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PERIODIC SOLUTIONS FOR NONAUTONOMOUS SYSTEMS WITH NONSMOOTH QUADRATIC OR SUPERQUADRATIC POTENTIAL

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Abstract. We study a semilinear nonautonomous second order periodic system with a nonsmooth potential function which exhibits a quadratic or superquadratic growth. We establish the existence of a solution, using minimax methods of the nonsmooth critical point theory.

1. Introduction

In this paper we study the following second order periodic system with a nonsmooth potential function:

$$(1.1)\qquad \begin{cases} -x''(t) - A(t)x(t) \in \partial j(t, x(t)) & \text{a.e. on } T = [0, b], \\ x(0) = x(b), \ x'(0) = x'(b). \end{cases}$$

Here $t \mapsto A(t)$ is a continuous map on $T = [0, b]$, for some $b > 0$, with values in the space of symmetric $N \times N$ matrices, $j(t, \cdot\,)$ is a nonsmooth locally Lipschitz function and $\partial j(t, \cdot\,)$ stands for its subdifferential in the sense of Clarke (see Section 2).

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When $A = 0$, the problem has been studied extensively and various existence results have been proved under the assumption that the potential function $j(t, \cdot)$ is smooth (i.e. a C^1-function). We refer to the works of Berger–Schechter [2], Mawhin–Willem [13], Long [9], Tang [17], Tang–Wu [18] (semilinear systems) and Manasevich–Mawhin [10], Mawhin [11], [12], Papageorgiou–Papageorgiou [15] (nonlinear systems driven by the ordinary vector p-Laplacian). The case where $A(t) = k^2\omega^2 I$, with $k \in \mathbb{N}$, $\omega = 2N/b$ and I the $N \times N$ identity matrix, was considered by Mawhin–Willem [13, p. 61] under the assumption that the right-hand side nonlinearity has the form $\nabla F(t, x)$, it is a Carathéodory function and it is monotone in $x \in \mathbb{R}^N$ (hence $F(t, \cdot)$ is a C^1, convex function). Their approach uses the dual action principle. In Mawhin–Willem [13, p. 88] we also find the general problem with $A(t)$ a general $N \times N$ matrix and the right-hand side nonlinearity $\nabla F(t, x)$ satisfying

$$|F(t,x)| \leq g(t) \quad \text{and} \quad \|\nabla F(t,x)\| \leq g(t)$$

for almost all $t \in T$ and all $x \in \mathbb{R}^N$, with $g \in L^1(T)_+$. The potential function $F(t, \cdot)$ is still a C^1 but it is no longer convex. Recently, Tang–Wu [19] extended the work of Mawhin–Willem to systems with subquadratic smooth potential, that is, they assumed that $F(t, \cdot) \in C^1(\mathbb{R}^N)$ and for almost all $t \in T$ and all $x \in \mathbb{R}^N$, we have

$$\|\nabla F(t,x)\| \leq g(t) + f(t)\|x\|^\alpha$$

with $g, f \in L^1(T)_+$ and $0 \leq \alpha < 1$.

Our work here complements the aforementioned work of Tang–Wu [19] by considering systems where the potential function is quadratic or superquadratic. In addition, our potential function is in general nonsmooth.

Our approach is variational based on the nonsmooth critical point theory as this was initially formulated by Chang [3] and extended more recently by Motreanu–Panagiotopoulos [14] and Kourogenis–Papageorgiou [8]. This theory is based on the subdifferential calculus for locally Lipschitz functions due to Clarke [4]. For the convenience of the reader in the next section we recall the basic definitions and facts from Clarke's theory and the nonsmooth critical point theory, which will be needed in the sequel.

2. Mathematical background

We start with the subdifferential theory for locally Lipschitz functions. For more details on this subject we refer to Clarke [4] and Denkowski–Migorski–Papageorgiou [5].

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X, X^*). A function $\varphi: X \to \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$, we can find an open set $U \subseteq X$ with $x \in U$ and a constant $k_U > 0$ (depending on U) such that

$$|\varphi(z) - \varphi(y)| \leq k_U \|z - y\| \quad \text{for all } z, y \in U.$$

From convex analysis we know that a function $\psi: X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$,which is convex, lower semicontinuous and not identically $+\infty$ is locally Lipschitz in the interior of its effective domain $\operatorname{dom}\psi = \{x \in X : \psi(x) < \infty\}$. So a continuous, convex function $\psi: X \to \mathbb{R}$ is locally Lipschitz. In particular, if X is finite dimensional, every convex function $\psi: X \to \mathbb{R}$ is locally Lipschitz.

Given a locally Lipschitz function $\varphi: X \to \mathbb{R}$ and $x, h \in X$, the generalized directional derivative of φ at $x \in X$ in the direction $h \in X$ is defined by

$$\varphi^0(x; h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

The function $h \mapsto \varphi^0(x; h)$ is sublinear, continuous and so it is the support function of the nonempty, convex and w^*-compact set $\partial\varphi(x)$ defined by

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \leq \varphi^0(x; h) \text{ for all } h \in X\}.$$

The multifunction $x \mapsto \partial\varphi(x)$ is known as the generalized (or Clarke) subdifferential of φ. This multifunction has a graph which is closed in $X \times X^*_{w^*}$ (here by $X^*_{w^*}$ we denote the Banach space X^* furnished with the w^*-topology). This fact follows easily from the upper semicontinuity of the map $(x, h) \mapsto \varphi^0(x; h)$.

If $\varphi, \psi: X \to \mathbb{R}$ are two locally Lipschitz functions, then

$$\partial(\varphi + \psi)(x) \subseteq \partial\varphi(x) + \partial\psi(x) \quad \text{and} \quad \lambda\partial\varphi(x) = \partial\varphi(\lambda x)$$

for all $x \in X$ and all $\lambda \in \mathbb{R}$. If $\varphi: X \to \mathbb{R}$ is continuous, convex (thus locally Lipschitz too), then the generalized subdifferential coincides with the subdifferential in the sense of convex analysis, given by

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq \varphi(y) - \varphi(x) \text{ for all } y \in X\}.$$

If $\varphi \in C^1(X)$ then $\partial\varphi(x) = \{\varphi'(x)\}$.

A point $x \in X$ is a critical point of the locally Lipschitz function $\varphi: X \to \mathbb{R}$ if $0 \in \partial\varphi(x)$. Then $c = \varphi(x)$ is a critical value of φ. It is easy to check that if $x \in X$ is a local extremum (i.e. a local minimum or a local maximum), then $0 \in \partial\varphi(x)$ (i.e. x is a critical point).

In the smooth critical point theory, a compactness-type condition, known as the Palais–Smale condition (PS-condition for short) plays a central role. In the present nonsmooth setting this condition takes the following form. A locally Lipschitz function $\varphi: X \to \mathbb{R}$ satisfies the nonsmooth PS-condition if every sequence

$\{x_n\}_{n\geq 1} \subset X$ such that $\{\varphi(x_n)\}_{n\geq 1}$ is bounded and $m(x_n) \to 0$ as $n \to \infty$, where

$$m(x_n) = \min\{\|x^*\| : x^* \in \partial\varphi(x_n)\}, \tag{2.1}$$

has a strongly convergent subsequence.

We will need some basic results from nonsmooth critical point theory, which can be found in Kandilakis–Kourogenis–Papageorgiou [7], Kourogenis–Papageorgiou [8] and Motreanu–Panagiotopoulos [14].

In all these results X is a reflexive Banach space and $\varphi: X \to \mathbb{R}$ is a Lipschitz functional satisfying the nonsmooth PS-condition.

The first result is the nonsmooth Mountain Pass Theorem.

Theorem 2.1. *If there exist $x_0, x_1 \in X$ and $\rho > 0$ such that $\|x_0 - x_1\| > \rho$ and*

$$\inf\{\varphi(x) : \|x\| = \rho\} = m_\rho > \max\{\varphi(x_0), \varphi(x_1)\},$$

then φ has a critical point $x \in X$ and $\varphi(x) \geq m_\varphi$.

The second result is known as the nonsmooth Generalized Mountain Pass Theorem.

Theorem 2.2. *If $X = Y \oplus V$ with $\dim Y < \infty$, $R > \rho > 0$, $e \in V$ with $\|e\| = \rho$,*

$$\begin{aligned} C &= \{x = y + \lambda e : \|x\| \leq R,\ y \in Y,\ \lambda \geq 0\}, \\ C_0 &= \{x = y + \lambda e : \|x\| = R,\ y \in Y,\ \lambda \geq 0 \text{ or } \|x\| \leq R,\ \lambda = 0\}, \\ D &= \{v \in V : \|v\| = \rho\} \end{aligned}$$

and

$$\inf_D \varphi = m_D > \max_{C_0} \varphi,$$

then φ has a critical point $x \in X$ and $\varphi(x) \geq m_D$.

The next result is a multiplicity result, known as the nonsmooth Local Linking Theorem.

Theorem 2.3. *If $X = Y \oplus V$ with $\dim Y < \infty$, φ is bounded below, $\inf \varphi < 0 = \varphi(0)$ and there exists $\rho > 0$ such that $\varphi(x) \leq 0$ for $x \in Y$, $\|x\| \leq \rho$ and $\varphi(x) \geq 0$ for $x \in V$, $\|x\| \leq \rho$, then φ has at least two nontrivial critical points.*

The final result is also a multiplicity result known as the nonsmooth Symmetric Mountain Pass Theorem.

