D_A u, 0) \quad \text{and} \quad \Lambda_2 u = (0, Pu(0)),$$

where $D_A : W^{1,p} \to L^p$ is the operator discussed in Theorem 2.3.

Since $\Lambda_2$ has finite rank and $\Lambda = \Lambda_1 + \Lambda_2, \Lambda$ is Fredholm if and only if $\Lambda_1$ is Fredholm and $\Lambda$ and $\Lambda_1$ have the same index. But clearly, $\ker \Lambda_1 = \ker D_A$ and $\operatorname{rge} \Lambda_1 = \operatorname{rge}\ D_A \times \{0\}$ so that $\operatorname{rge} \Lambda_1$ is closed if and only if $\operatorname{rge} D_A$ is closed.

If $\Lambda_1$ is Fredholm, $\operatorname{rge} \Lambda_1$ is closed and there exists a subspace $Z$ of $L^p \times X_1$ such that $\dim Z < \infty$ and $L^p \times X_1 = \operatorname{rge} \Lambda_1 \oplus Z = (\operatorname{rge} D_A \times \{0\}) \oplus Z$. Hence, the (finite dimensional) projection of $Z$ onto $L^p$ is a complement of $\operatorname{rge} D_A$ in $L^p$. Thus $D_A$ is Fredholm and so $\sigma_0(A) = \emptyset$ by Theorem 2.3.

Conversely, if $\sigma_0(A) = \emptyset$, we have that $\operatorname{rge} D_A = L^p$ by Theorem 2.3 and so $\operatorname{rge} \Lambda_1 = L^p \times \{0\}$ is closed and $\operatorname{codim} \operatorname{rge} \Lambda_1 = \dim X_1$. Since $\ker \Lambda_1 = \ker D_A = \{e^{-tA}\xi : \xi \in X_+\}$, it follows that $\Lambda_1$, and hence also $\Lambda$, is Fredholm with *ind* $\Lambda = \operatorname{ind} \Lambda_1 = \dim X_+ - \dim X_1$. Clearly

$$\begin{aligned}\ker \Lambda = \ker \Lambda_1 \cap \ker \Lambda_2 &= \{e^{-tA}\xi : \xi \in X_+\} \cap \{u \in W^{1,p} : Pu(0) = 0\}\\ &= \{e^{-tA}\xi : \xi \in X_+ \cap X_2\}.\end{aligned}$$

Suppose now that $(f, \eta) \in \operatorname{rge} \Lambda$. Then there exists $u \in W^{1,p}$ such that $\dot{u} + Au = f$ and $Pu(0) = \eta$. By Lemma 2.1, we must have

$$P_- u(0) = -\int_0^\infty e^{\tau A} P_- f(\tau)\, d\tau,$$

so that

$$u(0) = P_+ u(0) - \int_0^\infty e^{\tau A} P_- f(\tau) d\tau = \eta + (I - P)u(0)$$

since $\mathbb{R}^N = X_+ \oplus X_- = X_1 \oplus X_2$. Thus

$$\eta + \int_0^\infty e^{\tau A} P_- f(\tau) d\tau = P_+ u(0) - (I - P)u(0) \in X_+ + X_2,$$

showing that

$$\operatorname{rge} \Lambda \subset \left\{(f, \eta) \in L^p \times X_1 : \eta + \int_0^\infty e^{\tau A} P_- f(\tau)\, d\tau \in X_+ + X_2\right\}.$$

Conversely, if $(f, \eta) \in L^p \times X_1$ and

$$\eta + \int_0^\infty e^{\tau A} P_- f(\tau)\, d\tau \in X_+ + X_2,$$

there exist $\xi_+ \in X_+$ and $\xi_2 \in X_2$ such that

$$\eta + \int_0^\infty e^{\tau A} P_- f(\tau)\, d\tau = \xi_+ + \xi_2.$$

Let us set $\xi = \eta - \xi_2$ and consider the initial value problem

$$u \in W^{1,p}, \quad \dot{u} + Au = f \quad \text{and} \quad u(0) = \xi.$$

We note that

$$P_- \xi = P_-(\eta - \xi_2) = P_-\left(\xi_+ - \int_0^\infty e^{\tau A} P_- f(\tau)\, d\tau\right) = -\int_0^\infty e^{\tau A} P_- f(\tau)\, d\tau$$

and so it follows from Lemma 2.1 that this problem has a solution $u$.

On the other hand, $Pu(0) = P\xi = P(\eta - \xi_2) = \eta$ . Hence $\Lambda u = (f, \eta)$, showing that

$$\left\{(f, \eta) \in L^p \times X_1 : \eta + \int_0^\infty e^{\tau A} P_- f(\tau)\, d\tau \in X_+ + X_2\right\} \subset \operatorname{rge} \Lambda. \qquad \square$$

COROLLARY 2.5. *If $1 \le p < \infty$, the operator $\Lambda$ in (2.5) is an isomorphism of $W^{1,p}$ onto $L^p \times X_1$ if and only if $\sigma_0(A) = \emptyset$ and $\mathbb{R}^N = X_+ \oplus X_2$. In this case*

$$\Lambda^{-1}(f,\xi) = e^{-tA}\Pi\bigg(\xi + \int_0^\infty e^{(\tau-t)A}P_- f(\tau)\,d\tau\bigg) + \int_0^t e^{-(t-\tau)A}P_+ f(\tau)\,d\tau - \int_t^\infty e^{(\tau-t)A}P_- f(\tau)\,d\tau \tag{2.6}$$

*where $\Pi$ denotes the projection onto $X_+$ associated with the decomposition $\mathbb{R}^N = X_+ \oplus X_2$.*

PROOF. If $\Lambda$ is an isomorphism, it is a Fredholm operator of index zero with $\ker\Lambda = \{0\}$. By Theorem 2.4, this implies that $\sigma_0(A) = \emptyset$, that $\dim X_+ = \dim X_1$ and that $X_+ \cap X_2 = \{0\}$. In particular, $\mathbb{R}^N = X_+ \oplus X_2$.

If $\sigma_0(A) = \emptyset$ and $\mathbb{R}^N = X_+ \oplus X_2$, we have that $\Lambda$ is a Fredholm operator with $\operatorname{ind}\Lambda = \dim X_+ - \dim X_1 = \dim X_+ - \dim X_+ = 0$ and $\ker\Lambda = \{0\}$. This shows that $\Lambda$ is an isomorphism.

To obtain the formula for $\Lambda^{-1}$, we set $u = \Lambda^{-1}(f,\eta)$ and observe that $u \in W^{1,p}$, $D_A u = f$ and $Pu(0) = \eta$. It follows from Lemma 2.1 that

$$u(t) = e^{-tA}P_+u(0) + \int_0^t e^{-(t-\tau)A}P_+ f(\tau)d\tau - \int_t^\infty e^{(\tau-t)A}P_- f(\tau)d\tau \tag{2.7}$$

and

$$P_-u(0) = -\int_0^\infty e^{\tau A}P_- f(\tau)\,d\tau.$$

Since $u(0) = P_+u(0) + P_-u(0) = Pu(0) + (I-P)u(0)$, we have that $\Pi P_+u(0) + \Pi P_-u(0) = \Pi Pu(0) + \Pi(I-P)u(0)$, where $\Pi P_+u(0) = P_+u(0)$ and $\Pi(I-P)u(0) = 0$. Hence

$$P_+u(0) = \Pi\{Pu(0) - P_-u(0)\} = \Pi\bigg(\xi + \int_0^\infty e^{\tau A}P_- f(\tau)\,d\tau\bigg)$$

and we obtain (2.6) by substituting this expression in (2.7). $\square$

## 3. Smoothness of the Nemytskiĭ operator

Let $F = F(t,u)\colon [0,\infty) \times \mathbb{R}^N \to \mathbb{R}^N$ denote a function such that[1]

(3.1) $F \in C^0([0,\infty) \times \mathbb{R}^N, \mathbb{R}^N)$ and $F(\,\cdot\,,0) = 0$, $D_uF$ exists and is continuous on $[0,\infty) \times \mathbb{R}^N$,

and

(3.2) $D_uF(\,\cdot\,,0) \in L^\infty((0,\infty))$, $\{D_uF(t,\,\cdot\,)\}_{t\ge 0}$ is equicontinuous on $\mathbb{R}^N$.

[1] Although it would be simpler and little restrictive to assume that $F$ is $C^1$ on $[0,\infty)\times\mathbb{R}^N$, we shall occasionally need to replace $F(t,u)$ by $D_uF(t,0)u$, which *also* satisfies (3.1) when $F$ does, but is not $C^1$ when $F$ is $C^1$.

It follows from (3.2) that $\{D_uF(t,\,\cdot\,)\}_{t\geq 0}$ is uniformly equicontinuous on the compact subsets of $\mathbb{R}^N$: for any $\varepsilon > 0$ and any compact subset $K \subset \mathbb{R}^N$, there is a constant $\delta(\varepsilon, K) > 0$ such that

(3.3) $|D_uF(t,u) - D_uF(t,v)| \leq \varepsilon$ whenever $t \geq 0$ and $u, v \in K$, $|u-v| \leq \delta(\varepsilon, K)$.

The proof is similar to the proof that continuity and uniform continuity are the same on compact sets. Using $D_uF(\,\cdot\,,0) \in L^\infty$, we also have that $\{D_uF(t,\,\cdot\,)\}_{t\geq 0}$ is equibounded on compact subsets: for any $R \geq 0$, there exists a constant $C(R) \geq 0$ such that

$$|D_uF(t,u)| \leq C(R) \quad \text{for all } t \geq 0 \text{ and all } u \in \mathbb{R}^N \text{ with } |u| \leq R. \tag{3.4}$$

Without any loss of generality, we shall assume that $C(R)$ is a non-decreasing function of $R$.

Theorem 3.1. *Suppose that $F$ satisfies* (3.1) *and* (3.2). *If $1 \leq p < \infty$, then $F(\,\cdot\,,u) \in L^p$ for all $u \in W^{1,p}$ and*

$$|F(\,\cdot\,,u)|_{0,p} \leq C(R)|u|_{0,p}, \tag{3.5}$$

*whenever $|u|_{0,\infty} \leq R$. Furthermore, the Nemytskiĭ operator $\mathcal{F}$ defined by $\mathcal{F}(u) = F(\,\cdot\,,u)$ is of class $C^1$ from $W^{1,p}$ to $L^p$ and $D\mathcal{F}(u)v = D_uF(\,\cdot\,,u)v$ for all $u, v \in W^{1,p}$.*

Proof. Since $W^{1,p} \subset C^0([0,\infty),\mathbb{R}^N) \cap L^\infty$, we have

$$D_uF(\,\cdot\,,u) \in C^0([0,\infty),\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)) \cap L^\infty((0,\infty),\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N))$$

for every $u \in W^{1,p}$. Thus, for any $u \in W^{1,p}$, we can define an operator $M(u) \in \mathcal{L}(L^p, L^p)$ by setting

$$M(u)v = D_uF(\,\cdot\,,u)v. \tag{3.6}$$

A fortiori, $M(u) \in \mathcal{L}(W^{1,p}, L^p)$ and, for $u, v, w \in W^{1,p}$,

$$\begin{aligned} |(M(u) - M(w))v|_{0,p} &\leq \|D_uF(\,\cdot\,,u) - D_uF(\,\cdot\,,w)\|_{0,\infty}|v|_{0,p} \\ &\leq \|D_uF(\,\cdot\,,u) - D_uF(\,\cdot\,,w)\|_{0,\infty}\|v\|_{1,p}. \end{aligned}$$

Thus,

$$\|M(u) - M(w)\|_{\mathcal{L}(W^{1,p},L^p)} \leq \|D_uF(\,\cdot\,,u) - D_uF(\,\cdot\,,w)\|_{0,\infty}.$$

Since $W^{1,p}$ is continuously embedded in $L^\infty$, it follows from (3.3) that

$$M\colon W^{1,p} \to \mathcal{L}(W^{1,p}, L^p)$$

is continuous.

