APPROXIMATION AND LERAY–SCHAUDER
TYPE RESULTS FOR $\mathcal{U}_c^\kappa$ MAPS

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Abstract. The paper presents new approximation and fixed point results for $\mathcal{U}_c^\kappa$ maps in Hausdorff locally convex spaces.

1. Introduction

In 1969, Ky Fan [2] proved an interesting result that combined fixed point theory with the study of proximity maps. Its normed space version is stated as follows:

Let $C$ be a nonempty, compact, convex subset of a normed space $E$. Then for any continuous mapping $f$ from $C$ to $E$, there exists an $x_0 \in C$ with

$$
\|x_0 - f(x_0)\| = \inf_{y \in C} \|f(x_0) - y\|.
$$

During the last three decades, various multi-valued and single-valued versions of Fan’s result have been established by a number of authors; see, for instance, [1], [3], [5]–[7], [9], [10], [12], [13], [18], [19]. Recently, Lin and Park in [7] obtained a multivalued version of Ky Fan’s result for $\alpha$-condensing $\mathcal{U}_c^\kappa$ maps defined on a closed ball in a Banach space. More recently, O’Regan and Shahzad in [12] extended their result to countably condensing maps. The purpose of this paper is to prove some Ky Fan type approximation results for $\Phi$-condensing...
\( \mathcal{U}_c \) multimaps, where \( C \) is a closed convex subset of a Hausdorff locally convex space \( E \) with \( 0 \in \text{int}(C) \). Since every \( \alpha \)-condensing map \( F: C \to 2^E \) is \( \Phi \)-condensing if \( C \) is complete, the results of Lin and Park (see [7]) can be considered as special cases of our work. We also derive, as an application, the Leray–Schauder principle for \( \mathcal{U}_c \) multimaps, which was proved by Lin and Yu in [8]. The Leray–Schauder type results for compact admissible multimaps and approximable multimaps were obtained in [15] and [16].

2. Preliminaries

Let \( E \) be a Hausdorff locally convex space. For a nonempty set \( Y \subseteq E \), \( 2^Y \) denotes the family of nonempty subsets of \( Y \). If \( L \) is a lattice with a minimal element \( 0 \), a mapping \( \Phi: 2^E \to L \) is called a \textit{generalized measure of noncompactness} provided that the following conditions hold:

(a) \( \Phi(A) = 0 \) if and only if \( \overline{A} \) is compact.
(b) \( \Phi(\overline{\text{co}(A)}) = \Phi(A) \); here \( \overline{\text{co}(A)} \) denotes the closed convex hull of \( A \).
(c) \( \Phi(A \cup B) = \max\{\Phi(A), \Phi(B)\} \).

It is clear that if \( A \subseteq B \), then \( \Phi(A) \leq \Phi(B) \). Examples of the generalized measure of noncompactness are the Kuratowski–\( \check{\text{C}} \)ech measure and the Hausdorff measure of noncompactness (see [15]), which are defined below. Let \( C \) be a nonempty subset of a Banach space \( X \). The Kuratowski–\( \check{\text{C}} \)ech measure of noncompactness is the map \( \alpha: 2^C \to L \) defined by

\[
\alpha(A) = \inf \{ \varepsilon > 0 \mid A \text{ can be covered by a finite number of sets each of diameter less than } \varepsilon \},
\]

for \( A \in 2^C \). The Hausdorff measure of noncompactness is the map \( \chi: 2^C \to L \) defined by

\[
\chi(A) = \inf \{ \varepsilon > 0 \mid A \text{ can be covered by a finite number of balls with radius less than } \varepsilon \},
\]

for \( A \in 2^C \).

Let \( C \) be a nonempty subset of a Hausdorff locally convex space \( E \) and \( F: C \to 2^E \). Then \( F \) is called \( \Phi \)-\textit{condensing} provided that \( \Phi(A) = 0 \) for any \( A \subseteq C \) with \( \Phi(F(A)) \geq \Phi(A) \). Note that any compact map or any map defined on a compact set is \( \Phi \)-condensing.

