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APPROXIMATE SELECTIONS IN α-CONVEX METRIC SPACES AND TOPOLOGICAL DEGREE

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Abstract. The existence of continuous approximate selections is proved for a class of upper semicontinuous multifunctions taking closed α-convex values in a metric space equipped with an appropriate notion of α-convexity. The approach is based on the definition of pseudo-barycenter of an ordered n-tuple of points. As an application, a notion of topological degree for a class of α-convex multifunctions is developed.

1. Introduction

In linear spaces the notion of convexity plays a fundamental role in several problems of analysis, for instance, in the construction of continuous selections (Michael [22]), in fixed point theorems (Kakutani [19], Ky Fan [9]), in topological degree theory (Hukuhara [17], Cellina and Lasota [4], Ma [21], Petryshyn and Fitzpatrick [26]). A full account of the above and other subject-matters related to convexity can be found in the comprehensive monographs by Hu and Papageorgiou [16] and Repovš and Semenov [28].

In non linear spaces, in absence of a natural notion of convex set, different approaches to convexity have been developed so far.

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Michael [23] introduces, in a metric space Y, an axiomatically defined convex structure which permits one to take "convex combinations" of some, but not necessarily all, ordered n-tuples of points of Y; then by defining a convex set in the obvious way, Michael establishes a metric version of his classical continuous selection theorem. For similar ideas see also an earlier paper of Stone [29]. Further developments with application to selection theorems can be found in Michael [23], Curtis [5], [6] and Pasicki [25], [26].

Another axiomatic approach to convexity in non linear spaces has been developed by van de Val in [32], [33], (see also Bielawski, [1]) who defines convex the sets of a given family $\mathcal{C}$ of subsets of Y provided the following conditions are satisfied: $\mathcal{C}$ contains Y and the empty set, the intersection of every family of members of $\mathcal{C}$, and the union of every up-directed family of members of $\mathcal{C}$. Several applications, including a generalization of Michael's continuous selection theorem, are presented. Axiomatic convex structures of different type, also useful in selection problems, have been studied by Horvath in [15].

A further different viewpoint to convexity is due to Takahashi [30], who considers a metric space Y to be convex if there exists a function $w\colon Y \times Y \times [0,1] \to Y$ satisfying

$$d(z, w(y_1, y_2, t)) \leq (1-t)d(z, y_1) + td(z, y_2)$$

for all y_1, y_2, $z \in Y$ and $t \in [0,1]$, where d is the metric of Y. Then by defining convex any set $A \subset Y$ such that $w(y_1, y_2, t) \in A$, for every $y_1, y_2 \in A$ and $t \in [0,1]$, Takahashi proves some fixed point theorems for nonexpansive mappings in metric spaces. Related results in this direction can be found in Talman [31].

The approach to convexity we develop in the present paper is in the spirit of Michael [23]. As in Curtis [5], [6] and Pasicki [26], it actually rests on appropriate generalizations in a nonlinear space of the notions of a segment joining two points, and of a barycenter of a finite set of points. More precisely, we consider a metric space Y equipped with a continuous function $\alpha\colon Y \times Y \times [0,1] \to Y$ which satisfies the following conditions:

(i) $\alpha(y_0, y_0, t) = y_0$ for every $y_0 \in Y$ and $t \in [0,1]$,
(ii) $\alpha(y_1, y_2, 0) = y_1$, $\alpha(y_1, y_2, 1) = y_2$ for every $(y_1, y_2) \in Y \times Y$,
(iii) there is $0 < r_\alpha \leq +\infty$ such that for every (y_1, y_2), $(\overline{y}_1, \overline{y}_2) \in Y \times Y$, with $d(y_1, \overline{y}_1) < r_\alpha$, $d(y_2, \overline{y}_2) < r_\alpha$ one has

$$h(\Lambda_\alpha(y_1, y_2), \Lambda_\alpha(\overline{y}_1, \overline{y}_2)) \leq \max\{d(y_1, \overline{y}_1), d(y_2, \overline{y}_2)\}.$$

Here $\Lambda_\alpha(y_1, y_2) = \{\alpha(y_1, y_2, t) \mid t \in [0,1]\}$, and h is the Pompeiu–Hausdorff distance in the space of the non empty compact subsets of Y. Then Y, equipped with the mapping α, is called *Lipschitz α-convex metric space* ("α" stands for "arcwise"). Moreover, if α satisfies (i), (ii) and, instead of (iii), the weaker condition (iii) of Definition 3.1 below, then Y is called *α-convex*.

A subset A of Y is called *α-convex* if, for every $(y_1, y_2) \in A \times A$ and $t \in [0,1]$, one has $\alpha(y_1, y_2, t) \in A$.

When Y is normed, (i)–(iii) are trivially satisfied by letting $\alpha(y_1, y_2, t) = (1-t)y_1 + ty_2$, with $(y_1, y_2) \in Y \times Y$ and $t \in [0,1]$, and thus one recovers the usual notion of convex set.

In a normed space the notion of barycenter of a finite set of points enters naturally in approximation and selection problems for multifunctions, when partitions of unity are employed. In our α-convex metric space setting we introduce, for an ordered n-tuple of points, the notion of *pseudo-barycenter*. This retains only a few properties of the barycenter, yet it is still useful in approximation and selection problems. In fact, by using pseudo-barycenters and partition of unity techniques, we establish a metric version of Cellina's theorem [3], namely, the existence of approximate continuous selections, for Pompeiu–Hausdorff upper semicontinuous multifunctions with non empty closed bounded *α-convex* values. A metric version of Michael's selection theorem for lower semicontinuous multifunctions with α-convex values is proved in [7], by a similar approach.

It is worthwhile to point out that, in our axiomatic approach to convexity, we tried to identify a minimum set of readily verifiable conditions, under which a kind of barycentric calculus could be developed. Our conditions (i)–(iii) are perhaps questionable from the point of view of generality, yet they are easily verifiable, and also useful. In fact, condition (iii) makes possible to have that the pseudo-barycenter we define is actually stable in the sense of Proposition 4.12, a crucial property in approximation theory for multifunctions, which is introduced as an axiom by many authors.

The previous approximate selection result is used to define, as in [4], [17], the topological degree for compact vector fields $I - F$, where I is the identity and F is a Pompeiu–Hausdorff upper semicontinuous multifunction with non empty compact α-convex values. When F is compact and convex valued, the above reduces to the topological degree introduced by Hukuhara [17], and developed by Cellina and Lasota [4], Ma [21], Petryshyn and Fitzpatrick [27]. Fixed point theorems of Kakutani–Ky Fan type for multifunctions with α-convex values are considered as well.

For a fairly general class of multifunctions with compact non convex values, approximate continuous selections have been constructed, by a different method, by Górniewicz, Granas and Kryszewski [12] and Górniewicz and Lassonde [13] and hence used to develop an index theory. Moreover, for certain classes of multifunctions with non convex values, a non elementary degree theory was earlier constructed by Granas [14], and extended by Gęba and Granas

[10], Górniewicz [11], Borisovitch, Gelman, Myshkis, Obukhovskiĭ [2] (see [2], [10], [11] for further references), following a homology theory approach.

The paper is organized as follows. Section 2 contains notation and terminology. The notions of α-convex metric space, and pseudo-barycenter of a finite set of points, are considered in Sections 3 and 4, respectively. In Section 5 it is proved the existence of approximate continuous selections for α-convex valued multifunctions. The definition of the topological degree for compact α-convex valued vector fields is given in Section 6. A few properties of this degree including an application to fixed point theory are presented in Section 7.

2. Notation and preliminaries

Throughout Y is a nonempty metric space with distance d, and 2^Y the family of all nonempty subsets of Y. If $A \subset Y$, by $\operatorname{int} A$, $\overline{A}$, ∂A we denote the interior, closure, boundary of A.

For A, B nonempty subsets of Y, put

$$e(A,B) = \sup_{a\in A} d(a,B) \quad \text{where } d(a,B) = \inf_{b\in B} d(a,b).$$

The space of all nonempty closed bounded subsets of Y is equipped with the Pompeiu–Hausdorff metric

$$h(A,B) = \max\{e(A,B), e(B,A)\},$$

under which it is complete, if Y is so.

By $U(a,r)$, $U[a,r]$ we mean respectively an open, closed ball in Y with center a and radius r.

In the sequel, if a set $A \subset Y$ is considered as a metric space, it is tacitly assumed that A retains the metric of Y.

Unless the contrary is stated, the Cartesian product $Y \times \widetilde{Y}$ of two metric spaces Y, $\widetilde{Y}$, with distances d, $\widetilde{d}$ is always supposed to have distance given by

$$\max\{d(x,y), \widetilde{d}(\widetilde{x},\widetilde{y})\} \quad (x,\widetilde{x}), (y,\widetilde{y}) \in Y \times \widetilde{Y}.$$

Denote by M a metric space.

Definition 2.1. A multifunction $F\colon M \to 2^Y$ is called *Pompeiu–Hausdorff upper semicontinuous* (= h-u.s.c.) if, for every $x \in M$ and $\varepsilon > 0$, there exists $\delta = \delta(x,\varepsilon) > 0$ such that $x' \in U(x,\delta)$ implies $e(F(x'),F(x)) < \varepsilon$.

The *graph* of a multifunction $F\colon M \to 2^Y$ is the set, denoted graph F, given by

$$\operatorname{graph} F = \{(x,y) \in M \times Y \mid x \in M,\ y \in F(x)\}.$$

Definition 2.2. Given a multifunction $F: M \to 2^Y$ and $\varepsilon > 0$, then any continuous function $f_\varepsilon: M \to Y$ such that

$$e(\operatorname{graph} f_\varepsilon, \operatorname{graph} F) < \varepsilon$$

is called an *approximate continuous selection* of F.

Definition 2.3. A sequence $\{f_n\}$ of approximate continuous selections $f_n: M \to Y$ of $F: M \to 2^Y$ is said to be *graph-convergent* to F (for brevity, $f_n \overset{\text{gr}}{\to} F$) if

$$e(\operatorname{graph} f_n, \operatorname{graph} F) \to 0 \quad \text{as } n \to \infty.$$

For any nonempty set A we put $A^n = A \times \ldots \times A$, and denote by $(a_1, \ldots, a_n)$ an element of A^n, i.e. an ordered n-tuple of points $a_i \in A$, $i = 1, \ldots, n$.

Let $\mathbb{E}$ be a normed space.

The convex hull, and the closed convex hull of a set $A \subset \mathbb{E}$ are denoted, respectively, by $\operatorname{co} A$ and $\overline{\operatorname{co}} A$.

For $(p, q) \in \mathbb{E}^2$, we denote by $[p, q]$ (resp. (p, q)) the closed (resp. open) non oriented segment in $\mathbb{E}$ with end points p and q. When $p \neq q$ the segments $[p, q]$, (p, q) are called non degenerate.

3. α-convex metric spaces

In this section we introduce the notion of α-convexity in metric spaces and we consider some examples.

Set $J = [0, 1]$. For any map $\alpha: Y \times Y \times J \to Y$, and $(y_1, y_2) \in Y \times Y$, we agree to call (y_1, y_2)-*locus induced* by α the set $\Lambda_\alpha(y_1, y_2)$ given by

$$\Lambda_\alpha(y_1, y_2) = \{y \in Y \mid y = \alpha(y_1, y_2, t) \text{ for some } t \in J\}. \tag{3.1}$$

Definition 3.1. Let Y be a metric space, and let $\alpha: Y \times Y \times J \to Y$ be a continuous mapping satisfying the following conditions:

(a) $\alpha(y_0, y_0, t) = y_0$ for every $y_0 \in Y$ and $t \in J$,
(b) $\alpha(y_1, y_2, 0) = y_1$, $\alpha(y_1, y_2, 1) = y_2$ for every $(y_1, y_2) \in Y \times Y$,
(c) there is r_α, $0 < r_\alpha \leq +\infty$, such that, for every $0 < \varepsilon < r_\alpha$, there exists $0 < \eta \leq \varepsilon$ such that, whatever be (y_1, y_2), $(\overline{y}_1, \overline{y}_2) \in Y \times Y$, with $d(y_1, \overline{y}_1) < \varepsilon$ and $d(y_2, \overline{y}_2) < \eta$, one has

$$h(\Lambda_\alpha(y_1, y_2), \Lambda_\alpha(\overline{y}_1, \overline{y}_2)) < \varepsilon. \tag{3.2}$$

Then Y, equipped with the mapping α, is called α-convex metric space. If the continuous function α satisfies (a)–(c), the latter with $\eta = \varepsilon$, then Y is called strongly α-convex metric space.

