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COUNTING SOLUTIONS OF NONLINEAR ABSTRACT EQUATIONS

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Abstract. In this paper we use the topological degree to estimate the minimal number of solutions of the sections (defined by fixing a parameter) of the semi-bounded components of a general class of one-parameter abstract nonlinear equations by means of the *signature* of the semi-bounded component. A semi-bounded component is, roughly speaking, a component that is bounded along one direction of the parameter. The signature consists of the set of bifurcation values from the trivial state of the component together with their associated parity indices. The parity is a local invariant measuring the change of the local index of the trivial state.

1. Introduction

Suppose U is a real Banach space, denote by $\mathcal{L}(U)$ the set of linear continuous operators in U, and consider a continuous map $\mathfrak{F}\colon \mathbb{R} \times U \to U$ of the form

$$\mathfrak{F}(\lambda, u) = \mathfrak{L}(\lambda)u + \mathfrak{N}(\lambda, u),$$

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where

(HL) $\mathfrak{L}\colon \mathbb{R} \to \mathcal{L}(U)$ is a continuous mapping such that $\mathfrak{L}(\lambda) - I$ is compact for each $\lambda \in \mathbb{R}$; I being the identity operator on U.

(HN) $\mathfrak{N}\colon \mathbb{R} \times U \to U$ is a compact operator, such that

$$\lim_{u\to 0} \sup_{\lambda\in K} \frac{\|\mathfrak{N}(\lambda,u)\|}{\|u\|} = 0$$

for every compact set $K \subset \mathbb{R}$.

The main goal of this paper is to analyze some fine properties of the *semi-bounded components* of the set of non-trivial solutions of

$$\mathfrak{F}(\lambda,u) = 0. \tag{1.1}$$

Thanks to (HL) and (HN), $(\lambda,u) = (\lambda,0)$ solves equation (1.1) for each $\lambda \in \mathbb{R}$. This is why any solution of the form $(\lambda,0)$ will be called a *trivial solution*, while solutions of the form (λ,u) with $u \neq 0$ are referred to as non-trivial solutions. More precisely, although it may contain some trivial solution, the set of non-trivial solutions of equation (1.1) is defined by

$$\mathfrak{S} := \{(\lambda,u) \in \mathbb{R} \times (U \setminus \{0\}) : \mathfrak{F}(\lambda,u) = 0\} \cup (\Sigma \times \{0\}), \tag{1.2}$$

where Σ stands for the *spectrum of the family* $\mathfrak{L}(\lambda)$, i.e., the set of $\sigma \in \mathbb{R}$ such that $\mathfrak{L}(\sigma)$ has a non-trivial kernel. Since $\mathfrak{L}(\lambda)$ is Fredholm of index zero, by the open mapping theorem, it is an isomorphism if $\lambda \in \mathbb{R} \setminus \Sigma$. Combining this fact together with (HN), it is easily seen that Σ is a closed subset of $\mathbb{R}$ and that all bifurcation values of λ to non-trivial solutions of (1.1) from the trivial solution $(\lambda,0)$ must lie in Σ. In particular, by the continuity of $\mathfrak{F}$, $\mathfrak{S}$ is a closed subset of $\mathbb{R} \times U$ (cf. [6, Section 6.1]). The set $\mathfrak{S}$ consists of all non-trivial solutions of (1.1) plus all possible bifurcation points from the trivial solution curve $(\lambda,0)$. Although far from necessary, throughout this paper we assume that Σ is discrete. Then, one can introduce a *parity map* $P\colon \Sigma \to \{-1,0,1\}$, as follows

$$P(\lambda) := \frac{1}{2} \lim_{\varepsilon\downarrow 0} [\operatorname{Ind}(0,\mathfrak{L}(\lambda+\varepsilon)) - \operatorname{Ind}(0,\mathfrak{L}(\lambda-\varepsilon))]$$

where $\operatorname{Ind}(0,\,\cdot\,)$ stands for the local topological degree of zero — the index. So the parity vanishes if, and only if, the local topological degree does not change. In the sequel, when writing $\operatorname{Ind}(u,f)$, it entails that the Leray–Schauder degree $\operatorname{Deg}(f,B_R(u))$ is defined and independent for every $R > 0$ small enough, and $\operatorname{Ind}(u,f)$ equals this common value; $B_R(u)$ being the ball of radius R centered at $u \in U$.

As usual in global bifurcation theory, throughout this work by a *component* of a closed subset S of $\mathfrak{S}$ it is meant a maximal (for the inclusion) closed and connected subset of S. The most powerful result available in *global bifurcation* *theory* establishes that if $\mathfrak{C}$ is a bounded component of $\mathfrak{S}$, necessarily compact, with

$$\mathcal{B} := \mathfrak{C} \cap (\Sigma \times \{0\}) \neq \emptyset, \tag{1.4}$$

then

$$\sum_{\sigma \in \mathcal{P}_\lambda \mathcal{B}} P(\sigma) = 0, \tag{1.5}$$

where $\mathcal{P}_\lambda$ stands for the projection $\mathcal{P}_\lambda(\lambda, u) = \lambda$ (cf. P. H. Rabinowitz [9], [10], E. N. Dancer [1], [74], R. J. Magnus [7], J. Ize [4], [6, Chapter 6] and the references therein). As a consequence, if we denote by $\mathfrak{C}$ the component emanating from $(\lambda, 0)$ at a value $\lambda = \sigma \in \Sigma$ with $|P(\sigma)| = 1$, whose existence is guaranteed from the pioneering results by M. A. Krasnosel'skiĭ [5], then, some of the following alternatives occurs:

(1) $\mathfrak{C}$ is unbounded.
(2) $\mathfrak{C}$ contains another point of the form

$$(\widetilde{\sigma}, 0) \neq (\sigma, 0) \quad \text{with } P(\widetilde{\sigma})P(\sigma) = -1.$$

In particular, it must be unbounded if $\mathcal{B} = \{(\sigma, 0)\}$. This result, found in the pioneering paper of P. H. Rabinowitz [9], and usually refereed to as *Rabinowitz's alternative*, has been one of the paradigms of nonlinear analysis during the last three decades, because of its huge number of applications (cf. H. Amann [1], P. M. Fitzpatrick and J. Pejsachowicz [3], J. Mawhin [8], as well as the references therein). Precisely because of the great number of applications — as well as its really simple statement, easily retained by non-experts — Rabinowitz's alternative has not facilitated the development of sharper topological tools based on Leray–Schauder degree towards ascertaining further hidden properties of the solution components of nonlinear abstract equations. Concretely, it seems that, in practice, most of nonlinear analysts using these global bifurcation results are forgetting that the Leray–Schauder degree is a generalized counter of the number of zeros of $\mathfrak{F}(\lambda, \cdot)$, for each value of λ, and that Rabinowitz's Alternative exclusively expresses the fact that $\mathfrak{C}$ is unbounded if (1.5) fails.

In this paper we go back to the roots of the theory by using the topological degree to count the exact number of solutions of $\mathfrak{C}$ for each of the values of λ where $\mathcal{P}_\lambda \mathfrak{C} \neq \emptyset$. As a result, rather naturally, we are conducted towards the problem of analyzing the fine global topological structure of the semi-bounded and bounded components of $\mathfrak{S}$; an analysis which seems to be completely pioneering in the field.

To summarize the main results of this paper, we need to sketch the main methodology adopted in it. Throughout this paper, for any subset $S \subset \mathbb{R} \times U$

and $\lambda \in \mathbb{R}$ we will denote

$$S_\lambda := \{u \in U : (\lambda, u) \in S\}.$$

Suppose $\mathfrak{C}$ is a component of $\mathfrak{S}$, not necessarily bounded, set

$$\mathcal{B} := \mathfrak{C} \cap (\Sigma \times \{0\}),$$

pick $\Lambda \in \mathbb{R} \setminus \mathcal{P}_\lambda \mathcal{B}$ and consider $J \in \{(-\infty, \Lambda], [\Lambda, \infty)\}$. Let $\{\mathfrak{C}_\alpha\}_{\alpha \in A}$ be the family of components of $\mathfrak{C} \cap (J \times U)$ and set

$$\mathcal{B}_\alpha := \mathfrak{C}_\alpha \cap (\Sigma \times \{0\}), \quad \alpha \in A.$$

Then, $\{\mathcal{B}_\alpha\}_{\alpha \in A}$ is a family of disjoint subsets with union $\mathfrak{C} \cap (J \times \{0\})$. Except for a countable set of $\alpha \in A$, $\mathcal{B}_\alpha = \emptyset$, since Σ is countable. Actually, if $\mathcal{B} \cap (J \times \{0\})$ is finite, then $\mathcal{B}_\alpha = \emptyset$ except for a finite set of $\alpha \in A$. In general, for each $\lambda \in J$, one has that

$$\operatorname{Card} \mathfrak{C}_\lambda = \sum_{\alpha \in A} \operatorname{Card} (\mathfrak{C}_\alpha)_\lambda.$$

The main goal of this paper is to get optimal general estimates for $\operatorname{Card} (\mathfrak{C}_\alpha)_\lambda$. For each $\alpha \in A$ some of the following alternatives occurs:

(1) $\mathfrak{C}_\alpha$ is bounded in $J \times U$.
(2) $\mathfrak{C}_\alpha$ is unbounded in $J \times U$.

Suppose alternative (2) occurs and set

$$J_\alpha := \mathcal{P}_\lambda \mathfrak{C}_\alpha.$$

Then, $\operatorname{Card} (\mathfrak{C}_\alpha)_\lambda \geq 1$ for each $\lambda \in J_\alpha$, while $\operatorname{Card} (\mathfrak{C}_\alpha)_\lambda = 0$ if $\lambda \in J \setminus J_\alpha$. In general, $\operatorname{Card} (\mathfrak{C}_\alpha)_\lambda$ might be arbitrarily large for some $\lambda \in J_\alpha$, and hence 1 can be regarded as the minimal cardinal of $(\mathfrak{C}_\alpha)_\lambda$.

Now, suppose $\mathfrak{C}_\alpha$ satisfies alternative (1); an $\mathfrak{C}_\alpha$ arising in this way is called a semi-bounded component. The main result of this paper establishes that for any open isolating neighbourhood of $\mathfrak{C}_\alpha$ in $J \times U$, say Ω_α (cf. Definition 2.1), and any $\lambda^* \in J \setminus \mathcal{P}_\lambda \mathcal{B}_\alpha$ there exists $\rho^* > 0$ such that for any $0 < \rho \leq \rho^*$

$$\operatorname{Deg}(\mathfrak{F}(\lambda^*, \cdot\,), (\Omega_\alpha)_{\lambda^*} \setminus \overline{B}_\rho) = 2 \operatorname{sign} J^* \sum_{\sigma \in J^* \cap \mathcal{P}_\lambda \mathcal{B}_\alpha} P(\sigma), \tag{1.6}$$

where

$$J^* := \begin{cases} (\lambda^*, \infty) & \text{if } J = [\Lambda, \infty), \\ (-\infty, \lambda^*) & \text{if } J = (-\infty, \Lambda], \end{cases} \qquad \operatorname{sign} J^* := \begin{cases} 1 & \text{if } J = [\Lambda, \infty), \\ -1 & \text{if } J = (-\infty, \Lambda]. \end{cases}$$

Besides providing (1.5) and, hence, all existing abstract global bifurcation results (cf. Theorem 2.3), equation (1.6) also gives, in a rather natural manner, the minimal number of solutions of $(\mathfrak{C}_\alpha)_{\lambda^*}$ under some additional non-degeneracy conditions which are satisfied in many special circumstances of great interest.

The knowledge of the minimal number of solutions of $(\mathfrak{C}_\alpha)_{\lambda^*}$ in these special cases conducts rather naturally towards the problem of analyzing all admissible structures that $\mathfrak{C}_\alpha$ may have in the special case when $\mathfrak{C}_\alpha$ consists of a finite number of compact arcs of continuous curves in terms of the *signature* of $\mathfrak{C}_\alpha$ (cf. Definition 3.4); the signature being defined as the set $\mathcal{P}_\lambda\mathcal{B}_\alpha$ together with the parity value at these points. It seems this is the first work where this problem has been addressed in the context of global bifurcation theory. We refrain from giving more details herein.