By (3.4), if $u \in W^{1,p}$, then $F(\,\cdot\,,u) \in L^p$ and

$$|F(\,\cdot\,,u)|_{0,p} \leq C(|u|_{0,\infty})|u|_{0,p} \leq C(R)|u|_{0,p}$$

since $C(\,\cdot\,)$ is non-decreasing. Next, for $h \in \mathbb{R} \setminus \{0\}$ and $u, v \in W^{1,p}$,

$$\begin{aligned}\frac{\mathcal{F}(u+hv)(t) - \mathcal{F}u(t)}{h} &- M(u(t))v(t) \\ &= \frac{1}{h}\bigg\{\int_0^h (D_uF(t, u(t) + sv(t)) - D_uF(t, u(t)))\,ds\bigg\}v(t)\end{aligned}$$

and hence

$$\begin{aligned}\bigg|\frac{\mathcal{F}(u+hv) - \mathcal{F}u}{h} &- M(u)v\bigg|_{0,p} \\ &\leq \sup_{t\geq 0, s\in[0,h]} \|D_uF(t, u(t) + sv(t)) - D_uF(t, u(t))\| |v|_{0,p}\end{aligned}$$

Since $u, v \in W^{1,p} \subset L^\infty$, it follows from (3.3) that

$$\sup_{t\geq 0, 0\leq s\leq h} \|D_uF(t, u(t) + sv(t)) - D_uF(t, u(t))\| \to 0 \quad \text{as } h \to 0,$$

which shows that $\mathcal{F}: W^{1,p} \to L^p$ is Gâteaux differentiable at $u$ with $D\mathcal{F}(u)v = M(u)v$ for all $u, v \in W^{1,p}$. But we have already shown that $M: W^{1,p} \to \mathcal{L}(W^{1,p}, L^p)$ is continuous, so that $\mathcal{F}$ is $C^1$. $\square$

Remark 3.2. A similar result holds when the interval $(0, \infty)$ is replaced by the whole line $\mathbb{R}$.

## 4. Fredholm and properness properties

If $F$ satisfies (3.1) and (3.2) and $1 \leq p < \infty$, it follows from Theorem 3.1 that we can define a function $\Phi: W^{1,p} \to L^p \times X_1$ by

$$\Phi(u) = (\dot{u} + \mathcal{F}(u), Pu(0))$$

and that $\Phi \in C^1(W^{1,p}, L^p \times X_1)$ with

$$D\Phi(u)v = (\dot{v} + D_uF(\,\cdot\,, u)v, Pv(0)) \quad \text{for } u, v \in W^{1,p}.$$

Below we suppose, in addition, that there exists $A^\infty \in \mathcal{L}(\mathbb{R}^N)$ such that

$$\lim_{t\to\infty} D_uF(t, 0) = A^\infty \quad \text{in } \mathcal{L}(\mathbb{R}^N). \tag{4.1}$$

Theorem 4.1. *Let $F$ satisfy* (3.1), (3.2) *and* (4.1). *Then, $D\Phi(u): W^{1,p} \to L^p \times X_1$ is a Fredholm operator for all $u \in W^{1,p}$ if and only if $\sigma_0(A^\infty) = \emptyset$. If so, its index is* 0 *if and only if $\dim X_+^\infty = \dim X_1$, where $X_+^\infty$ is the positive eigenspace of $A^\infty$.*

Proof. Setting $\Lambda u = (\dot{u} + A^\infty u, Pu(0))$, we have that, for any $u, v \in W^{1,p}$,

$$D\Phi(u) - \Lambda = (D_uF(\,\cdot\,, u) - A^\infty, 0): W^{1,p} \to L^p \times X_1.$$

Since $u(t) \to 0$ as $t \to \infty$, it follows from (3.2) and (4.1) that

$$D_u F(t, u(t)) - A^\infty = D_u F(t, u(t)) - D_u F(t, 0) + D_u F(t, 0) - A^\infty \to 0$$

as $t \to \infty$ and it is easily seen that this implies that $D\Phi(u) - \Lambda: W^{1,p} \to L^p \times X_1$ is a compact linear operator. Thus, $D\Phi(u): W^{1,p} \to L^p \times X_1$ is a Fredholm operator (of index 0) for all $u \in W^{1,p}$ if and only if $\Lambda: W^{1,p} \to L^p \times X_1$ is a Fredholm operator (of index 0), so that the result follows from Theorem 2.4. $\square$

We now introduce an assumption stronger than (4.1).

(4.2) There exists a function $F^\infty \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ with $F^\infty(0) = 0$ such that

$$\lim_{t \to \infty} D_u F(t, u) = DF^\infty(u),$$

uniformly for $u$ in bounded subsets of $\mathbb{R}^N$.

The hypotheses (3.1) and (4.2) imply that $F^\infty(0) = 0$ and

$$\begin{aligned} F(t, u) &= \int_0^1 \frac{d}{ds} F(t, su) ds = \int_0^1 D_u F(t, su) u \, ds \\ &\to \int_0^1 DF^\infty(su) u ds = \int_0^1 \frac{d}{ds} F^\infty(su) \, ds = F^\infty(u) \end{aligned} \tag{4.3}$$

uniformly for $u$ in bounded subsets of $\mathbb{R}^N$. From (3.1) and (4.2) it also follows that $\lim_{t \to \infty} D_u F(t, 0) = DF^\infty(0)$, so that condition (4.1) holds with $A^\infty = DF^\infty(0)$. As it turns out, condition (3.2) also holds, see [12, Lemma 4.1]. By (4.3) and $F^\infty(0) = 0$, we find that

$$F(t, u(t)) \to 0 \quad \text{as } t \to \infty \quad \text{for any } u \in W^{1,p}. \tag{4.4}$$

The following characterization of the properness of $\Phi$ on closed bounded subsets is proved in Theorem 4.4 of [12][2] when $p = 2$. The proof for $1 < p < \infty$ is identical.

Theorem 4.2. *Let $F$ satisfy* (3.1) *and* (4.2). *Then,*

(a) *$F$ also satisfies* (3.2),
(b) *if $\sigma_0(DF^\infty(0)) = \emptyset$ and $1 < p < \infty$, the operator $\Phi: W^{1,p} \to L^p \times X_1$ is proper on the closed bounded subsets of $W^{1,p}$ if and only if the equation $\dot{u} + F^\infty(u) = 0$ has no nonzero solution $u \in W^{1,p}(\mathbb{R}, \mathbb{R}^N)$.*

We emphasize that the criterion given in Theorem 4.2(b) involves functions $u$ defined on the *whole* line, not merely $(0, \infty)$. Also, the arguments of [12] use the fact that the embedding of $W^{1,2}(I)$ into $C^0(\overline{I})$ is compact when $I$ is a bounded open interval. This remains true with $W^{1,p}(I)$ if $1 < p < \infty$, but not if $p = 1$.

[2] As pointed out in Section 3 of [12], all the results of that paper, given when $f$ is $C^1$, remain true under the slightly more general assumptions made here.

Generalizations of Theorems 4.1 and 4.2 without (4.1) or (4.2) have recently been found by Morris [7]: For Theorem 4.1, this amounts to assuming that $D_u F(\,\cdot\,,0)$ satisfies a suitable exponential dichotomy and, in Theorem 4.2, $\dot{u}+F^\infty(u)=0$ must be replaced by *all* the equations $\dot{u}+G(\,\cdot\,,u)=0$ where $G=G(t,u)$ is any accumulation point (in some topology) of the family of translates $(F(t+\tau,u))_{\tau\geq 0}$ when $\tau\to\infty$.

## 5. Continuation

Combining Theorems 3.1, 4.1 and 4.2, we obtain the following result (see Theorem 5.1 of [12] for more details when $p=2$)

Theorem 5.1. *Suppose that $F$ satisfies* (3.1) *and* (4.2) *and that, for some $1<p<\infty$,*

$$\{u\in W^{1,p}(\mathbb{R},\mathbb{R}^N):\dot{u}+F^\infty(u)=0\}=\{0\}.$$

*Suppose also that $\sigma_0(DF^\infty(0))=\emptyset$ and that $\dim X_1=\dim X_+^\infty$ where $X_+^\infty$ denotes the positive generalized eigenspace of $DF^\infty(0)$. Then the operator $\Phi$ defined by* (1.7) *has the following properties.*

(a) $\Phi\in C^1(W^{1,p},L^p\times X_1)$.
(b) $D\Phi(u)\colon W^{1,p}\to L^p\times X_1$ *is Fredholm index* $0$ *for all* $u\in W^{1,p}$.
(c) $\Phi\colon W^{1,p}\to L^p\times X_1$ *is proper on closed bounded subsets of* $W^{1,p}$.

Thanks to Theorem 5.1, we can use a degree theory argument to reduce the problem of proving the existence of solutions of (1.1)–(1.2) to that of finding *a priori* bounds for the possible solutions. This degree may be either the $\mathbb{Z}$-valued degree for proper $C^1$ Fredholm mappings of index 0 of [10], or the much older "mod 2" degree of Caccioppoli [3], with values in $\mathbb{Z}/2\mathbb{Z}=\{0,1\}$ (also discussed in [10] as a special case). Our results below are phrased with the latter to avoid introducing base points.

Under the hypotheses of Theorem 5.1, the mod 2 degree $\deg(\Phi,\Omega,d)$ is defined for any bounded open subset $\Omega$ of $W^{1,p}$ and any $d\in L^p\times X_1$ such that $d\notin\Phi(\partial\Omega)$. Furthermore, if $D\Phi(0)$ is an isomorphism and $B_r=\{u\in W^{1,p}:\|u\|_{1,p}<r\}$, then $\deg(D\Phi(0),B_r,0)=1$ for any $r>0$.

Given $(f,\xi)\in L^p\times X_1$, choose some $d\in C^0([0,1],L^p\times X_1)$ with $d(0)=(0,0)$ and $d(1)=(f,\xi)$ and then define $H\colon[0,1]\times W^{1,p}\to L^p\times X_1$ by

$$H(s,u)=\Phi(u)-d(s).$$

If there exists $r>0$ such that

$$\{u\in W^{1,p}:H(s,u)=0\text{ for some }s\in[0,1]\}\subset B_r,$$

then it follows from the homotopy invariance of the mod 2 degree that

$$\deg(\Phi,B_r,0)=\deg(H(0,\,\cdot\,),B_r,d(0))=\deg(H(1,\,\cdot\,),B_r,0)=\deg(\Phi,B_r,(f,\xi)).$$

Furthermore, if $\Phi^{-1}(0) = \{0\}$ and $D\Phi(0) \in GL(W^{1,p}, L^p \times X_1)$, we have that $\deg(\Phi, B_r, 0) = \deg(D\Phi(0), B_r, 0) = 1$.

Note that the condition $\Phi^{-1}(0) = \{0\}$ can be expressed as

$$\{u \in W^{1,p} : \dot{u} + F(\,\cdot\,, u) = 0 \text{ and } Pu(0) = 0\} = \{0\}.$$

while, by Theorem 5.1(b), $D\Phi(0) \in GL(W^{1,p}, L^p \times X_1)$ if and only if

$$\{u \in W^{1,p} : \dot{u} + D_uF(\,\cdot\,, 0)u = 0 \text{ and } Pu(0) = 0\} = \{0\}.$$

Summarizing this discussion we conclude that the existence of *a priori* bounds implies the existence of a solution of (1.1)–(1.2) in the following sense.

