A SET-VALUED APPROACH TO HEMIVARIATIONAL INEQUALITIES

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Abstract. Let $X$ be a Banach space, $X^*$ its dual and let $T: X \rightarrow L^p(\Omega, \mathbb{R}^k)$ be a linear, continuous operator, where $p, k \geq 1$, $\Omega$ being a bounded open set in $\mathbb{R}^N$. Let $K$ be a subset of $X$, $A: K \rightarrow X^*$, $G: K \times X \rightarrow \mathbb{R}$ and $F: \Omega \times \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}$ set-valued maps with nonempty values. Using mainly set-valued analysis, under suitable conditions on the involved maps, we shall guarantee solutions to the following inclusion problem:

Find $u \in K$ such that, for every $v \in K$

$$
\sigma(A(u), v - u) + G(u, v - u) + \int_\Omega F(x, Tu(x), Tv(x) - Tu(x)) \, dx \subseteq \mathbb{R}_+.
$$

In particular, well-known variational and hemivariational inequalities can be derived.

1. Introduction

Let $K$ be a nonempty subset of $H_0^1(\Omega)$, where $\Omega$ is a bounded open subset of $\mathbb{R}^N$ with $C^1$ boundary, $N \geq 1$. Many papers treat inclusion problems of the form:

Find $u \in K$ such that

$$(1.1) \quad -\Delta u \in G(x, u(x)) \quad \text{in} \ \Omega,
$$

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where \( G: \Omega \times \mathbb{R} \rightharpoonup \mathbb{R} \) is a set-valued map with nonempty values, satisfying some growth and continuity conditions, see for instance [6] and [11]. In these papers critical point arguments were used.

Here, we suppose that \( G \) has the form

\[
G(x, u(x)) = H(x, u(x)) - b(x)u(x), \quad x \in \Omega, \ u \in K,
\]

where \( b \in L^\infty(\Omega) \), and \( H: \Omega \times \mathbb{R} \rightharpoonup \mathbb{R} \) satisfies for all \( x \in \Omega \) the following inclusion:

\[
H(x, u(x)) \cdot v(x) = \{ h \cdot v(x) : h \in H(x, u(x)) \} \subseteq [-g(x, u(x), v(x)), \infty),
\]

where \( g(\cdot, u(\cdot), v(\cdot)) \in L^1(\Omega) \) for every \( u \in K, v \in H^1_0(\Omega) \).

Multiplying (1.1) by \((v - u)\), integrating over \( \Omega \) and applying the Gauss–Green formula, from (1.2) and (1.3) we obtain:

\[
\int_\Omega \nabla u \cdot \nabla (v - u) \, dx + \int_\Omega b(x)u(x)(v(x) - u(x)) \, dx
+ \int_\Omega [g(x, u(x), v(x) - u(x)), \infty) \, dx \subseteq \mathbb{R}_+ \tag{1.4}
\]

for all \( v \in K \), where the last term from the left hand side is the integral of a set-valued map in the sense of Aumann (see [2]).

If \( H \) has the form

\[
H(x, u(x)) = -\partial j(x, u(x)), \quad x \in \Omega,
\]

where \( j: \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) is a Carathéodory function such that \( j(x, \cdot) \) is locally Lipschitz continuous and \( \partial \) denotes the generalized gradient, then (1.3) is verified if we take \( g(x, y, z) = j^0_y(x; y; z), j^0_y \) being the (partial) generalized directional derivative, supposing that \( j \) satisfies a growth condition (see Section 4). In this situation, (1.4) reduces to the following classical hemivariational inequality, see for instance Motreanu and Panagiotopoulos [8], Naniewicz and Panagiotopoulos (see [9]):

\[
(HV \geq ) \text{ Find } u \in K \text{ such that, for all } v \in K
\]

\[
\int_\Omega \nabla u \cdot \nabla (v - u) \, dx + \int_\Omega b(x)u(x)(v(x) - u(x)) \, dx
+ \int_\Omega j^0_y(x, u(x); v(x) - u(x)) \, dx \geq 0.
\]

So, it seems natural to study the following general problem.