This paper is organized as follows. In Section 2 we prove (1.6). In Section 3 we use (1.6) to estimate the number of solutions of $(\mathfrak{C}_\alpha)_{\lambda^*}$ and, then, introduce the concept of signature of $\mathfrak{C}_\alpha$ and the concept of minimal number of solutions. Finally, in Sections 4 and 5 we analyze the case when $\mathfrak{C}_\alpha$ consists of compact arcs of continuous curves under certain regularity assumptions. Although the analysis of this special situation is very simple, it is certainly the beginning of a general abstract theory that should provide us with the minimal topological structure that $\mathfrak{C}_\alpha$ should have in order to be an admissible semi-bounded component of $\mathfrak{C}$.

2. The main theorem

Throughout this section we suppose that

$$J \in \{\mathbb{R}, (-\infty, \Lambda], [\Lambda, \infty)\} \tag{2.1}$$

for some $\Lambda \in \mathbb{R}$, and that $\mathfrak{C}$ is a bounded (hence compact) component of $\mathfrak{S} \cap (J \times U)$. The main goal of this section is to study general properties about the number of elements of $\mathfrak{C}_\lambda$ for each $\lambda \in J$. Most of the results of this section are new even in the classical case when $J = \mathbb{R}$. Subsequently, we denote by $\mathcal{B}$ the set of bifurcation points from $J \times \{0\}$ of $\mathfrak{C}$, i.e.

$$\mathcal{B} := \mathfrak{C} \cap (\Sigma \times \{0\}). \tag{2.2}$$

The set $\mathcal{B}$ is compact and discrete, so finite, possibly empty. The following concept will play a crucial role in the subsequent analysis.

Definition 2.1. A bounded open set $\Omega \subset \mathbb{R} \times U$ is said to be an *open isolating neighbourhood* of $\mathfrak{C}$ in $J \times U$ if the following conditions are satisfied:

$$\mathfrak{C} \subset \Omega, \quad \partial\Omega \cap \mathfrak{S} \subset (\mathbb{R} \setminus J) \times U, \quad \mathcal{B} = \Omega \cap (\Sigma \times \{0\}), \tag{2.3}$$

where $\mathcal{B}$ is the set defined by (2.2). In the special case when $J = \mathbb{R}$, the second relation of (2.3) should be read as $\partial\Omega \cap \mathfrak{S} = \emptyset$.

The following result establishes the existence of open isolating neighbourhoods satisfying some adequate properties to calculate the topological degree of $\mathfrak{F}$, which will provide us with the desired multiplicity results. Subsequently, we denote by $B_R(\lambda,u)$ the ball of radius $R>0$ centered at $(\lambda,u)\in\mathbb{R}\times U$, and by B_R the ball of radius R centered at the origin in U.

Proposition 2.2. *Suppose* (2.1) *and let* $\mathfrak{C}$ *be a bounded component of* $\mathfrak{S}\cap(J\times U)$. *Then, for each* $\beta>0$, $\mathfrak{C}$ *possesses an open isolating neighbourhood in* $J\times U$, *say* Ω, *with the property* $\Omega\subset\mathfrak{C}+B_\beta(0,0)$. *Moreover, for any open isolating neighbourhood* Ω *of* $\mathfrak{C}$ *in* $J\times U$ *and any* $\varepsilon>0$ *there exists* $\rho^*=\rho^*(\varepsilon)>0$ *such that for any* $0<\rho\le\rho^*$ *and any*

$$\lambda\in J_\varepsilon:=J\setminus\bigcup_{\sigma\in\mathcal{P}_\lambda\mathcal{B}}(\sigma-\varepsilon,\sigma+\varepsilon)$$

some of the following alternatives occurs:

(a) $\overline{B}_\rho\cap\Omega_\lambda=\emptyset$.
(b) $\{u\in\overline{B}_\rho:\mathfrak{F}(\lambda,u)=0\}=\{0\}$.

Proof. First, we will construct an open isolating neighbourhood. Fix $\alpha>0$ small so that $\mathcal{U}:=\mathfrak{C}+B_\alpha(0,0)$ satisfies

$$\overline{\mathcal{U}}\cap(\Sigma\times\{0\})=\mathcal{B}. \tag{2.4}$$

If $\partial\mathcal{U}\cap\mathfrak{S}\subset(\mathbb{R}\setminus J)\times U$, then $\mathcal{U}$ provides us with an open isolating neighbourhood of $\mathfrak{C}$ in $J\times U$, but, in general, this will not be the case. So, suppose

$$\partial\mathcal{U}\cap\mathfrak{S}\cap(J\times U)\neq\emptyset$$

and set

$$M:=\overline{\mathcal{U}}\cap\mathfrak{S}\cap(J\times U),\quad A:=\mathfrak{C},\quad B:=\partial\mathcal{U}\cap\mathfrak{S}\cap(J\times U).$$

Then, M is a compact metric space and A,B are two disjoint compact non-empty subsets of M. Moreover, by the maximality of $\mathfrak{C}$, no subcontinuum of M connects A with B. Thus, by a well-known result going back to G. T. Whyburn [11], there exist two disjoint compact subsets of M, say, M_A and M_B, such that

$$A\subset M_A,\quad B\subset M_B,\quad M=M_A\cup M_B.$$

Then, the open set neighbouring $\Omega:=M_A+B_\eta(0,0)$ provides us with an open isolating neighbourhood of $\mathfrak{C}$ in $J\times U$ for any sufficiently small $\eta>0$. Indeed, by (2.4), we have $M_A\cap(\Sigma\times\{0\})=\mathcal{B}$ and hence, for any sufficiently small $\eta>0$,

$$\Omega\cap(\Sigma\times\{0\})=\mathcal{B}.$$

Moreover, $\mathfrak{C}\subset M_A\subset\Omega$. Thus, it remains to check that

$$\partial\Omega\cap\mathfrak{S}\subset(\mathbb{R}\setminus J)\times U. \tag{2.5}$$

Since $\operatorname{dist}(M_A, M_B) > 0$, reducing $\eta > 0$, if necessary, one has that $\partial\Omega \cap M = \emptyset$, since $M = M_A \cup M_B$. Moreover, since $M_A \subset \mathcal{U}$, for any sufficiently small $\eta > 0$, $\partial\Omega \subset \mathcal{U}$. Thus,

$$\emptyset = \partial\Omega \cap M = \partial\Omega \cap \overline{\mathcal{U}} \cap \mathfrak{S} \cap (J \times U) = \partial\Omega \cap \mathfrak{S} \cap (J \times U).$$

This concludes the proof of (2.5) and shows that Ω is an open isolating neighbourhood. Moreover, by construction,

$$\Omega \subset \overline{\mathfrak{C} + B_\alpha(0,0)} + B_\eta(0,0)$$

and $\alpha, \eta > 0$ can be arbitrarily small. So for any $\beta > 0$ we can obtain Ω an open isolating neighbourhood of $\mathfrak{C}$ in $J \times U$ such that $\Omega \subset \mathfrak{C} + B_\beta(0,0)$.

Now, let Ω be any open isolating neighbourhood of $\mathfrak{C}$. To show the existence of $\rho^* > 0$ satisfying all the requirements of the statement we will argue by contradiction. Assume that there exists $\varepsilon > 0$ such that for each integer $n \geq 1$ there exist $\rho_n > 0$, $\lambda_n \in J_\varepsilon$, and $u_n \in \overline{B}_{\rho_n} \setminus \{0\}$ such that

$$\lim_{n\to\infty} \rho_n = 0, \quad \overline{B}_{\rho_n} \cap \Omega_{\lambda_n} \neq \emptyset, \quad \mathfrak{F}(\lambda_n, u_n) = 0.$$

If $\{\lambda_n\}_{n\in\mathbb{N}}$ is unbounded, then $\Omega_{\lambda_k} = \emptyset$ for some k, which is impossible, since $\overline{B}_{\rho_n} \cap \Omega_{\lambda_n} \neq \emptyset$. Then, $\{\lambda_n\}_{n\in\mathbb{N}}$ must be bounded. Thus, by extracting an adequate subsequence, labeled again by n, one can suppose that

$$\lim_{n\to\infty} \lambda_n = \sigma \in J_\varepsilon.$$

Since $u_n \neq 0$ and $\lim_{n\to\infty}(\lambda_n, u_n) = (\sigma, 0)$, necessarily $\sigma \in \Sigma$. Moreover, $\sigma \notin \mathcal{P}_\lambda\mathcal{B}$, since $\sigma \in J_\varepsilon$. Thus, $\sigma \in \Sigma \setminus \mathcal{P}_\lambda\mathcal{B}$ and, hence, $(\sigma, 0) \notin \Omega$, by the definition of open isolating neighbourhood. Now, pick a $v_n \in \overline{B}_{\rho_n} \cap \Omega_{\lambda_n}$ for each $n \geq 1$. Then, for each $n \geq 1$, $(\lambda_n, v_n) \in \Omega$ and

$$\lim_{n\to\infty} (\lambda_n, v_n) = (\sigma, 0).$$

Thus, $(\sigma, 0) \in \overline{\Omega} \setminus \Omega = \partial\Omega$, and therefore, $(\sigma, 0) \in \partial\Omega \cap \mathfrak{S} \cap (J \times U)$, which is impossible, since $\partial\Omega \cap \mathfrak{S} \subset (\mathbb{R} \setminus J) \times U$. This contradiction concludes the proof.□

Now, we can give the main result of this section.

THEOREM 2.3. *Suppose* (2.1) *and* $\mathfrak{C}$ *is a bounded component of* $\mathfrak{S} \cap (J \times U)$. *Let* Ω *be any open isolating neighbourhood of* $\mathfrak{C}$ *in* $J \times U$. *Then, for each* $\lambda^* \in J \setminus \mathcal{P}_\lambda\mathcal{B}$ *there exists* $\rho^* > 0$ *such that for any* $0 < \rho \leq \rho^*$

$$\operatorname{Deg}(\mathfrak{F}(\lambda^*, \cdot\,), \Omega_{\lambda^*} \setminus \overline{B}_\rho) = 2 \operatorname{sign} J^* \sum_{\sigma \in J^* \cap \mathcal{P}_\lambda\mathcal{B}} P(\sigma), \tag{2.6}$$

where

$$J^* := \begin{cases} (\lambda^*, \infty) & \text{if } J = [\Lambda, \infty), \\ (-\infty, \lambda^*) & \text{if } J = (-\infty, \Lambda], \end{cases} \qquad \operatorname{sign} J^* := \begin{cases} 1 & \text{if } J = [\Lambda, \infty), \\ -1 & \text{if } J = (-\infty, \Lambda], \end{cases}$$

and $J^* \in \{(-\infty, \lambda^*), (\lambda^*, \infty)\}$ *if* $J = \mathbb{R}$. *Actually, if* $J = \mathbb{R}$, *then*

$$\operatorname{Deg}(\mathfrak{F}(\lambda^*, \cdot\,), \Omega_{\lambda^*} \setminus \overline{B}_\rho) = 2 \sum_{\sigma \in (\lambda^*, \infty) \cap \mathcal{P}_\lambda \mathcal{B}} P(\sigma) = -2 \sum_{\sigma \in (-\infty, \lambda^*) \cap \mathcal{P}_\lambda \mathcal{B}} P(\sigma) \tag{2.7}$$

and, therefore,

$$\sum_{\sigma \in \mathcal{P}_\lambda \mathcal{B}} P(\sigma) = 0. \tag{2.8}$$

In any of those cases, when the set over which the summation runs is empty the sum should be taken as zero.

