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LOCAL FIXED POINT THEORY INVOLVING THREE OPERATORS IN BANACH ALGEBRAS

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Abstract. The present paper studies the local versions of a fixed point theorem of Dhage (1987) in Banach algebras. An application of the newly developed fixed point theorem is also discussed for proving the existence results to a nonlinear functional integral equation of mixed type.

1. Introduction

Fixed point theory constitutes an important and the core part of the subject of nonlinear functional analysis and is useful for proving the existence theorems for nonlinear differential and integral equations. The local fixed point theory is useful for proving the existence of the local solutions of the problems governed by nonlinear differential or integral equations. In the present paper we shall obtain the local versions of the known fixed point theorem of Dhage [9] and discuss some of their applications to functional integral equations.

Throughout this paper, let X denote Banach algebra with a norm $\|\cdot\|$. Let $a \in X$ and let r be a positive real number. Then by $\mathcal{B}_r(a)$ and $\overline{\mathcal{B}}_r(a)$ we denote respectively an open and closed balls in X centered at the point $a \in X$ of radius r. A mapping $A\colon X \to X$ is called *$\mathcal{D}$-Lipschitzian* if there exists a continuous nondecreasing function $\phi_A\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\|Ax - Ay\| \le \phi_A(\|x - y\|) \tag{1.1}$$

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for all $x, y \in X$ with $\phi_A(0) = 0$. Sometimes we call the function ϕ to be a $\mathcal{D}$-*function* of the mapping A on X. In the special case when $\phi_A(r) = \alpha r$, $\alpha > 0$, A is called a *Lipschitzian* with a Lipschitz constant α. In particular if $\alpha < 1, A$ is called a *contraction* with a contraction constant α. Further if $\phi_A(r) < r$ for $r > 0$, then A is called a nonlinear contraction on X.

A local version of the well-known Banach fixed point theorem is

THEOREM 1.1 ([11, p. 10–11]). *Let $T: X \to X$ be a contraction with contraction constant α. If T satisfies*

$$\|a - Ta\| \leq (1 - \alpha)r \tag{1.2}$$

for some $a \in X$ and $r > 0$, then T has a unique fixed point in $\overline{\mathcal{B}}_r(a)$.

REMARK 1.2. In Theorem 1.1, we could replace $T: \overline{\mathcal{B}}_r(a) \to X$.

A mapping $T: X \to X$ is called *compact* if $\overline{T(X)}$ is compact subset of X and *totally bounded* if $T(S)$ is a totally bounded subset of X for any bounded subset of X. Again a map $T: X \to X$ is called *completely continuous* if it is continuous and totally bounded on X. It is clear that every compact operator is totally bounded, but the converse may not be true, however the two notions are equivalent on bounded subsets of X.

A local version of the famous Schauder fixed point theorem may be given as follows.

THEOREM 1.3. *Let $a \in X$ and r a positive real number. If $T: \overline{\mathcal{B}}_r(a) \to \overline{\mathcal{B}}_r(a)$ is a compact and continuous operator, then T has a fixed point.*

Theorems 1.1 and 1.3 have been extensively used in the literature for proving the existence of the solutions of nonlinear differential and integral equations in the neighborhood of a point in the function spaces in question. The next important combination of a metric and a topological fixed point theorem involving three operators in a Banach algebra in its improved form than that appeared in Dhage [1] is

THEOREM 1.4 (Dhage, [2]). *Let S be a closed convex and bounded subset of a Banach algebra X and let $A, B, C: S \to X$ be three operators such that*

(a) *A and C are Lipschitzian with Lipschitz constants α and β, respectively,*
(b) *B is completely continuous, and*
(c) *$AxBx + Cx \in S$ for each $x \in S$.*

Then the operator equation $AxBx + Cx = x$ has a solution whenever $\alpha M + \beta < 1$, where $M = \|B(S\| = \sup\{\|Bx\| : x \in S\}$.

Like Krasnosel'skiĭ [12], Theorem 1.4 and its variants are also useful in the study of nonlinear integral equations of mixed type which arise in the inversion of the perturbed differential equations in Banach algebras. See Dhage and

O'Regan [10] and the references therein. The proof of Theorem 1.4 involves the use of the measures of noncompactness and condensing mappings in Banach spaces, but the following re-formulation of Theorem 1.4 is recently proved by the present author without using the notions of measures of noncompactness.