Remark 3.2. An α-convex metric space is contractible, hence arcwise connected. Observe also that the (y_1, y_2)-locus $\Lambda_\alpha(y_1, y_2)$ is not necessarily α-convex, and one can have $\Lambda_\alpha(y_1, y_2) \neq \Lambda_\alpha(y_2, y_1)$.

Definition 3.3. Let Y be a metric space and let $\alpha: Y \times Y \times J \to Y$ be a continuous function satisfying (a), (b) of Definition 3.1 and, instead of (c), the following:

(c)$'$ There is r_α, $0 < r_\alpha \leq +\infty$, such that for every (y_1, y_2), $(\overline{y}_1, \overline{y}_2) \in Y \times Y$, with $d(y_1, \overline{y}_1) < r_\alpha$, $d(y_2, \overline{y}_2) < r_\alpha$, one has

$$h(\Lambda_\alpha(y_1, y_2), \Lambda_\alpha(\overline{y}_1, \overline{y}_2)) \leq \max\{d(y_1, \overline{y}_1), d(y_2, \overline{y}_2)\}.$$

Then Y, equipped with the mapping α, is called *Lipschitz α-convex metric space.*

Definition 3.4. Let Y be a metric space and let $\alpha: Y \times Y \times J \to Y$ be a continuous function satisfying (a), (b) of Definition 3.1 and, instead of (c), the following condition (c)$''$ (resp. (c)$'''$):

(c)$''$ There is r_α, $0 < r_\alpha \leq +\infty$, such that, for every $0 < \varepsilon < r_\alpha$, there exists $0 < \eta \leq \varepsilon$ such that, whatever be (y_1, y_2), $(\overline{y}_1, \overline{y}_2) \in Y \times Y$, with $d(y_1, \overline{y}_1) < \varepsilon$ and $d(y_2, \overline{y}_2) < \eta$, one has

$$d(\alpha(y_1, y_2, t), \alpha(\overline{y}_1, \overline{y}_2, t)) < \varepsilon \quad \text{for every } t \in J.$$

(c)$'''$ There is r_α, $0 < r_\alpha \leq +\infty$, such that, for every (y_1, y_2), $(\overline{y}_1, \overline{y}_2) \in Y \times Y$, with $d(y_1, \overline{y}_1) < r_\alpha$, $d(y_2, \overline{y}_2) < r_\alpha$, one has

$$d(\alpha(y_1, y_2, t), \alpha(\overline{y}_1, \overline{y}_2, t)) \leq \max\{d(y_1, \overline{y}_1), d(y_2, \overline{y}_2)\} \quad \text{for every } t \in J.$$

Then Y, equipped with the mapping α, is called geodesically α-convex metric space (resp. Lipschitz geodesically α-convex metric space).

In the above definitions, α is also called the convexity mapping, or α-mapping, of Y.

Remark 3.5. The notion of geodesically α-convex space is similar to the notion of geodesic structure, introduced by Michael in [23], where α is continuous in t and satisfies some additional conditions which include (a), (b), and (c)$''$. It is worthwhile to observe that, from (c)$''$ and the continuity of α in t, it follows that α is actually continuous in (y_1, y_2, t), as required in Definition 3.4.

Remark 3.6. A normed space $\mathbb{E}$ (with norm $\|\cdot\|$) is Lipschitz α-convex, and geodesically α-convex, if it is endowed with the natural convexity mapping $\alpha_0: \mathbb{E} \times \mathbb{E} \times J \to \mathbb{E}$, given by

$$\alpha_0(y_1, y_2, t) = (1 - t)y_1 + ty_2. \tag{3.3}$$

Clearly $\Lambda_{\alpha_0}(y_1,y_2)=\Lambda_{\alpha_0}(y_2,y_1)=[y_1,y_2]$. Thus we have

$$h(\Lambda_\alpha(y_1,y_2),\Lambda_\alpha(\overline{y}_1,\overline{y}_2))\leq \max\{\|y_1-\overline{y}_1\|,\|y_2-\overline{y}_2\|\}$$

for every (y_1,y_2), $(\overline{y}_1,\overline{y}_2)\in\mathbb{E}^2$.

Proposition 3.7.

(a$_1$) *Y is Lipschitz geodesically α-convex $\Rightarrow$ Y is Lipschitz α-convex $\Rightarrow$ Y is strongly α-convex $\Rightarrow$ Y is α-convex.*

(a$_2$) *Y is geodesically α-convex $\Rightarrow$ Y is α-convex.*

Proof. (a$_1$) is obvious.

(a$_2$) With r_α, ε, η, (y_1,y_2), $(\overline{y}_1,\overline{y}_2)$ as in (c)$''$, and $t\in J$, one has

$$d(\alpha(y_1,y_2,t),\Lambda_\alpha(\overline{y}_1,\overline{y}_2))\leq \max_{s\in J} d(\alpha(y_1,y_2,s),\alpha(\overline{y}_1,\overline{y}_2,s))<\varepsilon,$$

and hence $e(\Lambda_\alpha(y_1,y_2),\Lambda_\alpha(\overline{y}_1,\overline{y}_2))<\varepsilon$. Combining this with the analogous inequality obtained by interchanging $\Lambda_\alpha(y_1,y_2)$ and $\Lambda_\alpha(\overline{y}_1,\overline{y}_2)$, (3.2) follows, and thus Y is α-convex. □

Proposition 3.8.

(a$_1$) *Let Y be an α-convex metric space, and let r_α correspond. Then for each $0<\varepsilon<r_\alpha$ there exists $0<\eta\leq\varepsilon$ such that, if $a\in Y$ and $y_1,y_2\in U(a,\eta)$, then one has $\Lambda_\alpha(y_1,y_2)\subset U(a,\varepsilon)$.*

(a$_2$) *Let Y be a strongly α-convex metric space, and let r_α correspond. Then, for every $a\in Y$ and $0<\varepsilon<r_\alpha$, the set $U(a,\varepsilon)$ is α-convex, hence contractible.*

Proof. (a$_1$) Let y_1, $y_2\in U(a,\eta)$ and let η be as in Definition 3.1. Since $a=\Lambda_\alpha(a,a)$, one has

$$\sup_{t\in J} d(\alpha(y_1,y_2,t),a)=e(\Lambda_\alpha(y_1,y_2),\Lambda_\alpha(a,a))=h(\Lambda_\alpha(y_1,y_2),\Lambda_\alpha(a,a))<\varepsilon,$$

and (a_1) holds. The above argument, with $\eta=\varepsilon$, proves also (a_2). □

The following Example 3.9 (resp. Example 3.10) below shows that there exist metric spaces Y which are α-convex (resp. Lipschitz α-convex) and not geodesically α-convex (resp. Lipschitz geodesically α-convex).

Example 3.9. Let $Y=\mathbb{R}^2$, and let $\{(a_n,b_n)\}$, $\{(\overline{a}_n,\overline{b}_n)\}\subset Y\times Y$ be sequences such that $\|a_n\|=n$, $b_n=a_n+c$, $\overline{a}_n=a_n+c/n$, $\overline{b}_n=b_n+c/n$, $n\in\mathbb{N}$, where $\|c\| = 1$. Set

$$\varphi(a,b,t) = \begin{cases} 0 & \text{if } (a,b,t) \in D_0 = \{(a,b,t) \in Y \times Y \times J \mid t = 0\}, \\ 1 & \text{if } (a,b,t) \in D_1 = \{(a,b,t) \in Y \times Y \times J \mid t = 1\}, \\ t & \text{if } (a,b,t) \in S_n = \{(a_n,b_n,t) \in Y \times Y \times J \mid t \in J\} \\ & \text{for some } n \in \mathbb{N}, \\ t^2 & \text{if } (a,b,t) \in T_n = \{(\overline{a}_n,\overline{b}_n,t) \in Y \times Y \times J \mid t \in J\} \\ & \text{for some } n \in \mathbb{N}. \end{cases}$$

Y is equipped with distance d induced by the Euclidean norm $\|\cdot\|$ of $\mathbb{R}^2$. As φ is continuous on $D_0 \cup D_1 \cup (\bigcup_{n\in\mathbb{N}}(S_n \cup T_n))$, a closed set, and takes values in J, by Tietze's theorem [8, p. 149], it admits a continuous extension, say φ, defined on $Y \times Y \times J$, with values in J.

Now define $\alpha: Y \times Y \times J \to Y$ by

$$\alpha(y_1,y_2,t) = (1-\varphi(y_1,y_2,t))y_1 + \varphi(y_1,y_2,t)y_2.$$

Clearly α is continuous, and satisfies conditions (a) and (b) of Definition 3.3. Further, for arbitrary (y_1,y_2), $(\overline{y}_1,\overline{y}_2) \in Y \times Y$, one has $\Lambda_\alpha(y_1,y_2) = [y_1,y_2]$, $\Lambda_\alpha(\overline{y}_1,\overline{y}_2) = [\overline{y}_1,\overline{y}_2]$, and hence

$$h(\Lambda_\alpha(y_1,y_2),\Lambda_\alpha(\overline{y}_1,\overline{y}_2)) = h([y_1,y_2],[\overline{y}_1,\overline{y}_2]) \leq \max\{\|y_1-\overline{y}_1\|,\|y_2-\overline{y}_2\|\}.$$

Thus Y is Lipschitz α-convex (with $r_\alpha = \infty$), and so α-convex. On the other hand Y is not geodesically α-convex, since (c)$''$ fails. In fact for any $n \in \mathbb{N}$ one has

$$\begin{aligned} \alpha(a_n,b_n,t) - \alpha(\overline{a}_n,\overline{b}_n,t) &= (1-t)a_n + tb_n - (1-t^2)\overline{a}_n - t^2\overline{b}_n \\ &= (1-t)(a_n-\overline{a}_n) + t(b_n-\overline{b}_n) + (t-t^2)(\overline{b}_n-\overline{a}_n) \\ &= -\frac{c}{n} + (t-t^2)c. \end{aligned}$$

Whence $\|\alpha(a_n,b_n,t) - \alpha(\overline{a}_n,\overline{b}_n,t)\| \geq t - t^2 - 1/n$, for each $t \in J$. For $t = 1/2$ and all $n \geq 8$, it follows

$$\|\alpha(a_n,b_n,1/2) - \alpha(\overline{a}_n,\overline{b}_n,1/2)\| \geq 1/8,$$

which shows that (c)$''$ fails, as $d(a_n,\overline{a}_n) = d(b_n,\overline{b}_n) = 1/n$.

Example 3.10. Set $Y = \{(u_1,u_2) \in \mathbb{R}^2 | u_1 \leq 0 \text{ or } u_2 \leq 0\}$, and equip Y with metric d induced by the Euclidean norm $\|\cdot\|$ of $\mathbb{R}^2$. The non-negative u_1-axis, u_2-axis of $\mathbb{R}^2$ are denoted by γ_1, γ_2. For $u \in \mathbb{R}^2$, $u \neq 0$, l_u denotes the line which contains 0 and is orthogonal to u. If λ', λ'' are non opposite half-lines issuing from 0, by $\widehat{\lambda'\lambda''}$ we mean the closed convex angle which is determined by λ', λ''.

For $(p,q) \in Y^2$, with $p,q \neq 0$, set $T(p,q) = \|p\|/(\|p\|+\|q\|)$. Clearly, for every $(p,q) \in Y^2$, with $[p,q] \not\subset Y$, we have p, $q \neq 0$, and hence $0 < T(p,q) < 1$.

Now define $\alpha: Y \times Y \times J \to Y$ $(J = [0,1])$ by:

$$(p,q,t) = (1-t)p + tq, \qquad t \in J, \qquad \text{if } [p,q] \subset Y,$$

$$\alpha(p,q,t) = \begin{cases} \left(1 - \dfrac{t}{T(p,q)}\right)p & t \in [0, T(p,q)], \\ \dfrac{t - T(p,q)}{1 - T(p,q)q} & t \in [T(p,q), 1], \end{cases} \qquad \text{if } [p,q] \not\subset Y. \tag{3.4}$$

It will be shown that Y, endowed with the mapping α, is a Lipschitz α-convex metric space (with $r_\alpha = \infty$).

