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APPROXIMATION AND LERAY–SCHAUDER TYPE RESULTS FOR $\mathcal{U}_c^\kappa$ MAPS

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Abstract. The paper presents new approximation and fixed point results for $\mathcal{U}_c^\kappa$ maps in Hausdorff locally convex spaces.

1. Introduction

In 1969, Ky Fan [2] proved an interesting result that combined fixed point theory with the study of proximity maps. Its normed space version is stated as follows:

Let C be a nonempty, compact, convex subset of a normed space E. Then for any continuous mapping f from C to E, there exists an $x_0 \in C$ with

$$\|x_0 - f(x_0)\| = \inf_{y \in C} \|f(x_0) - y\|.$$

During the last three decades, various multi-valued and single-valued versions of Fan's result have been established by a number of authors; see, for instance, [1], [3], [5]–[7], [9], [10], [12], [13], [18], [19]. Recently, Lin and Park in [7] obtained a multivalued version of Ky Fan's result for α-condensing $\mathcal{U}_c^\kappa$ maps defined on a closed ball in a Banach space. More recently, O'Regan and Shahzad in [12] extended their result to countably condensing maps. The purpose of this paper is to prove some Ky Fan type approximation results for Φ-condensing

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$\mathcal{U}_c^\kappa$ multimaps, where C is a closed convex subset of a Hausdorff locally convex space E with $0 \in \operatorname{int}(C)$. Since every α-condensing map $F: C \to 2^E$ is Φ-condensing if C is complete, the results of Lin and Park (see [7]) can be considered as special cases of our work. We also derive, as an application, the Leray–Schauder principle for $\mathcal{U}_c^\kappa$ multimaps, which was proved by Lin and Yu in [8]. The Leray–Schauder type results for compact admissible multimaps and approximable multimaps were obtained in [15] and [16].

2. Preliminaries

Let E be a Hausdorff locally convex space. For a nonempty set $Y \subseteq E$, 2^Y denotes the family of nonempty subsets of Y. If L is a lattice with a minimal element 0, a mapping $\Phi: 2^E \to L$ is called a *generalized measure of noncompactness* provided that the following conditions hold:

(a) $\Phi(A) = 0$ if and only if $\overline{A}$ is compact.
(b) $\Phi(\overline{\mathrm{co}}(A)) = \Phi(A)$; here $\overline{\mathrm{co}}(A)$ denotes the closed convex hull of A.
(c) $\Phi(A \cup B) = \max\{\Phi(A), \Phi(B)\}$.

It is clear that if $A \subseteq B$, then $\Phi(A) \leq \Phi(B)$. Examples of the generalized measure of noncompactness are the Kuratowskiĭ measure and the Hausdorff measure of noncompactness (see [15]), which are defined below. Let C be a nonempty subset of a Banach space X. The Kuratowskiĭ measure of noncompactness is the map $\alpha: 2^C \to L$ defined by

$$\alpha(A) = \inf\{\varepsilon > 0 \mid A \text{ can be covered by a finite number of sets each of diamter less than } \varepsilon\},$$

for $A \in 2^C$. The Hausdorff measure of noncompactness is the map $\chi: 2^C \to L$ defined by

$$\chi(A) = \inf\{\varepsilon > 0 \mid A \text{ can be covered by a finite number of balls with radius less than } \varepsilon\},$$

for $A \in 2^C$.

Let C be a nonempty subset of a Hausdorff locally convex space E and $F: C \to 2^E$. Then F is called Φ-*condensing* provided that $\Phi(A) = 0$ for any $A \subseteq C$ with $\Phi(F(A)) \geq \Phi(A)$. Note that any compact map or any map defined on a compact set is Φ-condensing.

Let X and Y be subsets of Hausdorff topological vector spaces E_1 and E_2 respectively. Let $F: X \to K(Y)$; here $K(Y)$ denotes the family of nonempty compact subsets of Y. Then F is *Kakutani* if F is upper semicontinuous with convex values. A nonempty topological space is called acyclic if all its reduced Čech homology groups over the rationals are trivial. Now F is *acyclic* if F is upper semicontinuous with acyclic values. The map F is said to be an *O'Neill* map if F is continuous and if the values of F consist of one or m acyclic components (here m is fixed).