THEOREM 1.5 (Dhage, [9]). *Let S be a closed convex and bounded subset of a Banach algebra X and let $A, C\colon X \to X$ and $B\colon S \to X$ be three operators such that*

(a) *A and C are $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-functions ϕ_A and ϕ_C respectively,*
(b) *B is completely continuous, and*
(c) *if $x = AxBy + Cx$ then $x \in S$ for all $y \in S$.*

Then the operator equation $AxBx + Cx = x$ has a solution whenever $M\phi_A(r) + \phi_C(r) < r$, where $M = \|B(S\| = \sup\{\|Bx\| : x \in S\}$.

In this paper we shall prove another formulation of Theorem 1.5 by modifying hypothesis (c) in a different way which ultimately yields a local version of Theorem 1.5.

2. Local fixed point theory

THEOREM 2.1. *Let $a \in X$ and r a positive real number. Let $A, C\colon X \to X$ and $B\colon \overline{\mathcal{B}}_r(a) \to X$ be three operators such that*

(a) *A and C are Lipschitzian with Lipschitz constants α and β respectively,*
(b) *B is completely continuous with $M = \|B(\overline{\mathcal{B}}_r(a))\|$, and*
(c) *$\|a-(AaBy+Ca)\| \leq [1-(\alpha M+\beta)]r$ for each $y \in \overline{\mathcal{B}}_r(a)$ with $\alpha M+\beta<1$.*

Then the operator equation $AxBx + Cx = x$ has a solution in $\overline{\mathcal{B}}_r(a)$.

PROOF. Let $y \in \overline{\mathcal{B}}_r(a)$ be fixed and define a mapping A_y on $\overline{\mathcal{B}}_r(a)$ by

$$A_y(x) = AxBy + Cx.$$

We show that A_y is a contraction on $\overline{\mathcal{B}}_r(a)$. Let $x_1, x_2 \in \overline{\mathcal{B}}_r(a)$ be arbitrary. Then we have

$$\begin{aligned}\|A_y(x_1) - A_y(x_2)\| &\leq \|Ax_1 - Ax_2\|\,\|By\| + \|Cx_1 - Cx_2\| \\ &\leq (\alpha M + \beta)\|x_1 - x_2\|\end{aligned}$$

where $0 < \alpha M + \beta < 1$, and so A_y is a contraction on $\overline{\mathcal{B}}_r(a)$. By hypothesis (c),

$$\|a - A_y(a)\| = \|a - (AaBy + Ca)\| \leq [1 - (\alpha M + \beta)]r.$$

Hence by Theorem 1.1 there is a unique point x^* in $\overline{\mathcal{B}}_r(a)$ such that

$$x^* = Ax^*By + Cx^*.$$

Define an operator $N\colon \overline{\mathcal{B}}_r(a)) \to \overline{\mathcal{B}}_r(a)$ by

$$Ny = z \tag{2.1}$$

where z is a unique solution of the operator equation $z = AzBy + Cz$ in $\overline{\mathcal{B}}_r(a)$. We show that N is a continuous operator on $\overline{\mathcal{B}}_r(a))$. Let $\{y_n\}$ be a sequence in $\overline{\mathcal{B}}_r(a)$ converging to a point $y \in \overline{\mathcal{B}}_r(a)$. Then we have

$$\begin{aligned}
\|Ny_n - Ny\| &\le \|Az_nBy_n - AzBy\| + \|Cz_n - Cz\| \\
&\le \|Az_nBy_n - AzBy_n\| + \|AzBy_n - AzBy\| + \|Cz_n - Cz\| \\
&\le \|Az_n - Az\|\|By_n\| + \|Az\|\|By_n - By\| + \|Cz_n - Cz\| \\
&\le (\alpha M + \beta)\|z_n - z\| + \|Az\|\|By_n - By\| \\
&\le (\alpha M + \beta)\|Ny_n - Ny\| + \|Az\|\|By_n - By\|.
\end{aligned}$$

Taking limit supremum in the above inequality yields

$$\lim_{n\to\infty} \|Ny_n - Ny\| = 0.$$

This proves that N is a continuous operator on $\overline{\mathcal{B}}_r(a)$. Next we show that N is a compact operator on $\overline{\mathcal{B}}_r(a)$. Now for any $z \in \overline{\mathcal{B}}_r(a)$ we have

$$\|Az\| \le \|Aa\| + \|Az - Aa\| \le \|Aa\| + \alpha\|z - a\| \le c,$$

where $c = \|Aa\| + \alpha r$.