Since conditions (a), (b) of Definition 3.1 are trivially satisfied it suffices to prove:

(j) α is continuous on $Y \times Y \times J$;

(jj) for every (p,q), $(\overline{p},\overline{q}) \in Y^2$, setting $\Lambda = \Lambda_\alpha(p,q)$, $\overline{\Lambda} = \Lambda_\alpha(\overline{p},\overline{q})$, we have

$$h(\Lambda, \overline{\Lambda}) \leq \max\{d(p,\overline{p}), d(q,\overline{q})\}. \tag{3.5}$$

Consider (j). Let $(p,q,t) \in Y \times Y \times J$, and let $\{(p_n,q_n,t_n)\} \subset Y \times Y \times J$ be an arbitrary sequence converging to (p,q,t). In view of (3.4), α is continuous at each point (p,q,t) with $[p,q] \not\subset Y$.

Suppose $[p,q] \subset Y$, and let $0 \in (p,q)$ (the argument is similar if $0 \notin (p,q)$). For some $\theta < 0$ we have $q = \theta p$ and thus, setting $T = T(p,q)$, we have $T = 1/(1+|\theta|)$. Assume $t < T$. Let $\{[p_{n_k}, q_{n_k}]\}$ (resp. $\{[p_{m_k}, q_{m_k}]\}$) be the infinite subsequence, if exists, consisting of all $[p_n,q_n] \not\subset Y$ (resp. $[p_n,q_n] \subset Y$). Consider $\{[p_{n_k}, q_{n_k}]\}$. Since $t_{n_k} \to t$, $T_{n_k} \to T$, where $T_{n_k} = T(p_{n_k}, q_{n_k})$, there is $k_0 \in \mathbb{N}$ such that $t_{n_k} < T_{n_k}$ for all $k \geq k_0$. By virtue of (3.4), we have $\lim_{k\to\infty} \alpha(p_{n_k}, q_{n_k}, t_{n_k}) = \alpha(p,q,t)$, because $(1-t/T)p = (1-t-|\theta|t)p = (1-t)p+tq$. Likewise, for $\{[p_{m_k}, q_{m_k}]\}$ we have $\lim_{k\to\infty} \alpha(p_{m_k}, q_{m_k}, t_{m_k}) = \alpha(p,q,t)$, and thus $\lim_{n\to\infty} \alpha(p_n,q_n,t_n) = \alpha(p,q,t)$. A similar reasoning shows that the latter equality remains valid when $t > T$, or $t = T$. Whence (j) is true.

(jj) Let (p,q), $(\overline{p},\overline{q}) \in Y^2$. Clearly (3.5) holds when $\Lambda = [p,q]$, $\overline{\Lambda} = [\overline{p},\overline{q}]$. If $\Lambda = [0,p] \cup [0,q]$, $\overline{\Lambda} = [0,\overline{p}] \cup [0,\overline{q}]$, with $[p,q] \not\subset Y$, $[\overline{p},\overline{q}] \not\subset Y$ then (3.5) is satisfied, because $h(\Lambda,\overline{\Lambda}) \leq \max\{h([0,p],[0,\overline{p}]), h([0,q],[0,\overline{q}])\}$, and $h([0,p],[0,\overline{p}]) \leq d(p,\overline{p})$, $h([0,q],[0,\overline{q}]) \leq d(q,\overline{q})$.

It remains to consider the cases:

$$\begin{cases} \Lambda = [p,q], \\ \overline{\Lambda} = [0,\overline{p}] \cup [0,\overline{q}], \quad [\overline{p},\overline{q}] \not\subset Y \end{cases} \tag{a$_1$}$$

$$\begin{cases} \Lambda = [0,p] \cup [0,q], \quad [p,q] \not\subset Y, \\ \overline{\Lambda} = [\overline{p},\overline{q}]. \end{cases} \tag{a$_2$}$$

In (a$_1$) (resp. (a$_2$)) we assume, without loss of generality, $0 \notin [p,q]$ (resp. $0 \notin [\overline{p},\overline{q}]$) for, when $0 \in [p,q]$ (resp. $0 \in [\overline{p},\overline{q}]$), (3.5) follows at once if one writes $[p,q] = [0,p] \cup [0,q]$ (resp. $[\overline{p},\overline{q}] = [0,\overline{p}] \cup [0,\overline{q}]$).

Claim. *In either case* (a$_1$), (a$_2$) *we have*

$$(3.6)\qquad \{p,q\} \cap E(\Lambda) \neq \phi, \quad \textit{where } E(\Lambda) = \{e \in \Lambda \mid d(e,\overline{\Lambda}) = \max_{x\in\Lambda} d(x,\overline{\Lambda})\}.$$

Observe that, if $\overline{\Lambda}$ is compact convex, the corresponding distance function $d(\,\cdot\,,\overline{\Lambda})$ is convex (in general not strictly convex), and thus (3.6) is true, if also Λ is compact convex.

Set $Q_1 = \{(u_1,u_2) \in \mathbb{R}^2 \mid u_1 > 0,\ u_2 \le 0\}$, $Q_2 = \{(u_1,u_2) \in \mathbb{R}^2 \mid u_1 \le 0, u_2 > 0\}$ and, for $u \in Q_1 \cup Q_2$, put $\lambda_u = \overline{l_u \cap \operatorname{int} Y}$.

Let Λ, $\overline{\Lambda}$ be as in (a_1), with $0 \notin \Lambda$. Since $[\overline{p},\overline{q}] \not\subset Y$, $\overline{p}$ and $\overline{q}$ are linearly independent, and either $\overline{p} \in Q_1$ and $\overline{q} \in Q_2$, or $\overline{p} \in Q_2$ and $\overline{q} \in Q_1$. Clearly $\operatorname{int}\widehat{\lambda_{\overline{p}}\lambda_{\overline{q}}} \neq \phi$. Let $\overline{p} \in Q_1$, $\overline{q} \in Q_2$ (the proof is analogous in the other case). For every $x \in \widehat{\gamma_1\lambda_{\overline{p}}}$ we have $d(x,[0,\overline{p}]) \le \|x\| = d(x,[0,\overline{q}])$, where the equality holds, since x and $\overline{q}$ make an angle greater or equal to $\pi/2$. Likewise we have $d(x,[0,\overline{p}]) \le \|x\| = d(x,[0,\overline{p}])$ for every $x \in \widehat{\gamma_2\lambda_{\overline{q}}}$, and $d(x,[0,\overline{p}]) = \|x\| = d(x,[0,\overline{q}])$ for every $x \in \widehat{\lambda_{\overline{p}}\lambda_{\overline{q}}}$. Therefore,

$$(3.7)\qquad d(x,\overline{\Lambda}) = \begin{cases} d(x,[0,\overline{p}]) & \text{if } x \in \widehat{\gamma_1\lambda_{\overline{p}}}, \\ d(x,[0,\overline{p}]) = \|x\| = d(x,[0,\overline{q}]) & \text{if } x \in \widehat{\lambda_{\overline{p}}\lambda_{\overline{q}}}, \\ d(x,[0,\overline{q}]) & \text{if } x \in \widehat{\gamma_2\lambda_{\overline{q}}}. \end{cases}$$

Put $A_1 = \widehat{\gamma_1\lambda_{\overline{p}}} \cup \widehat{\lambda_{\overline{p}}\lambda_{\overline{q}}}$, $A_2 = \widehat{\gamma_2\lambda_{\overline{q}}} \cup \widehat{\lambda_{\overline{p}}\lambda_{\overline{q}}}$. If Λ, which is compact convex, satisfies $\Lambda \subset A_1$ or $\Lambda \subset A_2$, then (3.6) holds, in view of (3.7). Suppose $\Lambda \not\subset A_1$ and $\Lambda \not\subset A_2$, thus we have either $p \in (\widehat{\gamma_2\lambda_{\overline{q}}}) \setminus \lambda_{\overline{q}}$ and $q \in (\widehat{\gamma_1\lambda_{\overline{p}}}) \setminus \lambda_{\overline{p}}$, or viceversa. Since $0 \notin \Lambda$, it follows that $\Lambda \cap \operatorname{int}\widehat{\lambda_{\overline{p}}\lambda_{\overline{q}}} \neq \phi$. Consequently $\Lambda = \Lambda_1 \cup \Lambda_2$, where the segments $\Lambda_1 = \Lambda \cap A_1$ and $\Lambda_2 = \Lambda \cap A_2$ have intersection which is a non degenerate segment. By (3.7) the function $d(\,\cdot\,,\Lambda)$ restricted to Λ_1, Λ_2 is convex, and thus it is actually convex also on Λ, proving (3.6).

Let Λ, $\overline{\Lambda}$ be as in (a_2), with $0 \notin \overline{\Lambda}$. Likewise in case (a_1), suppose $p \in Q_1$, $q \in Q_2$ (when $p \in Q_2$, $q \in Q_1$ the argument is similar). We have $0 \notin E(\Lambda)$. Suppose the contrary, i.e. $d(0,\overline{\Lambda}) = \max_{x\in\Lambda} d(x,\overline{\Lambda})$, and let $u \in \overline{\Lambda}$, be such that $\|u\| = d(0,\overline{\Lambda})$. Let π be the closed half-plane containing u, determined by l_u, and set $\pi' = \mathbb{R}^2 \setminus \pi$. Observe that $\overline{\Lambda} \subset u + \pi$, otherwise, for some $u' \in \overline{\Lambda} \cap (u+\pi')$, we have $\|u'\| < \|u\| = d(0,\overline{\Lambda})$.

Suppose $u \in Y \setminus (Q_1 \cup Q_2)$. The points p, q cannot lie both in π, since this implies $[p,q] \subset \pi \subset Y$, against the assumption. Then one of them lies in π' and so, for some $x' \in \Lambda \cap \pi'$ close enough to 0, we have $d(x',\overline{\Lambda}) \ge d(x',u+\pi) > \|u\| =$

$\max_{x\in\Lambda} d(x,\overline{\Lambda})$, a contradiction. When $u \in Q_1 \cup Q_2$, an analogous contradiction follows. Whence $0 \notin E(\Lambda)$.

Now let $e \in E(\Lambda)$. Suppose $e \in [0,p]$ (if $e \in [0,q]$ the argument is similar). Since $d(e,\overline{\Lambda}) = \max_{x\in[0,p]} d(x,\overline{\Lambda})$, we have $\{0,p\} \cap E(\Lambda) \neq \phi$. Thus $p \in E(\Lambda)$, for $0 \notin E(\Lambda)$, showing that (3.6) holds also in case (a_2).

In either case (a_1), (a_2), in view of (3.6), we have

$$\max_{x\in\Lambda} d(x,\overline{\Lambda}) = \max\{d(p,\overline{\Lambda}), d(q,\overline{\Lambda})\} \leq \max\{d(p,\overline{p}), d(q,\overline{q})\},$$

and hence $e(\Lambda,\overline{\Lambda}) \leq \max\{d(p,\overline{p}), d(q,\overline{q})\}$. By the Claim, the latter inequality remains valid by interchanging Λ and $\overline{\Lambda}$. Thus (3.5) holds, and also (jj) is proved.

Observe that the family of the α-convex subsets of Y contains, among other sets, those of the form $C \cap Y$, where C is any convex subset of $\mathbb{R}^2$ containing the origin. Moreover, each convex (in the usual sense) subset of Y is also α-convex, but not conversely. For instance, the set which is the union of the triangles with vertices $(0,0)$, $(a,0)$, $(a,-a)$ and $(0,0)$, $(0,a)$, $(-a,a)$, $a > 0$, is α-convex but not convex. Further, for each $(p,q) \in Y^2$, $\Lambda_\alpha(p,q)$ is convex and $\Lambda_\alpha(p,q) = \Lambda_\alpha(q,p)$.

Remark 3.11. The space Y in Example 3.10 is Lipschitz α-convex, with $r_\alpha = \infty$, but not Lipschitz geodesically α-convex. In fact, for $n \in \mathbb{N}$, by taking $p_n = (-1,3)/n$, $q_n = (3,-1)/n$, $\overline{p}_n = (4,-1)/n$, $\overline{q}_n = (8,-5)/n$, we have $\alpha(p_n,q_n,1/2) = 0$, $\alpha(\overline{p}_n,\overline{q}_n,1/2) = (6,-3)/n$, and hence $d(\alpha(p_n,q_n,1/2), \alpha(\overline{p}_n,\overline{q}_n,1/2)) = \sqrt{45}/n$. Moreover, $d(p_n,\overline{p}_n) = d(q_n,\overline{q}_n) = \sqrt{41}/n$, and thus

$$d(\alpha(p_n,q_n,1/2), \alpha(\overline{p}_n,\overline{q}_n,1/2)) > \max\{d(p_n,\overline{p}_n), d(q_n,\overline{q}_n)\},$$

for every $n \in \mathbb{N}$. Since $d(p_n,\overline{p}_n)$, $d(q_n,\overline{q}_n)$ vanish as $n \to \infty$, it follows that there is no $r_\alpha > 0$ for which condition (c)$'''$ of Definition 3.4 is satisfied.