Let \( X \) and \( Y \) be subsets of Hausdorff topological vector spaces \( E_1 \) and \( E_2 \) respectively. Let \( F: X \to K(Y) \); here \( K(Y) \) denotes the family of nonempty compact subsets of \( Y \). Then \( F \) is \( \text{Kakutani} \) if \( F \) is upper semicontinuous with convex values. A nonempty topological space is called acyclic if all its reduced \( \check{\text{C}} \)ech homology groups over the rationals are trivial. Now \( F \) is \( \text{acyclic} \) if \( F \) is upper semicontinuous with acyclic values. The map \( F \) is said to be an \( \text{O’Neill} \) map.
if $F$ is continuous and if the values of $F$ consist of one or $m$ acyclic components (here $m$ is fixed).

For our next definition let $X$ and $Y$ be metric spaces. A continuous single valued map $p: Y \to X$ is called a Vietoris map if the following two conditions hold:

(a) for each $x \in X$, the set $p^{-1}(x)$ is acyclic,
(b) $p$ is a proper map i.e., for every compact $A \subseteq X$ we have that $p^{-1}(A)$ is compact.

A multifunction $\phi: X \to K(Y)$ is admissible (strongly) in the sense of Górniewicz [4], if there exists a metric space $Z$ and two continuous maps $p: Z \to X$ and $q: Z \to Y$ such that

(a) $p$ is a Vietoris map, and
(b) $\phi(x) = q(p^{-1}(x))$, for any $x \in X$.

Let $X$ be a nonempty convex subset of a Hausdorff topological vector space $E$ and $Y$ a topological space. A polytope $P$ in $X$ is any convex hull of a nonempty finite subset of $X$; or a nonempty compact convex subset of $X$ contained in a finite dimensional subspace of $E$. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X,Y)$ denotes the set of maps $F: X \to 2^Y$ belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. A class $\mathcal{U}$ of maps is defined by the following properties:

(a) $\mathcal{U}$ contains the class $\mathcal{C}$ of single valued continuous functions,
(b) each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued,
(c) for any polytope $P$, $F \in \mathcal{U}_c(P,P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathcal{U}$.

An important class related to $\mathcal{U}_c(X,Y)$ is given below.

$F \in \mathcal{U}_c^\kappa(X,Y)$ if for any compact subset $K$ of $X$, there is a $G \in \mathcal{U}_c(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Examples of $\mathcal{U}_c^\kappa$ maps are the Kakutani maps, the acyclic maps, the O’Neill maps, and the maps admissible in the sense of Górniewicz. Note that $\mathcal{U}(X,Y) \subseteq \mathcal{U}_c(X,Y) \subseteq \mathcal{U}_c^\kappa(X,Y)$.

Let $Q$ be a subset of a Hausdorff topological space $X$. We let $\overline{Q}$ (respectively, $\partial(Q)$, $\operatorname{int}(Q)$) to denote the closure (respectively, boundary, interior) of $Q$.

Let $C$ be a subset of a Hausdorff topological vector space $E$ and $x \in X$. Then the inward set $I_C(x)$ is defined by

$I_C(x) = \{x + r(y - x) \mid y \in C, \ r \geq 0\}$.

If $C$ is convex and $x \in C$, then

$I_C(x) = x + \{r(y - x) \mid y \in C, \ r \geq 1\}$.

We shall need the following results in the sequel.
Lemma 2.1 ([14]). Let $C$ be a nonempty, convex subset of a Hausdorff locally convex space $E$. Suppose $F \in U^c_\kappa(C, C)$ is a compact map. Then $F$ has a fixed point in $C$.

Lemma 2.2 ([11]). Let $C$ be a nonempty, closed, convex subset of a Hausdorff topological vector space $E$. Suppose $G : C \to 2^C$ is a $\Phi$-condensing map. Then there exists a nonempty compact convex subset $K$ of $C$ such that $G(K) \subset K$.

Let $C$ be a convex subset of a Hausdorff locally convex space $E$ with $0 \in \text{int}(C)$. The Minkowski functional $p$ of $C$ is defined by

$$p(x) = \inf \{ r > 0 \mid x \in rC \}.$$

Now, we list some properties of the Minkowski functional:

(a) $p$ is continuous on $E$,
(b) $p(x + y) \leq p(x) + p(y)$, $x, y \in E$,
(c) $p(\lambda x) = \lambda p(x)$, $\lambda \geq 0$, $x \in E$,
(d) $0 \leq p(x) < 1$ if $x \in \text{int}(C)$,
(e) $p(x) > 1$, if $x \not\in \overline{C}$,
(f) $p(x) = 1$, if $x \in \partial C$.