Let $\varepsilon > 0$ be given. Since B is compact operator, $B(\overline{\mathcal{B}}_r(a))$ is totally bounded. Then there is a set $Y = \{y_1, \dots, y_n\}$ of points in $\overline{\mathcal{B}}_r(a)$ such that

$$B(\overline{\mathcal{B}}_r(a)) \subset \bigcup_{i=1}^{k} \mathcal{B}_\delta(w_i),$$

where $w_i = B(y_i)$ and $\delta = ((1 - (\alpha M + \beta))/c)\varepsilon$. So for any $y \in \overline{\mathcal{B}}_r(a)$, we have $y_k \in Y$ such that

$$\|By - By_k\| < \left(\frac{1 - (\alpha M + \beta)}{c}\right)\varepsilon.$$

Therefore

$$\begin{aligned}
\|Ny_k - Ny\| &\le \|Az_kBy_k - AzBy\| + \|Cz_k - Cz\| \\
&\le \|Az_kBy_k - AzBy_k\| + \|AzBy_k - AzBy\| + \|Cz_k - Cz\| \\
&\le \|Az_k - Az\|\|By_k\| + \|Az\|\|By_k - By\| + \|Cz_k - Cz\| \\
&\le (\alpha M + \beta)\|z_k - z\| + \|Az\|\|By_k - By\| \\
&\le (\alpha M + \beta)\|Ny_k - Ny\| + \|Az\|\|By_k - By\| \\
&\le \frac{c}{1 - (\alpha M + \beta)}\,\|By - By_k\| < \varepsilon.
\end{aligned}$$

Since $y \in \overline{\mathcal{B}}_r(a)$ was arbitrary,

$$N(\overline{\mathcal{B}}_r(a)) \subset \bigcup_{i=1}^{n} \mathcal{B}_\varepsilon(w_i)$$

where $w_i = N(y_i)$. As a result $N(\overline{\mathcal{B}}_r(a))$ is a totally bounded set in X. Since N is continuous, $N(\overline{\mathcal{B}}_r(a))$ is a compact subset of X. Hence N is continuous and compact operator on $\overline{\mathcal{B}}_r(a)$ into itself. Now an application of Schauder fixed point theorem yields that N has a fixed point in $\overline{\mathcal{B}}_r(a)$ and consequently the operator equation $AxBx + Cx = x$ has a solution in $\overline{\mathcal{B}}_r(a)$. This completes the proof. □

Taking $a = 0$, the origin of X, in Theorem 2.1 we obtain the following result useful in applications.

Corollary 2.2. *Let r be a positive real number and let $A, C: X \to X$ and $B: \overline{\mathcal{B}}_r(0) \to X$ be three operators such that*

(a) *A and C are Lipschitzian with Lipschitz constants α and β respectively,*
(b) *B is completely continuous with $M = \|\overline{\mathcal{B}}_r(0)\|$ and*
(c) *$\|A0By + C0\| \le [1 - (\alpha M + \beta)]r$ for all $y \in \overline{\mathcal{B}}_r(0)$, where $\alpha M + \beta < 1$.*

Then the operator equation $AxBx + Cx = x$ has a solution in $\overline{\mathcal{B}}_r(a)$.

Taking $C \equiv 0$ in above Theorem 2.1 we obtain

Theorem 2.3 (Dhage, [5]). *Let $a \in X$ and r a positive real number. Let $A: X \to X$ and $B: \overline{\mathcal{B}}_r(a) \to X$ be two operators such that*

(a) *A is Lipschitzian with Lipschitz constant α,*
(b) *B is completely continuous with $M = \|B(\overline{\mathcal{B}}_r(a))\|$, and*
(c) *$\|a - (AaBy)\| \le [1 - (\alpha M)]r$ for each $y \in \overline{\mathcal{B}}_r(a)$ with $\alpha M < 1$.*

Then the operator equation $AxBx = x$ has a solution $\overline{\mathcal{B}}_r(a)$.