4. Pseudo-barycenters in α-convex metric spaces

In this section, the notion of pseudo-barycenter in an α-convex metric space is introduced, and some of its properties are reviewed. Here we develop, in a different direction, some ideas which go back Michael [23] and Curtis [6] (see also Pasicki [26]).

Definition 4.1. A nonempty set A, contained in an α-convex metric space Y, is called *α-convex* if, for every $(y_1,y_2) \in A \times A$ and $t \in J$, one has $\alpha(y_1,y_2,t) \in A$.

Remark 4.2. The empty set is assumed to be α-convex. The intersection of a family of α-convex subsets of Y is α-convex. Furthermore, if $A \subset Y$ is α-convex, also its closure $\overline{A}$ is so.

Throughout Y stands for an α-convex metric space. Set:

$$\begin{aligned}\mathcal{K}_\alpha(Y) &= \{A \in 2^Y \mid A \text{ is compact and } \alpha\text{-convex}\},\\ \mathcal{C}_\alpha(Y) &= \{A \in 2^Y \mid A \text{ is closed bounded and } \alpha\text{-convex}\}.\end{aligned}$$

The spaces $\mathcal{K}_\alpha(Y)$, $\mathcal{C}_\alpha(Y)$ are equipped with the Pompeiu–Hausdorff metric h.

Remark 4.3. Each singleton subset of Y is in $\mathcal{K}_\alpha(Y)$ hence in $\mathcal{C}_\alpha(Y)$. Moreover, each set $A \in \mathcal{C}_\alpha(Y)$ is contractible.

Put:

$$\begin{aligned}Y^n &= \{(y_1,\dots,y_n) \mid y_i \in Y,\ i=1,\dots,n\}, && n \geq 1,\\ \Sigma^n &= \{(\lambda_1,\dots,\lambda_n) \mid 0 \leq \lambda_i \leq 1,\ i=1,\dots,n,\ \lambda_1+\dots+\lambda_n = 1\}, && n \geq 1,\\ \Sigma_0^n &= \{(\lambda_1,\dots,\lambda_n) \mid 0 \leq \lambda_i \leq 1,\\ &\qquad i=1,\dots,n-1,\ 0 \leq \lambda_n < 1, \lambda_1+\dots+\lambda_n = 1\} && n \geq 2.\end{aligned}$$

If $(y_1,\dots,y_n) \in Y^n$ and $(\lambda_1,\dots,\lambda_n) \in \Sigma^n$, we say that λ_i is the weight assigned to y_i, $i=1,\dots,n$, or for brevity, that $(\lambda_1,\dots,\lambda_n)$ is the weight assigned to $(y_1,\dots,y_n)$.

For $(y_1,\dots,y_n) \in Y^n$ and $(\lambda_1,\dots,\lambda_n) \in \Sigma^n$, we now define the corresponding pseudo-barycenter $b_n(y_1,\dots,y_n;\lambda_1,\dots,\lambda_n)$, an analogue of the usual notion of barycenter in a normed space.

For $y_1 \in Y^1$ and $\lambda_1 \in \Sigma^1$, i.e. $\lambda_1 = 1$, put

$$b_1(y_1,1) = y_1. \tag{4.1}$$

For $(y_1,y_2) \in Y^2$ and $(\lambda_1,\lambda_2) \in \Sigma^2$, set

$$b_2(y_1,y_2;\lambda_1,\lambda_2) = \alpha(y_1,y_2,\lambda_2). \tag{4.2}$$

Clearly, the maps $b_1: Y \times \Sigma^1 \to Y$, $b_2: Y^2 \times \Sigma^2 \to Y$ given by (4.1), (4.2) are continuous on $Y \times \Sigma^1$, $Y^2 \times \Sigma^2$, respectively. Now suppose that $b_{n-1}: Y^{n-1} \times \Sigma^{n-1} \to Y$, for some $n-1 \geq 2$, has been constructed and that it is continuous on $Y^{n-1} \times \Sigma^{n-1}$. We will define $b_n: Y^n \times \Sigma^n \to Y$ and show that it is continuous on $Y^n \times \Sigma^n$.

To this end, for $(y_1,\dots,y_n) \in Y^n$ and $(\lambda_1,\dots,\lambda_n) \in \Sigma_0^n$, $n \geq 3$, set

$$\begin{aligned}&\widetilde{b}_n(y_1,\dots,y_n;\lambda_1,\dots,\lambda_n)\\ &\qquad = \alpha\bigg(b_{n-1}\bigg(y_1,\dots,y_{n-1};\frac{\lambda_1}{1-\lambda_n},\dots,\frac{\lambda_{n-1}}{1-\lambda_n}\bigg), y_n, \lambda_n\bigg).\end{aligned} \tag{4.3}$$

By the induction assumption the map $\widetilde{b}_n: Y^n \times \Sigma_0^n \to Y$, given by (4.3), is continuous on $Y^n \times \Sigma_0^n$. Further, setting $p = (y_1,\dots,y_n;\lambda_1,\dots,\lambda_n)$, $p_0 =$

$(y_1^0,\dots,y_n^0;0,\dots,1)$, we have

$$\lim_{\substack{p\to p_0\\ p\in Y^n\times\Sigma_0^n}} \widetilde{b}_n(y_1,\dots,y_n;\lambda_1,\dots,\lambda_n)=y_n^0. \tag{4.4}$$

In the contrary case, there exists $\varepsilon>0$ and a sequence $\{(y_1^k,\dots,y_n^k;\lambda_1^k,\dots,\lambda_n^k)\}\subset Y^n\times\Sigma_0^n$ converging to $(y_1^0,\dots,y_n^0;0,\dots,1)$, as $k\to\infty$, such that

$$d(\widetilde{b}_n(y_1^k,\dots,y_n^k;\lambda_1^k,\dots,\lambda_n^k),y_n^0)\geq\varepsilon \quad \text{for every } k\in\mathbb{N}. \tag{4.5}$$

Set $a_k=b_{n-1}(y_1^k,\dots,y_{n-1}^k;\lambda_1^k/(1-\lambda_n^k),\dots,\lambda_{n-1}^k/(1-\lambda_n^k))$. Since

$$\left\{\left(\frac{\lambda_1^k}{1-\lambda_n^k},\dots,\frac{\lambda_{n-1}^k}{1-\lambda_n^k}\right)\right\}\subset\Sigma^{n-1},$$

a compact set, passing to subsequences, without changing notation, we can assume that there is $(\mu_1,\dots,\mu_{n-1})\in\Sigma^{n-1}$ such that

$$\left(y_1^k,\dots,y_{n-1}^k;\frac{\lambda_1^k}{1-\lambda_n^k},\dots,\frac{\lambda_{n-1}^k}{1-\lambda_n^k}\right)$$

converges to $(y_1^0,\dots,y_{n-1}^0;\mu_1,\dots,\mu_{n-1})$, as $k\to\infty$. Since b_{n-1} is continuous on $Y^{n-1}\times\Sigma^{n-1}$, one has that for $k\to\infty$, a_k converges to $a=b_{n-1}(y_1^0,\dots,y_{n-1}^0;\mu_1,\dots,\mu_{n-1})$. By the continuity of α, it follows that $\alpha(a_k,y_n^k,\lambda_n^k)$ converges to $\alpha(a,y_n^0,1)$, when $k\to\infty$. But $\alpha(a_k,y_n^k,\lambda_n^k)=\widetilde{b}_n(y_1^k,\dots,y_n^k;\lambda_1^k,\dots,\lambda_n^k)$ and $\alpha(a,y_n^0,1)=y_n^0$, thus a contradiction to (4.5) follows, and (4.4) holds.

Define $b_n\colon Y^n\times\Sigma^n\to Y$ by

$$b_n(y_1,\dots,y_n;\lambda_1,\dots,\lambda_n)=\begin{cases}\widetilde{b}_n(y_1,\dots,y_n;\lambda_1,\dots,\lambda_n) & \text{if } (\lambda_1,\dots\lambda_n)\in\Sigma_0^n,\\ y_n & \text{if } (\lambda_1,\dots,\lambda_n)=(0,\dots,1).\end{cases} \tag{4.6}$$

In view of (4.3) and (4.4), the function b_n is continuous on $Y^n\times\Sigma^n$.

Definition 4.4. For $(y_1,\dots,y_n)\in Y^n$ and $(\lambda_1,\dots,\lambda_n)\in\Sigma^n$, $n\geq 1$, the point $b_n(y_1,\dots,y_n;\ \lambda_1,\dots,\lambda_n)$ given by (4.1) if $n=1$, by (4.2) if $n=2$, and by (4.6) if $n\geq 3$, is called pseudo-barycenter of $(y_1,\dots,y_n)$ with weight $(\lambda_1,\dots,\lambda_n)$.

By the previous argument one has:

Proposition 4.5. *For each $n\in\mathbb{N}$ the pseudo-barycenter function $b_n\colon Y^n\times\Sigma^n\to Y$ is continuous on $Y^n\times\Sigma^n$. Furthermore, $b_n(y_0,\dots,y_0;\lambda_1,\dots,\lambda_n)=y_0$ for every $y_0\in Y$ and $(\lambda_1,\dots,\lambda_n)\in\Sigma^n$, and $b_n(y_1,\dots,y_n;1,\dots,0)=y_1$, $b_n(y_1,\dots,y_n;0,1,\dots,0)=y_2,\dots,b_n(y_1,\dots,y_n;0,\dots,1)=y_n$.*

Remark 4.6. Let $C\subset Y$ be α-convex. Then, for every $(y_1,\dots,y_n)\in C^n$ and $(\lambda_1,\dots,\lambda_n)\in\Sigma^n$, $n\in\mathbb{N}$, the pseudo-barycenter $b_n(y_1,\dots,y_n;\lambda_1,\dots,\lambda_m)$ lies in C.

Remark 4.7. The pseudo-barycenter depends on the ordered n-tuple of points $(y_1,\dots, y_n)$ in the sense that, if $(y_{i_1},\dots,y_{i_n})$ and $(\lambda_{i_1},\dots,\lambda_{i_n})$ are arbitrary permutations of $(y_1,\dots,y_n)$ and $(\lambda_1,\dots,\lambda_n)$, it can happen that

$$b_n(y_{i_1},\dots,y_{i_n};\lambda_{i_1},\dots,\lambda_{i_n}) \neq b_n(y_1,\dots,y_n;\lambda_1,\dots,\lambda_n).$$

Remark 4.8. Let $\mathbb{E}$ be a normed space equipped with its natural convexity mapping α_0 given by (3.3). In this case the α_0-convex sets are the usual convex sets, and we set

$$\mathcal{K}(\mathbb{E}) = \mathcal{K}_{\alpha_0}(\mathbb{E}), \quad \mathcal{C}(\mathbb{E}) = \mathcal{C}_{\alpha_0}(\mathbb{E}).$$

Moreover, for each $(y_1,\dots,y_n)\in\mathbb{E}^n$ with weight $(\lambda_1,\dots,\lambda_n)\in\Sigma^n$, the pseudo-barycenter reduces to the barycenter, i.e.

$$b_n(y_1,\dots,y_n;\lambda_1,\dots,\lambda_n) = \lambda_1 y_1 + \ldots + \lambda_n y_n.$$

In view of Proposition 4.5 one has:

Proposition 4.9. *Let $(y_1,\dots,y_n)\in Y^n$, and let $\lambda_i\colon M\to[0,1]$, $i=1,\dots,n$, be n continuous functions defined on a metric space M, such that $\lambda_1(x)+\ldots+\lambda_n(x)=1$ for every $x\in M$. Then the function $\Phi\colon M\to Y$ given by*

$$\Phi(x) = b_n(y_1,\dots,y_n;\lambda_1(x),\dots,\lambda_n(x))$$

is continuous.