Proof. We shall give all details of the proof in the case when $J = [\Lambda, \infty)$. Since Ω is bounded, there exists $b \in J^* \setminus \Sigma$ such that $\mathcal{P}_\lambda \Omega \subset (-\infty, b)$. Now, we have to distinguish two different cases. Suppose

$$J^* \cap \mathcal{P}_\lambda \mathcal{B} = \emptyset. \tag{2.9}$$

Then, thanks to Proposition 2.2, there exists $\rho^* > 0$ such that for each $\rho \in (0, \rho^*]$ and $\lambda \geq \lambda^*$ some of the following alternatives occurs: either $\overline{B}_\rho \cap \Omega_\lambda = \emptyset$, or $\{u \in \overline{B}_\rho : \mathfrak{F}(\lambda, u) = 0\} = \{0\}$. Thus, for each $\lambda \geq \lambda^*$, the topological degree $\operatorname{Deg}(\mathfrak{F}(\lambda, \cdot\,), \Omega_\lambda \setminus \overline{B}_\rho)$ is well defined. Moreover, the invariance by homotopy shows that

$$\operatorname{Deg}(\mathfrak{F}(\lambda^*, \cdot\,), \Omega_{\lambda^*} \setminus \overline{B}_\rho) = \operatorname{Deg}(\mathfrak{F}(b, \cdot\,), \Omega_b \setminus \overline{B}_\rho) = 0,$$

since $\Omega_b = \emptyset$, which concludes the proof of the theorem under condition (2.9). Now, suppose

$$J^* \cap \mathcal{P}_\lambda \mathcal{B} = \{\sigma_1, \ldots, \sigma_M\}, \tag{2.10}$$

instead of (2.9), where $\sigma_i < \sigma_j$ if $1 \leq i < j \leq M$. Let $\delta > 0$ be such that $\mathcal{B} + B_\delta(0,0) \subset \Omega$,

$$\lambda^* < \sigma_1 - \delta < \sigma_1 + \delta < \sigma_2 - \delta < \ldots < \sigma_M + \delta < b,$$

and, if $\mathcal{P}_\lambda \mathcal{B} \setminus J^* \neq \emptyset$, $\sup[\mathcal{P}_\lambda \mathcal{B} \setminus J^*] + \delta < \lambda^*$. Pick $0 < \rho \leq \rho^*$, where $\rho^* = \rho^*(\delta/2)$ is the value whose existence was shown by Proposition 2.2. By the homotopy invariance of the degree, for each $1 \leq j \leq M$, $\operatorname{Deg}(\mathfrak{F}(\lambda, \cdot\,), \Omega_\lambda)$ is constant in the interval $\lambda \in (\sigma_j - \delta, \sigma_j + \delta)$. Indeed, if $\lambda \in (\sigma_j - \delta, \sigma_j + \delta)$ for some $1 \leq j \leq M$, and $(\lambda, u) \in \partial\Omega$, then $(\lambda, u) \notin \mathfrak{S}$, since $\partial\Omega \cap \mathfrak{S} \subset (\mathbb{R} \setminus J) \times U$, by (2.3), and $\lambda \in J$. Moreover, $(\lambda, 0) \in \mathcal{B} + B_\delta(0,0) \subset \Omega$, and, hence, $(\lambda, 0) \notin \partial\Omega$ either. Thus, $\mathfrak{F}(\lambda, u) \neq 0$ and, therefore,

$$\operatorname{Deg}(\mathfrak{F}(\sigma_j - \delta/2, \cdot\,), \Omega_{\sigma_j - \delta/2}) = \operatorname{Deg}(\mathfrak{F}(\sigma_j + \delta/2, \cdot\,), \Omega_{\sigma_j + \delta/2}), \quad \text{for } 1 \leq j \leq M. \tag{2.11}$$

Also, again by the homotopy invariance of the degree, there exists $(d_0, \dots, d_M) \in \mathbb{Z}^{M+1}$ such that

$$
(2.12)\qquad
\begin{aligned}
\operatorname{Deg}(\mathfrak{F}(\lambda,\,\cdot\,),\Omega_\lambda\setminus\overline{B}_\rho) &= d_j, && \lambda\in[\sigma_j+\delta/2,\sigma_{j+1}-\delta/2],\\
& && 1\le j\le M-1,\\
\operatorname{Deg}(\mathfrak{F}(\lambda,\,\cdot\,),\Omega_\lambda\setminus\overline{B}_\rho) &= d_M, && \lambda\in[\sigma_M+\delta/2,b],\\
\operatorname{Deg}(\mathfrak{F}(\lambda,\,\cdot\,),\Omega_\lambda\setminus\overline{B}_\rho) &= d_0, && \lambda\in[\lambda^*,\sigma_1-\delta/2].
\end{aligned}
$$

In order to prove (2.12) it suffices to show that $\mathfrak{F}(\lambda, u) \neq 0$ if

$$\lambda\in[\lambda^*,\sigma_1-\delta/2]\cup[\sigma_M+\delta/2,b]\cup\bigcup_{j=1}^{M-1}[\sigma_j+\delta/2,\sigma_{j+1}-\delta/2]$$

and $u \in \partial(\Omega_\lambda \setminus \overline{B}_\rho)$. Pick (λ, u) satisfying those requirements. Obviously, $u \neq 0$. Moreover, by Proposition 2.2, either $\overline{B}_\rho \cap \Omega_\lambda = \emptyset$ or $\{u \in \overline{B}_\rho : \mathfrak{F}(\lambda, u) = 0\} = \{0\}$. Suppose

$$\overline{B}_\rho\cap\Omega_\lambda=\emptyset.$$

Then $\Omega_\lambda \setminus \overline{B}_\rho = \Omega_\lambda$, and, hence, $(\lambda, u) \in \partial\Omega \cap (J \times U)$ and $(\lambda, u) \notin \mathfrak{S}$, by (2.3). Thus, $\mathfrak{F}(\lambda, u) \neq 0$, since $u \neq 0$. Now, suppose

$$(2.13)\qquad \{u\in\overline{B}_\rho:\mathfrak{F}(\lambda,u)=0\}=\{0\}.$$

Clearly, $\partial(\Omega_\lambda \setminus \overline{B}_\rho) \subset \partial\Omega_\lambda \cup \partial B_\rho$. Due to (2.13), $u \in \partial B_\rho$ implies $\mathfrak{F}(\lambda, u) \neq 0$, whereas if $u \in \partial\Omega_\lambda$ then $(\lambda, u) \in \partial\Omega \cap (J \times U)$ and, hence, thanks to (2.3), $(\lambda, u) \notin \mathfrak{S}$. Therefore, $\mathfrak{F}(\lambda, u) \neq 0$, since $u \neq 0$. This concludes the proof of (2.12). Further, note that, since $\Omega_b = \emptyset$, $d_M = 0$.

On the other hand, for each sufficiently small $\rho > 0$ and $1 \le j \le M$, we have that

$$
\begin{aligned}
\operatorname{Deg}(\mathfrak{F}(\sigma_j-\delta/2,\,\cdot\,),\Omega_{\sigma_j-\delta/2}) &= d_{j-1}+\operatorname{Ind}(0,\mathfrak{L}(\sigma_j-\delta/2)),\\
\operatorname{Deg}(\mathfrak{F}(\sigma_j+\delta/2,\,\cdot\,),\Omega_{\sigma_j+\delta/2}) &= d_j+\operatorname{Ind}(0,\mathfrak{L}(\sigma_j+\delta/2)),
\end{aligned}
$$

and, hence, using (1.3), the identity (2.11) can be written in the form

$$d_{j-1}-d_j=\operatorname{Ind}(0,\mathfrak{L}(\sigma_j+\delta/2))-\operatorname{Ind}(0,\mathfrak{L}(\sigma_j-\delta/2))=2P(\sigma_j),\quad 1\le j\le M.$$

Thus, adding these equalities and using the fact that $d_M = 0$ we obtain that

$$d_0=\sum_{j=1}^{M}(d_{j-1}-d_j)=2\sum_{j=1}^{M}P(\sigma_j).$$

Therefore, thanks to (2.12),

$$(2.14)\qquad \operatorname{Deg}(\mathfrak{F}(\lambda^*,\,\cdot\,),\Omega_{\lambda^*}\setminus\overline{B}_\rho)=2\sum_{j=1}^{M}P(\sigma_j).$$

Thanks to (2.10), (2.14) provides us with (2.6) when $J = [\Lambda, \infty)$.

Now, we will explain the changes that one has to implement to get the result in the case when $J = (-\infty, \Lambda]$. First, pick $a \in J^* \setminus \Sigma$ such that $\mathcal{P}_\lambda \Omega \subset (a, \infty)$. As in the previous case, if $J^* \cap \mathcal{P}_\lambda \mathcal{B} = \emptyset$, then the invariance by homotopy of the degree gives, for small $\rho > 0$,

$$\operatorname{Deg}(\mathfrak{F}(\lambda^*, \cdot\,), \Omega_{\lambda^*} \setminus \overline{B}_\rho) = \operatorname{Deg}(\mathfrak{F}(a, \cdot\,), \Omega_a \setminus \overline{B}_\rho) = 0,$$

which concludes the proof. So, suppose (2.10), let $\delta > 0$ satisfy $\mathcal{B} + B_\delta(0,0) \subset \Omega$,

$$a < \sigma_1 - \delta < \sigma_1 + \delta < \sigma_2 - \delta < \ldots < \sigma_M + \delta < \lambda^*,$$

and, if $\mathcal{P}_\lambda \mathcal{B} \setminus J^* \neq \emptyset$, $\lambda^* < \inf[\mathcal{P}_\lambda \mathcal{B} \setminus J^*] - \delta$. Pick $0 < \rho \leq \rho^*$, where $\rho^* = \rho^*(\delta/2)$ is the value whose existence was shown by Proposition 2.2. Then, adapting the argument of the previous case, one is led to the identity

$$\operatorname{Deg}(\mathfrak{F}(\lambda^*, \cdot\,), \Omega_{\lambda^*} \setminus \overline{B}_\rho) = -2 \sum_{j=1}^{M} P(\sigma_j),$$

which is (2.6) for this special case.

Finally, suppose $J = \mathbb{R}$, let $\mathfrak{C}$ be a bounded component of $\mathfrak{S}$, Ω an open isolating neighbourhood of $\mathfrak{C}$ in $\mathbb{R} \times U$, and $\lambda^* \in \mathbb{R} \setminus \mathcal{P}_\lambda \mathcal{B}$. Pick up a $\Lambda \leq \lambda^*$ such that $\Omega \subset [\Lambda, \infty) \times U$. Then, $\mathfrak{C}$ is a component of $\mathfrak{S} \cap ([\Lambda, \infty) \times U)$, and Ω is an open isolating neighbourhood of $\mathfrak{C}$ in $[\Lambda, \infty) \times U$. Therefore, thanks to the first part of the proof,

$$\operatorname{Deg}(\mathfrak{F}(\lambda^*, \cdot\,), \Omega_{\lambda^*} \setminus \overline{B}_\rho) = 2 \sum_{\sigma \in (\lambda^*, \infty) \cap \mathcal{P}_\lambda \mathcal{B}} P(\sigma).$$

Analogously, we get the second relation of (2.7), and, as an immediate consequence, (2.8). This completes the proof of the theorem. $\square$

3. Practical consequences of the main theorem

In this section we use Theorem 2.3 to estimate the number of elements of $\mathfrak{C}_\lambda$.

Theorem 3.1. *Suppose* (2.1) *and* $\mathfrak{C}$ *is a bounded component of* $\mathfrak{S} \cap (J \times U)$. *Suppose, in addition, that there exist* $\lambda^* \in J \setminus \mathcal{P}_\lambda \mathcal{B}$, *and* $n \in \mathbb{N}$ *different isolated zeros of* $\mathfrak{F}(\lambda^*, \cdot\,)$, $u_j \in \mathfrak{C}_{\lambda^*}$, $1 \leq j \leq n$, *such that*

$$\operatorname{Ind}(u_j, \mathfrak{F}(\lambda^*, \cdot\,)) \in \{-1, 0, 1\}, \quad 1 \leq j \leq n.$$

Set

$$n_\pm := \operatorname{Card}\{j \in \{1, \ldots, n\} : \operatorname{Ind}(u_j, \mathfrak{F}(\lambda^*, \cdot\,)) = \pm 1\}.$$

Then,

$$\operatorname{Card} \mathfrak{C}_{\lambda^*} \geq n + 1 \quad \textit{whenever } 2 \Big| \sum_{\sigma \in J^* \cap \mathcal{P}_\lambda \mathcal{B}} P(\sigma) \Big| \neq \Big| n_+ - n_- \Big|. \tag{3.1}$$

In particular, $\operatorname{Card}\mathfrak{C}_{\lambda^*} \geq n+1$ *if* $n_+ + n_- \in 2\mathbb{N}+1$.