Theorem 2.4 (Dhage, [4]). *Let $a \in X$ and r a positive real number. Let $A: X \to X$ and $B: \overline{\mathcal{B}}_r(a) \to X$ be two operators such that*

(a) *A is Lipschitzian with Lipschitz constant α,*
(b) *B is completely continuous, and*
(c) *$\|a - (Aa + By)\| \le \big(1 - \alpha\big)r$ for each $y \in \overline{\mathcal{B}}_r(a))$, where $\alpha < 1$.*

Then the operator equation $Ax + Bx = x$ has a solution $\overline{\mathcal{B}}_r(a)$.

3. Nonlinear functional integral equations

Let $\mathbb{R}$ be the line and let $J = [0, 1] \subset \mathbb{R}$. Consider the nonlinear functional integral equation (in short FIE)

$$x(t) = k(t, x(\mu(t))) + [f(t, x(\theta(t)))]\bigg(q(t) + \int_0^{\sigma(t)} g(s, x(\eta(s)))\, ds\bigg) \tag{3.1}$$

for $t \in J$, where $q\colon J \to \mathbb{R}$, $\mu, \theta, \sigma, \eta\colon J \to J$ and $f, g, k\colon J \times \mathbb{R} \to \mathbb{R}$.

The FIE (3.1) and its special case have been studied in the literature very extensively via different fixed point methods for various aspects of the solutions. See Dhage [7], [9], Dhage and O'Regan [10] and the references therein. Here we shall prove the existence of local solutions of FIE (3.1) by an application of the abstract fixed point theorem of previous section under some suitable conditions different from Dhage [3].

Let $M(J,\mathbb{R})$ and $B(J,\mathbb{R})$ denote respectively the spaces of all measurable and bounded real-valued functions on J. We shall seek the solution of FIE (3.1) in the space $BM(J,\mathbb{R})$ of bounded and measurable real-valued functions on J. Define a norm

$$\|x\|_{BM} = \max_{t \in J} |x(t)|. \tag{3.2}$$

Clearly $BM(J,\mathbb{R})$ is a Banach algebra with this maximum norm. Let $L(J,\mathbb{R})$ denote the space of Lebesgue integrable real-valued functions on J with a norm $\|\cdot\|_{L^1}$ defined by

$$\|x\|_{L^1} = \int_0^1 |x(t)|\, dt. \tag{3.3}$$

We need the following definition in the sequel.

Definition 3.1. A mapping $\beta\colon J \times \mathbb{R} \to \mathbb{R}$ is said to satisfy a condition of L^1-Carathéodory or simply is called L^1-Carathéodory if

(a) $t \mapsto \beta(t,x)$ is measurable for each $x \in \mathbb{R}$,
(b) $x \mapsto \beta(t,x)$ is continuous almost everywhere for $t \in J$, and
(c) for each real number $r > 0$ there exists a function $h_r \in L^1(J,\mathbb{R})$ such that
$$|\beta(t,x)| \le h_r(t) \quad \text{a.e. } t \in J$$
for all $x \in \mathbb{R}$ with $|x| \le r$.

We consider the following set of assumptions:

(H_0) The functions $\mu, \theta, \sigma, \eta\colon J \to J$ are continuous.
(H_1) The function $q\colon J \to \mathbb{R}$ is continuous.
(H_2) The function $k\colon J \times \mathbb{R} \to \mathbb{R}$ is continuous and there is a function $\beta_1 \in B(J,\mathbb{R})$ with bound $\|\beta_1\|$ such that, for all $x, y \in \mathbb{R}$,
$$|k(t,x) - k(t,y)| \le \beta_1(t)|x-y| \quad \text{a.e. } t \in J.$$
(H_3) The function $f\colon J \times \mathbb{R} \to \mathbb{R} - \{0\}$ is continuous and there is a function $\alpha_1 \in B(J,\mathbb{R})$ with bound $\|\alpha_1\|$ such that, for all $x, y \in \mathbb{R}$,
$$|f(t,x) - f(t,y)| \le \alpha_1(t)|x-y| \quad \text{a.e. } t \in J.$$
(H_4) The function g is L^1-Carathéodory.