Let \( X \) be a Banach space, \( X^* \) its dual, and let \( T: X \rightarrow L^p(\Omega, \mathbb{R}^k) \) be a linear continuous operator, where \( 1 \leq p < \infty, k \geq 1, \Omega \) being a bounded open set in \( \mathbb{R}^N \).
Let $K$ be a subset of $X$, let $A: K \rightrightarrows X^*$, $G: K \times X \rightrightarrows \mathbb{R}$ and $F: \Omega \times \mathbb{R}^k \times \mathbb{R}^k \rightrightarrows \mathbb{R}$ be set-valued maps with nonempty values, such that

$$(H_1) \; x \in \Omega \rightrightarrows F(x, Tu(x), Tv(x) - Tu(x))$$

is a measurable set-valued map for all $u, v \in K$.

$$(H_2) \; \text{There exist } h_1 \in L^{p/(p-1)}(\Omega, \mathbb{R}_+) \text{ and } h_2 \in L^\infty(\Omega, \mathbb{R}_+) \text{ such that}$$

$$\text{dist}(0, F(x, y, z)) \leq (h_1(x) + h_2(x)|y|^{p-1})|z| \text{ for a.e. } x \in \Omega,$$

for every $y, z \in \mathbb{R}^k$.

The aim of this paper is to study the following hemivariational inclusion problem:

$$(HV_{\subseteq}) \; \text{Find } u \in K \text{ such that, for all } v \in K$$

$$\sigma(A(u), v - u) + G(u, v - u) + \int_{\Omega} F(x, Tu(x), Tv(x) - Tu(x))dx \subseteq \mathbb{R}_+.$$ 

We denoted by $\sigma(A(u), \cdot)$ the support function of $A(u)$, that is

$$\sigma(A(u), h) = \sup_{x^* \in A(u)} \langle x^*, h \rangle \text{ for all } h \in X.$$ 

The euclidean norm in $\mathbb{R}^k$ and the duality pairing between the Banach space and its dual is denoted by $| \cdot |$, respectively $\langle \cdot, \cdot \rangle$.

2. Preliminaries

We need some definitions and notions in order to state existence results concerning the problem $(HV_{\subseteq})$.

Let $J: \Omega \rightrightarrows \mathbb{R}$ be a measurable set-valued map with nonempty closed values, see [1, p. 307]. Define the set

$$J = \{ j \in L^1(\Omega, \mathbb{R}) : j(x) \in J(x) \text{ a.e. in } \Omega \}.$$ 

DEFINITION 2.1 (see [2]). The integral of $J$ on $\Omega$ is the set of integrals of integrable selections of $J$, i.e.

$$\int_{\Omega} J(x)dx = \left\{ \int_{\Omega} j(x)dx : j \in J \right\}.$$ 

From the above definition we clearly have

LEMMA 2.2. Let $J_1, J_2: \Omega \rightrightarrows \mathbb{R}$ be two measurable set-valued maps with closed values. Then the following assertions hold:

(a) If $J_1(x) \subseteq J_2(x)$ a.e. $x \in \Omega$, then $\int_{\Omega} J_1(x)dx \subseteq \int_{\Omega} J_2(x)dx$.

(b) $\int_{\Omega} J_1(x)dx + \int_{\Omega} J_2(x)dx \subseteq \int_{\Omega} (J_1 + J_2)(x)dx$.

(c) $\lambda \int_{\Omega} J_1(x)dx \subseteq \int_{\Omega} \lambda J_1(x)dx$ for all $\lambda \in \mathbb{R}$.
Definition 2.3. Let $X$ be a Banach space, and let $K$ be a nonempty subset of $X$. A set-valued map $A: K \rightrightarrows X^*$ with bounded values is said to be upper demicontinuous at $u_0 \in K$ (u.d.c. at $u_0 \in K$) if, for any $h \in X$, the real-valued function

$$u \in K \mapsto \sigma(A(u), h) = \sup_{x^* \in A(u)} \langle x^*, h \rangle$$

is upper semicontinuous at $u_0$. $A$ is upper demicontinuous on $K$ (u.d.c. on $K$) if it is udc at every $u \in K$.

Remark 2.4. If $A(u) = \{A(u)\}$ for all $u \in K$, that is, if $A$ is a single-valued map, then $A$ is u.d.c. at $u_0 \in K$ if and only if the map $A: K \to X^*$ is w*-demicontinuous at $u_0 \in K$, i.e. for each sequence $\{u_n\}$ in $K$ converging to $u_0$ (in the strong topology), the image sequence $\{A(u_n)\}$ converges to $A(u_0)$ in the weak*-topology of $X^*$.