PROOF. Suppose $\mathfrak{C}_{\lambda^*} = \{u_1, \ldots , u_n\}$. Necessarily $u_j \neq 0$ for each $1 \leq j \leq n$, since $\lambda^* \in J \setminus \mathcal{P}_\lambda\mathcal{B}$. Thus, $\mathfrak{C}_{\lambda^*} \cap \overline{B}_\rho = \emptyset$ for sufficiently small $\rho > 0$.

The set $\mathfrak{C}$ is compact and has empty intersection with

$$Z := \{\lambda^*\} \times \{u \in U \setminus \mathfrak{C}_{\lambda^*} : \mathfrak{F}(\lambda^*, u) = 0\},$$

which is closed due to the fact that each u_j, $1 \leq j \leq n$ is an isolated zero of $\mathfrak{F}(\lambda^*, \cdot\,)$. Therefore, there exists $\beta > 0$ such that $(\mathfrak{C} + B_\beta(0,0)) \cap Z = \emptyset$. By Proposition 2.2, there exists Ω an open isolating neighbourhood of $\mathfrak{C}$ in $J \times U$ such that $\Omega \cap Z = \emptyset$. Then,

$$\mathfrak{C}_{\lambda^*} = \{u \in \Omega_{\lambda^*} \setminus \overline{B}_\rho : \mathfrak{F}(\lambda^*, u) = 0\}.$$

Thus, the additivity property of the degree gives

$$\operatorname{Deg}(\mathfrak{F}(\lambda^*, \cdot\,), \Omega_{\lambda^*} \setminus \overline{B}_\rho) = \sum_{j=1}^{n} \operatorname{Ind}(u_j, \mathfrak{F}(\lambda^*, \cdot\,)).$$

Hence, it follows from (2.6) that

$$2 \operatorname{sign} J^* \sum_{\sigma \in J^* \cap \mathcal{P}_\lambda\mathcal{B}} P(\sigma) = \sum_{j=1}^{n} \operatorname{Ind}(u_j, \mathfrak{F}(\lambda^*, \cdot\,)),$$

and, therefore,

$$2 \left| \sum_{\sigma \in J^* \cap \mathcal{P}_\lambda\mathcal{B}} P(\sigma) \right| = |n_+ - n_-|.$$

This completes the proof of (3.1).

Finally, if $n_+ + n_-$ is odd, then $n_+ - n_-$ is as well odd, and, hence,

$$2 \left| \sum_{\sigma \in J^* \cap \mathcal{P}_\lambda\mathcal{B}} P(\sigma) \right| \neq |n_+ - n_-|.$$

This concludes the proof. $\square$

Subsequently, we will use the following concept.

DEFINITION 3.2. Suppose $\mathfrak{C}$ is a component of $\mathfrak{S} \cap (J \times U)$ and $\lambda^* \in J \setminus \mathcal{P}_\lambda\mathcal{B}$. Then, λ^* is said to be a regular parameter value of $\mathfrak{C}$ if $\mathfrak{C}_{\lambda^*}$ consists in exactly $n \in \mathbb{N}$ points, u_j, $1 \leq j \leq n$, isolated zeros of $\mathfrak{F}(\lambda^*, \cdot\,)$, such that

$$\operatorname{Ind}(u_j, \mathfrak{F}(\lambda^*, \cdot\,)) \in \{-1, 0, 1\}, \quad 1 \leq j \leq n.$$

In such case, we set

$$n_\pm := \operatorname{Card}\{j \in \{1, \ldots , n\} : \operatorname{Ind}(u_j, \mathfrak{F}(\lambda^*, \cdot\,)) = \pm 1\}.$$

Further, λ^* is said to be a strongly regular parameter value of $\mathfrak{C}$ if, in addition, $n = n_+ + n_-$.

Note that the regular-value formula of the degree ensures that $\lambda^* \in J \setminus \mathcal{P}_\lambda\mathcal{B}$ is a strongly regular parameter value of $\mathfrak{C}$ when $D_u\mathfrak{F}(\lambda^*, u)$ exists and is an isomorphism for all $u \in \mathfrak{C}_{\lambda^*}$.

Theorem 3.3. *Suppose* (2.1), $\mathfrak{C}$ *is a bounded component of* $\mathfrak{S} \cap (J \times U)$, *and* $\lambda^* \in J \setminus \mathcal{P}_\lambda\mathcal{B}$ *is a regular parameter value of* $\mathfrak{C}$. *Then,*

$$\operatorname{Card}\mathfrak{C}_{\lambda^*} \geq 2\Bigg|\sum_{\sigma\in J^*\cap\mathcal{P}_\lambda\mathcal{B}} P(\sigma)\Bigg|. \tag{3.2}$$

Moreover, $\operatorname{Card}\mathfrak{C}_{\lambda^*} \in 2\mathbb{N}$ *if* λ^* *is a strongly regular parameter value of* $\mathfrak{C}$.

Proof. It has been assumed that $\mathfrak{C}_{\lambda^*}$ is finite. Moreover, $0 \notin \mathfrak{C}_{\lambda^*}$ since $\lambda^* \notin \mathcal{P}_\lambda\mathcal{B}$. Set $n := \operatorname{Card}\mathfrak{C}_{\lambda^*}$ and suppose that $\mathfrak{C}_{\lambda^*} = \{u_1^*, \dots, u_n^*\}$. Then, reasoning as in the proof of Theorem 3.1 gives

$$2\operatorname{sign} J^* \sum_{\sigma\in J^*\cap\mathcal{P}_\lambda\mathcal{B}} P(\sigma) = \sum_{j=1}^{n} \operatorname{Ind}(u_j, \mathfrak{F}(\lambda^*, \cdot\,)).$$

Thus,

$$\begin{aligned} 2\Bigg|\sum_{\sigma\in J^*\cap\mathcal{P}_\lambda\mathcal{B}} P(\sigma)\Bigg| &= \Bigg|\sum_{j=1}^{n} \operatorname{Ind}(u_j, \mathfrak{F}(\lambda^*, \cdot\,))\Bigg| \\ &\leq \sum_{j=1}^{n} |\operatorname{Ind}(u_j, \mathfrak{F}(\lambda^*, \cdot\,))| = n_+ + n_- \leq \operatorname{Card}\mathfrak{C}_{\lambda^*} \end{aligned}$$

and, therefore, (3.2) holds.

Now, suppose $\operatorname{Card}\mathfrak{C}_{\lambda^*} = n_+ + n_-$. The last assertion of Theorem 3.1 gives $n_+ + n_- \in 2\mathbb{N}$. This concludes the proof. □

Strongly motivated by Theorem 3.3, we give the following fundamental concepts.

Definition 3.4 (Signature and minimal cardinal). Suppose (2.1) and $\mathfrak{C}$ is a bounded component of $\mathfrak{S} \cap (J \times U)$. Consider $\mathcal{B} := \mathfrak{C} \cap (J \times \{0\})$. If $J \neq \mathbb{R}$ suppose $\mathfrak{C}_\Lambda \neq \emptyset$ and $\Lambda \notin \mathcal{P}_\lambda\mathcal{B}$. Then:

(a) When $\mathcal{P}_\lambda\mathcal{B} = \{\sigma_1, \dots, \sigma_N\}$ with $\sigma_i < \sigma_{i+1}$, the *signature* of $\mathfrak{C}$ in $J \times U$ is defined by

$$\operatorname{Signature}[\mathfrak{C}; J \times U] := \begin{pmatrix} \sigma_1 & \dots & \sigma_N \\ P(\sigma_1) & \dots & P(\sigma_N) \end{pmatrix},$$

where $P(\sigma_j)$ stands for the parity of $\sigma_j \in \Sigma$, $1 \leq j \leq N$, whereas the signature of $\mathfrak{C}$ in $J \times U$ is said to be empty if $\mathcal{B} = \emptyset$.

(b) When $\mathcal{P}_\lambda\mathcal{B} = \{\sigma_1, \dots, \sigma_N\}$ with $\sigma_i < \sigma_{i+1}$, the *minimal cardinal* of $\mathfrak{C}$ in $J \times U$ is the map $\mathrm{MC}_{[\mathfrak{C};J\times U]}\colon J \setminus \{\sigma_1, \dots, \sigma_N\} \to \mathbb{N}$ defined by

$$\mathrm{MC}_{[\mathfrak{C};J\times U]}(\lambda) := \begin{cases} 2\max\left\{1, \left|\sum_{j=1}^{N} P(\sigma_j)\right|\right\} & \text{if } \lambda \in [\Lambda, \sigma_1), \\ 2\max\left\{1, \left|\sum_{j=i+1}^{N} P(\sigma_j)\right|\right\} & \text{if } \lambda \in (\sigma_i, \sigma_{i+1}), \\ & 1 \le i \le N-1, \\ 0 & \text{if } \lambda \in (\sigma_N, \infty), \end{cases}$$

if $J = [\Lambda, \infty)$, by

$$\mathrm{MC}_{[\mathfrak{C};J\times U]}(\lambda) := \begin{cases} 2\max\left\{1, \left|\sum_{j=1}^{N} P(\sigma_j)\right|\right\} & \text{if } \lambda \in (\sigma_N, \Lambda], \\ 2\max\left\{1, \left|\sum_{j=1}^{i} P(\sigma_j)\right|\right\} & \text{if } \lambda \in (\sigma_i, \sigma_{i+1}),\ 1 \le i \le N-1, \\ 0 & \text{if } \lambda \in (-\infty, \sigma_1), \end{cases}$$

if $J = (-\infty, \Lambda]$, and by

$$\mathrm{MC}_{[\mathfrak{C};J\times U]}(\lambda) := \begin{cases} 0 & \text{if } \lambda \in (-\infty, \sigma_1) \cup (\sigma_N, \infty), \\ 2\max\left\{1, \left|\sum_{j=1}^{i} P(\sigma_j)\right|\right\} & \text{if } \lambda \in (\sigma_i, \sigma_{i+1}),\ 1 \le i \le N-1, \end{cases}$$

if $J = \mathbb{R}$, whereas we take $\mathrm{MC}_{[\mathfrak{C};J\times U]} := 0$ if $\mathcal{B} = \emptyset$.

Note that $\mathrm{MC}_{[\mathfrak{C};J\times U]}$ is uniquely determined from the Signature of $\mathfrak{C}$ in $J\times U$. With this concept Theorem 3.3 may be rephrased as follows: $\mathrm{MC}_{[\mathfrak{C};J\times U]}(\lambda)$ is a lower bound of $\operatorname{Card}\mathfrak{C}_\lambda$ for any $\lambda \in J\setminus\mathcal{P}_\lambda\mathcal{B}$ being a strongly regular parameter value of $\mathfrak{C}$. In the subsequent discussion we will assume that $J = [\Lambda, \infty)$.

In the case when the signature of $\mathfrak{C}$ in $J \times U$ is empty we have defined $\mathrm{MC}_{[\mathfrak{C};J\times U]} = 0$. Although $\mathfrak{C}_\lambda$ might have an arbitrarily large number of solutions for some set of values of the parameter $\lambda > \Lambda$, it can have zero for each $\lambda > \Lambda$. Indeed, this is the case if $\mathcal{P}_\lambda\mathfrak{C} = \{\Lambda\}$.

Now, suppose

$$\operatorname{Signature}[\mathfrak{C}; J \times U] \in \left\{ \begin{pmatrix} \sigma_1 \\ -1 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ 1 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ 0 \end{pmatrix} \right\}.$$

Then,

$$\mathrm{MC}_{[\mathfrak{C};J\times U]}(\lambda) = \begin{cases} 2 & \text{if } \lambda \in [\Lambda, \sigma_1), \\ 0 & \text{if } \lambda > \sigma_1. \end{cases}$$

Although Figure 3.1(a) shows a possible component exhibiting less than two solutions for a value $\lambda \in [\Lambda, \sigma_1)$, it should be noted that $\mathrm{MC}_{[\mathfrak{C};J\times U]}(\lambda)$ equals the minimum number of solutions of $\mathfrak{C}_\lambda$ guaranteed by Theorem 3.3 at any strongly regular parameter value $\lambda \in [\Lambda, \infty) \setminus \{\sigma_1\}$ of $\mathfrak{C}$. In Figure 3.1(b), (c) we have represented two typical situations where any $\lambda \in [\Lambda, \sigma_1)$ is a regular parameter value of $\mathfrak{C}$. It should be clear why zero is the minimal number of solutions of $\mathfrak{C}_\lambda$ if $\lambda > \sigma_1$, though the compact might have an arbitrarily large number of solutions for some range of λ.