Proposition 4.10. *Let $(y_1,\dots,y_n)\in Y^n$ and $(\lambda_1,\dots,\lambda_n)\in\Sigma^n$, $n\ge 2$. Let $(i_1,\dots,i_k)$, $1\le k\le n-1$, be a subset of $(1,\dots,n)$ with $i_1<\ldots<i_k$, such that*

$$\lambda_i>0 \quad \text{if } i\in\{i_1,\dots,i_k\}, \qquad \lambda_i=0 \quad \text{if } i\in\{1,\dots,n\}\setminus\{i_1,\dots,i_k\}.$$

Then one has:

$$(4.7)_n \qquad b_n(y_1,\dots,y_n;\lambda_1,\dots,\lambda_n) = b_k(y_{i_1},\dots,y_{i_k};\lambda_{i_1},\dots,\lambda_{i_k}).$$

Proof. The statement is true for $n=2$. If, for some $n\ge 3$, $(4.7)_{n-1}$ is true, it will be proved that also $(4.7)_n$ is so. Let $(\lambda_{i_1},\dots,\lambda_{i_k})$ be as in the statement.

Case 1. $i_k\le n-1$. Hence $\lambda_n=0$, and thus

$$\begin{aligned} b_n(y_1,\dots,y_n;\lambda_1,\dots,\lambda_n) &= \alpha(b_{n-1}(y_1,\dots,y_{n-1};\lambda_1,\dots,\lambda_{n-1}),y_n,0)\\ &= b_{n-1}(y_1,\dots,y_{n-1};\lambda_1,\dots,\lambda_{n-1})\\ &= b_k(y_{i_1},\dots,y_{i_k};\lambda_{i_1},\dots,\lambda_{i_k}), \end{aligned}$$

where the latter equality is obvious, if $k=n-1$, while it follows from the induction assumption, if $1\le k\le n-2$. Therefore, if $i_k\le n-1$, $(4.7)_n$ is true.

Case 2. $i_k = n$. Thus, $\lambda_n = \lambda_{i_k} > 0$. If $\lambda_{i_k} < 1$, then $k \geq 2$. We have

$$b_n(y_1, \dots, y_n; \lambda_1, \dots, \lambda_n) = \alpha\bigg(b_{n-1}\bigg(y_1, \dots, y_{n-1}; \frac{\lambda_1}{1-\lambda_n}, \dots, \frac{\lambda_{n-1}}{1-\lambda_n}\bigg), y_n, \lambda_n\bigg).$$

As $(\lambda_1/(1-\lambda_n), \dots, \lambda_{n-1}/(1-\lambda_n)) \in \Sigma^{n-1}$ and $1 \leq k-1 \leq n-1$, the induction hypothesis implies

$$b_{n-1}\bigg(y_1, \dots, y_{n-1}; \frac{\lambda_1}{1-\lambda_n}, \dots, \frac{\lambda_{n-1}}{1-\lambda_n}\bigg) = b_{k-1}\bigg(y_{i_1}, \dots, y_{i_{k-1}}; \frac{\lambda_{i_1}}{1-\lambda_{i_k}}, \dots, \frac{\lambda_{i_{k-1}}}{1-\lambda_{i_k}}\bigg).$$

Whence,

$$\begin{aligned} b_n(y_1, \dots, y_n; \lambda_1, \dots, \lambda_n) &= \alpha\bigg(b_{k-1}\bigg(y_{i_1}, \dots, y_{i_{k-1}}; \frac{\lambda_{i_1}}{1-\lambda_{i_k}}, \dots, \frac{\lambda_{i_{k-1}}}{1-\lambda_{i_k}}\bigg), y_{i_k}, \lambda_{i_k}\bigg) \\ &= b_k(y_{i_1}, \dots, y_{i_k}; \lambda_{i_1}, \dots, \lambda_{i_k}). \end{aligned} \tag{4.8}$$

If $\lambda_{i_k} = 1$, one has $k = 1$, for $\lambda_1 = \ldots = \lambda_{n-1} = 0$, and thus

$$b_n(y_1, \dots, y_n; \lambda_1, \dots, \lambda_n) = b_n(y_1, \dots, y_n; 0, \dots, 1) = y_n = b_1(y_{i_1}, 1). \tag{4.9}$$

In view of (4.8) and (4.9), $(4.7)_n$ is true, if $i_k = n$. Hence in both Cases 1 and 2, $(4.7)_n$ holds, completing the proof. □

Let Y be an α-convex metric space. Given an ordered n-tuple $(y_1, \dots, y_n) \in Y^n$, $n \geq 2$, the set $\Lambda_\alpha(y_1, \dots, y_n)$ given by

$$\Lambda_\alpha(y_1, \dots, y_n) = \{z \in Y \mid z = b_n(y_1, \dots, y_n; \lambda_1, \dots, \lambda_n) \text{ for some } (\lambda_1, \dots, \lambda_n) \in \Sigma^n\}.$$

we agree to call $(y_1, \dots, y_n)$*-locus induced by* b_n,

Remark 4.11. For $n = 2$, the (y_1, y_2)-locus induced by b_2 coincides with the (y_1, y_2)-locus induced by α, given by (3.1). Clearly $\Lambda_\alpha(y_1, \dots, y_n)$ is a compact set. Further, if $(y_{i_1}, \dots, y_{i_n})$ is an arbitrary permutation of $(y_1, \dots, y_n)$, one can have that $\Lambda_\alpha(y_{i_1}, \dots, y_{i_n}) \neq \Lambda_\alpha(y_1, \dots, y_n)$.

The last statement of the following proposition is a kind of stability property which is very useful in approximation problems for multifunctions.

Proposition 4.12. *Let Y be an α-convex metric space, and let r_α correspond as in Definition* 3.1. *Then, for each $0 < \varepsilon < r_\alpha$, there exists $0 < \eta \leq \varepsilon$ such that, for every $(y_1, \dots, y_n)$, $(z_1, \dots, z_n) \in Y^n$, $n \geq 2$ arbitrary, with $d(y_i, z_i) < \eta$, $i = 1, \dots, n$, one has*

$$h(\Lambda_\alpha(y_1, \dots, y_n), \Lambda_\alpha(z_1, \dots, z_n)) < \varepsilon. \tag{4.10_n}$$

Moreover, for every nonempty α-convex set $C \subset Y$, $(y_1, \dots, y_n) \in Y^n$ with $d(y_i, C) < \eta$, $i = 1, \dots, n$, and $(\lambda_1, \dots, \lambda_n) \in \Sigma^n$, $n \geq 2$, one has

$$d(b_n(y_1, \dots, y_n; \lambda_1, \dots, \lambda_n), C) < \varepsilon.$$

Proof. Let $0 < \varepsilon < r_\alpha$, and let $0 < \eta \leq \varepsilon$ be as in Definition 3.1.

To show $(4.10)_n$ is equivalent to prove that, for every $(\lambda_1, \dots, \lambda_n) \in \Sigma^n$, there exists $(\mu_1, \dots, \mu_n) \in \Sigma^n$ such that

$$(4.11)_n \qquad d(b_n(y_1, \dots, y_n; \lambda_1, \dots, \lambda_n), b_n(z_1, \dots, z_n; \mu_1, \dots, \mu_n)) < \varepsilon,$$

and, furthermore, that the analogous inequality, obtained by interchanging the roles of $(\lambda_1, \dots, \lambda_n)$ and $(\mu_1, \dots, \mu_n)$, holds as well.

It suffices to prove $(4.11)_n$, as the proof of the other inequality is similar.

For $n = 2$, $(4.11)_n$ holds, by Definition 3.1. It will be shown that $(4.11)_{n+1}$ is true provided that $(4.11)_n$ is so, for some $n \geq 2$.

Let $(y_1, \dots, y_{n+1}), (z_1, \dots, z_{n+1}) \in Y^{n+1}$, with $d(y_i, z_i) < \eta$, $i = 1, \dots, n+1$, be given, and let $(\lambda_1, \dots, \lambda_{n+1}) \in \Sigma^{n+1}$ be arbitrary.

Let $0 \leq \lambda_{n+1} < 1$. Setting

$$c_n = b_n(y_1, \dots, y_n; \lambda_1/(1-\lambda_{n+1}), \dots, \lambda_n/(1-\lambda_{n+1})),$$

one has

$$(4.12) \qquad b_{n+1}(y_1, \dots, y_{n+1}; \lambda_1, \dots, \lambda_{n+1}) = \alpha(c_n, y_{n+1}, \lambda_{n+1}).$$

Since $c_n \in \Lambda_\alpha(y_1, \dots, y_n)$, by the induction hypothesis there is a point $e_n = b_n(z_1, \dots, z_n;\ \theta_1, \dots, \theta_n)$, for some $(\theta_1, \dots, \theta_n) \in \Sigma^n$, such that $d(c_n, e_n) < \varepsilon$. From this and the hypothesis $d(y_{n+1}, z_{n+1}) < \eta$, in view of Definition 3.1(c), one has $h(\Lambda_\alpha(c_n, y_{n+1}), \Lambda_\alpha(e_n, z_{n+1})) < \varepsilon$. Further, $\alpha(c_n, y_{n+1}, \lambda_{n+1}) \in \Lambda_\alpha(c_n, y_{n+1})$ and thus there exists $0 \leq \rho \leq 1$ such that

$$(4.13) \qquad d(\alpha(c_n, y_{n+1}, \lambda_{n+1}), \alpha(e_n, z_{n+1}, \rho)) < \varepsilon.$$

As $\alpha(e_n, z_{n+1}, \cdot\,)$ is continuous, without loss of generality, one can assume that $\rho < 1$. Put now $(\mu_1, \dots, \mu_{n+1}) = ((1-\rho)\theta_1, \dots, (1-\rho)\theta_n, \rho)$, and observe that $(\mu_1, \dots, \mu_{n+1}) \in \Sigma^{n+1}$. Since

$$\mu_{n+1} < 1 \quad \text{and} \quad (\mu_1/(1-\mu_{n+1}), \dots, \mu_n/(1-\mu_{n+1})) = (\theta_1, \dots, \theta_n),$$

one has

$$\alpha\bigg(b_n\bigg(z_1, \dots, z_n; \frac{\mu_1}{1-\mu_{n+1}}, \dots, \frac{\mu_n}{1-\mu_{n+1}}\bigg), z_{n+1}, \mu_{n+1}\bigg) = \alpha(e_n, z_{n+1}, \rho),$$

that is,

$$(4.14) \qquad b_{n+1}(z_1, \dots, z_{n+1}; \mu_1, \dots, \mu_{n+1}) = \alpha(e_n, z_{n+1}, \rho).$$

Combining (4.13) with (4.12) and (4.14) gives

$$(4.15)\quad d(b_{n+1}(y_1,\dots,y_{n+1};\lambda_1,\dots,\lambda_{n+1}),\\ b_{n+1}(z_1,\dots,z_{n+1};\mu_1,\dots,\mu_{n+1}))<\varepsilon.$$

Let $\lambda_{n+1}=1$. Clearly, $(\lambda_1,\dots,\lambda_{n+1})=(0,\dots,1)$. Thus, by taking $(\mu_1,\dots,\mu_{n+1})=(0,\dots,1)$, (4.15) follows trivially, as the left hand side equals $d(y_{n+1},z_{n+1})$ and so it is strictly less than $\eta\le\varepsilon$. Therefore $(4.11)_n$ holds for every $n\ge 2$.

The second statement follows from the previous one and the α-convexity of C. This completes the proof. □

Remark 4.13. Let $(y_1,\dots,y_{n+1})\in Y^{n+1}$, where $y_k=y_{k+1}$ for some $1\le k\le n$, and let $(\lambda_1,\dots,\lambda_{n+1})\in\Sigma^{n+1}$ be arbitrary. Unlike the barycenter, for the pseudo-barycenter it can happen that $b_{n+1}(y_1,\dots,y_{n+1};\ \lambda_1,\dots,\lambda_{n+1})\neq b_n(y_1,\dots,y_k,y_{k+2},\dots,y_{n+1};\ \lambda_1,\dots,\lambda_k+\lambda_{k+1},\ \lambda_{k+2},\dots,\lambda_{n+1})$.

Remark 4.14. The definition of pseudo-barycenter and some of its properties, including Proposition 4.5, 4.9 and 4.10, remain valid if in Definition 3.1 the continuous mapping $a\colon Y\times Y\times J\to Y$ satisfies only conditions (a), (b). Condition (c) plays a crucial role in the proof of Proposition 4.12.