It is easy to verify that, for all $u \in K$, the function $h \in X \mapsto \sigma(A(u), h)$ is lower semicontinuous, subadditive and positive homogeneous. Moreover, due to Banach–Steinhaus theorem, we can state the following useful result.

Proposition 2.5. Let $K$ be a nonempty subset of a Banach space $X$, and let $A: K \rightrightarrows X^*$ be an upper demicontinuous set-valued map with bounded values. Then the function $u \in K \mapsto \sigma(A(u), v - u)$ is upper semicontinuous for all $v \in K$.

Definition 2.6. Let $W, Y$ be two metric spaces. A set-valued map (with nonempty values) $J: W \rightrightarrows Y$ is called lower semicontinuous at $w \in W$ (l.s.c. at $w$) if and only if for any $y \in J(w)$ and for any sequence $\{w_n\}$, converging to $w$, there exists a sequence $\{y_n\}$, $y_n \in J(w_n)$ converging to $y$. $J$ is said to be lower semicontinuous (l.s.c.) if it is lsc at every point $w \in W$.

Definition 2.7. Let $\{K_n\}$ be a sequence of subsets of a metric space $Y$. The set

$$\text{Liminf } K_n = \{y \in Y : \lim_{n \to \infty} \text{dist}(y, K_n) = 0\}$$

is the (Kuratowski) lower limit of the sequence $K_n$.

Remark 2.8. Liminf$_{n \to \infty}$ is the set of limits of sequences $y_n \in K_n$ (see [1, p. 18]).

Proposition 2.9 (see [1, p. 42]). Let $X$ be a normed space. A set-valued map $F: X \rightrightarrows \mathbb{R}$ is lower semicontinuous at $u \in X$ if and only if

$$F(u) \subseteq \text{Liminf } F(u_n)$$

for any sequence $\{u_n\}$ in $X$ converging to $u$. 

Lemma 2.10. Let $Y$ be a real normed space, and let $\{K_n\}, \{L_n\}$ be two sequences of subsets of $Y$. Then the following assertions hold:

(a) $\liminf_{n \to \infty} K_n + \liminf_{n \to \infty} L_n \subseteq \liminf_{n \to \infty} (K_n + L_n)$.
(b) If $K_n \subseteq L_n$ for all $n \in \mathbb{N}$, then $\liminf_{n \to \infty} K_n \subseteq \liminf_{n \to \infty} L_n$.

Definition 2.11. Let $W$, $Y$ be real normed spaces, $K \subset W$ be a convex subset. The set-valued map $J: K \to Y$ with nonempty values is convex if and only if $\forall w_1, w_2 \in K$, $\forall \lambda \in [0, 1] : \lambda J(w_1) + (1 - \lambda) J(w_2) \subseteq J(\lambda w_1 + (1 - \lambda) w_2)$.

Remark 2.12. $J: K \to Y$ is convex if and only if for all $w_i \in K$, for all $\lambda_i \geq 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$, $n \in \mathbb{N}$, we have

$$\sum_{i=1}^{n} \lambda_i J(w_i) \subseteq J\left(\sum_{i=1}^{n} \lambda_i w_i\right).$$

Finally, we recall the well-known result of Ky Fan.

Lemma 2.13 (see [5]). Let $X$ be a Hausdorff topological vector space, $K$ a subset of $X$ and for each $x \in K$, let $S(x)$ be a closed subset of $X$, such that

(a) there exists $x_0 \in K$ such that the set $S(x_0)$ is compact,
(b) $S$ is a KKM-map, i.e. for each $x_1, \ldots, x_n \in K$, $\text{co}\{x_1, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} S(x_i)$, where $\text{co}$ stands for the convex hull operator.

Then $\bigcap_{x \in K} S(x) \neq \emptyset$.

3. Main results

We need some additional hypotheses to obtain a solution for $(HV \subseteq)$.

(H3) $w \in X \rightharpoonup G(u, w)$ and $z \in \mathbb{R}^k \rightharpoonup F(x, y, z)$ are convex for all $u \in K$, $x \in \Omega$, $y \in \mathbb{R}^k$.

(H4) $G(u, 0) \subseteq \mathbb{R}^+$ and $F(x, y, 0) \subseteq \mathbb{R}^+$ for all $u \in K$, $x \in \Omega$, $y \in \mathbb{R}^k$.

(H5) $(u, w) \in K \times X \rightharpoonup G(u, w)$ is lower semicontinuous.