Theorem 2.4. *If φ is even and*

(a) *there exists a subspace V of X of finite codimension such that*

$$\varphi|_{\partial B_r \cap V} \geq \beta > 0 \quad (B_r = \{x \in X : \|x\| < r\}),$$

(b) *for any* $k \geq 1$ *there is a* k*-dimensional subspace* Y_k *of* X *such that* $\varphi|_{Y_k}$ *is anticoercive, i.e.*

$$\varphi(x) \to -\infty \quad as\ \|x\| \to \infty,\ x \in Y_k,$$

then φ *has a sequence* $\{(x_n, -x_n)\}_{n\geq 1}$ *of distinct pairs of nontrivial critical points.*

3. Existence theorems

In this section we prove the existence of solutions for problem (1.1). To do this we will need the following hypothesis on the mapping A in (1.1).

H(A) $A\colon T \to \mathbb{R}^{N\times N}$ is a continuous map such that for all $t \in T$, $A(t)$ is a symmetric $N \times N$ matrix.

Remark 3.1. From Linear Algebra we know that for every $t \in T$, the matrix $A(t)$ has real eigenvalues $\lambda_1(t) \geq \ldots \geq \lambda_N(t)$ and

$$\lambda_N(t)\|x\|^2 \leq (A(t)x, x)_{\mathbb{R}^N} \leq \lambda_1(t)\|x\|^2 \quad \text{for all } x \in \mathbb{R}^N.$$

Moreover, for every $k \in \{1, \ldots, N\}$, the map $t \mapsto \lambda_k(t)$ is continuous on T.

Let $W^{1,2}_{\rm per}((0,b), \mathbb{R}^N) = \{x \in W^{1,2}((0,b), \mathbb{R}^N) : x(0) = x(b)\}$ which is a Hilbert space endowed with the norm defined by

$$\|x\|^2 = \|x\|_2^2 + \|x'\|_2^2 \quad \text{for all } x \in W^{1,2}_{\rm per}((0,b), \mathbb{R}^N).$$

The space $W^{1,2}_{\rm per}((0,b), \mathbb{R}^N)$ is compactly embedded in $C(T, \mathbb{R}^N)$. For a later use, denote by $C > 0$ a constant satisfying

$$\|u\|_\infty \leq C\|u\| \quad \text{for all } x \in W^{1,2}_{\rm per}((0,b), \mathbb{R}^N). \tag{3.1}$$

Because of the continuous embedding of $W^{1,2}((0,b), \mathbb{R}^N)$ into $C(T, \mathbb{R}^N)$, we see that the pointwise evaluations at $t = 0$ and $t = b$ make sense. Let $\widehat{A} \in \mathcal{L}(C(T, \mathbb{R}^N), C(T, \mathbb{R}^N))$ be defined by

$$(\widehat{A}x)(t) = A(t)x(t) \quad \text{for all } t \in T \text{ and all } x \in C(T, \mathbb{R}^N).$$

As in Mawhin–Willem [13, p. 89] and Showalter [16, p. 78], using the spectral theorem for compact self-adjoint operators in a Hilbert space, for the differential operator $x \mapsto -x'' - \widehat{A}x$ we have a sequence of eigenfunctions which is an orthonormal basis for $L^2(T, \mathbb{R}^N)$ and an orthogonal basis for $W^{1,2}_{\rm per}((0,b), \mathbb{R}^N)$. Moreover, we have the orthogonal direct sum decomposition

$$W^{1,2}_{\rm per}((0,b), \mathbb{R}^N) = V_- \oplus V_0 \oplus V_+ \tag{3.2}$$

where

$$\begin{aligned} H_- &= \operatorname{span}\{x \in W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N) : -x'' - \widehat{A}x = \lambda x \text{ for some } \lambda < 0\}, \\ H_0 &= \ker(-x'' - \widehat{A}x), \\ H_+ &= \overline{\operatorname{span}}\{x \in W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N) : -x'' - \widehat{A}x = \lambda x \text{ for some } \lambda > 0\}. \end{aligned}$$

Remark that $\dim H_-$, $\dim H_0 < \infty$. Moreover, $\dim H_- = 0$ if and only if $A(t)$ is positive semidefinite for all $t \in T$ (see Mawhin–Willem [13, p. 89]).

We start by proving some useful inequalities satisfied by the elements in the component subspaces H_- and H_+.

Proposition 3.2.

(a) *There exists a* $\beta_1 > 0$ *such that*

$$\|x'\|_2^2 - \int_0^b (A(t)x(t), x(t))_{\mathbb{R}^N}\, dt \geq \beta_1 \|x\|^2 \quad \textit{for all } x \in H_+.$$

(b) *There exists a* $\beta_2 > 0$ *such that*

$$\|x'\|_2^2 - \int_0^b (A(t)x(t), x(t))_{\mathbb{R}^N}\, dt \leq -\beta_2 \|x\|^2 \quad \textit{for all } x \in H_-.$$

Proof. (a) Let $\psi: H_+ \to \mathbb{R}$ be a C^1-function defined by

$$\psi(x) = \|x'\|_2^2 - \int_0^b (A(t)x(t), x(t))_{\mathbb{R}^N}\, dt.$$

Evidently if $x \in H_+$, then $x'' \in L^2(T, \mathbb{R}^N)$ and so by Green's identity we have

$$\|x'\|_2^2 = \langle -x'', x \rangle$$

where by $\langle \cdot\,, \cdot \rangle$ we denote the duality brackets for the pair

$$(W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N), W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N)^*).$$

So we have

$$\psi(x) = \langle -x'', x \rangle - \langle \widehat{A}(x), x \rangle \geq 0$$

(recall the definition of H_+).

If the inequality in part (a) of the statement were not true, then exploiting the 2-homogeneity of ψ, we could find a sequence $\{x_n\}_{n \geq 1} \subset H_+$ with $\|x_n\| = 1$ such that

$$\psi(x_n) = \|x_n'\|_2^2 - \int_0^b (A(t)x_n(t), x_n(t))_{\mathbb{R}^N}\, dt \downarrow 0 \quad \text{as } n \to \infty.$$

By passing to a subsequence if necessary, we may assume that

$$x_n \xrightarrow{\mathrm{w}} x \quad \text{in } W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N) \quad \text{and} \quad x_n \to x \quad \text{in } L^2(T, \mathbb{R}^N)$$

(recall that $W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N)$ is embedded compactly in $L^2(T,\mathbb{R}^N)$). Because the norm functional in a Banach space is weakly lower semicontinuous, we obtain

$$\|x'\|_2^2 - \int_0^b (A(t)x(t),x(t))_{\mathbb{R}^N}\,dt \le 0 \quad \text{with } x \in H_+$$

which implies that

$$\|x'\|_2^2 - \int_0^b (A(t)x(t),x(t))_{\mathbb{R}^N}\,dt = 0,$$

and thus $x \equiv 0$. But then $x_n', x_n \to 0$ in $L^2(T,\mathbb{R}^N)$, hence

$$x_n \to 0 \quad \text{in } W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N) \quad \text{as } n \to \infty,$$

which contradicts the fact that $\|x_n\| = 1$ for all $n \ge 1$.

(b) The proof of the inequality for the elements in H_- is done similarly. $\square$

For the first existence result we need the following hypotheses on the nonsmooth potential $j(t,x)$:

H(j)$_1$ $j\colon T \times \mathbb{R}^N \to \mathbb{R}$ is a function such that $j(\,\cdot\,,0) \in L^1(T)$ and

- (a) for all $x \in \mathbb{R}^N$, $t \mapsto j(t,x)$ is measurable,
- (b) for almost all $t \in T$, $x \mapsto j(t,x)$ is locally Lipschitz,
- (c) for almost all $t \in T$, all $x \in \mathbb{R}^N$ and all $u \in \partial j(t,x)$, we have

$$\|u\| \le a(t) + c(t)\|x\|^{p-1} \quad \text{with } a, c \in L^1(T)_+,\ 1 \le p < \infty,$$

- (d) there exist $\mu > 2$, $M > 0$ and $c_0 > 0$ such that for almost all $t \in T$ and all $\|x\| \ge M$, we have

$$c_0 \le \mu j(t,x) \le -j^0(t,x;-x),$$

- (e) $\limsup_{x\to 0}(2j(t,x)/\|x\|^2) \le 0$ uniformly for almost all $t \in T$,
- (f) if $\lambda_m \le 0$ is the biggest nonpositive eigenvalue of $x \mapsto -x'' - \widehat{A}x$, then for almost all $t \in T$ and all $x \in \mathbb{R}^N$, we have $\lambda_m\|x\|^2/2 \le j(t,x)$.

Example 3.3. The following function satisfies hypotheses H(j)$_1$:

$$j(t,x) = \begin{cases} \dfrac{\lambda_m}{2}\,\|x\|^2 & \text{if } \|x\| \le 1, \\ \dfrac{a(t)}{p}\|x\|^p + \dfrac{\lambda_m}{2} - \dfrac{a(t)}{p} & \text{if } \|x\| > 1, \end{cases}$$

with $a \in L^1(T)$, $a(t) \ge a_0$ a.e. $t \in T$, for constants $a_0 > 0$ and $2 < p < \infty$. Assumptions H(j)$_1$ are verified taking $2 < \mu < p$ in H(j)$_1$(d).

The next lemma shows that hypotheses H(j)$_1$ imply that the potential $j(t,\,\cdot\,)$ is strictly superquadratic.