Proposition 4.15 (Dugundji [8, p. 83]). *Let X, Z be topological spaces. Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a covering of X, where the sets $A_\lambda\subset X$ are open, and let $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ be a family of continuous functions $\varphi_\lambda\colon A_\lambda\to Z$ such that for every λ, $\mu\in\Lambda$, with $A_\lambda\cap A_\mu\neq\phi$,*

$$\varphi_\lambda(x)=\varphi_\mu(x)\quad \text{for every } x\in A_\lambda\cap A_\mu.$$

Then, there is a unique continuous function $f\colon X\to Z$, which is an extension of each φ_λ, that is, for each $\lambda\in\Lambda$,

$$f(x)=\varphi_\lambda(x)\quad \text{for every } x\in A_\lambda.$$

5. Approximate selections

In this section we present an α-convex version of an approximate selection theorem established by Cellina [3] in linear spaces.

The *support* of a map $f\colon M\to\mathbb{R}$, M a metric space, is the closed set $\operatorname{supp} f=\overline{\{x\in M\mid f(x)\neq 0\}}$.

Proposition 5.1. *Let M be a metric space, Y an α-convex metric space, C an α-convex subset of Y. Let $F\colon M\to\mathcal{C}_\alpha(Y)$ be a Pompeiu–Hausdorff upper semicontinuous multifunction such that $F(x)\subset C$, for every $x\in M$. Then, for each $\varepsilon>0$, there exists a continuous function $f_\varepsilon\colon M\to C$ such that*

$$(5.1)\qquad e(\operatorname{graph} f_\varepsilon,\operatorname{graph} F)\le\varepsilon.$$

Proof. Let $0 < \varepsilon < r_\alpha$, where r_α is as in Definition 3.1. By Proposition 4.12 there exists $0 < \eta \leq \varepsilon$ such that, for every nonempty α-convex set $C \subset Y$, $(y_1, \ldots , y_n) \in Y^n$, with $d(y_i, C) < \eta$, $i = 1, \ldots , n$, and $(\lambda_1, \ldots , \lambda_n) \in \Sigma^n$, $n \geq 1$ arbitrary, one has

$$d(b_n(y_1, \ldots , y_n; \lambda_1, \ldots , \lambda_n), C) < \varepsilon. \tag{5.2}$$

Since F is h-u.s.c. for every $x \in M$ there exists a $\sigma(x)$, with

$$0 < \sigma(x) < \eta, \tag{5.3}$$

such that

$$x' \in U(x, \sigma(x)) \quad \text{implies} \quad e(F(x'), F(x)) < \eta. \tag{5.4}$$

Arguing as in Hu and Papageorgiou in [16, Theorem 4.11, p. 106], consider the family $\mathcal{U} = \{U(x, \sigma(x)/4)\}_{x\in M}$. $\mathcal{U}$ is an open covering of M, a paracompact space, and thus it admits an open neighbourhood finite refinement $\mathcal{V} = \{V_\beta\}_{\beta\in B}$. For every $V_\beta \in \mathcal{V}$, the set

$$\mathcal{F}(V_\beta) = \{U(x, \sigma(x)/4) \mid V_\beta \subset U(x, \sigma(x)/4)\}$$

is nonempty. In each $\mathcal{F}(V_\beta)$ fix one set, say $U(x_\beta, \sigma(x_\beta)/4)$. Furthermore, with each $V_\beta \in \mathcal{V}$, associate a point $(u_\beta, y_\beta) \in M \times Y$, where

$$u_\beta \in V_\beta \quad \text{and} \quad y_\beta \in F(u_\beta). \tag{5.5}$$

In view of Dugundji [8, p. 170], there is a partition $\{p_{V_\beta}\}_{\beta\in B}$ of unity subordinated to $\mathcal{V}$, i.e. a family of continuous functions $p_{V_\beta} : M \to [0, 1]$ such that:

(j) $\operatorname{supp} p_{V_\beta} \subset V_\beta$ for every $\beta \in B$,

(jj) $\{\operatorname{supp} p_{V_\beta}\}_{\beta\in B}$ is a neighbourhood finite closed covering of M,

(jjj) $\sum_{\beta\in B} p_{V_\beta}(x) = 1$ for every $x \in M$.

By Zermelo's theorem [8, p. 31], $\mathcal{V}$ admits a partial ordering $\prec$ which makes $\mathcal{V}$ into a well ordered set. In the sequel $\mathcal{V}$ is assumed to be equipped with the well ordering $\prec$.

Let $u \in M$ be arbitrary. Since $\mathcal{V}$ is neighbourhood finite, there exists an open neighbourhood W_u of u such that the family

$$\mathcal{V}_{W_u} = \{V_\beta \in \mathcal{V} \mid V_\beta \cap W_u \neq \phi\}$$

is nonempty and finite, say

$$\mathcal{V}_{W_u} = (V_{\beta_1}, \ldots , V_{\beta_k}) \quad \text{where } V_{\beta_1} \prec \ldots \prec V_{\beta_k},$$

for some $k \geq 1$. Let $(u_{\beta_1}, \ldots , u_{\beta_k})$, $(y_{\beta_1}, \ldots , y_{\beta_k})$ and $(U(x_{\beta_1}, \sigma(x_{\beta_1})/4), \ldots , U(x_{\beta_k}, \sigma(x_{\beta_k})/4))$ correspond. For $x \in W_u$, set

$$\varphi_{W_u}(x) = b_k(y_{\beta_1}, \ldots , y_{\beta_k}; p_{V_{\beta_1}}(x), \ldots , p_{V_{\beta_k}}(x)).$$

Clearly $\varphi_{W_u}(x) \in C$, thus the above equality defines a mapping $\varphi_{W_u}: W_u \to C$ which is continuous on W_u, in view of Proposition 4.9. It will be shown that

$$e(\operatorname{graph} \varphi_{W_u}, \operatorname{graph} F) < \varepsilon. \tag{5.6}$$

To this end, let $x \in W_u$ be arbitrary. The set $\mathcal{V}^x_{W_u}$ of all $V_\beta \in \mathcal{V}$ such that $p_{V_\beta}(x) > 0$ is a nonempty subset of $\mathcal{V}_{W_u}$, for $x \in \operatorname{supp} p_{V_\beta} \subset V_\beta$. Hence for some $1 \le p \le k$ and $1 \le i_1 < \ldots < i_p \le k$, one has

$$\mathcal{V}^x_{W_u} = (V_{\beta_{i_1}}, \ldots, V_{\beta_{i_p}}) \quad \text{where } V_{\beta_{i_1}} \prec \ldots \prec V_{\beta_{i_p}}, \tag{5.7}$$

and thus Proposition 4.10 implies

$$\varphi_{W_u}(x) = b_p(y_{\beta_{i_1}}, \ldots, y_{\beta_{i_p}}; p_{V_{\beta_{i_1}}}(x), \ldots, p_{V_{\beta_{i_p}}}(x)). \tag{5.8}$$

Clearly,

$$x \in \operatorname{supp} p_{V_{\beta_{i_s}}} \subset V_{\beta_{i_s}} \subset U(x_{\beta_{i_s}}, \sigma(x_{\beta_{i_s}})/4), \quad s = 1, \ldots, p. \tag{5.9}$$

Set $\sigma(x_{\beta_{i_m}}) = \max\{\sigma(x_{\beta_{i_s}}) \mid s = 1, \ldots, p\}$, for some $1 \le m \le p$. In view of (5.9), one has

$$\begin{aligned} d(x_{\beta_{i_s}}, x_{\beta_{i_m}}) &\le d(x_{\beta_{i_s}}, x) + d(x, x_{\beta_{i_m}}) \\ &< \frac{1}{4}\sigma(x_{\beta_{i_s}}) + \frac{1}{4}\sigma(x_{\beta_{i_m}}) \le \frac{1}{2}\sigma(x_{\beta_{i_m}}), \end{aligned} \tag{5.10}$$

for $s = 1, \ldots, p$. Whence,

$$V_{\beta_{i_s}} \subset U(x_{\beta_{i_s}}, \sigma(x_{\beta_{i_s}})/4) \subset U(x_{\beta_{i_m}}, \sigma(x_{\beta_{i_m}})) \tag{5.11}$$

for $s = 1, \ldots, p$, if $z \in U(x_{\beta_{i_s}}, \sigma(x_{\beta_{i_s}})/4)$, by virtue of (5.10) one has

$$d(z, x_{\beta_{i_m}}) \le d(z, x_{\beta_{i_s}}) + d(x_{\beta_{i_s}}, x_{\beta_{i_m}}) < \frac{1}{4}\sigma(x_{\beta_{i_s}}) + \frac{1}{2}\sigma(x_{\beta_{i_m}}) < \sigma(x_{\beta_{i_m}}).$$

From (5.11) it follows that $u_{\beta_{i_s}} \in U(x_{\beta_{i_m}}, \sigma(x_{\beta_{i_m}}))$, for $u_{\beta_{i_s}} \in V_{\beta_{i_s}}$. Then $e(F(u_{\beta_{i_s}}), F(x_{\beta_{i_m}})) < \eta$, by (5.4), and a fortiori

$$d(y_{\beta_{i_s}}, F(x_{\beta_{i_m}})) < \eta, \quad s = 1, \ldots, p,$$

for $y_{\beta_{i_s}} \in F(u_{\beta_{i_s}})$. In view of (5.2), one has

$$d(b_p(y_{\beta_{i_1}}, \ldots, y_{\beta_{i_p}}; p_{V_{\beta_{i_1}}}(x), \ldots, p_{V_{\beta_{i_p}}}(x)), F(x_{\beta_{i_m}})) < \varepsilon$$

and, by (5.8),

$$d(\varphi_{W_u}(x), F(x_{\beta_{i_m}})) < \varepsilon. \tag{5.12}$$

On the other hand $x \in \operatorname{supp} p_{V_{\beta_{i_s}}}$, and thus (5.11) yields $x \in U(x_{\beta_{i_m}}, \sigma(x_{\beta_{i_m}}))$. By (5.3), $\sigma(x_{\beta_{i_m}}) < \eta \le \varepsilon$, and hence

$$d(x, x_{\beta_{i_m}}) < \varepsilon. \tag{5.13}$$

From (5.12) and (5.13), since $x \in W_u$ is arbitrary, (5.6) follows.

Now define $f_\varepsilon\colon M \to C$ by

$$f_\varepsilon(x) = \varphi_{W_u}(x) \quad \text{if } x \in W_u \text{ for some } u \in M.$$

The family $\{W_u\}_{u\in M}$ is an open covering of M. The above definition is meaningful if one shows that, for every W_u, $W_{u'}$, $u, u' \in M$, with $W_u \cap W_{u'} \neq \phi$, one has

$$\varphi_{W_u}(x) = \varphi_{W_{u'}}(x) \quad \text{for every } x \in W_u \cap W_{u'}. \tag{5.14}$$

In fact, let $x \in W_u \cap W_{u'}$. Clearly

$$\mathcal{V}^x_{W_u} = \{V_\beta \in \mathcal{V}_{W_u} \mid p_{V_\beta}(x) > 0\} = \{V_\beta \in \mathcal{V}_{W_{u'}} \mid p_{V_\beta}(x) > 0\} = \mathcal{V}^x_{W_{u'}}.$$

Whence, in view of (5.7) one has $\mathcal{V}^x_{W_{u'}} = \mathcal{V}^x_{W_u} = (V_{\beta_{i_1}}, \ldots, V_{\beta_{i_p}})$, where $V_{\beta_{i_1}} \prec \ldots \prec V_{\beta_{i_p}}$. Therefore, if $u_{\beta_{i_s}} \in V_{\beta_{i_s}}$ and $y_{\beta_{i_s}} \in F(u_{\beta_{i_s}})$ correspond to $V_{\beta_{i_s}}$, $s = 1, \ldots, p$, according (5.5), by Proposition 4.10 one has

$$\varphi_{W_u}(x) = b_p(y_{\beta_1}, \ldots, y_{\beta_p}; p_{V_{\beta_1}}(x), \ldots, p_{V_{\beta_p}}(x)) = \varphi_{W_{u'}}(x),$$

and thus (5.14) follows, as $x \in W_u \cap W_{u'}$ is arbitrary. Since each φ_{W_u} is continuous, also f_ε is so, by Proposition 4.15. Furthermore, in view of (5.6), f_ε satisfies (5.1). This completes the proof. □

Corollary 5.2. *Let M be a metric space, Y an α-convex metric space, C a compact α-convex subset of Y. Let $F\colon M \to \mathcal{K}_\alpha(Y)$ be a h-u.s.c. multifunction such that $F(x) \subset C$, for every $x \in M$. Then, for each $\varepsilon > 0$, there exists a continuous and compact function $f_\varepsilon\colon M \to C$ such that*

$$e(\operatorname{graph} f_\varepsilon, \operatorname{graph} F) < \varepsilon. \tag{5.15}$$

Remark 5.3. Proposition 5.1 and Corollary 5.2 remain valid for multifunctions with α-convex bounded values, as one can easily see from the above proofs.