Theorem 5.2. *Suppose that the function $F$ satisfies* (3.1) *and* (4.2) *and that $\sigma_0(DF^\infty(0)) = \emptyset$. Suppose also that* $\dim X_+^\infty = \dim X_1$, *where $X_+^\infty$ denotes the positive generalized eigenspace for $DF^\infty(0)$, that*

$$\{u \in W^{1,p}(\mathbb{R}, \mathbb{R}^N) : \dot{u} + F^\infty(u) = 0\} = \{0\}, \tag{5.1}$$

$$\{u \in W^{1,p} : \dot{u} + F(t, u) = 0 \textit{ and } Pu(0) = 0\} = \{0\} \tag{5.2}$$

*and that*

$$\{u \in W^{1,p} : \dot{u} + D_uF(t, 0)u = 0 \textit{ and } Pu(0) = 0\} = \{0\}. \tag{5.3}$$

*Given $1 < p < \infty$ and $(f, \xi) \in L^p \times X_1$, suppose that there exists a map $d \in C^0([0,1], L^p \times X_1)$ with $d(0) = (0,0)$ and $d(1) = (f, \xi)$ such that, for some $r \geq 0$,*

$$\{u \in W^{1,p}, \Phi(u) = d(s) \textit{ for some } s \in [0,1]\} \Rightarrow \|u\|_{1,p} \leq r. \tag{5.4}$$

*Then the problem* (1.1)–(1.2) *has at least one solution in $W^{1,p}$.*

Remark 5.3.

(a) If $F(t, -u) = -F(t, u)$ for all $t \geq 0$ and $u \in \mathbb{R}^N$, the assumptions (5.2) and (5.3) can be dropped since in this case Borsuk's Theorem implies that $\deg(\Phi, B_r, 0)$ is odd.

(b) If $F(t, u) = F(u)$ is independent of $t$, then $F = F^\infty$ and (5.3) holds if and only if $\mathbb{R}^N = X_+^\infty \oplus X_2$ (Corollary 2.5).

The following corollary shows that, the hypotheses (5.1)–(5.3) can be ensured by suitable conditions about the function $F$.

Corollary 5.4. *Suppose that $F \in C^1([0,\infty) \times \mathbb{R}^N, \mathbb{R}^N)$*[3] *with $F(\,\cdot\,, 0) = 0$ satisfies* (4.2) *and that $\sigma_0(DF^\infty(0)) = \emptyset$. Suppose also that* $\dim X_1 = \dim X_+^\infty$, *that*

$$\langle F(0, u), u \rangle \leq 0 \quad \textit{for all } u \in X_2, \tag{5.5}$$

[3] Hence, $F$ satisfies (3.1).

*and*

(5.6) *there is* $b < 1$ *such that*

$$\langle D_t F(t,u) - D_u F(t,u) F(t,u), u\rangle \le b|F(t,u)|^2$$

*for all* $t \ge 0$ *and all* $u \in \mathbb{R}^N$.

*Given* $1 < p < \infty$ *and* $(f,\xi) \in L^p \times X_1$, *suppose that there exists a map* $d \in C^0([0,1], L^p \times X_1)$ *with* $d(0) = (0,0)$ *and* $d(1) = (f,\xi)$, *such that*

$$\{u \in W^{1,p}, \Phi(u) = d(s) \text{ for some } s \in [0,1]\} \Rightarrow \|u\|_{1,p} \le r. \tag{5.7}$$

*Then, the problem* (1.1)–(1.2) *has at least one solution in* $W^{1,p}$.

Proof. This will follow from Theorem 5.2 after checking that the conditions (5.1)–(5.3) hold. Let then $u \in W^{1,p}$ be such that $\dot u + F(t,u) = 0$ and $Pu(0) = 0$. For any $T > 0$, an integration by parts (justified since $F$ is $C^1$ and $u \in W^{1,p}$) yields

$$\begin{aligned}\int_0^T |F(t,u)|^2\,dt &= -\int_0^T \langle F(t,u), \dot u\rangle\,dt \\ &= -\langle F(T,u(T)), u(T)\rangle + \langle F(0,u(0)), u(0)\rangle \\ &\quad + \int_0^T \langle D_t F(t,u) + D_u F(t,u)\dot u, u\rangle\,dt.\end{aligned}$$

Since $Pu(0) = 0$, it follows from (5.5) that

$$\begin{aligned}&\int_0^T |F(t,u)|^2\,dt \\ &\quad\le -\langle F(T,u(T)), u(T)\rangle + \int_0^T \langle D_t F(t,u) + D_u F(t,u)\dot u, u\rangle\,dt \\ &\quad= -\langle F(T,u(T)), u(T)\rangle + \int_0^T \langle D_t F(t,u) - D_u F(t,u) F(t,u), u\rangle\,dt.\end{aligned}$$

Hence, by (5.6),

$$(1-b)\int_0^T |F(t,u)|^2\,dt \le -\langle F(T,u(T)), u(T)\rangle.$$

Now, $b < 1$ and $\lim_{T\to\infty}\langle F(T,u(T)), u(T)\rangle = 0$ by (4.4), so that $F(t,u(t)) = 0$ for all $t \ge 0$. Since $\dot u + F(t,u) = 0$, it follows that $u$ is constant. But $u \in W^{1,p}$, whence $u = 0$.

We pass to the verification of condition (5.1). We begin with a preliminary remark: Given $u \in \mathbb{R}^N$, set $g(t) = \langle F(t,u), u\rangle$. Then, $\lim_{t\to\infty} g(t) = \langle F^\infty(u), u\rangle$ by (4.3), so that there is a sequence $t_n \ge 0$ with $\lim t_n = \infty$ and $\lim \dot g(t_n) = 0$. In other words, $\langle D_t F(t_n,u), u\rangle \to 0$. Since also $D_u F(t_n,u) \to DF^\infty(u)$, it follows from (5.6) that

$$-\langle DF^\infty(u) F^\infty(u), u\rangle \le b|F^\infty(u)|^2 \quad \text{for all } u \in \mathbb{R}^N. \tag{5.8}$$

Let $u \in W^{1,p}(\mathbb{R}, \mathbb{R}^N)$ be such that $\dot{u} + F^\infty(u) = 0$. For any $S < T$,

$$\begin{aligned}\int_S^T |F^\infty(u)|^2\,dt &= -\int_S^T \langle F^\infty(u), \dot{u}\rangle\,dt \\ &= \langle F^\infty(u(S)), u(S)\rangle - \langle F^\infty(u(T)), u(T)\rangle + \int_S^T \langle DF^\infty(u)\dot{u}, u\rangle\,dt \\ &= \langle F^\infty(u(S)), u(S)\rangle - \langle F^\infty(u(T)), u(T)\rangle - \int_S^T \langle DF^\infty(u)F^\infty(u), u\rangle\,dt,\end{aligned}$$

so that

$$(1-b)\int_S^T |F^\infty(u)|^2\,dt \le \langle F^\infty(u(S)), u(S)\rangle - \langle F^\infty(u(T)), u(T)\rangle$$

by (5.8). Since

$$\lim_{S\to-\infty} \langle F^\infty(u(S)), u(S)\rangle = \lim_{T\to\infty} \langle F^\infty(u(T)), u(T)\rangle = 0,$$

we infer that $F^\infty(u(t)) = 0$ for all $t \in \mathbb{R}$. Thus, $\dot{u} = -F^\infty(u) = 0$, so that $u = 0$ since $u \in W^{1,p}(\mathbb{R}, \mathbb{R}^N)$.

To check (5.3), observe that the hypotheses of the corollary continue to hold when $F(t,u)$ is replaced by $D_uF(t,0)u$: First, $F^\infty(u)$ is changed into $DF^\infty(0)u$, so that $DF^\infty(0)$ is unchanged. Next, by replacing $u$ by $su$ in (5.5) and (5.6), dividing by $s^2$ and letting $s \to 0$, we obtain (5.5) and (5.6) for $D_uF(t,0)u$. Thus, by (5.2) with $F$ replaced by $D_uF(\,\cdot\,,0)$, the conditions $\dot{u} + D_uF(t,0)u = 0$ and $Pu(0) = 0 = 0$ for $u \in W^{1,p}$ imply $u = 0$, which is (5.3). $\square$

As we shall see in the next section, condition (5.7) can be ensured by complementing conditions (5.5) and (5.6).

## 6. A priori bounds

For a function $F \in C^1([0,\infty) \times \mathbb{R}^N, \mathbb{R}^N)$ with $F(\,\cdot\,,0) = 0$, we introduce the following conditions:

(6.1) For every $\varepsilon > 0$ and every $R \ge 0$, there is $\gamma(\varepsilon, R) \ge 0$ such that

$$\langle F(0,u), u\rangle \le \gamma(\varepsilon,R)^2 + \varepsilon|u|^2,$$

for all $u \in \mathbb{R}^N$ with $|Pu| \le R$.

(6.2) There is $b < 1$ such that

$$\langle D_tF(t,u) - D_uF(t,u)F(t,u), u\rangle \le b|F(t,u)|^2$$

for all $t \ge 0$ and $u \in \mathbb{R}^N$.

(6.3) There is $C > 0$ such that $|u| \le C|F(t,u)|$ for all $t \ge 0$ and $u \in \mathbb{R}^N$.

Remark 6.1. Observe that the conditions (6.2) and (5.6) are the same and that (6.1) implies (5.5) if and only if $\gamma(\varepsilon, 0) = 0$ may be chosen in (6.1).

Remark 6.2. In the autonomous case where $F(t,u) = F(u)$ is independent of $t$, the condition (6.3) is equivalent to

$$(6.4)\qquad 0 \notin \sigma(DF(0)), \quad \underline{\lim}_{|u|\to\infty} \frac{|F(u)|}{|u|} > 0 \quad \text{and} \quad F(u) \neq 0 \quad \text{for } u \neq 0.$$

The following theorem establishes the existence of *a priori* bounds in $W^{1,2}$. It is followed by a discussion of the same issue in $W^{1,p}$, $p \neq 2$.