Proposition 3.4. *If hypotheses* $\mathrm{H(j)}_1$ *hold, then there exist* $c_1, c_2 \in L^1(T)_+$ *with* $c_0/M^\mu \le c_1(t)$ *a.e. on* T *such that*

$$c_1(t)\|x\|^\mu - c_2(t) \le j(t,x) \quad \textit{for almost all } t \in T \textit{ and all } x \in \mathbb{R}^N.$$

Proof. Let N_0 be the Lebesque-null subset of T outside of which hypotheses $\mathrm{H(j)}_1$(b)–(f) hold. Let $t \in T \setminus N_0$ and $x \in \mathbb{R}^N$ with $\|x\| \ge M$. Set $\beta(t,r) = j(t,rx)$, $r \ge 1$. Clearly, $\beta(t,\,\cdot\,)$ is locally Lipschitz. Moreover, from the nonsmooth chain rule (see [4, p. 45], or [5, p. 610]), we have that

$$-r\partial\beta(t,r) \subseteq (\partial j(t,rx), -rx)_{\mathbb{R}^N}. \tag{3.3}$$

Recall that $\beta(t,\,\cdot\,)$ is differentiable almost everywhere on $\mathbb{R}$ and at every point of differentiability $r \in \mathbb{R}$, we have $d\beta(t,r)/dr \in \partial\beta(t,r)$. So from (3.3) and hypothesis $\mathrm{H(j)}_1$(d), we have

$$r\frac{d}{dr}\,\beta(t,r) \ge -j^0(t,rx;-rx) \ge \mu j(t,rx) = \mu\beta(t,r) \quad \text{for almost all } r \ge 1,$$

which implies that

$$\frac{\mu}{r} \le \frac{d\beta(t,r)/dr}{\beta(t,r)} \quad \text{for almost all } r \ge 1.$$

Integrating from 1 to $r > 1$, we obtain

$$\ln r^\mu \le \ln\frac{\beta(t,r)}{\beta(t,1)},$$

and thus we have $r^\mu\beta(t,1) \le \beta(t,r)$. So we have shown that for all $t \in T \setminus N_0$, all $x \in \mathbb{R}^N$ with $\|x\| \ge M$ and all $r \ge 1$, we have

$$r^\mu j(t,x) \le j(t,rx). \tag{3.4}$$

In view of (3.4), for $\|x\| \ge M$ we have

$$\begin{aligned} j(t,x) &= j\bigg(t, \frac{\|x\|}{M} M \frac{x}{\|x\|}\bigg) \ge \frac{\|x\|^\mu}{M^\mu} j\bigg(t, \frac{Mx}{\|x\|}\bigg) \\ &\ge \frac{\|x\|^\mu}{M^\mu}\,\min\{j(t,y) : \|y\| = M\} = c_1(t)\|x\|^\mu, \end{aligned}$$

with $c_1 \in L^1(T)_+$, $c_1(t) \ge c_0/M^\mu$ a.e. on T. On the other hand, if $\|x\| < M$, from hypothesis $\mathrm{H(j)}_1$(c) and the mean value theorem for locally Lipschitz functions, we have that

$$|j(t,x)| \le c_2(t) \quad \text{for a.a. } t \in T, \text{ with } c_2 \in L^1(T)_+.$$

Therefore finally we obtain that

$$j(t,x) \ge c_1(t)\|x\|^\mu - c_2(t) \quad \text{for a.a. } t \in T \text{ and all } x \in \mathbb{R}^N. \qquad \square$$

We are ready for the first existence theorem concerning problem (1.1).

Theorem 3.5. *If hypotheses* H(A) *and* $\mathrm{H(j)}_1$ *hold, then problem* (1.1) *has a nontrivial solution* $x \in C^1(T, \mathbb{R}^N)$.

Proof. Consider the locally Lipschitz Euler functional

$$\varphi: W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N) \to \mathbb{R}$$

for problem (1.1) defined by

$$\varphi(x) = \frac{1}{2}\|x'\|_2^2 - \frac{1}{2}\int_0^b (A(t)x(t), x(t))_{\mathbb{R}^N}\, dt - \int_0^b j(t, x(t))\, dt$$

for all $x \in W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N)$.

Claim 1. φ satisfies the nonsmooth PS-condition.

To this end, let $\{x_n\}_{n\ge 1} \subset W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N)$ be a sequence such that

$$|\varphi(x_n)| \le M_1 \text{ for some } M_1 > 0, \text{ all } n \ge 1 \text{ and } m(x_n) \to 0 \text{ as } n \to \infty$$

(see (2.1)). Since $\partial\varphi(x_n) \subseteq W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N)^*$ is nonempty, weakly compact and the norm functional in a Banach space is weakly lower semicontinuous, by Weierstrass theorem we can find $x_n^* \in \partial\varphi(x_n)$ such that $m(x_n) = \|x_n^*\|$, $n \ge 1$. So $\|x_n^*\| \to 0$ as $n \to \infty$. Note that

$$x_n^* = V(x_n) - \widehat{A}x_n - u_n, \quad n \ge 1,$$

where $V \in \mathcal{L}(W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N), W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N)^*)$ defined by

$$\langle V(x), y\rangle = \int_0^b (x'(t), y'(t))_{\mathbb{R}^N}\, dt \quad \text{for all } x, y \in W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N)$$

and $u_n \in L^1(T, \mathbb{R}^N)$ is such that $u_n(t) \in \partial j(t, x_n(t))$ a.e. on T (see $\mathrm{H(j)}_1$(c)). Evidently V is monotone, so it is maximal monotone (see [6, p. 37]). Let $\eta \in (2, \mu)$. From the choice of the sequence $\{x_n\}_{n\ge 1} \subset W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N)$, we have

$$\frac{\eta}{2}\,\|x_n'\|^2 - \frac{\eta}{2}\int_0^b (A(t)x_n(t), x_n(t))_{\mathbb{R}^N}\, dt - \int_0^b \eta j(t, x_n(t))\, dt \le \eta M_1 \tag{3.5}$$

and $|\langle x_n^*, x_n\rangle| \le \varepsilon_n\|x_n\|$ with $\varepsilon_n \downarrow 0$. Hence

$$-\|x_n'\|_2^2 + \int_0^b (A(t)x_n(t), x_n(t))_{\mathbb{R}^N}\, dt + \int_0^b (u_n(t), x_n(t))_{\mathbb{R}^N}\, dt \le \varepsilon_n\|x_n\|,$$

which implies that

$$\begin{aligned} -\|x_n'\|_2^2 + \int_0^b (A(t)x_n(t), x_n(t))_{\mathbb{R}^N}\, dt \\ - \int_0^b j^0(t, x_n(t); -x_n(t))\, dt \le \varepsilon_n\|x_n\|. \end{aligned} \tag{3.6}$$

Adding (3.5) and (3.6), we obtain

$$\tag{3.7} \begin{aligned}\Big(\frac{\eta}{2}-1\Big)\|x_n'\|_2^2 &- \Big(\frac{\eta}{2}-1\Big)\int_0^b (A(t)x_n(t),x_n(t))_{\mathbb{R}^N}\,dt \\ &- \int_0^b [\eta j(t,x_n(t)) + j^0(t,x_n(t);-x_n(t))]\,dt \le \varepsilon_n\|x_n\| + \eta M_1.\end{aligned}$$

Recall that $x_n = \overline{x}_n + x_n^0 + \widehat{x}_n$ with $\overline{x}_n \in H_-$, $x_n^0 \in H_0$ and $\widehat{x}_n \in H_+$ (see (3.2)). Set $v_n = \overline{x}_n + x_n^0$. Exploiting the orthogonality among the component spaces, from (3.7) we obtain

$$\tag{3.8} \begin{aligned}\Big(\frac{\eta}{2}-1\Big)&\Big[\|\widehat{x}_n'\|_2^2 - \int_0^b (A(t)\widehat{x}_n(t),\widehat{x}_n(t))_{\mathbb{R}^N}\,dt\Big] \\ &+ \Big(\frac{\eta}{2}-1\Big)\Big[\|v_n'\|_2^2 - \int_0^b (A(t)v_n(t),v_n(t))_{\mathbb{R}^N}\,dt\Big] \\ &- \int_0^b [\mu j(t,x_n(t)) + j^0(t,x_n(t);-x_n(t))]\,dt \\ &+ (\mu-\eta)\int_0^b j(t,x_n(t))\,dt \le \varepsilon_n\|x_n\| + \eta M_1.\end{aligned}$$