(H6) $(y, z) \in \mathbb{R}^k \times \mathbb{R}^k \rightharpoonup F(x, y, z)$ is lower semicontinuous for all $x \in \Omega$.

Remark 3.1. If $F: \Omega \times \mathbb{R}^k \times \mathbb{R}^k \rightharpoonup \mathbb{R}$ is a closed-valued Carathéodory map (i.e. for any $(y, z) \in \mathbb{R}^k \times \mathbb{R}^k$, $x \in \Omega \rightharpoonup F(x, y, z)$ is measurable and for any $x \in \Omega$, $(y, z) \in \mathbb{R}^k \times \mathbb{R}^k \rightharpoonup F(x, y, z)$ is continuous), then the hypotheses (H6) and (H1) hold automatically (see [1, p. 314]).

Now, we establish the main result of this paper.

Theorem 3.2. Let $K$ be a nonempty compact convex subset of a Banach space $X$. Let $F: \Omega \times \mathbb{R}^k \times \mathbb{R}^k \rightharpoonup \mathbb{R}$ and $G: K \times X \rightharpoonup \mathbb{R}$ be two set-valued
maps satisfying \((H_1)–(H_6)\), of which \(F\) is closed-valued. If \(A: K \rightrightarrows X^*\) is upper demicontinuous on \(K\) with bounded values, then \((HV)\subseteq\) has at least a solution.

**Proof.** For any \(v \in K\) we set
\[
S_v = \left\{ u \in K : \sigma(A(u), v - u) + G(u, v - u) + \int_{\Omega} F(x, Tu(x), Tv(x) - Tu(x)) \, dx \subseteq \mathbb{R}^+ \right\}.
\]

First, we prove that \(S_v\) is closed set for all \(v \in K\). Fix a \(v \in K\). Of course, \(S_v \neq \emptyset\), since \(v \in S_v\), due to \((H_4)\). Now, let \(\{u_n\}\) be a sequence in \(S_v\) which converges to \(u \in X\). We prove that \(u \in S_v\). Since \(T: X \rightarrow L^p(\Omega, \mathbb{R}^k)\) is continuous, it follows that
\[
Tu_n \rightarrow Tu \quad \text{in } L^p(\Omega, \mathbb{R}^k) \quad \text{as } n \rightarrow \infty.
\]

Clearly, there exists a subsequence \(\{u_{n_l}\}\) of \(\{u_n\}\), see Proposition 2.5, such that
\[
\limsup_{n \rightarrow \infty} \sigma(A(u_n), v - u_n) = \lim_{m \rightarrow \infty} \sigma(A(u_m), v - u_m).
\]

Moreover, by [12, Lemma A.1, p.133] there exists a subsequence \(\{Tu_{l}\}\) of \(\{Tu_{n}\}\) and \(g \in L^p(\Omega, \mathbb{R}^+)\) such that
\[
|Tu_l(x)| \leq g(x), \quad Tu_l(x) \rightarrow Tu(x) \quad \text{for a.e. } x \in \Omega.
\]

In the relation
\[
\sigma(A(u_l), v - u_l) + G(u_l, v - u_l) + \int_{\Omega} F(x, Tu_l(x), Tv(x) - Tu_l(x)) \, dx \subseteq \mathbb{R}^+,
\]
letting the lower limit and using Lemma 2.10 (with \(Y = \mathbb{R}\)) we obtain
\[
\liminf_{l \rightarrow \infty} \sigma(A(u_l), v - u_l) \subseteq \liminf_{l \rightarrow \infty} G(u_l, v - u_l) \subseteq \liminf_{l \rightarrow \infty} \mathbb{R}^+ = \mathbb{R}^+.
\]

Using Remark 2.8, relation (3.1) and Proposition 2.5, we obtain
\[
\liminf_{l \rightarrow \infty} \sigma(A(u_l), v - u_l) = \lim_{l \rightarrow \infty} \sigma(A(u_l), v - u_l) = \limsup_{n \rightarrow \infty} \sigma(A(u_n), v - u_n) \leq \sigma(A(u), v - u).
\]

From \((H_5)\) and Proposition 2.9 we obtain
\[
G(u, v - u) \subseteq \liminf_{l \rightarrow \infty} G(u_l, v - u_l).
\]