Theorem 6.3. *Let $F \in C^1([0,\infty)\times\mathbb{R}^N,\mathbb{R}^N)$ with $F(\,\cdot\,,0) = 0$ satisfy* (4.2) *and* (6.1) *to* (6.3). *Suppose also that there exist a subspace $W$ of $\mathbb{R}^N$ and $M > 0$ such that*

$$(6.5)\qquad |\langle D_u F(t,u)z, u\rangle| \leq M|F(t,u)|\,|z|,$$

*for all $t \geq 0$, all $u \in \mathbb{R}^N$ and all $z \in W$. Then, there exist a constant $a > 0$ and, for every $R \geq 0$, another constant $D(R) \geq 0$ such that*

$$(6.6)\qquad ||u||_{1,2} \leq a(|f|_{0,2} + D(R))$$

*for any solution $u \in W^{1,2}$ of* (1.1) *with $f \in L^2((0,\infty),W)$ and $\xi \in X_1$ satisfying $|\xi| \leq R$. In particular, if $f \in L^2((0,\infty),W)$ and $\xi \in X_1$, we have that*

$$\{u \in W^{1,2}, \Phi(u) = (sf, s\xi) \text{ for some } s \in [0,1]\} \Rightarrow \|u\|_{1,2} \leq r,$$

*where $r = a(|f|_{0,2} + D(|\xi|))$.*

Proof. Let $u \in W^{1,2}$ solve $\dot{u} + F(t,u) = f$. By the arguments of the proof of Corollary 5.4, we easily arrive at the relation

$$(1-b)\int_0^\infty |F(t,u)|^2\,dt \leq \int_0^\infty [\langle F(t,u),f\rangle + \langle D_u F(t,u)f,u\rangle]\,dt + \langle F(0,u(0)),u(0)\rangle$$

and so, using (6.5),

$$\begin{aligned}(1-b)|F(\,\cdot\,,u)|_{0,2}^2 &\leq \int_0^\infty (M+1)|F(t,u)|\,|f|dt + \langle F(0,u(0)),u(0)\rangle \\ &\leq (M+1)|F(\,\cdot\,,u)|_{0,2}|f|_{0,2} + \langle F(0,u(0)),u(0)\rangle \\ &\leq (M+1)\bigg\{\frac{1}{2\lambda}|F(\,\cdot\,,u)|_{0,2}^2 + \frac{\lambda}{2}|f|_{0,2}^2\bigg\} + \langle F(0,u(0)),u(0)\rangle,\end{aligned}$$

for any $\lambda > 0$. With the choice $\lambda = (M+1)/(1-b)$, we obtain

$$(1-b)^2|F(\,\cdot\,,u)|_{0,2}^2 \leq (M+1)^2|f|_{0,2}^2 + 2(1-b)\langle F(0,u(0)),u(0)\rangle.$$

Thus, by (6.1),

$$(1-b)^2|F(\,\cdot\,,u)|_{0,2}^2 \leq (M+1)^2|f|_{0,2}^2 + 2(1-b)[\gamma(\varepsilon,R)^2 + \varepsilon|u(0)|^2]$$

since $|Pu(0)| = |\xi| \leq R$. It follows that

$$|F(\,\cdot\,,u)|_{0,2}^2 \leq K\{|f|_{0,2}^2 + \gamma(\varepsilon,R)^2 + \varepsilon|u(0)|^2\},$$

with $K = \max\{(M+1)^2/(1-b)^2, 2/(1-b)\}$. Using (6.3), this yields

$$|u|_{0,2}^2 \le C^2 K\{|f|_{0,2}^2 + \gamma(\varepsilon,R)^2 + \varepsilon|u(0)|^2\}$$

whereas

$$\begin{aligned}|\dot u|_{0,2}^2 &= |f - F(\,\cdot\,,u)|_{0,2}^2 \le 2\{|f|_{0,2}^2 + |F(\,\cdot\,,u)|_{0,2}^2\}\\ &\le 2[(K+1)|f|_{0,2}^2 + K\{\gamma(\varepsilon,R)^2 + \varepsilon|u(0)|^2\}].\end{aligned}$$

Altogether, we get

$$||u||_{1,2}^2 \le (C^2K + 2K + 2)\{|f|_{0,2}^2 + \gamma(\varepsilon,R)^2 + \varepsilon|u(0)|^2\}.$$

Since the trace operator $v \in W^{1,2} \mapsto v(0) \in \mathbb{R}^N$ has norm 1, it follows that

$$||u||_{1,2}^2 \le (C^2K + 2K + 2)\{|f|_{0,2}^2 + \gamma(\varepsilon,R)^2 + \varepsilon||u||_{1,2}^2\}.$$

Above, $\varepsilon > 0$ is arbitrary. The choice $\varepsilon = 1/2(C^2K + 2K + 2)$ yields

$$||u||_{1,2}^2 \le 2(C^2K + 2K + 2)\{|f|_{0,2} + \gamma(\varepsilon,R)\}^2,$$

This shows that (6.6) is satisfied with $a = \sqrt{2(C^2K + 2K + 2)}$ and $D(R) = \gamma(1/(2(C^2+2)K+4), R)$. $\square$

When $f \in L^p$ with $1 < p < \infty$, $p \ne 2$, the method of proof of Theorem 6.1 can be followed to obtain *a priori* bounds in $W^{1,p}$, but the hypotheses about $F$ must be changed and become rather complicated. Everything boils down to finding a suitable estimate for $|F(\,\cdot\,,u)|_{0,p}$. This can be done by writing

$$\int_0^T |F(t,u)|^p\,dt = \int_0^T |F(t,u)|^{p-2}\langle F(t,u), f - \dot u\rangle\,dt$$

and integrating by parts. It then appears that conditions (6.1) and (6.2) must be modified. Specifically, (6.1) becomes (replace $\mathbb{R}^N$ by $\mathbb{R}^N \setminus \{0\}$ in (6.7) below if $1 < p < 2$):

(6.7) For every $\varepsilon > 0$ and every $R \ge 0$, there is $\gamma(\varepsilon,R) \ge 0$ such that

$$|F(0,u)|^{p-2}\langle F(0,u), u\rangle \le \gamma(\varepsilon,R)^2 + \varepsilon|u|^p,$$

for all $u \in \mathbb{R}^N$ with $|Pu| \le R$.

To formulate the proper variant of (6.2), we introduce the notation

$$H_p(t,u) = (p-2)\langle F(t,u), u\rangle F(t,u) + |F(t,u)|^2 u \quad \text{for all } t \ge 0 \text{ and } u \in \mathbb{R}^N.$$

Then, (6.2) should be replaced by

(6.8) There is $b < 1$ such that

$$\langle D_tF(t,u) - D_uF(t,u)F(t,u), H_p(t,u)\rangle \le b|F(t,u)|^4$$

for all $t \ge 0$ and $u \in \mathbb{R}^N$, which indeed reduces to (6.2) when $p = 2$. There is no need to modify (6.3), but (6.5) must be complemented by also requiring that

$$\langle D_u F(t,u)z, F(t,u)\rangle\langle F(t,u), u\rangle \leq M|F(t,u)|^3|z|, \tag{6.9}$$

for all $t \geq 0$, all $u \in \mathbb{R}^N$ and all $z \in W$, where $M > 0$ is a constant (which may be chosen the same as in (6.5)).

## 7. An existence theorem

The following existence theorem in $W^{1,2}$ follows at once from Corollary 5.4 and Theorem 6.3 (see also Remark 6.1 and Theorem 4.2(a)). A similar existence theorem in $W^{1,p}$ with $1 < p < \infty$ can be obtained by modifying the hypotheses as indicated at the end of the previous section when $p \neq 2$.

Theorem 7.1. *Suppose that $F \in C^1([0,\infty) \times \mathbb{R}^N, \mathbb{R}^N)$ with $F(\,\cdot\,, 0) = 0$ satisfies* (4.2), (6.1) *with $\gamma(\varepsilon, 0) = 0$,* (6.2), (6.3) *and also* (6.5) *for some subspace $W$ of $\mathbb{R}^N$. Suppose in addition that $\sigma_0(DF^\infty(0)) = \emptyset$ and that $\dim X_+^\infty = \dim X_1$, where $X_+^\infty$ denotes the positive generalized eigenspace for $DF^\infty(0)$. Then, for every $f \in L^2((0,\infty), W)$ and every $\xi \in X_1$, the problem* (1.1)–(1.2) *has at least one solution $u \in W^{1,2}$.*

When $X_1 = \mathbb{R}^N$ (classical initial value problem; see Example 1.1), then $P = I$, all the eigenvalues of $DF^\infty(0)$ must have positive real part and condition (6.1) always holds, with $\gamma(\varepsilon, 0) = 0$. No simplification occurs in the other hypotheses of Theorem 7.1.

When $X_1 = \{0\}$ (Example 1.2), then $P = 0$, all the eigenvalues of $DF^\infty(0)$ must have negative real part and condition (6.1) is that $\langle F(0,u), u\rangle \leq \gamma(\varepsilon, R)^2 + \varepsilon|u|^2$ for all $u \in \mathbb{R}^N$, where $R \geq 0$ is arbitrary. Since Theorem 7.1 also requires $\gamma(\varepsilon, 0) = 0$ with $\varepsilon > 0$ is arbitrary, (6.1) holds with $\gamma(\varepsilon, 0) = 0$ if and only if $\langle F(0,u), u\rangle \leq 0$ for all $u \in \mathbb{R}^N$. Once again, no simplification occurs in the other hypotheses of Theorem 7.1.

The remainder of this section is devoted to two examples illustrating the use of Theorem 7.1.

Example 7.2. In this example, $N = 2$ and the decomposition of $\mathbb{R}^2$ is given by $X_1 = \mathbb{R} \times \{0\}$, $X_2 = \{0\} \times \mathbb{R}$, so that $P(v,w) = v$. For simplicity, we confine attention to a problem where $F = F(u)$ independent of $t$, but it should be clear how the hypotheses can be modified to accommodate the general case. The bracket $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the euclidian inner product.

Let $g,h\colon\mathbb{R}\to\mathbb{R}$ be two real-valued functions having the following properties.

$$g(s)=0\Leftrightarrow s=0 \text{ and } h(s)=0\Leftrightarrow s=0, \tag{7.1}$$

$$g,h\in C^1(\mathbb{R}) \text{ with } g'\geq 0 \text{ and } h'\geq 0, \tag{7.2}$$

$$g''(0)>0 \text{ and } h'(0)>0, \tag{7.3}$$

$$\varliminf_{|s|\to\infty} g'(s)>0 \quad\text{and}\quad \varliminf_{|s|\to\infty} h'(s)>0, \tag{7.4}$$

$$\exists C>0,\ \exists r\in[0,2) \text{ such that } |h(s)|\leq C(1+|s|^r), \text{ for all } s\in\mathbb{R}. \tag{7.5}$$

We consider the system

$$\begin{cases} \dot v+h(w)=f_1,\\ \dot w+g(v)=f_2,\\ v(0)=\xi, \end{cases} \tag{7.6}$$

where $f_1,f_2\in L^2$ and $\xi\in\mathbb{R}$. Setting $u=(v,w)$, $F(u)=(h(w),g(v))$ and $f=(f_1,f_2)$, we see that (7.6) has the form (1.1) where (3.1) and (4.2) are satisfied with $F^\infty=F$ and

$$D_uF(u)z=DF(u)z=(h'(w)z_2,g'(v)z_1).$$

Since $F^\infty=F$, we have

$$DF^\infty(0)=DF(0)=\begin{pmatrix} 0 & h'(0)\\ g'(0) & 0\end{pmatrix}$$

and hence

$$\sigma(DF^\infty(0))=\sigma(DF(0))=\{\lambda,-\lambda\}\quad\text{where } \lambda=\sqrt{g'(0)h'(0)}>0.$$

Thus it follows that $\sigma_0(DF^\infty(0))=\emptyset$ and

$$X_+^\infty=\operatorname{span}\{(h'(0),\lambda)\},\quad X_-=\operatorname{span}\{(h'(0),-\lambda)\},$$

so that $\mathbb{R}^2=X_+^\infty\oplus X_2$.