From Proposition 3.2, we know that

$$\tag{3.9} \|\widehat{x}_n'\|_2^2 - \int_0^b (A(t)\widehat{x}_n(t),\widehat{x}_n(t))_{\mathbb{R}^N}\,dt \ge \xi_1\|\widehat{x}_n\|^2 \quad \text{for all } n\ge 1$$

with $\xi_1 > 0$. Also we have

$$\tag{3.10} \|v_n'\|_2^2 - \int_0^b (A(t)v_n(t),v_n(t))_{\mathbb{R}^N}\,dt \ge \lambda_1\|v_n\|_2^2 \ge -|\lambda_1|\|v_n\|_2^2,$$

where λ_1 is the first eigenvalue of $x \mapsto -x'' - \widehat{A}x$. Moreover, from hypotheses $\mathrm{H(j)}_1$(c) and (d) and the mean value theorem for locally Lipschitz functions, we have

$$\tag{3.11} \begin{aligned}-\int_0^b &[\mu j(t,x_n(t)) + j^0(t,x_n(t);-x_n(t))]\,dt \\ &= -\int_{\{\|x_n(t)\|\ge M\}} [\mu j(t,x_n(t)) + j^0(t,x_n(t);-x_n(t))]\,dt \\ &\quad - \int_{\{\|x_n(t)\|< M\}} [\mu j(t,x_n(t)) + j^0(t,x_n(t);-x_n(t))]\,dt \ge -\beta_1\end{aligned}$$

for some $\beta_1 > 0$ and all $n \ge 1$. Finally Proposition 3.4 implies that

$$\tag{3.12} (\mu-\eta)\int_0^b j(t,x_n(t))\,dt \ge (\mu-\eta)c\|x_n\|_\mu^\mu - (\mu-\eta)\|c_2\|_1$$

for some $c > 0$ and all $n \geq 1$. Returning to (3.8) and using (3.9)–(3.12), we obtain

$$\left(\frac{\eta}{2} - 1\right)[\xi_1 \|\widehat{x}_n\|^2 - |\lambda_1|\, \|v_n\|_2^2] + c_3 \|x_n\|_\mu^\mu \leq M_2 + \varepsilon_n \|x_n\|$$

for some $c_3, M_2 > 0$ and all $n \geq 1$. Because $H_- \oplus H_0$ is finite dimensional, all norms are equivalent and so

$$\left(\frac{\eta}{2} - 1\right)\xi_1 \|\widehat{x}_n\|^2 + c_3 \|x_n\|_\mu^\mu - c_4 \|v_n\|_\mu^2 \leq M_2 + \varepsilon_n \|x_n\|$$

for some $c_4 > 0$ and all $n \geq 1$. Since v_n is the projection of x_n on $H_- \oplus H_0$ and $\mu > 2$, we have $\|v_n\|_2 \leq \|x_n\|_2 \leq c_5 \|x_n\|_\mu$ for some $c_5 > 0$ and all $n \geq 1$. Using once again the fact that $H_- \oplus H_0$ is finite dimensional, we conclude that $\|v_n\|_\mu \leq c_6 \|x_n\|_\mu$ for some $c_6 > 0$ and all $n \geq 1$. Therefore

$$\left(\frac{\eta}{2} - 1\right)\xi_1 \|\widehat{x}_n\|^2 + c_3 \|v_n\|_\mu^\mu - c_7 \|v_n\|_\mu^2 \leq M_2 \quad \text{for all } n \geq 1 \tag{3.13}$$

for some $c_7 > 0$, a possibly smaller $\xi_1 > 0$ and a new constant $c_3 > 0$. Since $\eta > 2$ and $\mu > 2$, from (3.13) we infer that

$$\{\widehat{x}_n\}_{n\geq 1} \subset W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N) \quad \text{and} \quad \{v_n\}_{n\geq 1} \subset L^\mu(T, \mathbb{R}^N) \subset L^2(T, \mathbb{R}^N)$$

are both bounded sequences. Due to the finite dimensionality of $H_- \oplus H_0$, it follows that

$$\{v_n\}_{n\geq 1} \subset W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N) \text{ is bounded,}$$

and so

$$\{x_n\}_{n\geq 1} \subset W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N) \text{ is bounded.}$$

By passing to a suitable subsequence if necessary, we may assume that

$$x_n \xrightarrow{\mathrm{w}} x \quad \text{in } W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N) \quad \text{and} \quad x_n \to x \quad \text{in } L^2(T, \mathbb{R}^N).$$

Recall that

$$\left| \langle V(x_n), x_n - x\rangle - \int_0^b (Ax_n, x_n - x)_{\mathbb{R}^N}\, dt - \int_0^b (u_n, x_n - x)_{\mathbb{R}^N}\, dt \right| \leq \varepsilon_n \|x_n - x\|.$$

Note by hypotheses H(A) and H(j)$_1$(c) that

$$\int_0^b (Ax_n, x_n - x)_{\mathbb{R}^N}\, dt \to 0 \quad \text{and} \quad \int_0^b (u_n, x_n - x)_{\mathbb{R}^N}\, dt \to 0 \quad \text{as } n \to \infty$$

(see also (3.1)). It follows that $\langle V(x_n), x_n\rangle \to \langle V(x), x\rangle$, which implies that $\|x_n'\|_2 \to \|x'\|_2$. Recall that $x_n' \xrightarrow{\mathrm{w}} x'$ in $L^2(T, \mathbb{R}^N)$ and that Hilbert spaces have Kadec–Klee property. So we conclude that $x_n' \to x'$ in $L^2(T, \mathbb{R}^N)$, therefore $x_n \to x$ in $W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N)$.

Claim 2. There exist $\rho > 0$ and $\beta > 0$ such that $\varphi(x) \geq \beta$ for all $x \in H_+$, $\|x\| = \rho$.

Because of hypothesis $\mathrm{H(j)}_1$(e), given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon)$ such that

$$j(t,x) \leq \frac{\varepsilon}{2}\,\|x\|^2 \tag{3.14}$$

for almost all $t \in T$ and all $\|x\| \leq \delta$. On the other hand by the mean value theorem for locally Lipschitz functions, we have

$$|j(t,x) - j(t,y)| = |(u^*, x-y)_{\mathbb{R}^N}|$$

for almost all $t \in T$, all $x, y \in \mathbb{R}^N$ with $\|x\| \geq \delta = \|y\|$ and with $u^* \in \partial j(t,z)$ where $z \in\,]x,y[\, = \{z \in \mathbb{R}^N : z = \theta x + (1-\theta)y,\ 0 < \theta < 1\}$. So because of hypothesis $\mathrm{H(j)}_1$(c), we have

$$|j(t,x)| \leq a_1(t) + c_1(t)\|x\|^p + |j(t,y)|$$

for almost all $t \in T$, all $x, y \in \mathbb{R}^N$ with $\|x\| \geq \delta = \|y\|$ and with $a_1, c_1 \in L^1(T)_+$. We deduce that

$$|j(t,x)| \leq a_2(t) + c_1(t)\|x\|^p$$

for almost all $t \in T$, all $\|x\| \geq \delta = \|y\|$ and with $a_2 \in L^1(T)_+$. So we can find $\tau > 2$, $\beta_1 \in L^1(T)_+$ such that

$$j(t,x) \leq \beta_1(t)\|x\|^\tau \quad \text{for a.a. } t \in T \text{ and all } \|x\| \geq \delta. \tag{3.15}$$

From (3.14) and (3.15) it follows that

$$j(t,x) \leq \frac{\varepsilon}{2}\,\|x\|^2 + \beta_1(t)\|x\|^\tau \quad \text{for a.a. } t \in T \text{ and all } x \in \mathbb{R}^N. \tag{3.16}$$

Then, in view of Proposition 3.2 and (3.16), for $x \in H_+$ we have

$$\begin{aligned} \varphi(x) &= \frac{1}{2}\,\|x'\|_2^2 - \frac{1}{2}\int_0^b (A(t)x(t), x(t))_{\mathbb{R}^N}\,dt - \int_0^b j(t,x(t))\,dt \\ &\geq \frac{\xi_1}{2}\,\|x\|^2 - \int_0^b j(t,x(t))\,dt \geq \frac{\xi_1}{2}\,\|x\|^2 - \frac{\varepsilon c}{2}\,\|x\|^2 - c_8\|x\|^\tau \end{aligned}$$

for some $c, c_8 > 0$. Choosing $\varepsilon > 0$ sufficiently small, we obtain $\varphi(x) \geq c_9\|x\|^2 - c_8\|x\|^\tau$ for some $c_9 > 0$ and all $x \in H_+$. Because $\tau > 2$, we can find $\rho > 0$ small such that $\varphi(x) \geq \beta > 0$ for all $x \in H_+$ with $\|x\| = \rho$. This proves Claim 2.

Claim 3. $\varphi|_{H_- \oplus H_0} \leq 0$.

For $v \in H_- \oplus H_0$, we have by $\mathrm{H(j)}_1$(f) that

$$\varphi(v) \leq \frac{1}{2}\,\|v'\|_2^2 - \frac{1}{2}\int_0^b (A(t)v(t), v(t))_{\mathbb{R}^N}\,dt - \frac{\lambda_m}{2}\,\|v\|_2^2 \leq 0,$$

which proves Claim 3.