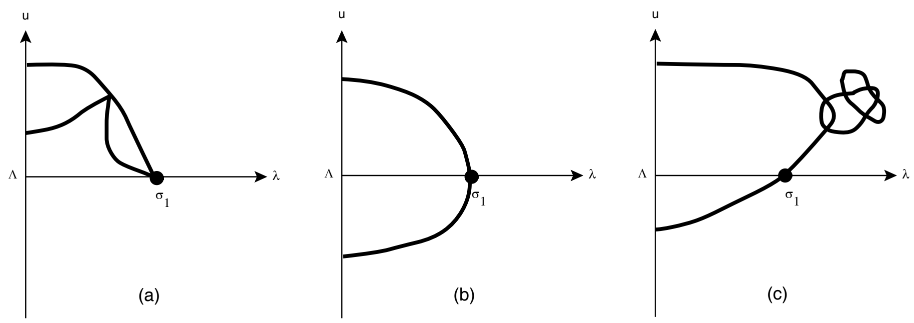

Figure 3.1. Three admissible components

In the case when

$$\text{Signature}\,[\mathfrak{C}; J \times U] \in \left\{ \begin{pmatrix} \sigma_1 & \sigma_2 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} \sigma_1 & \sigma_2 \\ 1 & 1 \end{pmatrix} \right\}, \tag{3.3}$$

then,

$$\mathrm{MC}_{[\mathfrak{C}; J \times U]}(\lambda) = \begin{cases} 4 & \text{if } \lambda \in [\Lambda, \sigma_1), \\ 2 & \text{if } \lambda \in (\sigma_1, \sigma_2), \\ 0 & \text{if } \lambda > \sigma_2. \end{cases} \tag{3.4}$$

Figure 3.2 shows some admissible components with signature (3.3) in the special case when any $\lambda \in [\Lambda, \infty) \setminus \{\sigma_1, \sigma_2\}$ is a strongly regular parameter value of $\mathfrak{C}$, and $\operatorname{Card} \mathfrak{C}_\Lambda = 4$.

It should be noted that, thanks to Theorem 3.3, $\mathfrak{C}_\lambda$ must possess an even number of solutions for each $\lambda \in [\Lambda, \infty) \setminus \{\sigma_1, \sigma_2\}$. The analysis carried out in the next section shows that all situations illustrated in Figure 3.2 are admissible. The five cases shown by Figure 3.2 correspond with each of the possible cases accordingly to the number of arcs of $\mathfrak{C}$ connecting $\{\Lambda\} \times \mathfrak{C}_\Lambda$ to $(\sigma_1, 0)$. Note that, (3.4) provides the minimal number of solutions in the general case. Of course, in all cases, $(\sigma_1, 0)$ and $(\sigma_2, 0)$ must be connected by some arc of curve, since $\mathfrak{C}$ is connected.

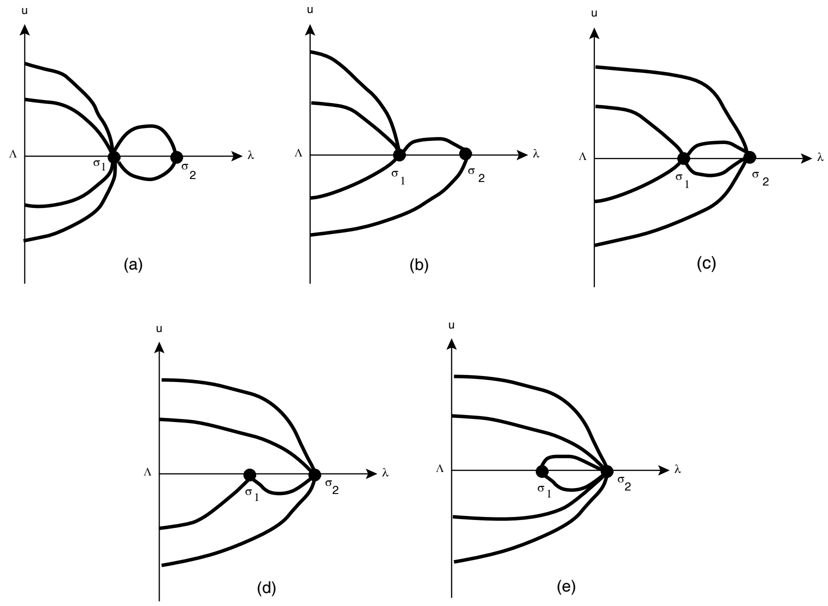

FIGURE 3.2. Some admissible components with Card $\mathfrak{C}_\Lambda = 4$

Now, suppose

$$\text{Signature}\,[\mathfrak{C}; J \times U] \in \left\{ \begin{pmatrix} \sigma_1 & \sigma_2 \\ \pm 1 & \mp 1 \end{pmatrix}, \begin{pmatrix} \sigma_1 & \sigma_2 \\ \pm 1 & 0 \end{pmatrix}, \begin{pmatrix} \sigma_1 & \sigma_2 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} \sigma_1 & \sigma_2 \\ 0 & 0 \end{pmatrix} \right\}. \tag{3.5}$$

Then,

$$\mathrm{MC}_{[\mathfrak{C}; J \times U]}(\lambda) = \begin{cases} 2 & \text{if } \lambda \in [\Lambda, \sigma_1) \cup (\sigma_1, \sigma_2), \\ 0 & \text{if } \lambda > \sigma_2. \end{cases} \tag{3.6}$$

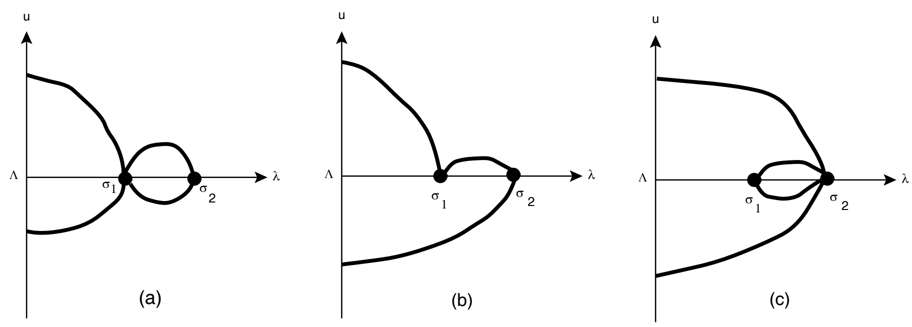

FIGURE 3.3. Some admissible components with Card $\mathfrak{C}_\Lambda = 2$

Figure 3.3 shows some admissible components with signature in the set (3.5) in the special case when any $\lambda \in [\Lambda, \infty) \setminus \{\sigma_1, \sigma_2\}$ is a strongly regular parameter value of $\mathfrak{C}$ and $\operatorname{Card} \mathfrak{C}_\Lambda = 2$.

Again, (3.6) provides the universal minimal number of solutions in all these cases.

When some degenerate points appear along any of these curves of $\mathfrak{C}$, the number of solutions might drastically increase, or decrease, of course, but it seems the previous diagrams provide us with the minimal topological patterns that $\mathfrak{C}$ should contain. We momentarily refrain from going further into the analysis we are carrying out; it will be completed in the next section.

3.1. The special case when $U = \mathbb{R}$. Although in one dimension it is very easy to construct examples of components satisfying all requirements of the theory developed in this section, one should take into account the following general result, which excludes many different cases which might appear in higher dimensions.

Theorem 3.5. *Suppose* (2.1) *and $\mathfrak{C}$ is a bounded component of $\mathfrak{S} \cap (J \times \mathbb{R})$ such that $\mathcal{P}_\lambda \mathcal{B} = \{\sigma_1, \ldots, \sigma_N\}$ with $\sigma_1 < \ldots < \sigma_N$ for some $N \geq 2$. Suppose, in addition, that there exist $1 \leq i < j \leq N$ such that $P(\sigma_i) P(\sigma_j) \neq 0$ and $P(\sigma_h) = 0$ for each $i < h < j$. Then $P(\sigma_i) = -P(\sigma_j)$, and therefore $\mathrm{MC}_{[\mathfrak{C}; J \times U]} \in \{0, 2\}$.*

Proof. It is clear that for each $1 \leq h \leq N-1$, either

$$\mathfrak{C}_\lambda \cap (0, \infty) \neq \emptyset, \quad \lambda \in (\sigma_h, \sigma_{h+1}) \tag{3.7}$$

or

$$\mathfrak{C}_\lambda \cap (-\infty, 0) \neq \emptyset, \quad \lambda \in (\sigma_h, \sigma_{h+1}). \tag{3.8}$$

Take $1 \leq h \leq N-1$ satisfying (3.8). Consider the family $\widetilde{\mathfrak{F}}\colon \mathbb{R}^2 \to \mathbb{R}$ defined as a reflection of $\mathfrak{F}$,

$$\widetilde{\mathfrak{F}}(\lambda, u) := \mathfrak{L}(\lambda) u + \widetilde{\mathfrak{N}}(\lambda, u), \quad \widetilde{\mathfrak{N}}(\lambda, u) := \begin{cases} \mathfrak{N}(\lambda, u) & \text{if } u \leq 0, \\ -\mathfrak{N}(\lambda, -u) & \text{if } u \geq 0. \end{cases}$$

Note that $\widetilde{\mathfrak{F}}$ satisfies assumptions (HL), (HN) of Section 1, it is an odd function of u, and the parity map P is the same for $\mathfrak{F}$ and for $\widetilde{\mathfrak{F}}$. Let $\widetilde{\mathfrak{S}}$ be the set of nontrivial solutions, as defined in (1.2), relatively to $\widetilde{\mathfrak{F}}$. Let $\widetilde{\mathfrak{C}}$ be the component of $\widetilde{\mathfrak{S}}$ containing $(\sigma_h, 0)$; note that it is bounded. Let $\widetilde{\mathfrak{C}}_1, \ldots, \widetilde{\mathfrak{C}}_m$ be the components of $\widetilde{\mathfrak{S}}$ with non-empty intersection with

$$S_h \times \{0\}, \quad S_h := (\sigma_h, \sigma_{h+1}) \cap P^{-1}(\{-1, 1\}).$$

Necessarily, they are bounded because they have empty intersection with $\widetilde{\mathfrak{C}}$ (recall that $U = \mathbb{R}$). Then,

$$\sum_{(\mu,0)\in\widetilde{\mathfrak{C}}_k} P(\mu) = 0, \quad 1 \le k \le m,$$

by Theorem 2.3. Thus,

$$0 = \sum_{k=1}^{m} \sum_{(\mu,0)\in\widetilde{\mathfrak{C}}_k} P(\mu) = \sum_{\mu\in S_h} P(\mu)$$

and $\operatorname{Card} S_h$ is even. An analogous reasoning for the case (3.7) proves that $\operatorname{Card} S_h$ is even for any $1 \le h \le N-1$. As a consequence, $(\sigma_i, \sigma_j) \cap P^{-1}(\{-1, 1\})$ is even and this concludes the proof. $\square$

Subsequently, we will suppose that $\sigma_1 < \sigma_2 < \sigma_3 < \sigma_4$, and, given $\alpha_i > 0$, $1 \le i \le 4$, $\alpha_2 < \alpha_3$, consider the function $F_1 \colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\begin{aligned} F_1(\lambda, u) = {} & [u^2 - \alpha_1(\lambda - \sigma_1)(\sigma_2 - \lambda)][u^2 - \alpha_2(\lambda - \sigma_2)(\sigma_3 - \lambda)] \\ & \cdot [u^2 - \alpha_3(\lambda - \sigma_2)(\sigma_3 - \lambda)][u^2 - \alpha_4(\lambda - \sigma_3)(\sigma_4 - \lambda)]u \end{aligned}$$

for each $(\lambda, u) \in \mathbb{R}^2$. The set of zeros of F_1 consists of the straight line $u = 0$ plus a bounded component, $\mathfrak{C}_1$ conformed by 8 elliptic arcs of curve looking like Figure 3.4(a). It should be noted that

$$D_u F_1(\lambda, 0) = \alpha_1\alpha_2\alpha_3\alpha_4(\lambda - \sigma_1)(\lambda - \sigma_2)^3(\lambda - \sigma_3)^3(\lambda - \sigma_4)$$

changes of sign at each σ_j, and, hence,

$$\text{Signature}\,[\mathfrak{C}_1; \mathbb{R}^2] = \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