For our next definition let X and Y be metric spaces. A continuous single valued map $p\colon Y \to X$ is called a Vietoris map if the following two conditions hold:

(a) for each $x \in X$, the set $p^{-1}(x)$ is acyclic,
(b) p is a proper map i.e., for every compact $A \subseteq X$ we have that $p^{-1}(A)$ is compact.

A multifunction $\phi\colon X \to K(Y)$ is *admissible* (strongly) in the sense of Górniewicz [4], if there exists a metric space Z and two continuous maps $p\colon Z \to X$ and $q\colon Z \to Y$ such that

(a) p is a Vietoris map, and
(b) $\phi(x) = q(p^{-1}(x))$, for any $x \in X$.

Let X be a nonempty convex subset of a Hausdorff topological vector space E and Y a topological space. A ploytope P in X is any convex hull of a nonempty finite subset of X; or a nonempty compact convex subset of X contained in a finite dimentional subpace of E. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X,Y)$ denotes the set of maps $F\colon X \to 2^Y$ belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. A class $\mathcal{U}$ of maps is defined by the following properties:

(a) $\mathcal{U}$ contains the class $\mathcal{C}$ of single valued continuous functions,
(b) each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued,
(c) for any polytope P, $F \in \mathcal{U}_c(P,P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathcal{U}$.

An important class related to $\mathcal{U}_c(X,Y)$ is given below.

$F \in \mathcal{U}_c^\kappa(X,Y)$ if for any compact subset K of X, there is a $G \in \mathcal{U}_c(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Examples of $\mathcal{U}_c^\kappa$ maps are the Kakutani maps, the acyclic maps, the O'Neill maps, and the maps admissible in the sense of Górniewicz. Note that $\mathcal{U}(X,Y) \subseteq \mathcal{U}_c(X,Y) \subseteq \mathcal{U}_c^\kappa(X,Y)$.

Let Q be a subset of a Hausdorff topological space X. We let $\overline{Q}$ (respectively, $\partial(Q)$, $\mathrm{int}(Q)$) to denote the closure (respectively, boundary, interior) of Q.

Let C be a subset of a Hausdorff topological vector space E and $x \in X$. Then the inward set $I_C(x)$ is defined by

$$I_C(x) = \{x + r(y-x) \mid y \in C,\ r \geq 0\}.$$

If C is convex and $x \in C$, then

$$I_C(x) = x + \{r(y-x) \mid y \in C,\ r \geq 1\}.$$

We shall need the following results in the sequel.

LEMMA 2.1 ([14]). *Let C be a nonempty, convex subset of a Hausdorff locally convex space E. Suppose $F \in \mathcal{U}_c^\kappa(C, C)$ is a compact map. Then F has a fixed point in C.*

LEMMA 2.2 ([11]). *Let C be a nonempty, closed, convex subset of a Hausdorff topological vector space E. Suppose $G: C \to 2^C$ is a Φ-condensing map. Then there exists a nonempty compact convex subset K of C such that $G(K) \subset K$.*

Let C be a convex subset of a Hausdorff locally convex space E with $0 \in \operatorname{int}(C)$. The Minkowski functional p of C is defined by

$$p(x) = \inf\{r > 0 \mid x \in rC\}.$$

Now, we list some properties of the Minkowski functional:

(a) p is continuous on E,
(b) $p(x + y) \leq p(x) + p(y)$, $x, y \in E$,
(c) $p(\lambda x) = \lambda p(x)$, $\lambda \geq 0$, $x \in E$,
(d) $0 \leq p(x) < 1$ if $x \in \operatorname{int}(C)$,
(e) $p(x) > 1$, if $x \notin \overline{C}$,
(f) $p(x) = 1$, if $x \in \partial C$.

For $x \in E$, set $d_p(x, C) = \inf\{p(x - y) \mid y \in C\}$.