Let \(F_l = F(\cdot, Tu_l(\cdot), Tv(\cdot) - Tu_l(\cdot))\). From \((H_1)\), \(F_l\) is measurable, for any \(l\).
The function $x \in \Omega \mapsto \sup \dist(0, F_l(x))$ is integrable. Indeed, from (H2) and relation (3.2) we have

$$\dist(0, F_l(x)) \leq (h_1(x) + h_2(x))|Tu_1(x)|^{p-1}|Tv(x) - Tu_l(x)| \leq (h_1(x) + h_2(x) \cdot |g(x)|^{p-1})(|Tv(x)| + g(x)) \ a.e. \ x \in \Omega.$$ 

Let $h(x) = (h_1(x) + h_2(x) \cdot |g(x)|^{p-1})(|Tv(x)| + g(x))$. From Hölder’s inequality and from the conditions for $h_1$ and $h_2$ it follows that $h \in L^1(\Omega, \mathbb{R})$. Therefore, the function $x \in \Omega \mapsto \sup_l \dist(0, F_l(x))$ is integrable. Applying the Lebesgue dominated convergence theorem for set-valued maps (see [1, p. 331]), one has

$$(3.6) \quad \int_{\Omega} \liminf_{l \to \infty} F(x, Tu_l(x), Tv(x) - Tu_l(x)) \, dx \subseteq \liminf_{l \to \infty} \int_{\Omega} F(x, Tu_l(x), Tv(x) - Tu_l(x)) \, dx.$$ 

Of course, the first integrand is measurable (see [1, p. 312]). Using the hypothesis (H6) (therefore Proposition 2.9) and (3.2), one has

$$F(x, Tu(x), Tv(x) - Tu(x)) \subseteq \liminf_{l \to \infty} F(x, Tu_l(x), Tv(x) - Tu_l(x))$$ 

a.e. $x \in \Omega$. From Lemma 2.2(a) and (3.6), we obtain

$$(3.7) \quad \int_{\Omega} F(x, Tu(x), Tv(x) - Tu(x)) \, dx \subseteq \liminf_{l \to \infty} \int_{\Omega} F(x, Tu_l(x), Tv(x) - Tu_l(x)) \, dx.$$ 

Therefore, from (3.4), (3.5), (3.7) and (3.3) we obtain

$$\sigma(A(u), v - u) + G(u, v - u) + \int_{\Omega} F(x, Tu(x), Tv(x) - Tu(x)) \, dx \subseteq \mathbb{R}_+,$$ 

i.e. $u \in S_\nu$.

Finally, we prove that $S: K \to K$ is a KKM-map. To this end, let $\{v_1, \ldots, v_n\}$ be an arbitrary finite subset of $K$. We prove that $\co\{v_1, \ldots, v_n\} \subseteq \bigcup_{i=1}^n S_{\nu_i}$. Supposing the contrary, there exist $\lambda_i > 0 \ (i \in \{1, \ldots, n\})$ such that $\sum_{i=1}^n \lambda_i = 1$ and $v = \sum_{i=1}^n \lambda_i v_i \not\in S_{\nu_i}$ for all $i \in \{1, \ldots, n\}$. The above relations mean that for all $i \in \{1, \ldots, n\}$

$$\left[\sigma(A(\nu), v_i - \nu) + G(\nu, v_i - \nu) + \int_{\Omega} F(x, T\nu(x), Tv_i(x) - T\nu(x)) \, dx\right] \cap \mathbb{R}_+^* \neq \emptyset.$$ 

(Here, $\mathbb{R}_+^* = [-\infty, 0[^*$. Let $I = \{i \in \{1, \ldots, n\} : \lambda_i > 0\}$. From the above we obtain

$$0 \neq \left\{ \sum_{i \in I} \lambda_i \left[\sigma(A(\nu), v_i - \nu) + G(\nu, v_i - \nu) + \int_{\Omega} F(x, T\nu(x), Tv_i(x) - T\nu(x)) \, dx\right] \right\} \cap \mathbb{R}_+^*.$$
Using the sublinearity of the function $h \in X \mapsto \sigma(A(v), h)$, (H$_3$), Lemma 2.2, the linearity of $T$ and (H$_4$), we obtain

$$\emptyset \neq \left\{ \sigma(A(v), \sum_{i \in I} \lambda_i v_i - \sum_{i \in I} \lambda_i \bar{v}) + \sum_{i \in I} \lambda_i G(v_i - \bar{v}) \right\} \cap \mathbb{R}_+^*$$