Theorem 3.2. *Assume that the hypotheses* (H_0)–(H_4) *hold. Further if there exists a real number* $r > 0$ *such that*

$$\|\alpha_1\|\,(\|q\|_{BM} + \|h_r\|_{L^1}) + \|\beta_1\| < 1 \tag{3.4}$$

and

$$r > \frac{F(\|q\|_{BM} + \|h_r\|_{L^1}) + K}{1 - [\|\alpha_1\|(\|q\|_{BM} + \|h_r\|_{L^1}) + \|\beta_1\|]} \tag{3.5}$$

where $F = \sup\{|f(t,0)| : t \in J\}$ *and* $K = \sup\{|k(t,0)| : t \in J\}$, *then* FIE (3.1) *has a solution on* J *with* $\|u\| \le r$.

Proof. Consider the closed ball $\overline{B}_r(0)$ centered at origin of radius r, where the real number r satisfies the inequalities (3.4) and (3.5). Define three operators A, B and C on $BM(J,\mathbb{R})$ by

$$Ax(t) = f(t, x(\theta(t))), \qquad t \in J \tag{3.6}$$

$$Bx(t) = q(t) + \int_0^{\sigma(t)} g(s, x(\eta(s)))\,ds, \quad t \in J, \tag{3.7}$$

$$Cx(t) = k(t, x(\mu(t))), \qquad t \in J. \tag{3.8}$$

Then the FIE (3.1) is equivalent to the fixed point equation

$$Ax(t)Bx(t) + Cx(t) = x(t), \quad t \in J. \tag{3.9}$$

Hence the problem of the existence of solution to FIE (3.1) is just reduced to finding the solutions of the operator equation $AxBx+Cx = x$ in $BM(J,\mathbb{R})$. We shall show that the operators A, B and C satisfy all the conditions of Corollary 2.1. First we shall show that A Lipschitzian on $BM(J,\mathbb{R})$. Let $x, y \in BM(J,\mathbb{R})$. Then, by ($H_2$),

$$\begin{aligned}|Ax(t) - Ay(t)| &= |f(t, x(\theta(t))) - f(t, y(\theta(t)))|\\ &\le \alpha_1(t)|x(\theta(t)) - y(\theta(t))| \le \|\alpha_1\|\,\|x - y\|_{BM}.\end{aligned}$$

Taking the maximum over t,

$$\|Ax - Ay\|_{BM} \le \|\alpha_1\|\|x - y\|_{BM}.$$

This shows that A is a Lipschitzian with a Lipschitz constant $\|\alpha_1\|$. Similarly it is shown that C is a Lipschitzian with a Lipschitz constant $\|\beta_1\|$. Next we shall show that the operator B is continuous and compact on $\overline{B}_r(0)$. Since $g(t,x)$ is L^1-Carathéodory, by using the dominated convergence theorem (see Granas *et al* [11]), it can be shown that B is continuous on $BM(J,\mathbb{R})$. Let $\{x_n\}$ be a sequence in $\overline{\mathcal{B}}_r(0)$. Then we have $\|x_n\| \le r$ for each $n \in \mathbb{N}$. Then by (H_4),

$$\begin{aligned}|Bx_n(t)| &\le |q(t)| + \left| \int_0^{\sigma(t)} |g(s, x(\eta(s)))|\, ds \right| \\ &\le \|q\|_{BM} + \int_0^{\sigma(t)} h_r(s)\, ds \le \|q\|_{BM} + \|h_r\|_{L^1}\end{aligned}$$

which further yields that $\|Bx_n\| \le \|q\|_{BM} + \|h_r\|_{L^1}$ for each $n \in \mathbb{N}$. As a result $\{Bx_n : n \in \mathbb{N}\}$ is a uniformly bounded set in $\overline{\mathcal{B}}_r(0)$. Let $t_1, t_2 \in J$. Then by the definition of B,

$$\begin{aligned}|Bx_n(t_1) - Bx_n(t_2)| &\le |q(t_1) - q(t_2)| + \left| \int_{\sigma(t_2)}^{\sigma(t_1)} h_r(s)\, ds \right| \\ &\le |q(t_1) - q(t_2)| + |p(t_1) - p(t_2)|\end{aligned}$$