Now let $e \in C^1(T, \mathbb{R}^N)$ be an eigenfunction corresponding to the eigenvalue $\lambda_{m+1} > 0$ with $\|e\| = \rho$. Set $w = v + re$, $v \in H_- \oplus H_0$, $r > 0$. Exploiting the orthogonality of the component spaces and using Proposition 3.4 and (3.1), we obtain

$$\begin{aligned}\varphi(w) &= \frac{1}{2}\,\|w'\|_2^2 - \frac{1}{2}\int_0^b (Aw, w)_{\mathbb{R}^N}\,dt - \int_0^b j(t, w(t))\,dt \\ &\le \frac{1}{2}\,\|v'\|_2^2 - \frac{1}{2}\int_0^b (Av, v)_{\mathbb{R}^N}\,dt \\ &\quad + \frac{r^2}{2}\,\|e'\|_2^2 - \frac{r^2}{2}\int_0^b (Ae, e)_{\mathbb{R}^N}\,dt - c_{10}\|w\|_\mu^\mu + \|c_2\|_1 \\ &\le \frac{r^2\lambda_{m+1}}{2}\,\|e\|_2^2 - c_{10}\|w\|_\mu^\mu + \|c_2\|_1\end{aligned}$$

for some $c_{10} > 0$. Since $\mu > 2$, we conclude that $\varphi(w) \to -\infty$ as $\|w\| \to \infty$ (since $H_- \oplus H_0$ is of finite dimension). Therefore we can find $R > \rho$ large such that if $\|w\| = R$ then

$$\varphi(w) < 0 < \beta. \tag{3.17}$$

We consider the half-ball

$$Q = \{w = v + re : v \in H_- \oplus H_0,\ \|w\| \le R,\ r \ge 0\}.$$

Then $\partial Q = \{w = v + re : v \in H_- \oplus H_0,\ \|w\| = R,\ r \ge 0 \text{ or } \|w\| \le R,\ r = 0\}$. Because of Claims 1–3 and relation (3.19) we can use Theorem 2.2 and obtain $x \in W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N)$ such that

$$\varphi(0) \le 0 < \beta \le \varphi(x) \quad \text{and} \quad 0 \in \partial\varphi(x).$$

From the inequality, we see that $x \ne 0$. From the inclusion we obtain

$$V(x) - \widehat{A}x - u = 0 \quad \text{with } u \in L^1(T, \mathbb{R}^N),\ u(t) \in \partial j(t, x(t)) \text{ a.e. on } T \tag{3.18}$$

(see Clarke [4, p. 80]). From the representation theorem for the elements of $W^{-1,2}((0,b), \mathbb{R}^N) = W_0^{1,2}((0,b), \mathbb{R}^N)^*$ (see [5, p. 362]), we know that $x'' \in W^{-1,2}((0,b), \mathbb{R}^N)$. Let $\langle\,\cdot\,,\,\cdot\,\rangle_0$ denote the duality brackets for the pair

$$(W_0^{1,2}((0,b), \mathbb{R}^N), W^{-1,2}((0,b), \mathbb{R}^N)).$$

Then because of (3.18), using as a test function $\psi \in C_c^1((0,b), \mathbb{R}^N)$ and Green's identity, we have that

$$\langle -x'', \psi\rangle_0 - \int_0^b (A(t)x(t), \psi(t))_{\mathbb{R}^N}\,dt = \int_0^b (u(t), \psi(t))_{\mathbb{R}^N}\,dt. \tag{3.19}$$

Since $C_c^1((0,b), \mathbb{R}^N)$ is dense in $W_0^{1,2}((0,b), \mathbb{R}^N)$, from (3.19) it follows that

$$-x''(t) - A(t)x(t) = u(t) \quad \text{a.e. on } T,\ x(0) = x(b), \tag{3.20}$$

thus $x'' \in L^1(T,\mathbb{R}^N)$, and so $x \in C^1(T,\mathbb{R}^N) \cap W^{2,1}((0,b),\mathbb{R}^N)$. Then for $y \in W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N)$ we have

$$\langle V(x), y\rangle - \int_0^b (A(t)x(t), y(t))_{\mathbb{R}^N}\,dt = \int_0^b (u(t), y(t))_{\mathbb{R}^N}\,dt.$$

We obtain

$$\begin{aligned}(x'(b), y(b))_{\mathbb{R}^N} &- (x'(0), y(0))_{\mathbb{R}^N} \\ &- \int_0^b (x''(t), y(t))_{\mathbb{R}^N}\,dt - \int_0^b (A(t)x(t), y(t))_{\mathbb{R}^N}\,dt = \int_0^b (u(t), y(t))_{\mathbb{R}^N}\,dt.\end{aligned}$$

By (3.20) we get

$$(x'(0), y(0))_{\mathbb{R}^N} = (x'(b), y(b))_{\mathbb{R}^N} \quad \text{for all } y \in W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N),$$

which yields $x'(0) = x'(b)$. Therefore $x \in C^1(T,\mathbb{R}^N)$ is a nontrivial solution of problem (1.1). □

We can weaken the hypotheses on the nonsmooth potential $j(t,x)$ if we assume that $\dim(H_- \oplus H_0) = 0$. In this case $\lambda_1 > 0$, the linear differential operator $x \mapsto -x'' - \widehat{A}x$ is maximal monotone, coercive and for all $c \in \mathbb{R}^N \setminus \{0\}$ we have $\int_0^b (A(t)c, c)_{\mathbb{R}^N}\,dt < 0$ (hence if $A(t) \equiv A$ for all $t \in T$, then A is negative definite).

Now the hypotheses on the nonsmooth potential $j(t,x)$ are the following:

$\mathrm{H(j)}_2$ $j\colon T \times \mathbb{R}^N \to \mathbb{R}$ is a function such that $j(\,\cdot\,,0) = 0$ almost everywhere on T and

(a) for all $x \in \mathbb{R}^N$, $t \mapsto j(t,x)$ is measurable,
(b) for almost all $t \in T$, $x \mapsto j(t,x)$ is locally Lipschitz,
(c) for almost all $t \in T$, all $x \in \mathbb{R}^N$ and all $u \in \partial j(t,x)$, we have

$$\|u\| \le a(t) + c(t)\|x\|^{r-1} \quad \text{with } a, c \in L^1(T)_+,\ 1 \le r < \infty,$$

(d) there exists $\mu > 2$ such that for almost all $t \in T$ and all $x \in \mathbb{R}^N$, we have

$$\mu j(t,x) \le -j^0(t,x;-x),$$

(e) there exists $\theta \in L^\infty(T)_+$ such that $\theta(t) \le \lambda_1$ a.e. on T with strict inequality on a set of positive measure and

$$\limsup_{x\to 0} \frac{2j(t,x)}{\|x\|^2} \le \theta(t) \quad \text{uniformly for almost all } t \in T,$$

(f) there exists $x_0 \in \mathbb{R}^N \setminus \{0\}$ such that $\int_0^b j(t,x_0)\,dt > 0$.

Example 3.6. The following function satisfies hypotheses $\mathrm{H(j)}_2$. For simplicity we drop the t-dependence:

$$j(x) = \begin{cases} -\dfrac{1}{p}\,\|x\|^p & \text{if } \|x\| \le 1, \\ \dfrac{1}{\mu}\,\|x\|^\mu - \dfrac{1}{\mu} - \dfrac{1}{p} & \text{if } \|x\| > 1, \end{cases}$$

with $p < 2 < \mu$.

In the case of $\mathrm{H(j)}_2$ the hypotheses allow both subquadratic and superquadratic potentials.

Theorem 3.7. *If hypotheses* H(A) *and* $\mathrm{H(j)}_2$ *hold and* $\dim(H_- \oplus H_0) = 0$, *then problem* (1.1) *has a nontrivial solution* $x \in C^1(T, \mathbb{R}^N)$.

Proof. Again we consider the locally Lipschitz functional φ defined for $x \in W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N)$ by

$$\varphi(x) = \frac{1}{2}\,\|x'\|_2^2 - \frac{1}{2}\int_0^b (A(t)x(t), x(t))_{\mathbb{R}^N}\,dt - \int_0^b j(t, x(t))\,dt.$$

Claim 1. φ satisfies the nonsmooth PS-condition.

Let a sequence $\{x_n\}_{n\ge1} \subset W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N)$ be such that

$$|\varphi(x_n)| \le M_1 \quad \text{for some } M_1 > 0, \text{ all } n \ge 1 \quad \text{and} \quad m(x_n) \to 0 \quad \text{as } n \to \infty.$$

As before we can find $x_n^* \in \partial\varphi(x_n)$ such that $m(x_n) = \|x_n^*\|$, $n \ge 1$. Then

$$|\langle x_n^*, x_n\rangle| \le \varepsilon_n \|x_n\| \quad \text{with } \varepsilon_n \downarrow 0.$$

So we obtain

$$\left(\frac{\mu}{2} - 1\right)\|x_n'\|_2^2 - \left(\frac{\mu}{2} - 1\right)\int_0^b (A(t)x_n(t), x_n(t))_{\mathbb{R}^N}\,dt \\ + \int_0^b [(u_n(t), x_n(t))_{\mathbb{R}^N} - \mu j(t, x_n(t))]\,dt \le \varepsilon_n\|x_n\| + \mu M_1.$$

By Proposition 3.2(a) it follows that

$$\left(\frac{\mu}{2} - 1\right)\xi_1\|x_n\|^2 + \left(\frac{\mu}{2} - 1\right)\int_0^b [-j^0(t, x_n(t); -x_n(t)) - \mu j(t, x_n(t))]\,dt \\ \le \varepsilon_n\|x_n\| + \mu M_1.$$

Taking into account hypothesis $\mathrm{H(j)}_2$(d) we deduce

$$\left(\frac{\mu}{2} - 1\right)\xi_1\|x_n\|^2 \le M_2 \quad \text{for some } M_2 > 0, \text{ all } n \ge 1$$

and thus $\{x_n\}_{n\ge1} \subset W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N)$ is bounded. From this as in the proof of Theorem 3.5 we conclude that φ satisfies the nonsmooth PS-condition.

Claim 2. There exist $\rho > 0$ and $\beta > 0$ such that $\varphi(x) \geq \beta$ for all $x \in W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N)$ with $\|x\| = \rho$.