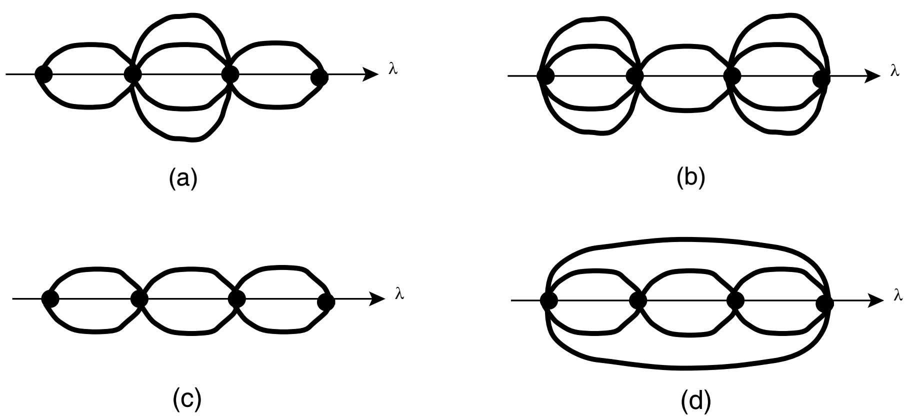

Figure 3.4. Some admissible one-dimensional components

In Figure 3.4(b) we have represented the connected set $\mathfrak{C}_2$ of nontrivial solutions of

$$\begin{aligned} F_2(\lambda,u) = {}&[u^2-\alpha_1(\lambda-\sigma_1)(\sigma_2-\lambda)][u^2-\alpha_2(\lambda-\sigma_1)(\sigma_2-\lambda)]\\ &\cdot[u^2-\alpha_3(\lambda-\sigma_2)(\sigma_3-\lambda)][u^2-\alpha_4(\lambda-\sigma_3)(\sigma_4-\lambda)]\\ &\cdot[u^2-\alpha_5(\lambda-\sigma_3)(\sigma_4-\lambda)]u, \end{aligned}$$

where $\alpha_i>0$, $1\le i\le 5$, with $\alpha_1<\alpha_2$ and $\alpha_4<\alpha_5$. Now,

$$\operatorname{Signature}[\mathfrak{C}_2;\mathbb{R}^2] = \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ 0 & -1 & 1 & 0 \end{pmatrix}.$$

In Figure 3.4(c) we have represented the connected set $\mathfrak{C}_3$ of nontrivial solutions of

$$\begin{aligned} F_3(\lambda,u) = {}&[u^2-\alpha_1(\lambda-\sigma_1)(\sigma_2-\lambda)][u^2-\alpha_2(\lambda-\sigma_2)(\sigma_3-\lambda)]\\ &\cdot[u^2-\alpha_3(\lambda-\sigma_3)(\sigma_4-\lambda)]u, \end{aligned}$$

where $\alpha_i>0$, $1\le i\le 3$. Now,

$$\operatorname{Signature}[\mathfrak{C}_3;\mathbb{R}^2] = \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

Finally, in Figure 3.4(d) we have represented the set $\mathfrak{C}_4$ of nontrivial solutions of

$$F_4(\lambda,u) = [u^2-\alpha(\lambda-\sigma_1)(\sigma_4-\lambda)]F_3(\lambda,u),$$

where $\alpha>0$ is sufficiently large. Now,

$$\operatorname{Signature}[\mathfrak{C}_4;\mathbb{R}^2] = \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

4. Counting the exact number of solutions

Throughout this section we suppose that J satisfies (2.1), and $\mathfrak{C}$ is a bounded component of $\mathfrak{S}\cap(J\times U)$. When $J\ne\mathbb{R}$ we also assume $\Lambda\notin\mathcal{P}_\lambda\mathcal{B}$ and $\mathfrak{C}_\Lambda\ne\emptyset$. Moreover, we impose the following non-degeneracy conditions:

(1) $D_u\mathfrak{F}(\lambda,u)$ exists and, for each $(\lambda,u)\in\mathfrak{C}\cap[J\times(U\setminus\{0\})]$, is an isomorphism

(2) There exists an open neighbourhood $\mathcal{O}$ of $\mathfrak{C}\setminus(\Sigma\times\{0\})$ such that the map

$$\mathcal{O}\to\mathcal{L}(U),\quad (\lambda,u)\mapsto D_u\mathfrak{F}(\lambda,u)$$

is continuous.

Our main goal is to analyze how $\operatorname{Card}\mathfrak{C}_\lambda$ changes as λ varies along the whole interval J. Many results are stated and proved for the case $J=[\Lambda,\infty)$, but it should be clear which modifications are to be implemented for the other cases of (2.1).

THEOREM 4.1. *Under the general assumptions of this section,* $\operatorname{Card}\mathfrak{C}_\lambda$ *is finite and locally constant in* $J \setminus \mathcal{P}_\lambda\mathcal{B}$. *Consequently, it is constant on each connected component of* $J \setminus \mathcal{P}_\lambda\mathcal{B}$.

To prove Theorem 4.1 we use the following version of the implicit function theorem.

THEOREM 4.2. *Let* Z, U, V *be Banach spaces,* Ω *an open subset of* $Z \times U$, *and* $G\colon \Omega \to V$ *a continuous map with the property that* D_uG *exists and it is continuous in* Ω. *Suppose there exists a point* $(z_0, u_0) \in \Omega$ *such that* $G(z_0, u_0) = 0$ *and* $D_uG(z_0, u_0)$ *is an isomorphism. Then, there are open balls* $B_r(z_0) \subset Z$ *and* $B_s(u_0) \subset U$ *such that, for each* $z \in B_r(z_0)$, *there is a unique* $u = u(z) \in B_s(u_0)$ *satisfying*

$$G(z, u(z)) = 0.$$

Moreover, the mapping $z \mapsto u(z)$ *is continuous.*

PROOF OF THEOREM 4.1. To prove that $\mathfrak{C}_\lambda$ is finite for any $\lambda \in J \setminus \mathcal{P}_\lambda\mathcal{B}$ it suffices to show that it is compact and discrete. As $\mathfrak{C}$ itself is compact, any section $\mathfrak{C}_\lambda$, $\lambda \in J$, must be compact. The fact that $\mathfrak{C}_\lambda$ is discrete follows at once from Theorem 4.2, since $D_u\mathfrak{F}(\lambda, u)$ is an isomorphism for each $(\lambda, u) \in \mathfrak{C}$ with $\lambda \in J \setminus \mathcal{P}_\lambda\mathcal{B}$.

We now show that $\operatorname{Card}\mathfrak{C}_\lambda$ is locally constant on $J \setminus \mathcal{P}_\lambda\mathcal{B}$. Pick up $\lambda^* \in J \setminus \mathcal{P}_\lambda\mathcal{B}$ and suppose

$$r := \operatorname{Card}\mathfrak{C}_{\lambda^*}, \quad \mathfrak{C}_{\lambda^*} = \{u_1^*, \dots, u_r^*\}.$$

If $r = 0$, the compactness of $\mathfrak{C}$ shows that $\mathfrak{C}_\lambda = \emptyset$ for $\lambda \simeq \lambda^*$. Suppose, then, $r > 0$. We have that $u_i^* \neq 0$ for each $1 \le i \le r$. Moreover, thanks to Theorem 4.2, there exist $\delta > 0$ and r continuous maps $u_i\colon (\lambda^* - \delta, \lambda^* + \delta) \to U$, $1 \le i \le r$, such that

$$\mathfrak{F}(\lambda, u_i(\lambda)) = 0, \qquad u_i(\lambda^*) = u_i^*,$$

if $|\lambda - \lambda^*| < \delta$ and $1 \le i \le r$. Moreover, if $\mathfrak{F}(\lambda, u) = 0$ with $|\lambda - \lambda^*| < \delta$ and $|u - u_i^*| < \delta$ for some $1 \le i \le r$, then $u = u_i(\lambda)$. Furthermore, reducing δ, if necessary, one can assume that each of the r curves $(\lambda, u_i(\lambda))$, $|\lambda - \lambda^*| < \delta$, $1 \le i \le r$, is bounded away from $\mathbb{R} \times \{0\}$, that $u_i(\lambda) \neq u_j(\lambda)$ if $i \neq j$, and that

$$(\lambda^* - \delta, \lambda^* + \delta) \cap \mathcal{P}_\lambda\mathcal{B} = \emptyset.$$

Thus, since $\mathfrak{C}$ is a component of $\mathfrak{S} \cap (J \times U)$,

$$\bigcup_{i=1}^{r} \{(\lambda, u_i(\lambda)) : |\lambda - \lambda^*| < \delta,\ \lambda \in J\} \subset \mathfrak{C}$$

and

$$\operatorname{Card}\mathfrak{C}_\lambda \ge \operatorname{Card}\mathfrak{C}_{\lambda^*} = r \quad \text{if } \lambda \in (\lambda^* - \delta, \lambda^* + \delta) \cap J.$$

To complete the proof of the local constancy of $\operatorname{Card}\mathfrak{C}_\lambda$, we will argue by contradiction. Suppose that there are sequences $\{\lambda_n\}_{n\geq 1}\subset(\lambda^*-\delta,\lambda^*+\delta)\cap J$ and $\{v_n\}_{n\geq 1}\subset U$, such that

$$\lim_{n\to\infty}\lambda_n=\lambda^*,\quad v_n\in\mathfrak{C}_{\lambda_n}\setminus\{u_1(\lambda_n),\dots,u_r(\lambda_n)\},\ n\geq 1.$$

Necessarily,

$$|v_n-u_i^*|\geq\delta,\quad 1\leq i\leq r,\ n\geq 1.$$

Moreover, by compactness, one can extract a subsequence, labeled again by n, such that

$$\lim_{n\to\infty}(\lambda_n,v_n)=(\lambda^*,u^*)\in\mathfrak{C}$$

for some $u^*\in\mathfrak{C}_{\lambda^*}$. Necessarily,

$$|u^*-u_i^*|\geq\delta,\quad 1\leq i\leq r,$$

and, hence, $\operatorname{Card}\mathfrak{C}_{\lambda^*}\geq r+1$, which is impossible. This completes the proof of the theorem. □

As an immediate consequence of Theorem 4.1, one gets the result that a change of $\operatorname{Card}\mathfrak{C}_\lambda$ entails the existence of some bifurcation value from the trivial solution. Moreover, from the regular-value formula of the degree, every $\lambda\in J\setminus\mathcal{P}_\lambda\mathcal{B}$ is a strongly regular parameter value of $\mathfrak{C}$. The following result is then obtained.