For $x \in E$, set $d_p(x, C) = \inf \{ p(x - y) \mid y \in C \}$.

3. Main results

Theorem 3.1. Let $C$ be a closed, convex subset of a Hausdorff locally convex space $E$ with $0 \in C$ and $U$ a convex open neighbourhood of $0$. Suppose $F \in U^c_\kappa(U \cap C, C)$ is a $\Phi$-condensing map. Then there exist $x_0 \in U \cap C$ and $y_0 \in F(x_0)$ with

$$p(y_0 - x_0) = d_p(y_0, U \cap C) = d_p(y_0, \overline{U}(x_0) \cap C),$$

here $p$ is the Minkowski functional of $U$. More precisely, either

(a) $F$ has a fixed point $x_0 \in U \cap C$, or
(b) there exist $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0)$ with

$$0 < p(y_0 - x_0) = d_p(y_0, U \cap C) = d_p(y_0, \overline{U}(x_0) \cap C).$$

Here $\partial_C(U)$ denotes the boundary of $U$ relative to $C$.

Proof. Let $r : E \to \overline{U}$ be defined by

$$r(x) = \begin{cases} x & \text{if } x \in U, \\ x/p(x) & \text{if } x \not\in U, \end{cases}$$

that is

$$r(x) = \frac{x}{\max\{1, p(x)\}}, \text{ for } x \in E.$$
Since $0 \in U = \text{int}(U)$, $p$ is continuous and so $r$ is continuous. Let $f$ be the restriction of $r$ to $C$. Since $C$ is convex and $0 \in C$, it follows that $f(C) \subseteq \overline{U} \cap C$. Also $f \in C(U \cap C)$. Since $\mathcal{U}_C$ is closed under composition, $f \circ F \in \mathcal{U}_C(U \cap C)$. Let $G = f \circ F$. We show that $G$ is $\Phi$-condensing. Let $A$ be a subset of $\overline{U} \cap C$ such that $\Phi(A) \subseteq \Phi(G(A))$. Then $G(A) \subseteq \text{co}(\{0\} \cup F(A))$ and so

\[
\Phi(A) \leq \Phi(G(A)) \leq \Phi(\text{co}(\{0\} \cup F(A))) \leq \Phi(\{0\} \cup F(A)) \leq \max\{\Phi(\{0\}), \Phi(F(A))\} = \Phi(F(A)),
\]

which gives $\overline{A}$ is compact. This shows that $G$ is $\Phi$-condensing and so, by Lemma 2.2, there exists a nonempty compact convex subset $K$ of $\overline{U} \cap C$ such that $G(K) \subseteq K$. Since $G \in \mathcal{U}_C(U \cap C, \overline{U} \cap C)$ and $K$ is compact, there exists $T \in \mathcal{U}_C(K, \overline{U} \cap C)$ such that $T(x) \subseteq G(x)$ for all $x \in K$. This implies that $T(K) \subseteq G(K) \subseteq K$ and $T$ is compact. Since $T \in \mathcal{U}_C(K, K)$, by Lemma 2.1, $T$ has a fixed point $x_0 \in K$, that is, $x_0 \in T(x_0) \subseteq G(x_0)$. Clearly $x_0 \in \overline{U} \cap C$. Therefore, there exists some $y_0 \in F(x_0)$ with $x_0 = f(y_0)$. Now, we consider two cases:

(a) $y_0 \in \overline{U} \cap C$ or
(b) $y_0 \in C \setminus \overline{U}$.

Suppose $y_0 \in \overline{U} \cap C$. Then $x_0 = f(y_0) = y_0$. As a result

\[p(y_0 - x_0) = 0 = d_p(0, \overline{U} \cap C)\]

and $x_0$ is a fixed point of $F$. On the other hand, if $y_0 \in C \setminus \overline{U}$, then

\[x_0 = f(y_0) = \frac{y_0}{p(y_0)}.
\]

So, for any $x \in \overline{U} \cap C$,

\[p(y_0 - x_0) = p\left(y_0 - \frac{y_0}{p(y_0)}\right) = \left(\frac{p(y_0)}{p(y_0)} - 1\right)p(y_0) = p(y_0) - 1 \leq p(y_0) - p(x) = p((y_0 - x) + x) - p(x) \leq p(y_0 - x),\]

which gives

\[p(y_0 - x_0) = \inf\{p(y_0 - z) \mid z \in \overline{U} \cap C\} = d_p(y_0, \overline{U} \cap C).
\]

Since $p(y_0 - x_0) = p(y_0) - 1$, we have $p(y_0 - x_0) > 0$.