As in the proof of Theorem 3.5, from hypothesis $\mathrm{H(j)}_2$(e), we see that given $\varepsilon > 0$, we can find $c_\varepsilon \in L^1(T)_+$ such that

$$j(t,x) \leq \frac{1}{2}\,(\theta(t)+\varepsilon)\,\|x\|^2 + c_\varepsilon(t)\|x\|^\tau \tag{3.21}$$

for almost all $t \in T$, all $x \in \mathbb{R}^N$ and with $\tau > 2$, $c_\varepsilon \in L^1(T)_+$. Then for all $x \in W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N)$ we have

$$\begin{aligned}\varphi(x) &= \frac{1}{2}\,\|x'\|_2^2 - \frac{1}{2}\int_0^b (A(t)x(t),x(t))_{\mathbb{R}^N}\,dt - \int_0^b j(t,x(t))\,dt \\ &\geq \frac{1}{2}\,\|x'\|_2^2 - \frac{1}{2}\int_0^b (A(t)x(t),x(t))_{\mathbb{R}^N}\,dt \\ &\quad - \frac{1}{2}\int_0^b \theta(t)\|x(t)\|^2\,dt - \frac{\varepsilon}{2}\,\|x\|_2^2 - c_1\|x\|^\tau\end{aligned} \tag{3.22}$$

for some $c_1 > 0$ (see (3.21)). For all $x \in W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N)$ let

$$\psi(x) = \|x'\|_2^2 - \int_0^b (A(t)x(t),x(t))_{\mathbb{R}^N}\,dt - \int_0^b \theta(t)\|x(t)\|^2\,dt.$$

We will show that

$$\psi(x) \geq \xi_3\|x\|^2 \quad \text{for some } \xi_3 > 0 \text{ and all } x \in W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N). \tag{3.23}$$

By virtue of the hypothesis $\mathrm{H(j)}_2$(d) on θ, we have $\psi \geq 0$. Due to the 2-homogeneity of ψ, arguing by contradiction, we can find a sequence $\{x_n\}_{n\geq 1} \subset W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N)$ such that

$$\psi(x_n) \downarrow 0 \quad \text{in } W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N) \text{ with } \|x_n\| = 1 \text{ for all } n \geq 1.$$

We may assume that

$$x_n \xrightarrow{\mathrm{w}} x \quad \text{in } W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N) \quad \text{and} \quad x_n \to x \quad \text{in } C(T,\mathbb{R}^N).$$

So in the limit as $n \to \infty$, from $\psi(x_n) \downarrow 0$, we obtain

$$\|x'\|_2^2 - \int_0^b (A(t)x(t),x(t))_{\mathbb{R}^N}\,dt - \int_0^b \theta(t)\|x(t)\|^2\,dt \leq 0, \tag{3.24}$$

which implies that

$$\|x'\|_2^2 - \int_0^b (A(t)x(t),x(t))_{\mathbb{R}^N}\,dt \leq \int_0^b \theta(t)\|x(t)\|^2\,dt \leq \lambda_1\|x\|_2^2,$$

and so

$$\|x'\|_2^2 - \int_0^b (A(t)x(t),x(t))_{\mathbb{R}^N}\,dt = \lambda_1\|x\|_2^2.$$

Hence $x \in C^1_{\mathrm{per}}(T, \mathbb{R}^N)$ is an eigenfunction corresponding to the eigenvalue $\lambda_1 > 0$. This means that $x(t) \neq 0$ a.e. on T and so from (3.24) we infer that

$$\|x'\|_2^2 - \int_0^b (A(t)x(t), x(t))_{\mathbb{R}^N}\, dt < \lambda_1 \|x\|_2^2.$$

which is a contradiction. This proves (3.23). Using (3.23) in (3.22) and choosing $\varepsilon > 0$ sufficiently small we obtain

$$\varphi(x) \geq \frac{\xi_3}{2}\, \|x\|^2 - \frac{\varepsilon}{2}\, \|x\|_2^2 - c_1 \|x\|^\tau \geq c_2 \|x\|^2 - c_1 \|x\|^\tau$$

for some $c_2 > 0$. Because $\tau > 2$, we can find $\rho > 0$ small such that

$$\varphi(x) \geq \beta > 0 \quad \text{for all } x \in W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N) \text{ with } \|x\| = \rho.$$

Claim 3. For almost all $t \in T$, all $x \in \mathbb{R}^N$ and all $r \geq 1$ we have

$$r^\mu j(t,x) \leq j(t, rx).$$

On $\mathbb{R}_+ \setminus \{0\}$ the function $r \mapsto 1/r^\mu$ is continuous convex, hence locally Lipschitz. So for almost all $t \in T$, $r \mapsto (1/r^\mu) j(t, rx)$ is locally Lipschitz and

$$\partial_r \bigg(\frac{1}{r^\mu}\, j(t, rx) \bigg) \subseteq -\frac{\mu}{r^{\mu+1}}\, j(t, rx) + \frac{1}{r^\mu} (\partial j(t, rx), x)_{\mathbb{R}^N}$$

(see [4, p. 48], or [5, p. 612]). In the above inclusion by ∂_r we denote the subdifferential with respect to $r \geq 1$. Using the mean value theorem for locally Lipschitz functions, we can find $\lambda \in (1, r)$ (depending in general on t) with $r > 1$ such that

$$\frac{1}{r^\mu}\, j(t, rx) - j(t, x) = \frac{r-1}{\lambda^{\mu+1}}\, (-\mu j(t, \lambda x) + (u^*, \lambda x)_{\mathbb{R}^N})$$

where $u^* \in \partial j(t, \lambda x)$. Because of hypothesis $\mathrm{H(j)}_2$(d) we have that

$$-\mu j(t, \lambda x) + (u^*, \lambda x)_{\mathbb{R}^N} \geq -\mu j(t, \lambda x) - j^0(t, \lambda x; -\lambda x)) \geq 0$$

for almost all $t \in T$, which implies that $j(t, rx) \geq r^\mu j(t, x)$ for almost all $t \in T$, all $x \in \mathbb{R}^N$ and all $r \geq 1$. This proves Claim 3.

Using the definition of φ and Claim 3 for $r \geq 1$ we have

$$\varphi(rx_0) \leq -\frac{r^2}{2} \int_0^b (A(t)x_0(t), x_0(t))_{\mathbb{R}^N}\, dt - r^\mu \int_0^b j(t, x_0)\, dt. \tag{3.25}$$

By $\mathrm{H(j)}_2$(f), (3.25) and since $\mu > 2$ it follows that

$$\varphi(rx_0) \to -\infty \quad \text{as } r \to \infty.$$

So for $r \geq 1$ large, we will have

$$\varphi(rx_0) \leq \varphi(0) = 0 < \beta \leq \inf\{\varphi(x) : \|x\| = \rho\}$$

(see Claim 2). This and Claim 1 permit the use of Theorem 2.1. So we obtain $x \in W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N)$ such that

$$\varphi(0) = 0 < \beta \le \varphi(x) \quad \text{and} \quad 0 \in \partial\varphi(x).$$

From the inequality we see that $x \ne 0$, while from the inclusion, as in the proof of Theorem 3.5, we conclude that $x \in C^1(T,\mathbb{R}^N)$ is a solution of problem (1.1). $\square$

In the case of Theorem 3.7 the kernel of the linear differential operator $x \mapsto -x'' - \widehat{A}x$ with periodic boundary conditions is trivial. This convenient situation allowed us to incorporate in our framework both subquadratic and superquadratic systems. In the next existence theorem, we still require that $\dim H_- = 0$, but now $\dim H_0 > 0$ (i.e. the linear differential operator $x \mapsto -x'' - \widehat{A}x$ with periodic boundary conditions has a nontrivial kernel). Now our hypotheses on $j(t,x)$ incorporate in our setting quadratic or superquadratic systems.

The hypotheses on the nonsmooth potential are the following:

H(j)$_3$ $j\colon T \times \mathbb{R}^N \to \mathbb{R}$ is a function such that $j(t,0) = 0$ a.e. on T and

(a) for all $x \in \mathbb{R}^N$, $t \mapsto j(t,x)$ is measurable,
(b) for almost all $t \in T$, $x \mapsto j(t,x)$ is locally Lipschitz,
(c) for almost all $t \in T$, all $x \in \mathbb{R}^N$ and all $u \in \partial j(t,x)$, we have

$$\|u\| \le a(t) + c(t)\|x\|^{r-1} \quad \text{with } a, c \in L^1(T)_+,\ 1 \le r < \infty,$$

(d) there exists $\theta \in L^1(T)_+$ such that $\theta(t) \le 0$ a.e. on T with strict inequality on a set of positive measure and

$$\limsup_{\|x\|\to\infty} \frac{2j(t,x)}{\|x\|^2} \le \theta(t) \quad \text{uniformly for almost all } t \in T,$$

(e) there exists $x_0 \in H_0$ such that $\int_0^b j(t,x_0(t))\,dt > 0$.

Example 3.8. The following locally Lipschitz function satisfies hypotheses H(j)$_3$:

$$j(t,x) = \begin{cases} \dfrac{1}{r}\,\|x\|^r & \text{if } \|x\| \le 1, \\[2mm] \dfrac{\theta(t)}{2}\,\|x\|^2 - \dfrac{\theta(t)}{2} + \dfrac{1}{r} & \text{if } \|x\| > 1, \end{cases}$$

with $\theta \in L^1(T)_+$ as in hypothesis H(j)$_3$(d) and $1 \le r < \infty$. Here to verify H(j)$_3$(e) we choose $x_0 \in H_0$ with $0 < \|x_0\| < 1$.

Theorem 3.9. *If hypotheses* H(A) *and* $\mathrm{H(j)}_3$ *hold and* $\dim H_- = 0$*, then problem* (1.1) *has a nontrivial solution* $x \in C^1(T,\mathbb{R}^N)$.