Corollary 5.4. *Let M be a metric space, and $\mathbb{E}$ a Banach space.*

(a) *If $F\colon M \to \mathcal{C}(\mathbb{E})$ is h-u.s.c. multifunction, then, for each $\varepsilon > 0$ there exists a continuous function $f_\varepsilon\colon M \to \mathbb{E}$ which satisfies* (5.15).

(b) *If $F\colon M \to \mathcal{K}(\mathbb{E})$ is a h-u.s.c. multifunction with precompact range $R = \bigcup_{x\in M} F(x)$ then, for each $\varepsilon > 0$, there exists a continuous and compact function $f_\varepsilon\colon M \to \mathbb{E}$ satisfying* (5.15), *with values $f_\varepsilon(x) \in \overline{\text{co}}\, R$, for every $x \in M$.*

Proof. (a) follows from Proposition 5.1 and Remark 4.7. (b) follows from Corollary 5.2, by taking $C = \overline{\text{co}}\, R$, a convex compact set by Mazur's theorem.

The following proposition is known yet, for the sake of completeness, the proof is included. □

Proposition 5.5. *Let M be a metric space, and Y an α-convex metric space. Let $F: M \to \mathcal{C}_\alpha(Y)$ be h-u.s.c. and let $\{f_n\}$ be a sequence of continuous functions $f_n: M \to Y$ such that $f_n \xrightarrow{\mathrm{gr}} F$ as $n \to \infty$. If $x_n \to x$ and $f_n(x_n) \to y$ as $n \to \infty$, then one has $y \in F(x)$.*

Proof. Let $\varepsilon > 0$, and denote by d, d_1, ρ the metric of M, Y, $M \times Y$, respectively. Take $n_0 \in \mathbb{N}$ so that $n \geq n_0$ implies

$$\rho((z, f_n(z)), \operatorname{graph} F) \leq e(\operatorname{graph} f_n, \operatorname{graph} F) < \varepsilon/2 \quad \text{for every } z \in M.$$

Thus, for each $n \geq n_0$, there exists (ξ_n, η_n), where $\xi_n \in M$ and $\eta_n \in F(\xi_n)$, such that $\rho((x_n, f_n(x_n)), (\xi_n, \eta_n)) < \varepsilon/2$. Whence,

$$d(x_n, \xi_n) < \varepsilon/2, \quad d_1(f_n(x_n), \eta_n) < \varepsilon/2 \quad \text{for every } n \geq n_0.$$

Since $x_n \to x$ and $f_n(x_n) \to y$, as $n \to \infty$, for n large enough, say $n \geq n_1 \geq n_0$, one has

$$\begin{aligned} d(\xi_n, x) &\leq d(\xi_n, x_n) + d(x_n, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon \\ d_1(\eta_n, y) &\leq d_1(\eta_n, f_n(x_n)) + d_1(f_n(x_n), y) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Consequently $(\xi_n, \eta_n) \to (x, y)$, as $n \to \infty$. Since $\eta_n \in F(\xi_n)$, and F is h-u.s.c. with closed values, letting $n \to \infty$ gives $y \in F(x)$, completing the proof. □

6. Topological degree

In this section we use the approximate continuous selection result established in Section 5 in order to define the topological degree for a class of multifunctions with α-convex values. When the values are convex, this reduces to the topological degree defined by Hukuhara in [17] and Cellina and Lasota in [4].

Throughout $\mathbb{E}$ is a real Banach space, D a nonempty open bounded subset of $\mathbb{E}$, and p a point of $\mathbb{E}$. I is the identity mapping on $\mathbb{E}$, and O the origin of $\mathbb{E}$.

Furthermore, Y is a closed subset of $\mathbb{E}$ containing O, equipped with a convexity mapping $\alpha: Y \times Y \times J \to Y$, i.e. Y is an α-convex metric space in the sense of Definition 3.1.

Denote by $\mathcal{F}(\overline{D}, \mathcal{K}_\alpha(Y))$ the set of all multifunctions $F: \overline{D} \to \mathcal{K}_\alpha(Y)$ such that:

- (j) F is h-u.s.c.
- (jj) F is compact, i.e. there is a compact α-convex set $A \subset Y$ (A depending on F) such that $F(x) \subset A$, for every $x \in \overline{D}$.

Occasionally, the set A in (jj) corresponding to F is denoted by A_F.

Remark 6.1. Since $O \in Y$ and, by Remark 4.3, $\mathcal{K}_\alpha(Y)$ contains all singleton subsets of Y, it follows that the mapping Θ defined by $\Theta(x) = \{0\}$, for every $x \in \overline{D}$, is an element of $\mathcal{F}(\overline{D}, \mathcal{K}_\alpha(Y))$ and, obviously, of $\mathcal{F}(\overline{D}, \mathcal{K}(\mathbb{E}))$.

For $F \in \mathcal{F}(\overline{D}, \mathcal{K}_\alpha(Y))$, put:

$$\mathcal{A}_F(\overline{D}, A) = \{\{f_n\} \mid f_n: \overline{D} \to A \text{ is continuous compact, } f_n \xrightarrow{\text{gr}} F\},$$

where $\{f_n\}$ stands for $\{f_n\}_{n=1}^\infty$. By Corollary 5.2, one has:

Proposition 6.2. *$\mathcal{A}_F(\overline{D}, A)$ is nonempty, for every $F \in \mathcal{F}(\overline{D}, \mathcal{K}_\alpha(Y))$.*

Definition 6.3. Let $F \in \mathcal{F}(\overline{D}, \mathcal{K}_\alpha(Y))$, and let $p \notin \bigcup_{x \in \partial D}(I - F)(x)$. Let $\{f_n\} \in \mathcal{A}_F(\overline{D}, A)$. The topological degree $\mathrm{Deg}(I - F, D, p)$ of $I - F$ at p relative to D is defined by

$$\mathrm{Deg}(I - F, D, p) = \lim_{n \to \infty} \deg(I - f_n, D, p), \tag{6.1}$$

where $\deg(I - f_n, D, p)$ denotes the Leray–Schauder topological degree of $I - f_n$ at p relative to D.

In the sequel we shall use some properties of the Leray–Schauder topological degree, that can be found in Istrătescu [18] and Llyod [20].

The above definition is meaningful by virtue of the following proposition.

Proposition 6.4. *Let $F \in \mathcal{F}(\overline{D}, \mathcal{K}_\alpha(Y))$, and let $p \notin \bigcup_{x \in \partial D}(I - F)(x)$. Let $\{f_n\}$, $\{g_n\} \in \mathcal{A}_F(\overline{D}, A)$. Then one has*

(a) *There exists $n_0 \in \mathbb{N}$ such that*

$$\deg(I - f_n, D, p) = \deg(I - f_m, D, p) \quad \text{for all } n, m \geq n_0. \tag{6.2}$$

(b) *There exists $n_0 \in \mathbb{N}$ such that*

$$\deg(I - f_n, D, p) = \deg(I - g_n, D, p) \quad \text{for all } n \geq n_0. \tag{6.3}$$

Proof. By Proposition 6.2, the set $\mathcal{A}_F(\overline{D}, A)$ is nonempty.

(b) Let $\{f_n\} \in \mathcal{A}_F(\overline{D}, A)$. Define $H_{n,m}: \overline{D} \times [0,1] \to Y$ by

$$H_{n,m}(x, t) = \alpha(f_n(x), f_m(x), t).$$

Clearly $H_{n,m}$ is well defined, continuous and compact.

There is $n_0 \in \mathbb{N}$ such that

$$p \notin \bigcup_{n,m \geq n_0} \bigcup_{(x,t) \in \partial D \times [0,1]} (x - H_{n,m}(x, t)). \tag{6.4}$$

Supposing the contrary, there exist subequences $\{n_k\}$, $\{m_k\} \subset \mathbb{N}$ and a sequence $\{(x_k, t_k)\} \subset \partial D \times [0,1]$, such that

$$p = x_k - \alpha(f_{n_k}(x_k), f_{m_k}(x_k), t_k) \quad \text{for every } k \in \mathbb{N}. \tag{6.5}$$

Hence, for all $k \in \mathbb{N}$, one has $x_k \in p + \alpha(A, A, [0,1])$, where the latter is a compact set, since A and $[0,1]$ are so, and α is continuous. Passing to subsequences, without changing notation, for some $t \in [0,1]$, $x \in \partial D$, and $y, z \in A$ one has

$t_k \to t$, $x_k \to x$, $f_{n_k}(x_k) \to y$, and $f_{m_k}(x_k) \to z$, as $k \to \infty$. Since F is h-u.s.c. with compact α-convex values, Proposition 5.5 implies that y, $z \in F(x)$, and hence $\alpha(y,z,t) \in F(x)$. From (6.5), letting $k \to \infty$, one has $p = x - \alpha(y,z,t)$, and thus $p \in x - F(x)$. This contradicts the hypothesis, consequently, for some $n_0 \in \mathbb{N}$, (6.5) holds.

For n, $m \geq n_0$ each $H_{n,m}$ is continuous compact and satisfies (6.4).

Whence, by the homotopy property of the Leray–Schauder degree, (6.2) follows and (a) is proved.

(b) Let $\{f_n\}$, $\{g_n\} \in \mathcal{A}_F(\overline{D}, A)$. Define $H_n\colon \overline{D} \times [0,1] \to Y$ by

$$H_n(x,t) = \alpha(f_n(x), g_n(x), t).$$

H_n is well defined continuous and compact. There exists $n_0 \in \mathbb{N}$ such that

$$p \notin \bigcup_{n \geq n_0} \bigcup_{(x,t) \in \partial D \times [0,1]} (x - H_n(x,t)). \tag{6.6}$$

In the contrary case, there exist a subsequence $\{n_k\} \subset \mathbb{N}$ and a sequence $\{(x_k,t_k)\} \subset \partial D \times [0,1]$, such that

$$p = x_k - \alpha(f_{n_k}(x_k), g_{n_k}(x_k), t_k) \quad \text{for every } k \in \mathbb{N}. \tag{6.7}$$

As before, passing to subsequences, without changing notation, for some $t \in [0,1]$, $x \in \partial D$, and $y, z \in A$, one has $t_k \to k$, $x_k \to x$, $f_{n_k}(x_k) \to y$, and $g_{n_k}(x_k) \to z$, when $k \to \infty$, and hence $\alpha(y,z,t) \in F(x)$. Letting $k \to \infty$, (6.7) gives $p = x - \alpha(y,z,t)$, and thus $p \in x - F(x)$. Since this contradicts the hypothesis, there exists $n_0 \in \mathbb{N}$ for which (6.6) holds.

For $n \geq n_0$ each H_n is continuous compact and satisfies (6.6). Hence (6.3) follows, proving (b). □

Proposition 6.5. *Let $F \in \mathcal{F}(\overline{D}, \mathcal{K}_\alpha(Y))$ and let $p \notin \bigcup_{x \in \partial D}(I - F)(x)$. Then, the topological degree* $\mathrm{Deg}(I - F, D, p)$ *of $I - F$ at p relative to D is well defined.*

Proof. By Proposition 6.2, the set $\mathcal{A}_F(\overline{D}, A)$ is nonempty. Furthermore, by Proposition 6.4, the limit (6.1) exists and it is independent of the sequence $\{f_n\} \in \mathcal{A}_F(\overline{D}, A)$. □

Remark 6.6. If $F \in \mathcal{F}(\overline{D}, \mathcal{K}(\mathbb{E}))$ and $p \notin \bigcup_{x \in \partial D}(I - F)(x)$, then $\mathrm{Deg}(I - F, D, p)$ reduces to the topological degree of $I - F$ at p relative to D defined by Hukuhara in [17] and, in particular, to Leray–Schauder's degree, when F is single valued.

7. Properties of the topological degree

In this section, we present a few properties of the topological degree introduced before, including an application to fixed point theory.