3. Main results

THEOREM 3.1. *Let C be a closed, convex subset of a Hausdorff locally convex space E with $0 \in C$ and U a convex open neighbourhood of 0. Suppose $F \in \mathcal{U}_c^\kappa(\overline{U} \cap C, C)$ is a Φ-condensing map. Then there exist $x_0 \in \overline{U} \cap C$ and $y_0 \in F(x_0)$ with*

$$p(y_0 - x_0) = d_p(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}(x_0)} \cap C),$$

here p is the Minkowski functional of U. More precisely, either

(a) *F has a fixed point $x_0 \in \overline{U} \cap C$, or*
(b) *there exist $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0)$ with*

$$0 < p(y_0 - x_0) = d_p(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}(x_0)} \cap C).$$

Here $\partial_C(U)$ denotes the boundary of U relative to C.

PROOF. Let $r: E \to \overline{U}$ be defined by

$$r(x) = \begin{cases} x & \text{if } x \in \overline{U}, \\ x/p(x) & \text{if } x \notin \overline{U}, \end{cases}$$

that is

$$r(x) = \frac{x}{\max\{1, p(x)\}}, \quad \text{for } x \in E.$$

Since $0 \in U = \text{int}(U)$, p is continuous and so r is continuous. Let f be the restriction of r to C. Since C is convex and $0 \in C$, it follows that $f(C) \subseteq \overline{U} \cap C$. Also $f \in \mathcal{C}(C, \overline{U} \cap C)$. Since $\mathcal{U}_c^\kappa$ is closed under composition, $f \circ F \in \mathcal{U}_c^\kappa(\overline{U} \cap C, \overline{U} \cap C)$. Let $G = f \circ F$. We show that G is Φ-condensing. Let A be a subset of $\overline{U} \cap C$ such that $\Phi(A) \leq \Phi(G(A))$. Then $G(A) \subseteq co(\{0\} \cup F(A))$ and so

$$\begin{aligned}\Phi(A) \leq \Phi(G(A)) &\leq \Phi(co(\{0\} \cup F(A))) \leq \Phi(\{0\} \cup F(A)) \\ &= \max\{\Phi(\{0\}), \Phi(F(A))\} = \Phi(F(A)),\end{aligned}$$

which gives $\overline{A}$ is compact. This shows that G is Φ-condensing and so, by Lemma 2.2, there exists a nonempty compact convex subset K of $\overline{U} \cap C$ such that $G(K) \subset K$. Since $G \in \mathcal{U}_c^\kappa(\overline{U} \cap C, \overline{U} \cap C)$ and K is compact, there exists $T \in \mathcal{U}_c(K, \overline{U} \cap C)$ such that $T(x) \subset G(x)$ for all $x \in K$. This implies that $T(K) \subset G(K) \subset K$ and T is compact. Since $T \in \mathcal{U}_c(K, K)$, by Lemma 2.1, T has a fixed point $x_0 \in K$, that is, $x_0 \in T(x_0) \subset G(x_0)$. Clearly $x_0 \in \overline{U} \cap C$. Therefore, there exists some $y_0 \in F(x_0)$ with $x_0 = f(y_0)$. Now, we consider two cases:

(a) $y_0 \in \overline{U} \cap C$ or
(b) $y_0 \in C \setminus \overline{U}$.

Suppose $y_0 \in \overline{U} \cap C$. Then $x_0 = f(y_0) = y_0$. As a result

$$p(y_0 - x_0) = 0 = d_p(y_0, \overline{U} \cap C)$$

and x_0 is a fixed point of F. On the other hand, if $y_0 \in C \setminus \overline{U}$, then

$$x_0 = f(y_0) = \frac{y_0}{p(y_0)}.$$

So, for any $x \in \overline{U} \cap C$,

$$\begin{aligned}p(y_0 - x_0) &= p\left(y_0 - \frac{y_0}{p(y_0)}\right) = \left(\frac{p(y_0) - 1}{p(y_0)}\right) p(y_0) \\ &= p(y_0) - 1 \leq p(y_0) - p(x) = p((y_0 - x) + x) - p(x) \leq p(y_0 - x),\end{aligned}$$

which gives

$$p(y_0 - x_0) = \inf\{p(y_0 - z) \mid z \in \overline{U} \cap C\} = d_p(y_0, \overline{U} \cap C).$$

Since $p(y_0 - x_0) = p(y_0) - 1$, we have $p(y_0 - x_0) > 0$.