Proof. We consider the locally Lipschitz Euler functional φ. We claim that φ is coercive. We argue indirectly. So suppose the claim is not true. Then we can find $\{x_n\}_{n\geq 1} \subset W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N)$ such that

$$\varphi(x_n) \leq M_1 \quad \text{for some } M_1 > 0, \text{ all } n \geq 1 \text{ and } \|x_n\| \to \infty.$$

We have $x_n = x_n^0 + \widehat{x}_n$ with $x_n^0 \in H_0$, $\widehat{x}_n \in H_+$, $n \geq 1$.

First assume that

$$\frac{\|\widehat{x}_n\|}{\|x_n\|} \to 0 \quad \text{as } n \to \infty. \tag{3.26}$$

Set $y_n = x_n/\|x_n\|$, $n \geq 1$. We may assume that

$$y_n \xrightarrow{\mathrm{w}} y \quad \text{in } W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N) \quad \text{and} \quad y_n \to y \quad \text{in } C(T,\mathbb{R}^N), \text{ with } y \in H_0.$$

Then we have

$$\frac{\varphi(x_n)}{\|x_n\|^2} = \frac{1}{2}\,\|y_n'\|_2^2 - \frac{1}{2}\int_0^b (Ay_n(t),y_n(t))_{\mathbb{R}^N}\,dt - \int_0^b \frac{j(t,x_n(t))}{\|x_n\|^2}\,dt \leq \frac{M_1}{\|x_n\|^2},$$

which implies by passing to the limit that

$$\frac{1}{2}\,\|y'\|_2^2 - \frac{1}{2}\int_0^b (Ay(t),y(t))_{\mathbb{R}^N}\,dt \leq \liminf_{n\to\infty} \int_0^b \frac{j(t,x_n(t))}{\|x_n\|^2}\,dt.$$

Since $\|y'\|_2^2 = \int_0^b (A(t)y(t),y(t))_{\mathbb{R}^N}\,dt$ (because $y \in H_0$), we obtain

$$0 \leq \liminf_{n\to\infty} \int_0^b \frac{j(t,x_n(t))}{\|x_n\|^2}\,dt. \tag{3.27}$$

Claim.

$$\limsup_{n\to\infty} \int_0^b \frac{j(t,x_n(t))}{\|x_n\|^2}\,dt < 0.$$

From Tang–Wu [18] (see the proof of Lemma 3 in that paper), we know that given $\varepsilon_1 > 0$, we can find $m_1 = m_1(\varepsilon_1) > 0$ such that

$$|\{t \in T : \|v(t)\| < m_1\|v\|\}|_1 < \varepsilon_1 \quad \text{for all } v \in H_0 \setminus \{0\}$$

(by $|\cdot|_1$ we denote the Lebesgue measure on $\mathbb{R}$). In a similar way, we can show that given $\varepsilon_2 > 0$, we can find $m_2 = m_2(\varepsilon_2) > 0$ such that

$$|\{t \in T : \|w(t)\| > m_2\|w\|\}|_1 < \varepsilon_2 \quad \text{for all } w \in H_+ \setminus \{0\}$$

(see also Bartolo–Benci–Fortunato [1]). For every $n \geq 1$, we introduce the sets

$$\begin{aligned} E_{1n} &= \{t \in T : \|x_n^0(t)\| \geq m_1\|x_n^0\|\}, \\ E_{2n} &= \{t \in T : \|w_n(t)\| \leq m_2\|w_n\|\}. \end{aligned}$$

The convergence in (3.26) ensures that $x_n^0 \neq 0$ for n sufficiently large.

Clearly we have $|T\setminus E_{1n}|<\varepsilon_1$ and $|T\setminus E_{2n}|<\varepsilon_2$. If $\varepsilon_1+\varepsilon_2<b$ and because $(T\setminus E_{1n})\cup(T\setminus E_{2n})=T\setminus(E_{1n}\cap E_{2n})$, we infer that $E_{1n}\cap E_{2n}\neq\emptyset$. For all $n\geq 1$ and all $t\in E_{1n}\cap E_{2n}$, we have

$$\frac{\|x_n(t)\|}{\|x_n\|}\geq\frac{\|x_n^0(t)\|}{\|x_n\|}-\frac{\|w_n(t)\|}{\|x_n\|}\geq\frac{m_1\|x_n^0\|}{\|x_n\|}-\frac{m_2\|w_n\|}{\|x_n\|}. \tag{3.28}$$

Similarly for all $n\geq 1$ and all $t\in E_{1n}\cap E_{2n}$, using (3.1) it follows

$$\frac{\|x_n(t)\|}{\|x_n\|}\leq\frac{\|w_n(t)\|}{\|x_n\|}+\frac{\|x_n^0(t)\|}{\|x_n\|}\leq\frac{m_2\|w_n\|}{\|x_n\|}+\frac{C\|x_n^0\|}{\|x_n\|}. \tag{3.29}$$

By virtue of hypothesis $\mathrm{H(j)}_3$(c) and (d) and the mean value theorem for locally Lipschitz functions (see [4, p. 41] or [5, p. 609]), given $\varepsilon>0$ we can find $\xi_\varepsilon\in L^1(T)_+$ such that for almost all $t\in T$ and all $x\in\mathbb{R}^N$, we have

$$j(t,x)\leq\frac{1}{2}(\theta(t)+\varepsilon)\|x\|^2+\xi_\varepsilon(t). \tag{3.30}$$

Then for all $n\geq 1$, we have from (3.30) that

$$\begin{aligned}\int_{E_{1n}\cap E_{2n}}&\frac{j(t,x_n(t))}{\|x_n\|^2}\,dt\\&\leq\frac{1}{2}\int_{E_{1n}\cap E_{2n}}(\theta(t)+\varepsilon)\frac{\|x_n(t)\|^2}{\|x_n\|^2}\,dt+\frac{1}{\|x_n\|^2}\int_{E_{1n}\cap E_{2n}}\xi_\varepsilon(t)\,dt\\&\leq\frac{1}{2}\int_{E_{1n}\cap E_{2n}}\theta(t)\frac{\|x_n(t)\|^2}{\|x_n\|^2}\,dt+\frac{\varepsilon}{2}\int_{E_{1n}\cap E_{2n}}\frac{\|x_n(t)\|^2}{\|x_n\|^2}\,dt+\frac{\|\xi_\varepsilon\|_1}{\|x_n\|^2}.\end{aligned}$$

Recall that $\theta(t)\leq 0$ a.e. on T (see hypothesis $\mathrm{H(j)}_3$(d)). So using (3.28) and (3.29), we find

$$\begin{aligned}\int_{E_{1n}\cap E_{2n}}\frac{j(t,x_n(t))}{\|x_n\|^2}\,dt\leq{}&\frac{1}{2}\left(\frac{m_1\|x_n^0\|}{\|x_n\|}-\frac{m_2\|w_n\|}{\|x_n\|}\right)^2\int_{E_{1n}\cap E_{2n}}\theta(t)\,dt\\&+\frac{\varepsilon}{2}\left(\frac{m_2\|w_n\|}{\|x_n\|}+\frac{C\|x_n^0\|}{\|x_n\|}\right)^2|E_{1n}\cap E_{2n}|_1+\frac{\|\xi_\varepsilon\|_1}{\|x_n\|^2}.\end{aligned}$$

From (3.26) and taking into account that $\|x_n^0\|/\|x_n\|\to 1$ and $\|x_n\|\to\infty$, we can find $n_0=n_0(\varepsilon)\geq 1$ such that for all $n\geq n_0$ we have that

$$\int_{E_{1n}\cap E_{2n}}\frac{j(t,x_n(t))}{\|x_n\|^2}\,dt\leq\frac{1}{2}\,(m_1-\varepsilon)^2\int_{E_{1n}\cap E_{2n}}\theta(t)\,dt+\frac{\varepsilon}{2}(C+\varepsilon)^2b+\varepsilon. \tag{3.31}$$

Note that

$$\int_{E_{1n}\cap E_{2n}}\theta(t)\,dt=\int_0^b\theta(t)\,dt-\int_{(T\setminus E_{1n})\cup(T\setminus E_{2n})}\theta(t)\,dt. \tag{3.32}$$

Since $\theta\in L^\infty(T)$, $\theta(t)\leq 0$ a.e. on T and $|(T\setminus E_{1n})\cup(T\setminus E_{2n})|_1<\varepsilon_1+\varepsilon_2$, we have

$$-\int_{(T\setminus E_{1n})\cup(T\setminus E_{2n})}\theta(t)\,dt\leq\|\theta\|_\infty\,|(T\setminus E_{1n})\cup(T\setminus E_{2n})|_1<\|\theta\|_\infty(\varepsilon_1+\varepsilon_2). \tag{3.33}$$

Therefore using (3.32) and (3.33) in (3.31), for all $n \geq n_0$ it turns out

$$\int_{E_{1n}\cap E_{2n}} \frac{j(t,x_n(t))}{\|x_n\|^2}\,dt \leq \frac{1}{2}\,(m_1-\varepsilon)^2 \int_0^b \theta(t)\,dt + \frac{1}{2}\,(m_1-\varepsilon)^2\|\theta\|_\infty(\varepsilon_1+\varepsilon_2) + \frac{\varepsilon}{2}(C+\varepsilon)^2 b + \varepsilon. \tag{3.34}$$

On the other hand, for all $n \geq 1$ and all $t \in E_{2n} \setminus E_{1n}$, we have

$$\frac{\|x_n(t)\|}{\|x_n\|} \leq \frac{\|x_n^0(t)\|}{\|x_n\|} + \frac{\|w_n(t)\|}{\|x_n\|} < \frac{m_1\|x_n^0\|}{\|x_n\|} + \frac{m_2\|w_n\|}{\|x_n\|}. \tag{3.35}$$