Theorem 4.3. *Suppose $J=[\Lambda,\infty)$. Under the general assumptions of this section, $\mathcal{B}\neq\emptyset$. Moreover, if*

$$\mathcal{B}=\{(\sigma_1,0),\dots,(\sigma_N,0)\},\quad \sigma_i<\sigma_{i+1},\ 1\leq i\leq N-1,$$

then

(a) $\mathcal{P}_\lambda\mathfrak{C}=[\Lambda,\sigma_N]$ *and* $\mathfrak{C}_{\sigma_N}=\{0\}$.
(b) $\operatorname{Card}\mathfrak{C}_\lambda\in 2\mathbb{N}$ *is constant in each of the intervals* $[\Lambda,\sigma_1)$, (σ_i,σ_{i+1}), $1\leq i\leq N-1$.
(c) $\operatorname{Card}\mathfrak{C}_\lambda\geq \mathrm{MC}_{[\mathfrak{C};J\times U]}(\lambda)$ *for each* $\lambda\in[\Lambda,\sigma_N]\setminus\{\sigma_1,\dots,\sigma_N\}$.
(d) $\mathfrak{C}$ *can be expressed as*

$$\mathfrak{C}=\bigcup_{\gamma\in\Gamma}\gamma([a_\gamma,b_\gamma]),$$

where Γ is a finite set of continuous curves $\gamma\colon[a_\gamma,b_\gamma]\to\mathbb{R}\times U$ such that $a_\gamma<b_\gamma$, $\gamma(a_\gamma)\in(\{\Lambda\}\times\mathfrak{C}_\Lambda)\cup\mathcal{B}$, $\gamma(b_\gamma)\in\mathcal{B}$, $\mathcal{P}_\lambda(\gamma(\lambda))=\lambda$ for any $\lambda\in[a_\gamma,b_\gamma]$,

$$\gamma((a_\gamma,b_\gamma))\cap[\mathcal{B}\cup(\{\Lambda\}\times\mathfrak{C}_\Lambda)]=\emptyset,$$

and $\gamma([a_\gamma,b_\gamma])\cap\eta([a_\eta,b_\eta])\subset\mathcal{B}$ for $\gamma,\eta\in\Gamma$, $\gamma\neq\eta$.

Moreover, for each $i \in \{1, \dots, N-1\}$ *and* $s,\, t \in J$ *such that*

$$[s,t] \cap \mathcal{P}_\lambda \mathcal{B} = \{\sigma_i\}, \quad s < \sigma_i < t, \tag{4.1}$$

one has that

$$\operatorname{Card} \mathfrak{C}_t + \natural[s,\sigma_i] = \operatorname{Card} \mathfrak{C}_s + \natural[t,\sigma_i], \tag{4.2}$$

where

$$\begin{aligned} \natural[s,\sigma_i] &:= \operatorname{Card} \left\{\gamma \in \Gamma : a_\gamma < s < b_\gamma,\ b_\gamma = \sigma_i\right\}, \\ \natural[t,\sigma_i] &:= \operatorname{Card} \left\{\gamma \in \Gamma : a_\gamma = \sigma_i,\ a_\gamma < t < b_\gamma\right\}. \end{aligned}$$

Thus, $\natural[s,\sigma_i] + \natural[t,\sigma_i] \geq 2$ *is an even natural number. Also,* $\natural[t,\sigma_1] \geq 1$ *if* $N \geq 2$.

PROOF. Theorem 4.1 clearly implies that $\mathcal{B} \neq \emptyset$.

It is plain that $[\Lambda, \sigma_N] \subset \mathcal{P}_\lambda \mathfrak{C}$. Theorem 4.1 proves the equality immediately. The fact that $\mathfrak{C}_{\sigma_N} = \{0\}$ can be proved by contradiction by using a standard continuation argument based on the fact that $\mathcal{P}_\lambda \mathfrak{C} = [\Lambda, \sigma_N]$. This proves (a).

Parts (b), (c) are immediate consequences of Theorems 4.1 and 3.3. Part (d) is an easy consequence of the previous features using the continuation method of the proof of Theorem 4.1.

Now, suppose $s, t \in J$ satisfy (4.1). Then, exactly $\natural[s,\sigma_i]$ points of $\{s\} \times \mathfrak{C}_s$ are connected by $\natural[s,\sigma_i]$ curves of Γ to $(\sigma_i, 0)$. Therefore, by assumptions (1) and (2), using a standard global continuation argument based on the implicit function theorem, the remaining

$$R_s := \operatorname{Card} \mathfrak{C}_s - \natural[s,\sigma_i]$$

points are connected by R_s curves of Γ, separated away from $J \times \{0\}$, to R_s points of $\{t\} \times \mathfrak{C}_t$. The remaining points of $\{t\} \times \mathfrak{C}_t$,

$$R_t := \operatorname{Card} \mathfrak{C}_t - R_s,$$

must lie in R_t curves of Γ going back necessarily to $(\sigma_i, 0)$, because of assumptions (1) and (2). Moreover, no additional arcs of Γ can connect $(\sigma_i, 0)$ to $\{t\} \times \mathfrak{C}_t$. Therefore,

$$R_t = \natural[t,\sigma_i] = \operatorname{Card} \mathfrak{C}_t - \operatorname{Card} \mathfrak{C}_s + \natural[s,\sigma_i],$$

which concludes the proof of (4.2). The fact that $\natural[s,\sigma_i] + \natural[t,\sigma_i]$ is even follows easily from part (b) and (4.2); it cannot vanish, since $(\sigma_i, 0) \in \mathfrak{C}$ and $\mathfrak{C}$ is connected. Finally, if $i = 1$ (and, hence, $N \geq 2$) then $\natural[t,\sigma_1] \geq 1$, since otherwise, denoting Γ_1 the set of $\gamma \in \Gamma$ such that $(\sigma_1, 0) \in \gamma([a_\gamma, b_\gamma])$, then

$$\bigcup_{\gamma \in \Gamma_1} \gamma([a_\gamma, b_\gamma]), \qquad \bigcup_{\gamma \in \Gamma \setminus \Gamma_1} \gamma([a_\gamma, b_\gamma])$$

would be two disjoint non-empty closed subsets with union $\mathfrak{C}$, thus contradicting that $\mathfrak{C}$ is connect. This concludes the proof. $\square$

Note that in the statement of Theorem 4.3 the quantities $\natural[s,\sigma_i]$ and $\natural[s,\sigma_{i-1}]$ do not depend on s as soon as $s \in (\sigma_{i-1},\sigma_i)$. In the following discussion we assume $J = [\Lambda,\infty)$.

In general, $\mathrm{MC}_{[\mathfrak{C};J\times U]}$ provides us with an optimal estimate of the number of solutions of $\mathfrak{C}$. Indeed, under condition (3.3), the counter (3.4) provides us with the exact number of solutions of the components shown in Figure 3.2(a) and (b). Similarly, under condition (3.5), (3.6) gives the exact number of solutions of the components represented in Figure 3.3(a), (b), though in Figure 3.2(c)–(e) and Figure 3.3(c) one has that

$$\operatorname{Card}\mathfrak{C}_\lambda > \mathrm{MC}_{[\mathfrak{C};J\times U]}(\lambda), \quad \lambda \in (\sigma_1,\sigma_2).$$

Subsequently, we will focus our attention in the case when (3.3) is satisfied. The minimal number of solutions of $\mathfrak{C}_\Lambda$ in this case is 4. Suppose this is the case and pick $\Lambda \le s < \sigma_1 < t < \sigma_2$. Then, thanks to (4.2),

$$\operatorname{Card}\mathfrak{C}_t = \operatorname{Card}\mathfrak{C}_s + \natural[t,\sigma_1] - \natural[s,\sigma_1] = 4 + \natural[t,\sigma_1] - \natural[s,\sigma_1]. \tag{4.3}$$

On the other hand, accordingly to Theorem 4.3(c),

$$\operatorname{Card}\mathfrak{C}_t \ge \mathrm{MC}_{[\mathfrak{C};J\times U]}(t) = 2.$$

Suppose $\operatorname{Card}\mathfrak{C}_t = 2$. Then, it follows from (4.3) that $\natural[s,\sigma_1] - \natural[t,\sigma_1] = 2$, and, therefore, $\natural[s,\sigma_1] \ge 3$, since $\natural[t,\sigma_1] \ge 1$. Consequently, if $\natural[s,\sigma_1] \le 2$, then

$$\operatorname{Card}\mathfrak{C}_t \ge 4 > \mathrm{MC}_{[\mathfrak{C};J\times U]}(t) = 2.$$

Actually, if $\natural[s,\sigma_1] = 2$ (respectively, $\natural[s,\sigma_1] = 1$), then the minimal admissible value for $\natural[t,\sigma_1]$ is 2 (respectively, 1), and, in that case, $\operatorname{Card}\mathfrak{C}_t = 4$, whereas if $\natural[s,\sigma_1] = 0$, then the four arcs of curve starting at $\lambda = \Lambda$ must end at $(\sigma_2,0)$ and, since $(\sigma_1,0)$ must be connected with $(\sigma_2,0)$ as well, and solutions arise by pairs, the minimal admissible number of solutions of $\mathfrak{C}_t$ must be 6. This is the case already represented in Figure 3.2(e). More generally, one has the following general consequence of Theorem 4.3.

Corollary 4.4. *Suppose $J = [\Lambda,\infty)$. Under the general assumptions of this section, assume that*

$$\operatorname{Signature}[\mathfrak{C};J\times U] \in \left\{ \begin{pmatrix} \sigma_1 & \dots & \sigma_N \\ -1 & \dots & -1 \end{pmatrix}, \begin{pmatrix} \sigma_1 & \dots & \sigma_N \\ 1 & \dots & 1 \end{pmatrix} \right\}$$

for some $N \ge 2$, and pick $t \in (\sigma_1,\sigma_2)$. If $\operatorname{Card}\mathfrak{C}_\Lambda = 2N$, then $\mathfrak{C}$ must adjust to some of the following structural patterns:

(a) $\operatorname{Card}\mathfrak{C}_t = 2(N-1)$, $\natural[\Lambda,\sigma_1] \in \{3,4,\dots,2N\}$, $\natural[t,\sigma_1] = \natural[\Lambda,\sigma_1] - 2$.
(b) $\operatorname{Card}\mathfrak{C}_t = 2N$, $\natural[\Lambda,\sigma_1] = \natural[t,\sigma_1] \in \{1,\dots,2N\}$.
(c) $\operatorname{Card}\mathfrak{C}_t = 2k$ *with* $k \in \mathbb{N}$, $k > N$, $\natural[\Lambda,\sigma_1] \le 2N$, $\natural[t,\sigma_1] = 2(k-N) + \natural[\Lambda,\sigma_1]$.

Corollary 4.4 characterizes the admissible structural patterns of $\mathfrak{C}$ in the interval $[\Lambda, \sigma_1]$. Among the several possibilities, the number of solutions of $\mathfrak{C}$ may decrease by 2 when we cross each of σ_i's, as predicted by the minimal cardinal function. In Figure 4.1 we have represented three different admissible components in case $N = 3$, each of them satisfying one alternative of Corollary 4.4.

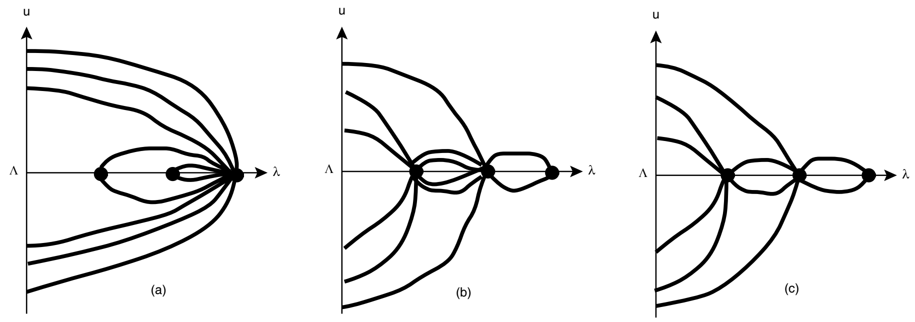

Figure 4.1. Some admissible components with $N = 3$ and Card $\mathfrak{C}_\Lambda = 6$

5. The case when $\mathfrak{C}$ consists of continuous arcs of curve

Instead of assumptions (1), (2) of Section 4, in this section we suppose the following:

(A) There exists an open neighbourhood $\mathcal{O}$ of $\mathfrak{C} \setminus (\Sigma \times \{0\})$ such that the map

$$\mathcal{O} \to \mathcal{L}(\mathbb{R} \times U; U), \quad (\lambda, u) \mapsto D\mathfrak{F}(\lambda, u)$$

is continuous.

(B) For each $(\lambda, u) \in \mathfrak{C} \cap [J \times (U \setminus \{0\})]$, $D\mathfrak{F}(\lambda, u)$ is surjective.

This section also assumes that $J = [\Lambda, \infty)$. The following result establishes that $\mathfrak{C}$ consists of a union of compact arcs of curve ending in $(\{\Lambda\} \times \mathfrak{C}_\Lambda) \cup \mathcal{B}$.