Let $z \in \mathcal{T}(x_0) \cap C \setminus (\overline{U} \cap C)$. Then there exists $y \in \overline{U}$ and $c \geq 1$ with

\[z = x_0 + c(y - x_0).
\]

Suppose that $p(y_0 - z) < p(y_0 - x_0)$.

The convexity of $C$ implies that

\[\frac{1}{c}z + \left(1 - \frac{1}{c}\right)x_0 \in C.
\]
Since \( \frac{1}{c} z + \left(1 - \frac{1}{c}\right)x_0 = y \in \mathcal{U} \),

it follows that

\[
p(y_0 - y) = p \left[ \frac{1}{c}(y_0 - z) + \left(1 - \frac{1}{c}\right)(y_0 - x_0) \right] \\
\leq \frac{1}{c} p(y_0 - z) + \left(1 - \frac{1}{c}\right)p(y_0 - x_0) < p(y_0 - x_0).
\]

This contradicts the choice of \( y_0 \). Consequently, we have

\[
p(y_0 - x_0) \leq p(y_0 - z) \quad \text{for all } z \in I_{\mathcal{U}}(x_0) \cap C.
\]

The continuity of \( p \) further implies that

\[
p(y_0 - x_0) \leq p(y_0 - z) \quad \text{for all } z \in I_{\mathcal{U}}(x_0) \cap C.
\]

Hence

\[
0 < p(y_0 - x_0) = d_p(y_0, \mathcal{U} \cap C) = d_p(y_0, I_{\mathcal{U}}(x_0) \cap C).
\]

If \( x_0 \in U \), then \( I_{\mathcal{U}}(x_0) = E \) and so \( d_p(y_0, I_{\mathcal{U}}(x_0) \cap C) = 0 \). Thus \( x_0 \in \partial_C(U) \).

Essentially the same reasoning as before yields the following result.

**Theorem 3.2.** Let \( C \) be a closed, convex subset of a Hausdorff locally space \( E \) with \( 0 \in \text{int}(C) \). Suppose \( F \in \mathcal{U}_c^\Phi(C, E) \) is a \( \Phi \)-condensing map. Then there exist \( x_0 \in C \) and \( y_0 \in F(x_0) \) with

\[
p(y_0 - x_0) = d_p(y_0, C) = d_p(y_0, I_C(x_0)),
\]

here \( p \) is the Minkowski functional of \( C \) in \( E \). More precisely, either

(a) \( F \) has a fixed point \( x_0 \in C \), or

(b) there exist \( x_0 \in \partial(C) \) and \( y_0 \in F(x_0) \) with

\[
0 < p(y_0 - x_0) = d_p(y_0, C) = d_p(y_0, I_C(x_0)).
\]

Since \( p(x) = \|x\|/R \) is the Minkowski functional on \( B_R \), we have the following result.

**Corollary 3.3.** Let \( E \) be a normed space. Suppose \( F \in \mathcal{U}_c^\Phi(B_R, E) \) is a \( \Phi \)-condensing map. Then there exist \( x_0 \in B_R \) and \( y_0 \in F(x_0) \) with

\[
\|y_0 - x_0\| = d(y_0, B_R) = d(y_0, I_{B_R}(x_0)).
\]

More precisely, either

(a) \( F \) has a fixed point \( x_0 \in B_R \) or

(b) there exist \( x_0 \in \partial(B_R) \) and \( y_0 \in F(x_0) \) with

\[
0 < \|y_0 - x_0\| = d(y_0, B_R) = d(y_0, I_{B_R}(x_0)).
\]
Remark 3.1. Theorem 1 of Lin and Park [7] and a result of Lin [6] can be considered as special cases of Corollary 3.3.

As applications of our approximation theorems, we now derive some fixed point results.