The conditions (7.1)–(7.4) show that $\varliminf_{|u|\to\infty}|F(u)|/|u|>0$ and $F(u)\neq 0$ for $u\neq 0$. Therefore, (6.3) follows from Remark 6.2 since $0\notin\sigma(DF(0))$. By (7.1) and (7.2), we have $sg(s)\geq 0$ and $sh(s)\geq 0$ for every $s\in\mathbb{R}$. Since also $D_tF=0$ and $\langle DF(u)F(u),u\rangle=vg(v)h'(w)+wh(w)g'(v)$, we infer that $\langle DF(u)F(u),u\rangle\geq 0$ for all $u\in\mathbb{R}^2$ and hence that (6.2) holds with $b=0$. Finally, by (7.5),

$$\begin{aligned}\langle F(u),u\rangle &= vh(w)+wg(v)\leq C|v|+C|v|\,|w|^r+|w|\,|g(v)|\\ &\leq C|v|+C\biggl\{\frac{r}{2}(\delta|w|^r)^{2/r}+\frac{2-r}{2}(\delta^{-1}|v|)^{2/(2-r)}\biggr\}+\frac{\delta}{2}w^2+\frac{1}{2\delta}g(v)^2\\ &= C\biggl\{|v|+\frac{1}{2\delta}g(v)^2+\frac{2-r}{2}(\delta^{-1}|v|)^{2/(2-r)}\biggr\}+\biggl\{C\frac{r}{2}\delta^{2/r}+\frac{\delta}{2}\biggr\}w^2,\end{aligned}$$

for any $\delta > 0$. Since $w^2 \leq |u|^2$, this shows that (6.1) holds by choosing $\delta > 0$ small enough and setting

$$\gamma(\varepsilon, R) = \max_{|v| \leq R} C\left\{|v| + \frac{1}{2\delta} g(v)^2 + \frac{2-r}{2}(\delta^{-1}|v|)^{2/(2-r)}\right\}.$$

Note that $\gamma(\varepsilon, 0) = 0$, as required in Theorem 7.1.

Theorem 7.3. *Let the conditions* (7.1)–(7.5) *be satisfied.*

(a) *For every $\xi \in \mathbb{R}$, the system*

$$\begin{cases} \dot{v} + h(w) = 0, \\ \dot{w} + g(v) = 0, \\ v(0) = \xi, \end{cases}$$

*has at least one solution $(v, w) \in W^{1,2}$ such that $v(0) = \xi$.*

(b) *Suppose that there exists $M > 0$ such that $|h'(s)| \leq M$ for all $s \in \mathbb{R}$. Then, for all $\xi \in \mathbb{R}$ and all $f_2 \in L^2$, the system*

$$\begin{cases} \dot{v} + h(w) = 0, \\ \dot{w} + g(v) = f_2, \\ v(0) = \xi, \end{cases}$$

*has at least one solution $(v, w) \in W^{1,2}$ such that $v(0) = \xi$.*

(c) *Suppose that there exists $M > 0$ such that $|g'(s)| \leq M$ and $|h'(s)| \leq M$ for all $s \in \mathbb{R}$. Then, for all $\xi \in \mathbb{R}$ and all $f_1, f_2 \in L^2$, the system*

$$\begin{cases} \dot{v} + h(w) = f_1, \\ \dot{w} + g(v) = f_2, \\ v(0) = \xi, \end{cases}$$

*has at least one solution $(v, w) \in W^{1,2}$ such that $v(0) = \xi$.*

Proof. It remains only to check condition (6.5) for a suitable subspace $W$ of $\mathbb{R}^2$ to obtain the desired result by Theorem 7.1. In case (a), we simply choose $W = \{0\}$, so that (6.5) holds trivially. In case (b) (resp. (c)), we let $W = X_2 = \{0\} \times \mathbb{R}$ (resp. $W = \mathbb{R}^2$). The relation

$$\langle DF(u)z, u\rangle = vz_2 h'(w) + wz_1 g'(v)$$

shows that

$$|\langle DF(u)z, u\rangle| \leq |vz_2 h'(w)| \leq M|u|\,|z|$$

in case (b) and

$$|\langle DF(u)z, u\rangle| \leq |vz_2 h'(w)| + |wz_1 g'(v)| \leq 2M|u|\,|z|$$

in case (c). Since (6.3) holds, $M|u|\,|z| \leq MC|F(u)||z|$, showing that (6.5) is satisfied in both cases. $\square$

Example 7.4. We (briefly) return to the second order problem discussed in Example 1.3. Assume first, with the notation of that example, that $G = G(v)$ is independent of $t$ and $w$. A more or less routine verification shows that the hypotheses of Theorem 7.1 are satisfied, with $W = X_2 = \{0\} \times \mathbb{R}^M$, if the following conditions hold:

$$G(v) = 0 \Leftrightarrow v = 0, \tag{7.7}$$

$$\varliminf_{|v|\to\infty} \frac{|G(v)|}{|v|} > 0, \tag{7.8}$$

$$\langle G(v), v\rangle \leq 0, \quad \text{for all } v \in \mathbb{R}^M, \tag{7.9}$$

$$0 \notin \sigma(DG(0)), \tag{7.10}$$

(7.11) there is a constant $\omega \geq 0$ such that

$$\langle DG(v)w, w\rangle \leq \omega|w|^2, \quad \text{for all } v, w \in \mathbb{R}^M.$$

These assumptions are exactly those of Theorem 8.5 of [12]. (That (7.10) implies $\sigma_0(DF^\infty(0)) = \emptyset$ is shown in [12, Lemma 7.1]. It should also be pointed out that if $\omega < 1$, then $b = \omega$ works in (6.2). If $\omega \geq 1$, a preliminary rescaling of the $t$ variable is needed to reduce the problem to the case when $\omega < 1$.) This is to say that Theorem 7.1 yields a generalization of Theorem 8.5 of [12] when $G = G(t, v, w)$ in Example 1.3.

## 8. Exponentially decaying right-hand sides

In this section, we discuss the properties of solutions $u$ of $\dot{u} + F(\,\cdot\,, u) = f$ for the problem when the right-hand side $f$ has exponential decay. The main question is whether $u$ inherits the exponential decay of $f$. For $1 \leq p \leq \infty$, we introduce the spaces

$$\begin{aligned} L^p_{\exp} &= \{f \in L^p : e^{\mu t} f \in L^p \text{ for some } \mu > 0\}, \\ W^{1,p}_{\exp} &= \{f \in W^{1,p} : e^{\mu t} f \in W^{1,p} \text{ for some } \mu > 0\}. \end{aligned}$$

In other words, $f \in L^p_{\exp}$ if and only if there are $\mu > 0$ and $g \in L^p$ such that $f = e^{-\mu t} g$ and $u \in W^{1,p}_{\exp}$ if and only if there are $\mu > 0$ and $v \in W^{1,p}$ such that $u = e^{-\mu t} v$. Evidently, $u \in W^{1,p}_{\exp}$ implies $u, \dot{u} \in L^p_{\exp}$. It is actually straightforward to check that the converse is true (just notice that if $u$ and $\dot{u}$ are in $L^p_{\exp}$, then $u$ and $\dot{u}$ can be written in the form $u = e^{-\mu t} v$ and $\dot{u} = e^{-\mu t} w$ for some $v, w \in L^p$ and the *same* $\mu > 0$).

If $F$ satisfies (3.1), (3.2) and (4.1) and if $1 \leq p < \infty$, we saw in Theorem 4.1 that the operator $\Phi: W^{1,p} \to L^p$ is Fredholm if and only if $\sigma_0(A^\infty) = \emptyset$. Since $\Phi$ and the operator $\dot{u} + \mathcal{F}(u) = \dot{u} + F(\,\cdot\,, u)$ differ only from the finite dimensional operator $Pu(0)$, they are simultaneously Fredholm. Hence, for $1 \leq p < \infty$,

$$u \in W^{1,p} \mapsto \dot{u} + F(\,\cdot\,, u) \in L^p \tag{8.1}$$

is Fredholm if and only if $\sigma_0(A^\infty) = \emptyset$. In turn, it follows readily from the general properties in [11] that, when $1 < p < \infty$ and (8.1) is a Fredholm operator, then

$$\text{if } u \in W^{1,p}, \dot{u} + F(\,\cdot\,, u) = f \in L^p_{\mathrm{exp}} \text{ then } u \in W^{1,p}_{\mathrm{exp}}. \tag{8.2}$$

While (8.2) follows from general properties of Fredholm operators in *reflexive* Banach spaces (whence the restriction $p > 1$) our first task will be to show that a stronger form is valid. The starting point is Theorem 2.1 of [11] in the linear case. Below, we only give the statement for the Fredholm operators of interest to us in this paper.

Theorem 8.1. *Let $B \in L^\infty([0,\infty), \mathcal{L}(\mathbb{R}^N))$ be given and suppose that, for some $1 < p < \infty$, the linear operator $D_B\colon u \in W^{1,p} \mapsto D_B u = \dot{u} + B(\,\cdot\,)u \in L^p$ is Fredholm (of any index). There is $\mu_0 > 0$ with the following property: If $f \in L^p$ is such that $e^{\mu t} f \in L^p$ for some $\mu > 0$ and if $u \in W^{1,p}$ solves $\dot{u} + B(\,\cdot\,)u = f$, then $u = e^{-\min(\mu,\mu_0)}v$ for some $v \in W^{1,p}$.*

Theorem 8.1 is the special case of Theorem 2.1 of [11] in which the semigroups $T(\mu)$ and $S(\mu)$ of that reference[4] are the multiplication by $e^{-\mu t}$ in the spaces $W^{1,p}$ and $L^p$, respectively. While not specifically pointed out in [11], it follows from the given proofs that the real number $\mu_0 = \mu_0(B)$ depends, roughly speaking, "continuously" upon $B$. More precisely, if Theorem 8.1 holds, choose any $\mu_0' < \mu_0$. If $A \in L^\infty([0,\infty), \mathcal{L}(\mathbb{R}^N))$ and $||B - A||_{L^\infty([0,\infty),\mathcal{L}(\mathbb{R}^N))}$ is small enough, then ($u \to \dot{u} + A(\,\cdot\,)u$ is Fredholm and) Theorem 8.1 holds with $B$ replaced by $A$ and $\mu_0$ replaced by $\mu_0'$. Thus, after changing $\mu_0'$ into $\mu_0$ for simplicity of notation, we have the following generalization of Theorem 8.1:

Theorem 8.2. *Let $B \in L^\infty([0,\infty), \mathcal{L}(\mathbb{R}^N))$ be given and suppose that, for some $1 < p < \infty$, the linear operator*

$$u \in W^{1,p} \mapsto \dot{u} + B(\,\cdot\,)u \in L^p \tag{8.3}$$

*is Fredholm (of any index). There are $\varepsilon > 0$ and $\mu_0 > 0$ with the following properties: Suppose that $A \in L^\infty([0,\infty), \mathcal{L}(\mathbb{R}^N))$ satisfies $||B - A||_{L^\infty([0,\infty),\mathcal{L}(\mathbb{R}^N))} < \varepsilon$ and that $f \in L^p$ is such that $e^{\mu t} f \in L^p$ for some $\mu > 0$. Then, every solution $u \in W^{1,p}$ of $\dot{u} + A(\,\cdot\,)u = f$ has the form $u = e^{-\min(\mu,\mu_0)}v$ for some $v \in W^{1,p}$.*

While Theorem 8.2 remains a special case of a completely general result about Fredholm operators in Banach spaces, the following corollary has no obvious analog in an abstract setting since its proof relies in various ways on the fact that $W^{1,p}$ and $L^p$ are function spaces:

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[4] In which $\mu$ is called $s$.