Let $z \in I_{\overline{U}}(x_0) \cap C \setminus (\overline{U} \cap C)$. Then there exists $y \in \overline{U}$ and $c \geq 1$ with $z = x_0 + c(y - x_0)$. Suppose that

$$p(y_0 - z) < p(y_0 - x_0).$$

The convexity of C implies that

$$\frac{1}{c} z + \left(1 - \frac{1}{c}\right) x_0 \in C.$$

Since

$$\frac{1}{c}z + \left(1 - \frac{1}{c}\right)x_0 = y \in \overline{U},$$

it follows that

$$\begin{aligned} p(y_0 - y) &= p\left[\frac{1}{c}(y_0 - z) + \left(1 - \frac{1}{c}\right)(y_0 - x_0)\right] \\ &\leq \frac{1}{c}p(y_0 - z) + \left(1 - \frac{1}{c}\right)p(y_0 - x_0) < p(y_0 - x_0). \end{aligned}$$

This contradicts the choice of y_0. Consequently, we have

$$p(y_0 - x_0) \leq p(y_0 - z) \quad \text{for all } z \in I_{\overline{U}}(x_0) \cap C.$$

The continuity of p further implies that

$$p(y_0 - x_0) \leq p(y_0 - z) \quad \text{for all } z \in \overline{I_{\overline{U}}(x_0)} \cap C.$$

Hence

$$0 < p(y_0 - x_0) = d_p(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}(x_0)} \cap C).$$

If $x_0 \in U$, then $\overline{I_{\overline{U}}(x_0)} = E$ and so $d_p(y_0, \overline{I_{\overline{U}}(x_0)} \cap C) = 0$. Thus $x_0 \in \partial_C(U)$. □

Essentially the same reasoning as before yields the following result.

Theorem 3.2. *Let C be a closed, convex subset of a Hausdorff locally space E with $0 \in \operatorname{int}(C)$. Suppose $F \in \mathcal{U}_c^k(C, E)$ is a Φ-condensing map. Then there exist $x_0 \in C$ and $y_0 \in F(x_0)$ with*

$$p(y_0 - x_0) = d_p(y_0, C) = d_p(y_0, \overline{I_C(x_0)}),$$

here p is the Minkowski functional of C in E. More precisely, either

(a) *F has a fixed point $x_0 \in C$, or*
(b) *there exist $x_0 \in \partial(C)$ and $y_0 \in F(x_0)$ with*

$$0 < p(y_0 - x_0) = d_p(y_0, C) = d_p(y_0, \overline{I_C(x_0)}).$$

Since $p(x) = \|x\|/R$ is the Minkowski functional on B_R, we have the following result.

Corollary 3.3. *Let E be a normed space. Suppose $F \in \mathcal{U}_c^k(B_R, E)$ is a Φ-condensing map. Then there exist $x_0 \in B_R$ and $y_0 \in F(x_0)$ with*

$$\|y_0 - x_0\| = d(y_0, B_R) = d(y_0, \overline{I_{B_R}(x_0)}).$$

More precisely, either

(a) *F has a fixed point $x_0 \in B_R$ or*
(b) *there exist $x_0 \in \partial(B_R)$ and $y_0 \in F(x_0)$ with*

$$0 < ||y_0 - x_0|| = d(y_0, B_R) = d(y_0, \overline{I_{B_R}(x_0)}).$$

Remark 3.1. Theorem 1 of Lin and Park [7] and a result of Lin [6] can be considered as special cases of Corollary 3.3.

As applications of our approximation theorems, we now derive some fixed point results.