Theorem 3.4. Let $C$ be a closed, convex subset of a Hausdorff locally convex space $E$ with $0 \in C$ and $U$ a convex open neighbourhood of $0$. Suppose $f \in \mathcal{U}(U \cap C, C)$ is a $\Phi$-condensing map. If $f$ satisfies any one of the following conditions for any $x \in \partial_C(U) \setminus f(x)$:

(a) for each $y \in f(x)$, $p(y - z) < p(y - x)$ for some $z \in \overline{T_U(x)} \cap C$,
(b) for each $y \in f(x)$, there exists $\lambda$ with $|\lambda| < 1$ such that $\lambda x + (1 - \lambda)y \in \overline{T_U(x)} \cap C$,
(c) $f(x) \subseteq \overline{T_U(x)} \cap C$,
(d) $f(x) \cap \{ \lambda x \mid \lambda > 1 \} = \emptyset$,
(e) for each $y \in f(x)$, $p(y - x) \neq p(y) - 1$,
(f) for each $y \in f(x)$, there exists $\alpha \in (1, \infty)$ such that

$$p^\alpha(y) - 1 \leq p^\alpha(y - x),$$

(g) for each $y \in f(x)$, there exists $\beta \in (0, 1)$ such that $p^\beta(y) - 1 \geq p^\beta(y - x)$,

then $f$ has a fixed point.

Proof. Theorem 3.1 guarantees that either

(1) $f$ has a fixed point in $U \cap C$ or
(2) there exist $x_0 \in \partial_C(U)$ and $y_0 \in f(x_0)$ with $x_0 = f(y_0)$ such that

$$0 < p(y_0) - 1 = p(y_0 - x_0) = d_p(y_0, U \cap C) = d_p(y_0, \overline{T_U(x_0)} \cap C),$$

where $p$ is the Minkowski functional of $U$ and $f$ is the restriction of the continuous retraction $r$ to $C$.

Suppose (2) holds (with some $x_0$ and $y_0$) and $x_0 \notin f(x_0)$. We shall show contradictions in all conditions (a)–(g).

If $f$ satisfies condition (a), then we have $p(y_0 - z) < p(y_0 - x_0)$, for some $z \in \overline{T_U(x_0)} \cap C$. This contradicts the choice of $x_0$.

If $f$ satisfies condition (b), then there exists $\lambda$ with $|\lambda| < 1$ such that $\lambda x_0 + (1 - \lambda)y_0 \in \overline{T_U(x_0)} \cap C$. This implies that

$$p(y_0 - x_0) \leq p(y_0 - (\lambda x_0 + (1 - \lambda)y_0)) = p(\lambda(y_0 - x_0)) = |\lambda|p(y_0 - x_0) < p(y_0 - x_0),$$

which is a contradiction.

The proof for condition (c) is obvious.
If $F$ satisfies condition (d), then $\lambda x_0 \neq y_0$ for each $\lambda > 1$. But we have $x_0 = f(y_0) = y_0/p(y_0)$. Therefore, $y_0 = \lambda_0 x_0$ with $\lambda_0 = p(y_0) > 1$, which is a contradiction.

If $F$ satisfies condition (e), then $p(y_0 - x_0) \neq p(y_0) - 1$ and this contradicts $p(y_0 - x_0) = p(y_0) - 1$.

If $F$ satisfies condition (f), then there exists $\alpha \in (1, \infty)$ with $p^\alpha(y_0) - 1 \leq p^\alpha(y_0 - x_0)$. Set $\lambda_0 = 1/p(y_0)$. Then $\lambda_0 \in (0, 1)$ and
\[
\frac{(p(y_0) - 1)^\alpha}{p^\alpha(y_0)} = (1 - \lambda_0)^\alpha < 1 - \lambda_0^\alpha = \frac{p^\alpha(y_0) - 1}{p^\alpha(y_0)} \leq \frac{p^\alpha(y_0 - x_0)}{p^\alpha(y_0)}.
\]
This implies that $p(y_0 - x_0) > p(y_0) - 1$. This contradicts the fact that $p(y_0 - x_0) = p(y_0) - 1$.

Finally if $F$ satisfies condition (g), then, as above (see the proof of (f)), we can get a contradiction to $p(y_0 - x_0) = p(y_0) - 1$.

\begin{remark}
We have derived the Leray–Schauder principle as an application of Theorem 3.1 (see Theorem 3.4(d)), which was established by Lin and Yu in [8].