where $p(t) = \int_0^{\sigma(t)} h_r(s)\, ds$. Since q and p are continuous on J, they are uniformly continuous and consequently

$$|Bx_n(t_1) - Bx_n(t_2))| \to 0 \quad \text{as } t_1 \to t_2.$$

Thus $\{Bx_n : n \in \mathbb{N}\}$ is an equi-continuous set in $B(\overline{\mathcal{B}}_r(0))$. Hence $B(\overline{\mathcal{B}}_r(0))$ is compact by Arzelà–Ascoli theorem for compactness. Thus B is a continuous and compact operator on $B(\overline{\mathcal{B}}_r(0))$.

Finally for any $y \in \overline{\mathcal{B}}_r(0)$ we have

$$\begin{aligned}\|A0By + C0\| &\le \|A0\|_{BM} \|By\|_{BM} + \|C0\|_{BM} \\ &\le \sup_{t\in J} |f(t,0)|\, [\|q\|_{BM} + \|h_r\|_{L^1}] + \sup_{t\in J} |k(t,0)| \\ &\le F(\|q\|_{BM} + \|h_r\|_{L^1}) + K \\ &\le [1 - \|\alpha_1\|(\|q\|_{BM} + \|h_r\|_{L^1}) - \|\beta_1\|]r.\end{aligned}$$

Now in view of Corollary 2.1 the FIE (3.1) has a solution in $\overline{\mathcal{B}}_r(0)$. This completes the proof. □

Below we show that Theorem 3.2 could also be used to discuss existence result for a certain differential equation in Banach algebras. Consider the following initial value problem of a first order functional differential equation (in short FDE)

$$\text{(3.10)} \qquad \begin{cases} \left(\dfrac{x(t) - k(t, x(\mu(t)))}{f(t, x(\theta(t)))} \right)' = g(s, x(\eta(t))) & \text{a.e. } t \in J, \\ x(0) = x_0 & \text{in } \mathbb{R}, \end{cases}$$

where $f\colon J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$, $k\colon J \times \mathbb{R} \to \mathbb{R}$ are continuous, $g\colon J \times \mathbb{R} \to \mathbb{R}$ and $\mu, \theta, \eta\colon J \to J$ are continuous with $\mu(0) = 0 = \theta(0)$.

By a solution of FDE (3.10) we mean a function $x \in AC(J, \mathbb{R})$ that satisfies the equation (3.10), where $AC(J, \mathbb{R})$ is the space of all absolutely continuous real-valued functions on J.

Notice that $AC(J, \mathbb{R}) \subset BM(J, \mathbb{R})$.

THEOREM 3.3. *Assume that the hypotheses* (H_2)–(H_3) *hold. Further suppose that there exists a real number* $r > 0$ *such that conditions* (3.4) *and* (3.5) *hold with*

$$\|q\|_{BM} = \left|\frac{x_0 - k(0, x_0)}{f(0, x_0)}\right|.$$

Then FDE (3.10) *has a solution* u *on* J *with* $\|u\|_{BM} \leq r$.

PROOF. Notice that the FDE (3.10) is equivalent to the integral equation

$$x(t) = k(t, x(\mu(t))) + [f(t, x(\theta(t)))]\left(\frac{x_0 - k(0, x_0)}{f(0, x_0)} + \int_0^t g(s, x(\eta(s)))\, ds\right), \tag{3.11}$$

for $t \in J$. Now an application of Theorem 3.2 with

$$q(t) = \left|\frac{x_0 - k(0, x_0)}{f(0, x_0)}\right| \quad \text{for all } t \in J,$$

yields that (3.10) has a solution in $\overline{B}_r(0)$, because $AC(J, \mathbb{R}) \subset BM(J, \mathbb{R})$. The proof is complete. □

REMARK 3.4. Applications of fixed point theory to nonlinear differential and integral equations is an art and it depends upon the clever selection of fixed point theorem suitable for the given data or conditions. Notice that the local solutions of the FIE (3.1) could also be obtained via a nonlinear alternative of Leray–Schauder type recently proved in Dhage [8]. But in that case, we need the integral equation to satisfy the boundary conditions. In the present situation this is not the case. At the present, we do not know which approach out of above two is better for dealing with the nonlinear equations.

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