Corollary 8.3. *Let $A^\infty \in \mathcal{L}(\mathbb{R}^N)$ be such that $\sigma_0(A^\infty) = \emptyset$. Given $1 < p < \infty$, there is $\mu_0 > 0$ with the following property: Suppose that $A \in C^0([0,\infty), \mathcal{L}(\mathbb{R}^N))$ satisfies $\lim_{t\to\infty} A(t) = A^\infty$ and that $f \in L^p$ is such that $e^{\mu t} f \in L^p$ for some $\mu > 0$. Then, every solution $u \in W^{1,p}$ of $\dot{u} + A(\,\cdot\,)u = f$, has the form $u = e^{-\min(\mu,\mu_0)} v$ for some $v \in W^{1,p}$.*

Proof. Let $\varepsilon > 0$ and $\mu_0 > 0$ be given by Theorem 8.2 with $B(t) = A^\infty$ (so that the operator (8.3) is Fredholm by Theorem 2.3). Since $\lim_{t\to\infty} A(t) = A^\infty$, let $T > 0$ be such that $\sup_{t\geq T} ||A(t) - A^\infty|| < \varepsilon$, so that $\widetilde{A} \in L^\infty([0,\infty), \mathcal{L}(\mathbb{R}^N))$ defined by

$$\widetilde{A}(t) = \begin{cases} A^\infty & \text{if } t \in [0,T), \\ A(t) & \text{if } t \geq T, \end{cases} \tag{8.4}$$

satisfies $||A^\infty - \widetilde{A}||_{L^\infty([0,\infty,\mathcal{L}(\mathbb{R}^N))} < \varepsilon$. Now, rewrite $\dot{u} + A(\,\cdot\,)u = f$ as

$$\dot{u} + \widetilde{A}(\,\cdot\,)u = \widetilde{f}, \tag{8.5}$$

where $\widetilde{f} = f + (\widetilde{A}(\,\cdot\,) - A(\,\cdot\,))u$. Since $e^{\mu t} f \in L^p$ and since $(\widetilde{A}(\,\cdot\,) - A(\,\cdot\,))u \in L^p$ has compact support by (8.4), it follows that $e^{\mu t}\widetilde{f} \in L^p$ and hence, by Theorem 8.2 for (8.5), that $u = e^{-\min(\mu,\mu_0)} v$ for some $v \in W^{1,p}$. □

The value of Corollary 8.3 is of course that $\mu_0$ depends only upon $A^\infty$ and not upon $A(t)$ satisfying $\lim_{t\to\infty} A(t) = A^\infty$. This yields at once the desired strengthening of (8.2) mentioned above:

Corollary 8.4. *Suppose that $F$ satisfies the conditions* (3.1), (3.2) *and* (4.1) *and that $\sigma_0(A^\infty) = \emptyset$. Given $1 < p < \infty$, there is $\mu_0 > 0$ with the following property: If $f \in L^p$ is such that $e^{\mu t} f \in L^p$ for some $\mu > 0$, every solution $u \in W^{1,p}$ of $\dot{u} + F(\,\cdot\,, u) = f$ has the form $u = e^{-\min(\mu,\mu_0)} v$ for some $v \in W^{1,p}$.*

Proof. We choose a solution $u \in W^{1,p}$ of $\dot{u} + F(\,\cdot\,, u) = f$ and construct a linear equation that is satisfied by this $u$. Write $F(t, u(t)) = A(t)u(t)$ with $A(t) = \int_0^1 D_u F(t, su(t))\,ds$, where $F(t,0) = 0$ was used. Since $\lim_{t\to\infty} u(t) = 0$, it follows easily from the equicontinuity of $\{DF(t,\,\cdot\,)\}_{t\geq 0}$ at 0 (see (3.2)) and from (4.1) that $\lim_{t\to\infty} A(t) = A^\infty$. Thus, $u$ solves $\dot{u} + A(t)u = f$ and the conclusion follows from Corollary 8.3. □

Remark 8.5. More information can be obtained about $\mu_0$ in Corollaries 8.3 and 8.4. Indeed, when $B = A^\infty$ is constant in Theorem 8.1, then Theorem 2.3 shows that $\dot{u} + B(\,\cdot\,)u$ is not only Fredholm but also surjective. As a result, when $B = A^\infty$, Theorem 8.1 follows from Lemma 2.1 of [11]. The proof of that lemma (in the special case of interest here) then shows that $\mu_0$ can be chosen to be any positive number such that the operator $\dot{u} + (A^\infty - \mu I)u$ is surjective for $\mu \in [0, \mu_0]$. Another application of Theorem 2.2 then shows that, in Theorem 8.1, $\mu_0$ can be any positive number in the interval $(0,\gamma)$ where $\gamma = \inf \operatorname{Re}\sigma_+(A^\infty)$. From the above arguments involved in deriving Theorem 8.2 from Theorem 8.1, the same choice of $\mu_0$ works in that theorem, provided that $\varepsilon > 0$ is small enough. Therefore, any $\mu_0 \in (0,\gamma)$ can be chosen in Corollaries 8.3 and 8.4. All this is consistent with Corollary VII.3–7 of [5], where the linear case with $f = 0$ is considered. Note also that $\gamma = \infty$ if $\sigma_+(A^\infty) = \emptyset$, i.e. if $X_1^+ = \{0\}$. In this case, $\mu_0 > 0$ is arbitrary, so that $\min(\mu,\mu_0) = \mu$: The solutions $u \in W^{1,p}$ of $\dot u + F(\,\cdot\,,u) = f$ inherit *all* the exponential decay of $f$.

## 9. Existence of exponentially decaying solutions

The results of the previous section justify looking for exponentially decaying solutions of the problem (1.1)–(1.2) whenever the right-hand side $f$ has exponential decay. In this section, we show that for such right-hand sides, the existence question can be settled under hypotheses rather different from those of Theorem 7.1 and that the condition $\sigma_0(DF^\infty(0)) = \emptyset$ is no longer essential.

The first lemma shows that the existence of exponentially decaying solutions can be proved after replacing $W^{1,p}$ bounds by suitable exponentially weighted $L^\infty$ bounds.

Lemma 9.1. *Suppose that $F$ satisfies the conditions* (3.1) *and* (4.2) *and that* $\dim X_+^\infty = \dim X_1$ *where $X_+^\infty$ denotes the positive generalized eigenspace for* $DF^\infty(0)$. *Suppose also that, for some $1 < p < \infty$,*

$$\{u \in W^{1,p} : \dot u + F(\,\cdot\,,u) = 0 \text{ and } Pu(0) = 0\} = \{0\}, \tag{9.1}$$

$$\{u \in W^{1,p} : \dot u + D_uF(\,\cdot\,,0)u = 0 \text{ and } Pu(0) = 0\} = \{0\}. \tag{9.2}$$

*Let $f \in L^p_{\exp}$ and $\xi \in X_1$ be given and suppose that there exist $\mu > 0$ and $R > 0$ such that*

$$\sup_{t\ge 0} e^{\mu t}|u(t)| \le R \quad \text{for all } u \in W^{1,p} \text{ with } (\dot u + F(\,\cdot\,,u), Pu(0)) = (sf, s\xi) \text{ for some } s \in [0,1]. \tag{9.3}$$

*Then, the problem* (1.1)–(1.2) *has at least one solution in $W^{1,p}_{\exp}$.*

Proof. We look for a solution $u$ of the form $u(t) = e^{-\lambda t}v(t)$ with $\lambda > 0$ and $v \in W^{1,p}$. Obviously, (1.1)–(1.2) is equivalent to the system

$$\begin{cases} \dot v + G(\,\cdot\,,v) = e^{\lambda t}f, \\ Pv(0) = \xi, \end{cases} \tag{9.4}$$

where

$$G(t,v) = e^{\lambda t}F(t,e^{-\lambda t}v) - \lambda v. \tag{9.5}$$

We prove the existence of a solution $v \in W^{1,p}$ of (9.4) using Theorem 5.2. The function $G$ above satisfies (3.1) and (4.2) with $G^\infty(v) = DF^\infty(0)v - \lambda v$, so $DG^\infty(0) = DF^\infty(0) - \lambda I$. Clearly, the condition

$$\sigma_0(DG^\infty(0)) = \emptyset \tag{9.6}$$

holds for all $\lambda > 0$ small enough (regardless of whether it also holds for $\lambda = 0$) and hence for some $\lambda \in (0, \mu)$ with $\mu$ from (9.3). In addition, since $f \in L^p_{\mathrm{exp}}$, it is not restrictive to assume that

$$e^{\lambda t} f \in L^p. \tag{9.7}$$

Noting that the equation $\dot{v} + G^\infty(v) = 0$ is linear with constant coefficients (since $G^\infty = DG^\infty(0)$ is linear), it follows from (9.6) (see Remark 2.2), that

$$\{v \in W^{1,p}(\mathbb{R}, \mathbb{R}^N) : \dot{v} + G^\infty(v) = 0\} = \{0\}.$$

Furthermore, if $v \in W^{1,p}, \dot{v} + G(\,\cdot\,, v) = 0$ and $Pv(0) = 0$, then $u(t) = e^{-\lambda t} v(t)$ is in $W^{1,p}$, $\dot{u} + F(\,\cdot\,, u) = 0$ and $Pu(0) = 0$, so $u = 0$ by (9.1). Thus $v = 0$.

Likewise, let $v \in W^{1,p}$ be such that $\dot{v} + D_v G(\,\cdot\,, 0)v = 0$, i.e. $\dot{v} + D_u F(\,\cdot\,, 0)v - \lambda v = 0$, and $Pv(0) = 0$. Then, $u(t) = e^{-\lambda t} v(t)$ is in $W^{1,p}$, $\dot{u} + D_u F(\,\cdot\,, 0)u = 0$ and $Pu(0) = 0$. Thus $u = 0$ by (9.2), so $v = 0$.

To apply Theorem 5.2, it remains to obtain *a priori* bounds in $W^{1,p}$ for the solutions $v \in W^{1,p}$ of

$$(\dot{v} + G(\,\cdot\,, v), Pv(0)) = (se^{\lambda t} f, s\xi). \tag{9.8}$$

For any such solution $v$, the function $u(t) = e^{-\lambda t} v(t)$ is in $W^{1,p}$ and solves $(\dot{u} + F(\,\cdot\,, u), Pu(0)) = (sf, s\xi)$. Therefore, $\sup_{t \ge 0} e^{\mu t} |u(t)| \le R$ by (9.3), which means that $|v(t)| \le R\, e^{-(\mu - \lambda)t}$. As a result,

$$|v|_{0,\infty} \le R \quad \text{and} \quad |v|_{0,p} \le \frac{R}{\{p(\mu - \lambda)\}^{1/p}}. \tag{9.9}$$

But then, by (9.7) and (9.8),

$$|\dot{v}|_{0,p} = |G(\,\cdot\,, v) - se^{\lambda t} f|_{0,p} \le |G(\,\cdot\,, v)|_{0,p} + |se^{\lambda t} f|_{0,p}.$$

Since

$$G(t, v(t)) = \bigg(\int_0^1 D_u F(t, \tau e^{-\lambda t} v(t))\, d\tau - \lambda I\bigg) v(t)$$

by (9.5) and $\{D_u F(t, \cdot)\}_{t \ge 0}$ is equibounded on compact subsets of $\mathbb{R}^N$ (see (3.4)), it follows from (9.9) that

$$|G(t, v(t))| \le C(R) |v(t)|$$

and hence

$$|G(\,\cdot\,, v)|_{0,p} \le C(R) |v|_{0,p}$$

for some constant $C(R) > 0$. Thus,

$$|\dot{v}|_{0,p} \leq \frac{RC(R)}{\{p(\mu - \lambda)\}^{1/p}} + |e^{\lambda t} f|_{0,p},$$

which, together with (9.9), gives the desired bound for $||v||_{1,p}$. □

Our next goal is to complement Lemma 9.1 by giving a sufficient criterion for the validity of (9.1) and (9.3). To do this, we will make use of the following elementary lemma.

LEMMA 9.2. *If $f \in L^p$ with $1 < p < \infty$ and $a > 0$, the function*

$$\widetilde{f}(t) = \int_0^t e^{a(\tau - t)} f(\tau)\, d\tau$$

*is continuous and bounded and*

$$|\widetilde{f}|_{0,\infty} \leq \left(\frac{p-1}{pa}\right)^{(p-1)/p} |f|_{0,p}.$$

PROOF. That $\widetilde{f}$ is continuous is obvious. The estimate for $|\widetilde{f}|_{0,\infty}$ follows from the Hölder inequality. □

Recall that $P$ is the projection on $X_1$ relative to the splitting $\mathbb{R}^N = X_1 \oplus X_2$, so that $I - P$ is the projection onto $X_2$. From now on, we set $Q = I - P$ for simplicity of notation.