Theorem 5.1. *Suppose assumptions* (A), (B) *are satisfied, and* $\mathfrak{C}$ *is a bounded component of* $\mathfrak{S} \cap (J \times U)$ *with* $\Lambda \notin \mathcal{P}_\lambda \mathcal{B}$. *Then,* $\mathfrak{C}$ *can be expressed as*

$$\mathfrak{C} = \bigcup_{\gamma \in \Gamma} \gamma([0,1]),$$

where Γ *is a set of continuous curves* $\gamma\colon [0,1] \to \mathbb{R} \times U$ *such that* $\gamma((0,1)) \cap \mathcal{B} = \emptyset$ *and*

$$\{\gamma(0), \gamma(1)\} \subset (\{\Lambda\} \times \mathfrak{C}_\Lambda) \cup \mathcal{B}.$$

Furthermore, $\gamma([0,1]) \cap \eta([0,1]) \subset \mathcal{B}$ *for* $\gamma, \eta \in \Gamma$, $\gamma \neq \eta$. *Moreover, for each* $\lambda \in J \setminus \mathcal{P}_\lambda \mathcal{B}$, *the set of* $\gamma \in \Gamma$ *satisfying* $\lambda \in \mathcal{P}_\lambda(\gamma([0,1]))$ *is finite.*

Proof. Pick $(\lambda_0, u_0) \in \mathfrak{C}$ with $u_0 \neq 0$. As $D_u\mathfrak{F}(\lambda_0, u_0)$ is Fredholm of index 0, then $D\mathfrak{F}(\lambda_0, u_0)$ is Fredholm of index 1 and surjective, so

$$\dim N[D\mathfrak{F}(\lambda_0, u_0)] = 1.$$

Pick up $\omega \in N[D\mathfrak{F}(\lambda_0, u_0)] \setminus \{0\}$. Subsequently we denote by $\langle \cdot\, , \cdot \rangle$ the duality between $\mathbb{R} \times U$ and $(\mathbb{R} \times U)'$. Let $\psi \in (\mathbb{R} \times U)'$ be such that $\langle \psi, \omega \rangle = 1$, and consider the operator $\mathfrak{G}: \mathbb{R} \times U \times \mathbb{R} \to \mathbb{R} \times U$ defined by

$$\mathfrak{G}(\lambda, u, t) := (\langle \psi, (\lambda - \lambda_0, u - u_0) \rangle - t, \mathfrak{F}(\lambda, u)).$$

By construction, $\mathfrak{G}(\lambda_0, u_0, 0) = 0$ and

$$D_{(\lambda,u)}\mathfrak{G}(\lambda_0, u_0, 0) = \begin{pmatrix} \psi \\ D\mathfrak{F}(\lambda_0, u_0) \end{pmatrix} \in \mathcal{L}(\mathbb{R} \times U), \tag{5.1}$$

which is an isomorphism, since

$$N[D_{(\lambda,u)}\mathfrak{G}(\lambda_0, u_0, 0)] = N[D\mathfrak{F}(\lambda_0, u_0)] \cap N[\psi] = \{0\}$$

and for any $(a, b) \in \mathbb{R} \times U$, there exists $(x, y) \in \mathbb{R} \times U$ such that

$$D\mathfrak{F}(\lambda_0, u_0)(x, y) = b,$$

and the image of $(x, y) + (a - \langle \psi, (x, y) \rangle)\omega$ under (5.1) is (a, b). Consequently, thanks to the implicit function theorem, there exists $T > 0$ and a continuous map

$$[-T, T] \xrightarrow{(\lambda,u)} \mathbb{R} \times U, \quad t \mapsto (\lambda(t), u(t))$$

such that $(\lambda(0), u(0)) = (\lambda_0, u_0)$ and

$$\mathfrak{G}(\lambda(t), u(t), t) = 0, \quad t \in [-T, T].$$

Moreover, those are the unique zeros of $\mathfrak{G}$ in a neighbourhood of $(\lambda_0, u_0, 0)$.

The uniqueness of the implicit function theorem allows us to conclude that there exists a maximal continuous curve $\gamma: I \to J \times U$, for some interval $I \subset \mathbb{R}$, such that $\gamma(0) = (\lambda_0, u_0)$ and $\mathfrak{G}(\gamma(t), t) = 0$, $t \in I$, maximal with the property

$$\gamma(I) \cap \mathcal{B} = \emptyset.$$

A simple continuation argument shows that γ can be extended with continuity to $\overline{I}$, the closure of I in $\mathbb{R} \cup \{-\infty, \infty\}$, and

$$\{\gamma(\inf \overline{I}), \gamma(\sup \overline{I})\} \subset (\{\Lambda\} \times \mathfrak{C}_\Lambda) \cup \mathcal{B}.$$

As $\mathfrak{C}$ is compact and connect, then $\gamma(\overline{I}) \subset \mathfrak{C}$. Reparametrizing, we can suppose that this γ is defined on $[0, 1]$. This proves the existence of Γ with the properties indicated by the statement of the theorem.

Now we prove the last part of the proof. Suppose there exist $\lambda_0 \in J \setminus \mathcal{P}_\lambda\mathcal{B}$ and a sequence $\gamma_n \in \Gamma$, $n \geq 1$, such that $\lambda_0 \in \mathcal{P}_\lambda(\gamma_n([0, 1]))$ for each $n \geq 1$.

Then, for each $n \geq 1$, there exists $u_n \in U$ such that $(\lambda_0, u_n) \in \gamma_n([0,1])$. By compactness, we can assume that

$$u_\infty := \lim_{n\to\infty} u_n.$$

Necessarily $u_\infty \neq 0$, since $\lambda_0 \notin \mathcal{P}_\lambda\mathcal{B}$ and $(\lambda_0, u_\infty) \in \mathfrak{C}$. The uniqueness of the implicit function theorem applied to the function $\mathfrak{G}$ concludes that γ_n is the same curve for any $n \geq n_0$ and some $n_0 \in \mathbb{N}$. This concludes the proof. □

Strongly motivated by Theorems 4.3 and 5.1, we give the following definition.

Definition 5.2. Suppose assumptions (A), (B) are satisfied, $\Lambda \in \mathbb{R}$, $J = [\Lambda, \infty)$, $\mathfrak{C}$ is a bounded component of $\mathfrak{S} \cap (J \times U)$ with $\Lambda \notin \mathcal{P}_\lambda\mathcal{B}$, $(\lambda_0, u_0) \in \mathfrak{C}$, $u_0 \neq 0$, and $D_u\mathfrak{F}(\lambda_0, u_0)$ is not an isomorphism. Consider the set Γ of the statement of Theorem 5.1 and take $\gamma \in \Gamma$ such that $\gamma(t_0) = (\lambda_0, u_0)$ for some $t_0 \in [0,1]$. Then, (λ_0, u_0) is said to be

(a) a *subcritical turning point* of $\mathfrak{C}$ if $\mathcal{P}_\lambda(\gamma(t)) \leq \lambda_0$ for t in some neighbourhood of t_0 in $[0,1]$,
(b) a *supercritical turning point* of $\mathfrak{C}$ if $\mathcal{P}_\lambda\gamma(t) \geq \lambda_0$ for t in some neighbourhood of t_0 in $[0,1]$,
(c) a *hysteresis point* of $\mathfrak{C}$ in any other case.

In Figure 5.1 we illustrate each of the situations described by Definition 5.2. Figure 5.1(a) shows a genuine subcritical turning point, Figure 5.1(b) shows a supercritical turning point (actually, a segment filled in with this kind of points), while Figure 5.1(c) shows a genuine hysteresis point.

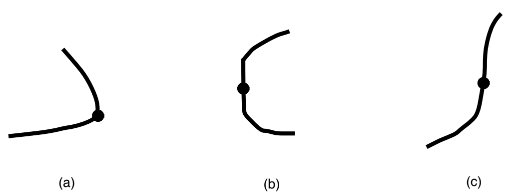

Figure 5.1. Turning and hysteresis points

It should be noted that in Theorem 5.1 the component $\mathfrak{C}$ might consist of a single point in $\{\Lambda\} \times U$, say (Λ, u_0). If this is the case, necessarily $D_u\mathfrak{F}(\Lambda, u_0)$ fails to be invertible. Also, Theorem 5.1 does not entail the number of curves conforming $\mathfrak{C}$ to be finite, as their number might grow to infinity as λ approaches some of the values of $\mathcal{P}_\lambda\mathcal{B}$. This is the situation illustrated by Figure 5.2(a), where there is an infinity family of closed loops shrinking to the unique point of $\mathcal{B}$.

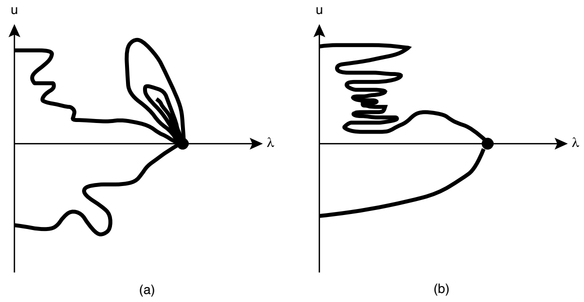

Figure 5.2. Two admissible semi-bounded components

Actually, $\operatorname{Card}\mathfrak{C}_\lambda$ might be infinity for some $\lambda \in J \setminus \mathcal{P}_\lambda\mathcal{B}$, as illustrated by Figure 5.2(b) where a curve exhibiting infinitely many turning points around some of those λ's can be shown. Therefore, under the general assumptions of Theorem 5.1 there is not, in general, limitation for the number of curves conforming $\mathfrak{C}$, or for $\operatorname{Card}\mathfrak{C}_\lambda$, though all results of Section 3 remain valid in this context.

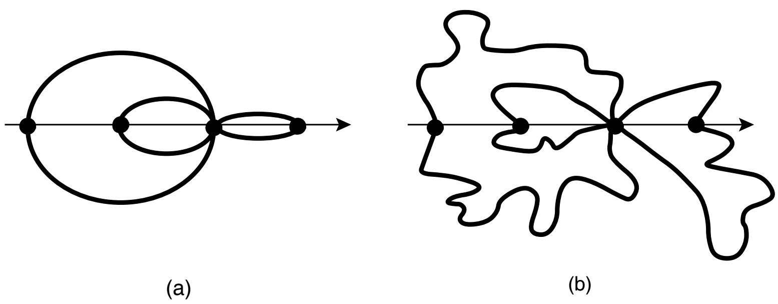

Figure 5.3. Two equivalent components satisfying assumptions (A) and (B)

At this stage of our analysis it should be clear that, under assumptions (A), (B), the integer number $\mathrm{MC}_{[\mathfrak{C};J\times U]}(\lambda)$ equals the minimal number of solutions if $\mathfrak{C} \setminus \mathcal{B}$ consists of strongly regular parameter values, though this is not the case, in general, because of the eventual formation of turning and hysteresis points when those components vary as a result of the variation of some additional parameter in the problem setting. Actually, from a topological point of view, the bounded components adjusted to the same signature $(-1,-1,1,1)$ shown in Figure 5.3 are admissible. In both cases, $\mathrm{MC}_{[\mathfrak{C};J\times U]}(\lambda)$, $\lambda \in J \setminus \mathcal{P}_\lambda\mathcal{B}$ provides us with a lower bound of the number of solutions of $\mathfrak{C}_\lambda$ and of the number of arcs passing through $\mathfrak{C}_\lambda$, estimations which are exact in case (a). Both components are equivalent from the point of view of graph theory, which strongly suggests

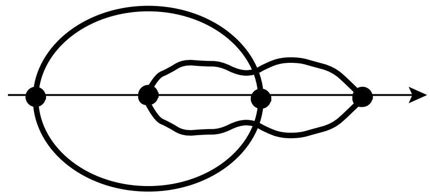

FIGURE 5.4. Unfolding of Figure 5.3(a)

the use of graph theory in order to classify all admissible components under assumptions (A) and (B).

A quite suggestive feature relies into the fact that Figure 5.3(a) might be unfolded, by adding some additional parameter, into a component of the type shown in Figure 5.4, but this analysis is outside the scope of this work.

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