Essentially the same reasoning as in Theorem 3.4 (with Theorem 3.2 replacing Theorem 3.1) yields the following result.
\end{remark}

\begin{theorem}
Let $C$ be a closed, convex subset of a Hausdorff locally convex space $E$ with $0 \in \operatorname{int}(C)$. Suppose $F \in \mathcal{K}(C, E)$ is a $\Phi$-condensing map. If $F$ satisfies any one of the following conditions for any $x \in \partial(C) \setminus F(x)$:
\begin{enumerate}[(a)]
\item for each $y \in F(x)$, $p(y - z) < p(y - x)$, for some $z \in I_C(x)$,
\item for each $y \in F(x)$, there exists $\lambda$ with $|\lambda| < 1$ such that $\lambda x + (1 - \lambda)y \in I_C(x)$,
\item $F(x) \subseteq \overline{I_C(x)}$,
\item $F(x) \cap \{x : \lambda > 1\} = \emptyset$,
\item for each $y \in F(x)$, $p(y - x) \neq p(y) - 1$,
\item for each $y \in F(x)$, there exists $\alpha \in (1, \infty)$ such that $p^\alpha(y) - 1 \leq p^\alpha(y - x)$,
\item for each $y \in F(x)$, there exists $\beta \in (0, 1)$ such that $p^\beta(y) - 1 \geq p^\beta(y - x)$,
\end{enumerate}
then $F$ has a fixed point.
\end{theorem}

\begin{corollary}
Let $E$ be a normed space. Suppose $F \in \mathcal{K}(B_R, E)$ is a $\Phi$-condensing map. If $F$ satisfies any one of the following conditions for any $x \in \partial(B_R) \setminus F(x)$:
\begin{enumerate}[(a)]
\item for each $y \in F(x)$, $\|y - z\| < \|y - x\|$, for some $z \in \overline{I_{B_R}(x)}$,
\end{enumerate}
then $F$ has a fixed point.
\end{corollary}
(b) for each \( y \in F(x) \), there exists \( \lambda \) with \( |\lambda| < 1 \) such that \( \lambda x + (1-\lambda)y \in T_B(x) \),
(c) \( F(x) \subseteq T_B(x) \),
(d) \( F(x) \cap \{ \lambda x \mid \lambda > 1 \} = \emptyset \),
(e) for each \( y \in F(x) \), \( \| y - x \| \neq \| y \| - R \),
(f) for each \( y \in F(x) \), there exists \( \alpha \in (1, \infty) \) such that \( \| y \|^{\alpha} - R \leq \| y - x \|^{\alpha} \),
(g) for each \( y \in F(x) \), there exists \( \beta \in (0, 1) \) such that \( \| y \|^{\beta} - R \geq \| y - x \|^{\beta} \),

then \( F \) has a fixed point.

**Remark 3.3.** Corollary 3.6 contains, as special cases, Theorem 2 of Lin and Park [7] as well as a result of Lin [6].

Essentially the same reasoning as above gives the following results in Hilbert spaces (here the retraction \( r \) is replaced by the proximity map \( p \)), which extend Theorem 3 and Theorem 4 of Lin and Park [7].

**Theorem 3.7.** Let \( C \) be a nonempty, closed, convex subset of a Hilbert space \( H \). Suppose \( F \in U^k_c(C, H) \) is a \( \Phi \)-condensing map. Then there exist \( x_0 \) and \( y_0 \in F(x_0) \) with

\[
\| y_0 - x_0 \| = d(y_0, C) = d(y_0, I_C(x_0)),
\]

here \( \| \cdot \| \) is the norm induced by the inner product. More precisely, either

(a) \( F \) has a fixed point \( x_0 \in C \) or
(b) there exist \( x_0 \in \partial(C) \) and \( y_0 \in F(x_0) \) with

\[
0 < \| y_0 - x_0 \| = d(y_0, C) = d(y_0, I_C(x_0)).
\]

**Theorem 3.8.** Let \( C \) be a nonempty, closed, convex subset of a Hilbert space \( H \). Suppose \( F \in U^k_c(C, H) \) is a \( \Phi \)-condensing map. If \( F \) satisfies any one of the following conditions for any \( x \in \partial(C) \setminus F(x) \):

(a) for each \( y \in F(x) \), \( \| y - z \| < \| y - x \| \), for some \( z \in I_C(x) \),
(b) for each \( y \in F(x) \), there exists \( \lambda \) with \( |\lambda| < 1 \) such that \( \lambda x + (1-\lambda)y \in I_C(x) \),
(c) \( F(x) \subseteq I_C(x) \),

then \( F \) has a fixed point.

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