$$\subseteq \left\{ \sigma(A(v), 0) + \sum_{i \in I} \lambda_i v_i - \sum_{i \in I} \lambda_i \bar{v} \right\} \cap \mathbb{R}_+^*$$

$$\subseteq \left\{ G(v, 0) + \int_{\Omega} F(x, T\bar{v}(x), 0) dx \right\} \cap \mathbb{R}_+^* \subseteq \{ \mathbb{R}_+ + \int_{\Omega} \mathbb{R}_+ dx \} \cap \mathbb{R}_+^* = \emptyset,$$

contradiction. This means that $S$ is a KKM-map. Since $K$ is compact, applying Lemma 2.13, we obtain $\cap_{v \in K} S_v \neq \emptyset$, i.e. (HV$\subseteq$) has at least a solution. $\square$

When $K$ is not compact, we can state the following result, using a coercivity assumption.

**Theorem 3.3.** Let $K$ be a nonempty closed, convex subset of a Banach space $X$. Let $A$, $G$ and $F$ be as in Theorem 3.2. In addition, suppose that there exists a compact subset $K_0$ of $K$ and an element $w_0 \in K_0$ such that

$$\left\{ \sigma(A(u), w_0 - u) + \int_{\Omega} F(x, Tu(x), Tw_0(x) - Tu(x)) dx \right\} \cap \mathbb{R}_+^* \neq \emptyset$$

for all $u \in K \setminus K_0$. Then (HV$\subseteq$) has at least a solution.

**Proof.** We define the map $S$ as in Theorem 3.2. Clearly, $S$ is a KKM-map and $S_v$ is closed for all $v \in K$, as seen above. Moreover, $S_{w_0} \subseteq K_0$. Indeed, supposing the contrary, there exists an element $u \in S_{w_0} \subseteq K$ such that $u \notin K_0$. But this contradicts (3.8). Since $K_0$ is compact, the set $S_{w_0}$ is also compact. Applying again Lemma 2.13, we obtain a solution for (HV$\subseteq$). $\square$

4. Consequences

First, we obtain a result of Browder concerning variational inequalities (see [3, Theorem 6]).
Corollary 4.1. Let $K$ be a nonempty compact convex subset of a Banach space $X$, and let $\mathcal{A}:K \rightrightarrows X^*$ be an upper demicontinuous set-valued map with bounded values. Then there exists $\overline{\alpha} \in K$ such that
\[ \sigma(\mathcal{A}(\overline{\alpha}), v - \overline{\alpha}) \geq 0 \text{ for all } v \in K. \]

Proof. Choose $F \equiv 0$ and $G \equiv 0$ in Theorem 3.2. \hfill $\square$

In particular, Corollary 4.1 reduces to a classical result of Hartman and Stampacchia [7] if $\mathcal{A}$ is a single-valued continuous operator and $X$ is of finite dimension.

Now, we give a solution for the hemivariational inequality treated by Panagiotopoulos, Fundo and Rădulescu (see [10]). Before to do this, we recall two elementary facts.

Lemma 4.2. Let $K$ be a nonempty subset of a normed space $X$, and let $j:K \to \mathbb{R}$ be a function. Define $J:K \rightrightarrows \mathbb{R}$ by $J(u) = [j(u), \infty)$ for all $u \in K$. If $j$ is upper semicontinuous on $K$, then $J$ is lower semicontinuous on $K$.

Lemma 4.3. If $h: \Omega \to \mathbb{R}$ is a measurable function, then $H: \Omega \rightrightarrows \mathbb{R}$ defined by $H(x) = [h(x), \infty)$ for all $x \in \Omega$, is also measurable (as set-valued map).

Let $\Omega$, $X$, $K$ and $T$ be as in the Introduction, let $\mathcal{A}:K \to X^*$ be an operator, and we suppose that $j: \Omega \times \mathbb{R}^k \to \mathbb{R}$ is a Carathéodory function which is locally Lipschitz continuous with respect to the second variable and which satisfies the following assumption:

(j) there exist $h_1$ and $h_2$ as in (H$_2$) such that
\[ |w| \leq h_1(x) + h_2(x)|y|^{p-1} \]
for a.e. $x \in \Omega$, every $y \in \mathbb{R}^k$ and $w \in \partial j(x, y)$.