Throughout $\mathbb{E}$ and Y are as in Section 6. Furthermore, D is a nonempty open bounded subset of $\mathbb{E}$, and p a point of $\mathbb{E}$.

Proposition 7.1 (Invariance under homotopy). *Let F_1, $F_2 \in \mathcal{F}(\overline{D}, \mathcal{K}_\alpha(Y))$, and suppose that the multifunction $H: \overline{D} \times [0,1] \to 2^Y$ given by*

$$H(x,t) = \alpha(F_1(x), F_2(x), t)$$

is such that $p \notin \bigcup_{(x,t)\in\partial D\times[0,1]}(x - H(x,t))$. Then, one has

$$\operatorname{Deg}(I - F_1, D, p) = \operatorname{Deg}(I - F_2, D, p). \tag{7.1}$$

Proof. Let $\{f_n^1\} \in \mathcal{A}_{F_1}(\overline{D}, A_1)$, $\{f_n^2\} \in \mathcal{A}_{F_2}(\overline{D}, A_2)$, where A_1, $A_2 \subset Y$ are compact α-convex sets corresponding to F_1, F_2, respectively. Define $K_n: \overline{D} \times [0,1] \to Y$ by

$$K_n(x,t) = \alpha(f_n^1(x), f_n^2(x), t). \tag{7.2}$$

K_n is well defined continuous and compact.

There exists $n_0 \in \mathbb{N}$ such that

$$p \notin \bigcup_{n\ge n_0} \bigcup_{(x,t)\in\partial D\times[0,1]} (x - K_n(x,t)). \tag{7.3}$$

Supposing the contrary, there exist subsequences $\{f_{n_k}^1\}$, $\{f_{n_k}^2\}$ and a sequence $\{(x_k,t_k)\} \subset \partial D \times [0,1]$, such that

$$p = x_k - \alpha(f_{n_k}^1(x_k), f_{n_k}^2(x_k), t_k) \quad \text{for every } k \in \mathbb{N}. \tag{7.4}$$

As $\{f_{n_k}^1(x_k)\} \subset A_1$, $\{f_{n_k}^2(x_k)\} \subset A_2$, $\{x_k\} \subset p + \alpha(A_1, A_2, [0,1])$, and $\{t_k\} \subset [0,1]$, passing to subsequences, whithout changing notation, for some $x \in \partial D$, $y_1 \in A_1$, $y_2 \in A_2$, and $t \in [0,1]$, one has $x_k \to x$, $f_{n_k}^1(x_k) \to y_1$, $f_{n_k}^2(x_k) \to y_2$, $t_k \to t$, when $k \to \infty$. Furthermore, $y_1 \in F_1(x)$, $y_2 \in F_2(x)$, by Proposition 5.5. Letting $k \to \infty$, (7.4) gives $p = x - \alpha(y_1, y_2, t)$, and thus $p \in x - H(x,t)$, a contradiction. Therefore, for some $n_0 \in \mathbb{N}$, (7.3) holds.

By the homotopy property of the Leray–Schauder degree, in view of (7.2) and (7.3), one has

$$\deg(I - f_n^1, D, p) = \deg(I - f_n^2, D, p) \quad \text{for all } n \ge n_0.$$

Hence, letting $n \to \infty$, (7.1) follows, completing the proof. □

Proposition 7.2 (Inclusions solving property). *Let* $F \in \mathcal{F}(\overline{D}, \mathcal{K}_\alpha(Y))$*, let* $p \notin \bigcup_{x\in\partial D}(I-F)(x)$*, and suppose that* $\operatorname{Deg}(I-F,D,p) \neq 0$*. Then, there exists* $x \in D$ *such that*

$$p \in x - F(x). \tag{7.5}$$

Proof. Let $\{f_n\} \in \mathcal{A}_F(\overline{D}, A)$. By Definition 6.3 and Proposition 6.4(a), there is $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $\deg(I - f_n, D, p) = \operatorname{Deg}(I-F,D,p)$. The latter is non zero, hence by a property of the Leray–Schauder degree, for each $n \geq n_0$ there exists $x_n \in D$ such that

$$p = x_n - f_n(x_n). \tag{7.6}$$

Since $\{f_n(x_n)\} \subset A$, a compact set, passing to subsequences, without changing notation, one can assume that $x_n \to x$, $f_n(x_n) \to y$, as $n \to \infty$, for some $x \in \overline{D}$ and $y \in A$. Furthermore $y \in F(x)$, by Proposition 5.5. Then from (7.6), letting $n \to \infty$, (7.5) follows and, clearly, $x \in D$. This completes the proof. □

Proposition 7.3 (Normalization). *If* $p \in D$ *then* $\operatorname{Deg}(I-\Theta,D,p) = 1$.

Proof. Since $\Theta \in \mathcal{F}(\overline{D}, \mathcal{K}_\alpha(Y))$, by Remark 6.1, $\operatorname{Deg}(I-\Theta,D,p)$ is defined. By Remark 6.6, $\operatorname{Deg}(I-\Theta,D,p) = \deg(I-\Theta,D,p)$ and, as the latter is 1, the statement follows. □

Proposition 7.4 (Continuity in p). *Let* $F \in \mathcal{F}(\overline{D}, \mathcal{K}_\alpha(Y))$*, and let* $p, q \in C$*, where* C *is an open component of* $\mathbb{E} \setminus \bigcup_{x\in\partial D}(I-F)(x)$*. Then one has:*

$$\operatorname{Deg}(I-F,D,p) = \operatorname{Deg}(I-F,D,q). \tag{7.7}$$

Proof. Let $\{f_n\} \in \mathcal{A}_F(\overline{D}, A)$. Let $\gamma\colon [0,1] \to C$ be a continuous path joining p and q. For $\varepsilon > 0$ put

$$\Gamma_\varepsilon = \bigcup_{t\in[0,1]} U(\gamma(t),\varepsilon),$$

where $U(\gamma(t),\varepsilon)$ denotes the open ball in $\mathbb{E}$ with center $\gamma(t)$ and radius $\varepsilon > 0$. Γ_ε is open connected and $\Gamma_\varepsilon \subset C$, if ε is small enough, say $\varepsilon < \varepsilon_0$.

There exist $0 < \varepsilon < \varepsilon_0$ and $n_0 \in \mathbb{N}$ such that

$$\Gamma_\varepsilon \subset \mathbb{E} \setminus \bigg(\bigcup_{n\geq n_0} \bigcup_{x\in\partial D} (I-f_n)(x) \bigg). \tag{7.8}$$

In the contrary case, there exist a subsequence $\{f_{n_k}\}$ and sequences $\{x_k\} \subset \partial D$, $\{t_k\} \subset [0,1]$, such that

$$\gamma(t_k) \in x_k - f_{n_k}(x_k) + \frac{1}{k} U \quad \text{for every } k \in \mathbb{N}, \tag{7.9}$$

where U stands for the open unit ball in $\mathbb{E}$. Since $\{\gamma(t_k)\} \subset \gamma([0,1])$, $\{f_{n_k}(x_k)\} \subset A$, and $\gamma([0,1])$, A are compact, passing to subsequences, without changing notation, one has $\gamma(t_k) \to z$, $f_{n_k}(x_k) \to y$, $x_k \to x$, for some $z \in \gamma([0,1])$, $y \in A$, and $x \in \partial D$. Furthermore $y \in F(x)$, by Proposition 5.5. Letting $k \to \infty$, (7.9) gives $z = x - y \in x - F(x)$. As $z \in \gamma([0,1]) \subset C$ and $x \in \partial D$, a contradiction follows and thus, for some $0 < \varepsilon < \varepsilon_0$ and $n_0 \in \mathbb{N}$, (7.8) holds.

Since Γ_ε is open connected, contains p and q, and satisfies (7.8), by a property of the Leray–Schauder degree one has $\deg(I - f_n, D, p) = \deg(I - f_n, D, q)$, for every $n \geq n_0$. Letting $n \to \infty$, (7.7) follows, completing the proof. □

Proposition 7.5. *Let D be a nonempty open bounded subset of $\mathbb{E}$, with $0 \in D \subset Y$. Suppose D is α-convex. Let $F: \overline{D} \to \mathcal{K}_\alpha(Y)$ be a h-u.s.c. and compact multifunction with corresponding set $A_F \subset D$. Then F has a fixed point.*

Proof. From the hypothesis, $F, \Theta \in \mathcal{F}(\overline{D}, \mathcal{K}_\alpha(Y))$. Define $H: \overline{D} \times [0,1] \to 2^Y$ by

$$H(x,t) = \alpha(0, F(x), t). \tag{7.10}$$

We have

$$0 \notin \bigcup_{(x,t) \in \partial D \times [0,1]} (x - H(x,t)). \tag{7.11}$$

In the contrary case, there are $x \in \partial D$ and $t \in [0,1]$ such that $x \in \alpha(0, F(x), t)$, and thus $x = \alpha(0, y, t)$, for some $y \in F(x)$. Since $0 \in D$ and $F(x) \subset A \subset D$, where D is α-convex, one has $\alpha(0, y, t) \in D$ and, from the contradiction, (7.11) follows.

In view of (7.11), Proposition 7.1 gives $\mathrm{Deg}(I - F, D, 0) = \mathrm{Deg}(I - \Theta, D, 0)$, where the latter is 1, by Proposition 7.3. Whence, by Proposition 7.2, $x \in F(x)$ for some $x \in D$, completing the proof. □

Definition 7.6. Let C be a nonempty open bounded subset of $\mathbb{E}$ with $C \subset Y$. Suppose C is α-convex. A point $a \in C$ is called absorbing for $\overline{C}$ if $\alpha(a, y, t) \in C$ for all $y \in \overline{C}$ and $t \in [0,1)$.

Remark 7.7. Suppose $\mathbb{E}$ is equipped with the (natural) convexity mapping α_0 given by (3.3), and let C be a nonempty open bounded convex subset of $\mathbb{E}$. Then, each point $a \in C$ is absorbing for $\overline{C}$. This is no longer true if convexity is replaced by α-convexity.

Proposition 7.8. *Let D be a nonempty open bounded subset of $\mathbb{E}$, with $0 \in D \subset Y$. Suppose that D is α-convex, and that 0 is an absorbing point of $\overline{D}$.*

Let $F:\overline{D}\to\mathcal{K}_\alpha(Y)$ be a h-u.s.c. and compact multifunction, with corresponding set $A_F\subset\overline{D}$. Then, F has a fixed point.

Proof. Without loss of generality, one can assume

$$0\notin\bigcup_{x\in\partial D}(I-F)(x). \tag{7.12}$$

Now, let $H:\overline{D}\times[0,1]\to 2^Y$ be given by (7.10), and observe that F, $\Theta\in\mathcal{F}(\overline{D},\mathcal{K}_\alpha(Y))$.

Under the above assumptions, (7.11) holds. In the contrary case, there are $x\in\partial D$ and $t\in[0,1]$ such that $x\in\alpha(0,F(x),t)$. The cases $t=0$, $t=1$ imply respectively $x=0$, $x\in F(x)$, which are excluded by the assumptions $0\in D$, and (7.12). Whence $x=\alpha(0,y,t)$, for some $0<t<1$ and $y\in F(x)$. Since $y\in\overline{D}$ and 0 is absorbing for $\overline{D}$, one has $\alpha(0,y,t)\in D$. As $x\in\partial D$, a contradiction follows, and so (7.11) holds. In view of (7.11), one can conclude as in the proof of Proposition 7.5. □

Remark 7.9. If, in Proposition 7.8, one takes $Y=\mathbb{E}$, with the natural convexity mapping α_0, and assumes $0\in D$, then the classical fixed point theorem of Kakutani–Ky Fan (see [19] and [9]) follows at once.

Proposition 7.10. *Let Y be a compact α-convex metric space. Then, every h-u.s.c. multifunction $F:Y\to\mathcal{K}_\alpha(Y)$ has a fixed point.*

Proof. By Corollary 5.2, there exists a sequence $\{f_n\}$ of continuous functions $f_n:Y\to Y$ such that $f_n\xrightarrow{\text{gr}}F$, as $n\to\infty$. By [7, Corollary 3.3], each f_n has a fixed point $x_n=f_n(x_n)$. Since Y is compact, passing to subsequences (without changing notation), one can assume that $x_n\to x$ and $f_n(x_n)\to x$ as $n\to\infty$, for some $x\in Y$. Then, by Proposition 5.5, one has $x\in F(x)$, completing the proof. □

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