LEMMA 9.3. *Suppose that $F$ satisfies* (3.1) *and* (4.1). *Suppose also that there exist constants $\delta > 0$, $\rho > 0$ and $M > 0$ such that, for all $t \geq 0$ and all $u \in \mathbb{R}^N$ with $|Pu| \leq \rho$,*

$$\langle Pu, PF(t,u)\rangle \geq \delta |Pu|^2, \tag{9.10}$$

$$\langle Qu, QF(t,u)\rangle \leq M|Pu||Qu|. \tag{9.11}$$

*Then, for any $1 \leq p < \infty$,*

$$\{u \in W^{1,p} : \dot{u} + F(\,\cdot\,,u) = 0 \text{ and } Pu(0) = 0\} = \{0\}, \tag{9.12}$$

$$\{u \in W^{1,p} : \dot{u} + D_uF(\,\cdot\,,0)u = 0 \text{ and } Pu(0) = 0\} = \{0\}. \tag{9.13}$$

*Furthermore, if $1 < p < \infty$, the following property holds for every $\mu \in (0,\delta)$: Given any $C \geq 0$, there is a constant $R = R(\rho, \mu, C) > 0$ such that*

$$|v|_{0,\infty} \leq R \text{ whenever } v \in W^{1,p} \text{ and } \dot{v} + e^{\mu t}F(t, e^{-\mu t}v) - \mu v = g$$
$$\text{with } |Qg|_{0,p} \leq C \text{ and } |Pv(0)| + \left(\frac{p-1}{p(\delta - \mu)}\right)^{(p-1)/p} |Pg|_{0,p} < \rho. \tag{9.14}$$

Proof. It follows from (9.11) that $\langle u, QF(t,u)\rangle \le 0$ for all $u \in X_2$ and all $t \ge 0$. In particular,

$$\{u \in X_2 \text{ and } F(t,u) \in X_2\} \Rightarrow \langle u, F(t,u)\rangle \le 0. \tag{9.15}$$

Suppose now that $u \in W^{1,p}$, $Pu(0) = 0$ and $\dot u + F(\,\cdot\,, u) = 0$, so that $u$ is $C^1$. By (9.10),

$$\frac{d|Pu|^2}{dt}(t) = -2\langle Pu(t), PF(t,u(t))\rangle \le 0,$$

for all $t > 0$ such that $|Pu(t)| \le \rho$. Clearly, since $Pu(0) = 0$, this implies that $Pu(t) = 0$ for all $t \ge 0$ and hence that $u(t) \in X_2$ for all $t \ge 0$. Then, $F(t,u(t)) = -\dot u(t) \in X_2$ for all. $t > 0$ so that, from (9.15), $\langle u(t), F(t,u(t))\rangle \le 0$ for all $t > 0$. Since

$$\frac{d|u|^2}{dt}(t) = -2\langle u(t), F(t,u(t))\rangle,$$

we infer that $|u|$ is nondecreasing on $[0,\infty)$ and since $\lim_{t\to\infty} u(t) = 0$ hence $u = 0$. This proves (9.12).

Next, (9.13) follows from (9.12) with $F(t,u)$ replaced by $D_uF(t,0)u$ since the hypotheses of the lemma are also satisfied by this function. This is obvious for (3.1) and (4.1) and the corresponding variants of (9.10) and (9.11) are obtained by replacing $u$ by $su$ in (9.10) and (9.11), dividing by $s^2$ and letting $s \to 0$.

The proof of (9.14) proceeds in two steps, corresponding to the boundedness of $|Pv(t)|$ and $|Qv(t)|$, respectively, where $v \in W^{1,p}$ satisfies the conditions required in (9.14). To prove the boundedness of $|Pv(t)|$, we show that

$$|Pv(t)| < \rho \quad \text{for all } t \ge 0. \tag{9.16}$$

Indeed, if not, let $T \ge 0$ be such that $|Pv(T)| = \rho$ and $|Pv(t)| < \rho$ for $0 \le t < T$. Then, $T > 0$ since $|Pv(0)| < \rho$ by hypothesis. By (9.10), the relation

$$\frac{d|Pv|^2}{dt}(t) = -2\langle Pv(t), e^{\mu t}PF(t, e^{-\mu t}v(t)) - \mu Pv(t) - Pg(t)\rangle,$$

for a.e. $t > 0$, shows that

$$\frac{d|Pv|^2}{dt} \le -2(\delta-\mu)|Pv|^2 + |Pv|\,|Pg| \quad \text{a.e. on } (0,T)$$

and hence

$$\frac{d|Pv|}{dt} + (\delta-\mu)|Pv| \le |Pg| \quad \text{a.e. on } (0,T).$$

This yields

$$|Pv(t)| \le |Pv(0)| + \int_0^t e^{(\delta-\mu)(\tau-t)}|Pg(\tau)|\,d\tau$$

for all $0 \le t \le T$. In particular,

$$|Pv(T)| \le |Pv(0)| + \left(\frac{p-1}{p(\delta-\mu)}\right)^{(p-1)/p}|Pg|_{0,p}$$

by Lemma 9.2. Since

$$|Pv(0)| + \left(\frac{p-1}{p(\delta-\mu)}\right)^{(p-1)/p} |Pg|_{0,p} < \rho$$

by hypothesis, it follows that $|Pv(T)| < \rho$, in contradiction with $|Pv(T)| = \rho$. This proves (9.16).

Since $|Pe^{-\mu t}v(t)| \leq |Pv(t)|$, it follows from (9.16) and (9.11) that

$$\begin{aligned}\frac{d|Qv|^2}{dt}(t) = & -2e^{2\mu t}\langle Qe^{-\mu t}v(t), QF(t, e^{-\mu t}v(t))\rangle \\ & + 2\mu|Qv(t)|^2 + 2\langle Qv(t), Qg(t)\rangle \\ \geq & -2M|Pv(t)||Qv(t)| + 2\mu|Qv(t)|^2 - 2|Qv(t)||Qg(t)| \\ \geq & -2M\rho|Qv(t)| + 2\mu|Qv(t)|^2 - 2|Qv(t)||Qg(t)|,\end{aligned}$$

a.e. in $(0,\infty)$. It follows that

$$\frac{d|Qv|}{dt}(t) - \mu|Qv(t)| \geq -M\rho - |Qg(t)| \quad \text{a.e. in } (0,\infty). \tag{9.17}$$

Indeed, it is clear from the above that (9.17) holds a.e. on the complement of the zero set $E$ of $Qv$. On the other hand, since $|Qv| \in W^{1,p}$, it is well-known that $d|Qv|/dt = 0$ a.e. on $E$ (see [2, p. 195]), so that (9.17) also holds a.e. on $E$.

Given $T \geq 0$, multiply both sides of (9.17) by $e^{-\mu(t-T)}$ and integrate to find

$$\begin{aligned}|Qv(t)| &\geq e^{\mu(t-T)}\left(|Qv(T)| - \frac{M\rho}{\mu} - \int_T^t e^{-\mu(\tau-T)}|Qg(\tau)|\,d\tau\right) \\ &\geq e^{\mu(t-T)}\left(|Qv(T)| - \frac{M\rho}{\mu} - \left(\frac{p-1}{p\mu}\right)^{(p-1)/p}|Qg|_{0,p}\right),\end{aligned} \tag{9.18}$$

for all $t \geq T$. This implies

$$|Qv(T)| \leq \frac{M\rho}{\mu} + \left(\frac{p-1}{p\mu}\right)^{(p-1)/p}|Qg|_{0,p},$$

for otherwise $\lim_{t\to\infty}|Qv(t)| = \infty$ by (9.18), in contradiction with $v \in W^{1,p}$. Since $T \geq 0$ is arbitrary, it follows that

$$|Qv|_{0,\infty} \leq \frac{M\rho}{\mu} + \left(\frac{p-1}{p\mu}\right)^{(p-1)/p}|Qg|_{0,p}$$

and hence

$$|Qv|_{0,\infty} \leq \frac{M\rho}{\mu} + C\left(\frac{p-1}{p\mu}\right)^{(p-1)/p}.$$

Together with (9.16), this yields $|v|_{0,\infty} \leq R$ with

$$R = \rho\left(1 + \frac{M}{\mu}\right) + C\left(\frac{p-1}{p\mu}\right)^{(p-1)/p}. \qquad \square$$

Theorem 9.4. *Suppose that $F$ satisfies the conditions* (3.1) *and* (4.2) *and that* $\dim X_+^\infty = \dim X_1$, *where* $X_+^\infty$ *denotes the positive generalized eigenspace for* $DF^\infty(0)$. *Suppose also that there are constants* $\delta > 0$, $\rho > 0$ *and* $M > 0$ *such that, for all* $t \geq 0$ *and all* $u \in \mathbb{R}^N$ *with* $|Pu| \leq \rho$,

$$\langle Pu, PF(t,u)\rangle \geq \delta |Pu|^2 \quad \text{and} \quad \langle Qu, QF(t,u)\rangle \leq M|Pu||Qu|. \tag{9.19}$$

*Given* $1 < p < \infty$, *if* $f \in L^p_{\exp}$ *and* $\xi \in X_1$ *are such that*

$$|\xi| + \left(\frac{p-1}{p(\delta-\mu)}\right)^{(p-1)/p} |Pe^{\mu t} f|_{0,p} < \rho, \tag{9.20}$$

*for some* $\mu > 0$, *the problem* (1.1)–(1.2) *has at least one solution* $u \in W^{1,p}_{\exp}$.

Proof. First, we prove the theorem under the additional assumption that

$$\sigma_0(DF^\infty(0)) = \emptyset.$$

Since (9.20) is unaffected by decreasing $\mu > 0$ and since $f \in L^p_{\exp}$, it is not restrictive to assume that $e^{\mu t} f \in L^p$ and that $\mu \in (0, \min(\delta,\gamma))$, where $\gamma = \inf \operatorname{Re} \sigma_+(DF^\infty(0))$. If so, it follows from Corollary 8.4 and Remark 8.5 that every $u \in W^{1,p}$ such that $\dot u + F(t,u) = sf$ and $Pu(0) = s\xi$ for some $s \in [0,1]$ has the form $u = e^{-\mu t} v$ with $v \in W^{1,p}$. Furthermore,

$$\dot v(t) + e^{\mu t} F(t, e^{-\mu t} v(t)) - \mu v(t) = s e^{\mu t} f \in L^p$$

and, with $\theta_p = ((p-1)/(p(\delta-\mu)))^{(p-1)/p}$ for simplicity,

$$|Pv(0)| + \theta_p |e^{\mu t} Psf|_{0,p} = s\{|\xi| + \theta_p |e^{\mu t} Pf|_{0,p}\} \leq \rho.$$

Since also $|Qse^{\mu t} f|_{0,p} \leq C = |Qe^{\mu t} f|_{0,p}$, Lemma 9.3 ensures that

$$\{u \in W^{1,p} : \dot u + F(\,\cdot\,, u) = 0 \text{ and } Pu(0) = 0\} = \{0\}$$

and there is a constant $R > 0$ independent of $s$ and of $v$ (hence of $u$) such that $|v|_{0,\infty} = \sup_{t\geq 0} |e^{\mu t} u(t)| \leq R$. Thus, the existence of solutions $u \in W^{1,p}_{\exp}$ of the problem (1.1)–(1.2) follows from Lemma 9.1.