Theorem 3.4. *Let C be a closed, convex subset of a Hausdorff locally convex space E with $0 \in C$ and U a convex open neighbourhood of 0. Suppose $F \in \mathcal{U}_c^k(\overline{U} \cap C, C)$ is a Φ-condensing map. If F satisfies any one of the following conditions for any $x \in \partial_C(U) \setminus F(x)$:*

(a) *for each $y \in F(x)$, $p(y-z) < p(y-x)$ for some $z \in \overline{I_{\overline{U}}(x)} \cap C$,*
(b) *for each $y \in F(x)$, there exists λ with $|\lambda| < 1$ such that $\lambda x + (1-\lambda)y \in \overline{I_{\overline{U}}(x)} \cap C$,*
(c) *$F(x) \subseteq \overline{I_{\overline{U}}(x)} \cap C$,*
(d) *$F(x) \cap \{\lambda x \mid \lambda > 1\} = \emptyset$,*
(e) *for each $y \in F(x)$, $p(y-x) \neq p(y) - 1$,*
(f) *for each $y \in F(x)$, there exists $\alpha \in (1, \infty)$ such that*

$$p^\alpha(y) - 1 \le p^\alpha(y-x),$$

(g) *for each $y \in F(x)$, there exists $\beta \in (0,1)$ such that $p^\beta(y) - 1 \ge p^\beta(y-x)$,*

then F has a fixed point.

Proof. Theorem 3.1 guarantees that either

(1) F has a fixed point in $\overline{U} \cap C$ or
(2) there exist $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0)$ with $x_0 = f(y_0)$ such that

$$0 < p(y_0) - 1 = p(y_0 - x_0) = d_p(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}(x_0)} \cap C),$$

where p is the Minkowski functional of U and f is the restriction of the continuous retraction r to C.

Suppose (2) holds (with some x_0 and y_0) and $x_0 \notin F(x_0)$. We shall show contradictions in all conditions (a)–(g).

If F satisfies condition (a), then we have $p(y_0 - z) < p(y_0 - x_0)$, for some $z \in \overline{I_{\overline{U}}(x_0)} \cap C$. This contradicts the choice of x_0.

If F satisfies condition (b), then there exists λ with $|\lambda| < 1$ such that $\lambda x_0 + (1-\lambda)y_0 \in \overline{I_{\overline{U}}(x_0)} \cap C$. This implies that

$$\begin{aligned} p(y_0 - x_0) &\le p(y_0 - (\lambda x_0 + (1-\lambda)y_0)) = p(\lambda(y_0 - x_0)) \\ &= |\lambda| p(y_0 - x_0) < p(y_0 - x_0), \end{aligned}$$

which is a contradiction.

The proof for condition (c) is obvious.

If F satisfies condition (d), then $\lambda x_0 \neq y_0$ for each $\lambda > 1$. But we have $x_0 = f(y_0) = y_0/p(y_0)$. Therefore, $y_0 = \lambda_0 x_0$ with $\lambda_0 = p(y_0) > 1$, which is a contradiction.

If F satisfies condition (e), then $p(y_0 - x_0) \neq p(y_0) - 1$ and this contradicts $p(y_0 - x_0) = p(y_0) - 1$.

If F satisfies condition (f), then there exists $\alpha \in (1, \infty)$ with $p^\alpha(y_0) - 1 \leq p^\alpha(y_0 - x_0)$. Set $\lambda_0 = 1/p(y_0)$. Then $\lambda_0 \in (0,1)$ and

$$\frac{(p(y_0) - 1)^\alpha}{p^\alpha(y_0)} = (1 - \lambda_0)^\alpha < 1 - \lambda_0^\alpha = \frac{p^\alpha(y_0) - 1}{p^\alpha(y_0)} \leq \frac{p^\alpha(y_0 - x_0)}{p^\alpha(y_0)}.$$

This implies that $p(y_0 - x_0) > p(y_0) - 1$. This contradicts the fact that $p(y_0 - x_0) = p(y_0) - 1$.

Finally if F satisfies condition (g), then, as above (see the proof of (f)), we can get a contradiction to $p(y_0 - x_0) = p(y_0) - 1$. □

Remark 3.2. We have derived the Leray–Schauder principle as an application of Theorem 3.1 (see Theorem 3.4(d)), which was established by Lin and Yu in [8].

Essentially the same reasoning as in Theorem 3.4 (with Theorem 3.2 replacing Theorem 3.1) yields the following result.