Here $\partial j(x, y)$ is the Clarke generalized gradient of $j$, i.e.
\[ \partial j(x, y) = \{ w \in \mathbb{R}^k : \langle w, z \rangle \leq j^0_y(x, y; z) \text{ for all } z \in \mathbb{R}^k \}, \]
where $j^0_y(x, y; z)$ is the (partial) generalized directional derivative of the locally Lipschitz continuous function $j(x, \cdot)$ at the point $y \in \mathbb{R}^k$ with respect to the direction $z \in \mathbb{R}^k$, where $x \in \Omega$, that is
\[ j^0_y(x, y; z) = \limsup_{y' \to y, t \to 0^+} \frac{j(x, y' + tz) - j(x, y')}{t}. \]

We consider the following hemivariational inequality problem:

(P) Find $\overline{\alpha} \in K$ such that
\[ \langle \mathcal{A}(\overline{\alpha}), v - \overline{\alpha} \rangle + \int_\Omega j^0_y(x, T\overline{\alpha}(x); T\overline{\alpha}(x) - T\overline{\alpha}(x)) \, dx \geq 0 \quad \text{for all } v \in K. \]
Corollary 4.4 (see [10]). Let K be a nonempty compact convex subset of a Banach space X, and let $j : \Omega \times \mathbb{R}^k \to \mathbb{R}$ satisfying the condition (j). If the operator $A : K \to X^*$ is $w^*$-demicontinuous, then (P) has at least a solution.

Proof. We choose $A(u) = \{A(u)\}$ for all $u \in K$, $G \equiv 0$ and $F : \Omega \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ as $F(x, y, z) = j^0_y(x, y; z)$ for all $(x, y, z) \in \Omega \times \mathbb{R}^k \times \mathbb{R}^k$. Due to Remark 2.4, the operator $A$ is upper demicontinuous (with bounded values). We will verify the hypotheses from Theorem 3.2 for $F$.

(H1) Using the linearity of $T$ and the measurability of

$$x \in \Omega \mapsto j^0_y(x, Tu(x); Tv(x) - Tu(x))$$

for all $u, v \in K$ (see [8, p. 15]), from Lemma 4.3 we obtain that $x \in \Omega \mapsto F(x, Tu(x), Tv(x) - Tu(x))$ is measurable.

(H2) Since $j^0_y(x, y; z) = \max\{\langle w, z \rangle : w \in \partial j(x, y)\} = \langle w_0, z \rangle$, for some $w_0 \in \partial j(x, y)$ (using (j)) we have

$$|j^0_y(x, y; z)| \leq |w_0| \cdot |z| \leq (h_1(x) + h_2(x)|y|^{p-1})|z|.$$  

Since $\text{dist}(0, F(x, y, z)) \leq |j^0_y(x, y; z)|$, we obtain the desired relation.

(H3) Since $z \in \mathbb{R}^k \mapsto j^0_y(x, y; z)$ is convex (see [4, p. 25]) we obtain that $z \in \mathbb{R}^k \mapsto F(x, y, z)$ is convex for all $x \in \Omega$ and all $y \in \mathbb{R}^k$.

(H4) Since $j^0_y(x, y; 0) = 0$, we have $F(x, y, 0) = \mathbb{R}_+$ for all $x \in \Omega$ and all $y \in \mathbb{R}^k$.

(H6) Since $(y, z) \in \mathbb{R}^k \times \mathbb{R}^k \mapsto j^0_y(x, y; z)$ is upper semicontinuous (see [4, p. 25]), and using Lemma 4.2 we obtain that $(y, z) \in \mathbb{R}^k \times \mathbb{R}^k \mapsto F(x, y, z)$ is lower semicontinuous for all $x \in \Omega$.

Therefore, from Theorem 3.2 we have a solution $\pi \in K$ such that

$$\langle A\pi, v - \pi \rangle + \int_\Omega F(x, T\pi(x), Tv(x) - T\pi(x))dx \subseteq \mathbb{R}_+$$

for all $v \in K$.

In particular, for the “lower” selection of $F(\cdot, T\pi(\cdot), Tv(\cdot) - T\pi(\cdot))$, i.e. for $j^0_y(\cdot, T\pi(\cdot); Tv(\cdot) - T\pi(\cdot))$, which is integrable due to (j), we have

$$\langle A\pi, v - \pi \rangle + \int_\Omega j^0_y(x, T\pi(x); Tv(x) - T\pi(x))dx \geq 0$$

for all $v \in K$,

i.e. $\pi$ is a solution for (P). \hfill \Box

References


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