Suppose now that (9.21) does not hold. We look for $u$ of the form $u(t) = e^{-\lambda t} w$ with $w \in W^{1,p}$ and $\lambda > 0$ small enough. For such a $\lambda$, $w$ must solve the problem

$$\begin{cases} \dot w + e^{\lambda t} F(t, e^{-\lambda t} w) - \lambda w = g, \\ Pw(0) = \xi, \end{cases} \tag{9.22}$$

where $g = e^{\lambda t} f \in L^p_{\exp}$ if $\lambda > 0$ is small enough.

We claim that the first part of the proof yields the existence of solutions $w \in W^{1,p}$ (even $W^{1,p}_{\exp}$) for (9.22), and therefore solutions $u \in W^{1,p}_{\exp}$ for (1.1)–(1.2). This amounts to showing that the hypotheses of the theorem *and* (9.21) are satisfied when $F(t,u)$ is replaced by $e^{\lambda t} F(t, e^{-\lambda t} u) - \lambda u$ and $f$ is replaced by

$e^{\lambda t}f$ (and $\lambda > 0$ is small enough). Since $DF^\infty(0)$ is changed into $DF^\infty(0) - \lambda I$, it is clear that the space $X_+^\infty$ is unchanged and that (9.21) holds for small $\lambda > 0$. Next, since $|Pu| \leq \rho$ implies $|Pe^{-\lambda t}u| \leq \rho$, it is readily checked that (9.19) holds for the modified function $F$ with $\delta$ replaced by $\delta - \lambda$ if $\lambda < \delta$. Lastly, the condition (9. 20) is unchanged when $\mu$ and $\delta$ are replaced by $\mu - \lambda$ ($> 0$ if $\lambda < \mu$) and $\delta - \lambda$, respectively, and $f$ is replaced by $e^{\lambda t}f$. This completes the proof. □

Theorem 9.4 is valid even when $\sigma_0(DF^\infty(0)) \neq \emptyset$, so that, by Theorem 4.1, the operator $\Phi$ of Theorem 5.1 is not be Fredholm.

Example 9.5. In Example 1.1 (classical initial value problem), $P = I$ and $Q = 0$ and the hypotheses of Theorem 9.1 reduce to assuming, in addition to (3.1) and (4.2), that $\langle u, F(t,u)\rangle \geq \delta |u|^2$ for all $t \geq 0$ and all $u \in \mathbb{R}^N$ with $|u| \leq \rho$. Note that this implies $\langle u, F^\infty(u)\rangle \geq \delta |u|^2$ if $|u| \leq \rho$ and hence $\langle u, DF^\infty(0)u\rangle \geq \delta |u|^2$ for all $u \in \mathbb{R}^N$, so that all the eigenvalues of $DF^\infty(0)$ have strictly positive real part. Thus, $\dim X_+^\infty = \dim X_1 = N$. Condition (9.20) is that

$$|\xi| + \left(\frac{p-1}{p(\delta - \mu)}\right)^{(p-1)/p} |e^{\mu t} f|_{0,p} \leq \rho.$$

Example 9.6. In Example 1.2, $P = 0$ and $Q = I$ and the hypotheses of Theorem 9.4 reduce to assuming, in addition to (3.1) and (4.2), that $\langle u, F(t,u)\rangle \leq 0$ for all $t \geq 0$ and all $u \in \mathbb{R}^N$. Note that this implies $\langle u, F^\infty(u)\rangle \leq 0$ and hence $\langle u, DF^\infty(0)u\rangle \leq 0$, so that all the eigenvalues of $DF^\infty(0)$ have nonpositive real part. Thus, $\dim X_+^\infty = \dim X_1 = 0$. Condition (9.20) is vacuous.

In fact, the solution $u \in W^{1,p}_{\exp}$ obtained by Theorem 9.4 is *unique*. If $u_1$ and $u_2$ are two such solutions, then $u_i(t) = e^{-\mu t} v_i$ with $v_i \in W^{1,p}$ for $i = 1, 2$ and some $\mu > 0$, and $\dot v_i + G(\,\cdot\,, v_i) = e^{\mu t} f$, $i = 1, 2$, where $G(t,u) = e^{\mu t}F(t, e^{-\mu t}u) - \mu u$. Since $\mu > 0$ may be chosen so that $e^{\mu t} f \in L^p$, it suffices to show that the problem $\dot u + G(\,\cdot\,, u) = g$ has at most one solution in $W^{1,p}$ irrespective of $\mu > 0$.

Indeed, notice that $\langle u, G(t,u)\rangle \leq -\mu |u|^2$ for all $t \geq 0$ and all $u \in \mathbb{R}^N$, whence $\langle v, D_uG(t,0)v\rangle \leq -\mu |v|^2$ for all $t \geq 0$ and all $v \in \mathbb{R}^N$. In turn, since $\{D_uG(t,u)\}_{t\geq 0}$ is equicontinuous at 0 by (3.2) (use Theorem 4.4(a)), this implies that

$$\langle v, D_uG(t,u)v\rangle \leq -\frac{\mu}{2}|v|^2$$

for all $t \geq 0$, all $u$ in some convex neighbourhood $\mathcal{U}$ of 0 and all $v \in \mathbb{R}^N$. Thus, $-G(t,\,\cdot\,)$ is monotone in $\mathcal{U}$ for all $t \geq 0$, which, as pointed out in Example 1.2, ensures the desired uniqueness property.

Example 9.7. This example shows that Theorem 9.4 is nontrivial even for linear problems with constant coefficients (although there are of course more direct and better ways to tackle this special case).

Let $A \in \mathcal{L}(\mathbb{R}^N)$ be diagonalizable, with complex eigenvalues $\lambda_1, \ldots, \lambda_N$, and let $S \in \mathcal{L}(\mathbb{C}^N)$ be invertible and such that $A = S^{-1}DS$ where $D$ is the multiplication by $\lambda_j$ along the $j$th coordinate axis. If $F(t,u) = Au$, the hypotheses of Theorem 9.4 are satisfied with $X_1 = X_+$ and $X_2 = X_0 \oplus X_-$ (notation of Section 2), provided that $\mathbb{R}^N$ is equipped with the inner product $\langle u, v\rangle = \operatorname{Re} Su \cdot Sv$, where the dot denotes the euclidian inner product of $\mathbb{C}^N$.

First, note that $\widetilde{P} = SPS^{-1}$ is the *orthogonal* projection (for the euclidian inner product) onto the positive eigenspace of $D$ and that $\widetilde{Q} = I - \widetilde{P} = SQS^{-1}$ projects onto the sum of the eigenspaces of $D$ corresponding to the eigenvalues of $A$ with nonpositive real part. Therefore,

$$\begin{aligned}\langle PAu, Pu\rangle &= \operatorname{Re} \widetilde{P}DSu \cdot \widetilde{P}Su = \operatorname{Re} D\widetilde{P}Su \cdot \widetilde{P}Su \\ &\geq \widetilde{\delta}(\widetilde{P}Su \cdot \widetilde{P}Su) = \widetilde{\delta}(SPu \cdot SPu)\end{aligned}$$

where $\widetilde{\delta} = \min_{\operatorname{Re}\lambda_j > 0} \operatorname{Re}\lambda_j$ and $\widetilde{\delta}(SPu \cdot SPu) \geq \delta|Pu|^2$ for some $\delta > 0$ since all the norms on $\mathbb{R}^N$ are equivalent and $S$ is invertible. Thus, $\langle PAu, Pu\rangle \geq \delta|Pu|^2$ for all $u \in \mathbb{R}^N$. (In particular, $\rho > 0$ can be chosen arbitrarily and hence condition (9.20) is vacuous.) Also,

$$\langle QAu, Qu\rangle = \operatorname{Re} \widetilde{Q}DSu \cdot \widetilde{Q}Su = \operatorname{Re} D\widetilde{Q}Su \cdot \widetilde{Q}Su \leq 0$$

for all $u \in \mathbb{R}^N$. This shows that (9.19) holds. The verification of the other hypotheses is trivial.

From Theorem 9.4, if $1 < p < \infty$, the linear problem

$$\begin{cases} \dot{u} + Au = f, \\ Pu = \xi, \end{cases}$$

has a solution $u \in W^{1,p}_{\exp}$ for every $f \in L^p_{\exp}$ and every $\xi \in X_1 = X_+$. If $\sigma_0(A) \neq \emptyset$, this does not follow from Theorem 8.1.

Example 9.8. Consider the system

$$\begin{cases} \dot{v} + av + R_1(v,w) = f_1, \\ \dot{w} + cv + dw + R_2(v,w) = f_2, \\ v(0) = \xi. \end{cases}$$

It is readily checked that the hypotheses of Theorem 9.4 are satisfied with $X_1 = \mathbb{R} \times \{0\}$, $X_2 = \{0\} \times \mathbb{R}$ and $M = |c|$ if $a, c, d \in \mathbb{R}$ satisfy $d \leq 0 < a$, $R_i$ is of class $C^1$, $R_i(0) = 0$, $\nabla R_i(0) = 0$, $i = 1, 2$, and there is $\rho > 0$ such that $vR_1(v,w) \geq 0$ and $wR_2(v,w) \leq 0$ for all $v \in [-\rho, \rho]$ and $w \in \mathbb{R}$. The condition (9.20) on the data amounts to

$$|\xi| + \left(\frac{p-1}{p(\delta - \mu)}\right)^{(p-1)/p} |e^{\mu t} f_1|_{0,p} \leq \rho$$

and does not limit $|f_2|_{0,p}$.

**Acknowledgements.** The authors are grateful to a referee for his careful reading of the paper and helpful remarks.

## References


[1] H. Amann, *Ordinary Differential Equations*, Studies in Mathematics, de Gruyter, Berlin, 1990.

[2] H. Brézis, *Analyse Fonctionnelle*, Masson, Paris, 1983.

[3] R. Caccioppoli, *Sulle corrispondenze funzionali inverse diramate: teoria generale e applicazioni ad alcune equazioni funzionali nonlineari e al problema di Plateau* I, II, Rend. Accad. Naz. Lincei **24** (1936), 258–263, 416–421; Opere Scelte, vol. 2, Edizioni Cremonese, Roma, 1963, pp. 157–177.

[4] M. Cecchi, M. Furi, M. Marini and M. P. Pera, *Fredholm linear operators associated with ordinary differential equations on noncompact intervals*, Electron. J. Differential Equations **44** (1999), 16 pp., (electronic).

[5] P.-O. Hsieh and Y. Sibuya, *Basic Theory of Ordinary Differential Equations*, Springer–Verlag, Berlin, 1999.

[6] J. L. Massera and J. J. Shäffer, *Linear Differential Equations and Function spaces*, Academic Press, New York, 1966.

[7] J. M. Morris, *Doctoral dissertation*, University of Pittsburgh, in preparation.

[8] K. J. Palmer, *Exponential dichotomies and transversal homoclinic points*, J. Differential Equations **55** (1984), 225–256.

[9] ———, *Exponential dichotomies and Fredholm operators*, Proc. Amer. Math. Soc. **104** (1988), 149–156.

[10] J. Pejsachowicz and P. J. Rabier, *Degree theory for $C^1$ Fredholm mappings of index* 0, J. Anal. Math. **76** (1998), 289–319.

[11] P. J. Rabier, *Semigroups, Fredholm operators and the asymptotic and boundary behavior of solutions of* PDEs, J. Differential Equations **193** (2003), 460–480.

[12] P. J. Rabier and C. A. Stuart, *A Sobolev space approach to boundary value problems on the half-line*, Comm. Contemp. Math. (to appear).


*Manuscript received January 30, 2004*

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