Theorem 3.5. *Let C be a closed, convex subset of a Hausdorff locally convex space E with $0 \in \operatorname{int}(C)$. Suppose $F \in \mathcal{U}_c^k(C, E)$ is a Φ-condensing map. If F satisfies any one of the following conditions for any $x \in \partial(C) \setminus F(x)$:*

(a) *for each $y \in F(x)$, $p(y - z) < p(y - x)$, for some $z \in \overline{I_C(x)}$,*
(b) *for each $y \in F(x)$, there exists λ with $|\lambda| < 1$ such that*

$$\lambda x + (1 - \lambda)y \in \overline{I_C(x)},$$

(c) *$F(x) \subseteq \overline{I_C(x)}$,*
(d) *$F(x) \cap \{\lambda x \mid \lambda > 1\} = \emptyset$,*
(e) *for each $y \in F(x)$, $p(y - x) \neq p(y) - 1$,*
(f) *for each $y \in F(x)$, there exists $\alpha \in (1, \infty)$ such that*

$$p^\alpha(y) - 1 \leq p^\alpha(y - x),$$

(g) *for each $y \in F(x)$, there exists $\beta \in (0,1)$ such that $p^\beta(y) - 1 \geq p^\beta(y - x)$,*

then F has a fixed point.

Corollary 3.6. *Let E be a normed space. Suppose $F \in \mathcal{U}_c^k(B_R, E)$ is a Φ-condensing map. If F satisfies any one of the following conditions for any $x \in \partial(B_R) \setminus F(x)$:*

(a) *for each $y \in F(x)$, $\|y - z\| < \|y - x\|$, for some $z \in \overline{I_{B_R}(x)}$,*

(b) *for each* $y \in F(x)$, *there exists* λ *with* $|\lambda| < 1$ *such that* $\lambda x + (1-\lambda)y \in \overline{I_{B_R}(x)}$,

(c) $F(x) \subseteq \overline{I_{B_R}(x)}$,

(d) $F(x) \cap \{\lambda x \mid \lambda > 1\} = \emptyset$,

(e) *for each* $y \in F(x)$, $\|y - x\| \neq \|y\| - R$,

(f) *for each* $y \in F(x)$, *there exists* $\alpha \in (1, \infty)$ *such that* $\|y\|^\alpha - R \leq \|y - x\|^\alpha$,

(g) *for each* $y \in F(x)$, *there exists* $\beta \in (0, 1)$ *such that* $\|y\|^\beta - R \geq \|y - x\|^\beta$,

then F *has a fixed point.*

Remark 3.3. Corollary 3.6 contains, as special cases, Theorem 2 of Lin and Park [7] as well as a result of Lin [6].

Essentially the same reasoning as above gives the following results in Hilbert spaces (here the retraction r is replaced by the proximity map p), which extend Theorem 3 and Theorem 4 of Lin and Park [7].

Theorem 3.7. *Let* C *be a nonempty, closed, convex subset of a Hilbert space* H. *Suppose* $F \in \mathcal{U}_c^k(C, H)$ *is a* Φ*-condensing map. Then there exist* x_0 *and* $y_0 \in F(x_0)$ *with*

$$\|y_0 - x_0\| = d(y_0, C) = d(y_0, \overline{I_C(x_0)}),$$

here $\|\cdot\|$ *is the norm induced by the inner product. More precisely, either*

(a) F *has a fixed point* $x_0 \in C$ *or*

(b) *there exist* $x_0 \in \partial(C)$ *and* $y_0 \in F(x_0)$ *with*

$$0 < \|y_0 - x_0\| = d(y_0, C) = d(y_0, \overline{I_C(x_0)}).$$

Theorem 3.8. *Let* C *be a nonempty, closed, convex subset of a Hilbert space* H. *Suppose* $F \in \mathcal{U}_c^k(C, H)$ *is a* Φ*-condensing map. If* F *satisfies any one of the following conditions for any* $x \in \partial(C) \setminus F(x)$:

(a) *for each* $y \in F(x)$, $\|y - z\| < \|y - x\|$, *for some* $z \in \overline{I_C(x)}$,

(b) *for each* $y \in F(x)$, *there exists* λ *with* $|\lambda| < 1$ *such that* $\lambda x + (1-\lambda)y \in \overline{I_C(x)}$,

(c) $F(x) \subseteq \overline{I_C(x)}$,

then F *has a fixed point.*

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