Moreover. by (3.30) we can write for all $n \geq 1$

$$\int_{E_{2n}\setminus E_{1n}} \frac{j(t,x_n(t))}{\|x_n\|^2}\,dt \leq \frac{\|\xi_\varepsilon\|_1}{\|x_n\|^2} + \frac{\varepsilon}{2}\int_{E_{2n}\setminus E_{1n}} \frac{\|x_n(t)\|^2}{\|x_n\|^2}\,dt.$$

Because of (3.35), we see that

$$\int_{E_{2n}\setminus E_{1n}} \frac{j(t,x_n(t))}{\|x_n\|^2}\,dt < \frac{\|\xi_\varepsilon\|_1}{\|x_n\|^2} + \frac{\varepsilon}{2}\left(\frac{m_1\|x_n^0\|}{\|x_n\|} + \frac{m_2\|w_n\|}{\|x_n\|}\right)^2 |E_{2n}\setminus E_{1n}|_1.$$

Again from (3.26), $\|x_n^0\|/\|x_n\| \to 1$ and $\|x_n\| \to \infty$, we can find $n_1 = n_1(\varepsilon) \geq 1$ such that for all $n \geq n_1$ we have

$$\begin{aligned}\int_{E_{2n}\setminus E_{1n}} \frac{j(t,x_n(t))}{\|x_n\|^2}\,dt &\leq \varepsilon + (m_1+\varepsilon)^2\,\frac{\varepsilon}{2}\,|E_{2n}\setminus E_{1n}|_1 \\ &\leq \varepsilon + (m_1+\varepsilon)^2\,\frac{\varepsilon}{2}\,|T\setminus E_{1n}|_1 \leq \varepsilon + (m_1+\varepsilon)^2\,\frac{\varepsilon}{2}\,\varepsilon_1.\end{aligned} \tag{3.36}$$

Finally for all $n \geq 1$, we have from (3.30) and (3.1) that

$$\begin{aligned}\int_{T\setminus E_{2n}} \frac{j(t,x_n(t))}{\|x_n\|^2}\,dt &\leq \frac{\|\xi_\varepsilon\|_1}{\|x_n\|^2} + \frac{\varepsilon}{2}\int_{T\setminus E_{2n}} \frac{\|x_n(t)\|^2}{\|x_n\|^2}\,dt \\ &\leq \frac{\|\xi_\varepsilon\|_1}{\|x_n\|^2} + \frac{\varepsilon}{2}\int_{T\setminus E_{2n}} \frac{\|x_n^0(t)\|^2 + \|w_n(t)\|^2}{\|x_n\|^2}\,dt \\ &\leq \frac{\|\xi_\varepsilon\|_1}{\|x_n\|^2} + \frac{\varepsilon}{2}\,C^2\,|T\setminus E_{2n}|_1 + \frac{\varepsilon}{2}\,\frac{\|w_n\|_2^2}{\|x_n\|^2} \\ &\leq \frac{\|\xi_\varepsilon\|_1}{\|x_n\|^2} + \frac{\varepsilon}{2}\,C^2\,\varepsilon_2 + \frac{\varepsilon}{2}\,\frac{\|w_n\|^2}{\|x_n\|^2}.\end{aligned}$$

Because of (3.26), we see that we can find $n_2 = n_2(\varepsilon) \geq 1$ such that for all $n \geq n_2 \geq 1$ we have

$$\int_{T\setminus E_{2n}} \frac{j(t,x_n(t))}{\|x_n\|^2}\,dt \leq \varepsilon + \frac{\varepsilon}{2}\,C^2\,\varepsilon_2. \tag{3.37}$$

From (3.34), (3.36) and (3.37), we see that for $n \geq n_3 = \max\{n_1, n_2, n_3\}$ we have

$$\int_0^b \frac{j(t, x_n(t))}{\|x_n\|^2}\, dt \leq \frac{1}{2}\,(m_1 - \varepsilon)^2 \int_0^b \theta(t)\, dt + \frac{1}{2}\,(m_1 - \varepsilon)^2 \|\theta\|_\infty (\varepsilon_1 + \varepsilon_2) + \frac{\varepsilon}{2}(C + \varepsilon)^2 b + (m_1 + \varepsilon)^2 \frac{\varepsilon}{2} \varepsilon_1 + \frac{\varepsilon}{2}\, C^2\, \varepsilon_2 + 3\varepsilon.$$

Recall that $\varepsilon, \varepsilon_2 > 0$ were arbitrary. So we let $\varepsilon, \varepsilon_2 \downarrow 0$. It follows that

$$\limsup_{n\to\infty} \int_0^b \frac{j(t, x_n(t))}{\|x_n\|^2}\, dt \leq \frac{1}{2}\, m_1^2 \int_0^b \theta(t)\, dt + \frac{1}{2}\, m_1^2 \|\theta\|_\infty \varepsilon_1. \tag{3.38}$$

Because $\varepsilon_1 > 0$ was arbitrary and $\int_0^b \theta(t)\, dt < 0$, we choose $\varepsilon_1 > 0$ small enough so that

$$\varepsilon_1 \|\theta\|_\infty < -\int_0^b \theta(t)\, dt.$$

Then from (3.38) we conclude that the claim is true when $w_n \neq 0$ for n sufficiently large as admitted.

Comparing the claim with (3.27), we reach a contradiction.

Next assume that

$$\frac{\|\widehat{x}_n\|}{\|x_n\|} \to \eta \in (0, 1] \quad \text{as } n \to \infty.$$

Then $\|y_n^0\|^2 \to 1 - \eta^2$. Because of the orthogonality of the component spaces H_0 and H_+ and the fact that $\theta \leq 0$, we have

$$\varphi(x_n) \geq \frac{1}{2}\, \|\widehat{x}_n'\|_2^2 - \frac{1}{2} \int_0^b (A(t)\widehat{x}_n(t), \widehat{x}_n(t))_{\mathbb{R}^N}\, dt - \frac{\varepsilon}{2}\, \|x_n\|^2 - \|\xi_\varepsilon\|_1$$

(see (3.30)), which implies that

$$\begin{aligned} \frac{M_1}{\|x_n\|^2} &\geq \frac{1}{2}\, \|\widehat{y}_n'\|_2^2 - \frac{1}{2} \int_0^b (A(t)\widehat{y}_n(t), \widehat{y}_n(t))_{\mathbb{R}^N}\, dt - \frac{\varepsilon}{2}\, \|y_n\|^2 - \frac{\|\xi_\varepsilon\|_1}{\|x_n\|^2} \\ &\geq \frac{\xi_1 - \varepsilon}{2}\, \|\widehat{y}_n\|^2 - \frac{\varepsilon}{2}\, \|y_n^0\|^2 - \frac{\|\xi_\varepsilon\|_1}{\|x_n\|^2} \end{aligned}$$

(ξ_1 is the first positive eigenvalue). Passing to the limit as $n \to \infty$, we obtain

$$0 \geq \frac{\xi_1 - \varepsilon}{2}\, \eta^2 - \frac{\varepsilon}{2}\, (1 - \eta^2) = \frac{\xi_1}{2}\, \eta^2 - \frac{\varepsilon}{2}.$$

Choose $\varepsilon < \xi_1 \eta^2$ to reach a contradiction.

Therefore we have proved that φ is coercive. Also it is easy to see that it is weakly lower semicontinuous on $W^{1,2}_{\text{per}}((0, b), \mathbb{R}^N)$. Involving the Weierstrass theorem, we can find $x \in W^{1,2}_{\text{per}}((0, b), \mathbb{R}^N)$ such that

$$\varphi(x) = \inf \varphi$$

and thus $0 \in \partial\varphi(x)$. From the inclusion we infer that $x \in C^1(T, \mathbb{R}^N)$ is a solution of problem (1.1). Moreover, if $x_0 \in H_0$ is as in hypothesis $\mathrm{H(j)}_3$(e), then

$$\varphi(x_0) = -\int_0^b j(t, x_0(t))\, dt < 0$$

and so $\varphi(x) \leq \varphi(x_0) < 0 = \varphi(0)$, hence $x \neq 0$. □

Next we pass to multiplicity results. For the first multiplicity result we require that $\dim H_- = 0$ and we impose the following conditions on the nonsmooth potential $j(t, x)$:

$\mathrm{H(j)}_4$ $j\colon T \times \mathbb{R}^N \to \mathbb{R}$ is a function such that $j(t, 0) = 0$ a.e. on T and

(a) for all $x \in \mathbb{R}^N$, $t \mapsto j(t, x)$ is measurable,

(b) for almost all $t \in T$, $x \mapsto j(t, x)$ is locally Lipschitz,

(c) for almost all $t \in T$, all $x \in \mathbb{R}^N$ and all $u \in \partial j(t, x)$, we have

$$\|u\| \leq a(t) + c(t)\|x\|^{r-1} \quad \text{with } a, c \in L^1(T)_+,\ 1 \leq r < \infty,$$

(d) there exists $\theta \in L^\infty(T)$ such that $\theta(t) \leq 0$ a.e. on T with strict inequality on a set of positive measure and

$$\limsup_{\|x\| \to \infty} \frac{2j(t, x)}{\|x\|^2} \leq \theta(t) \quad \text{uniformly for almost all } t \in T,$$

(e) there exists $\eta \in L^\infty(T)_+$ such that if $\lambda_m > 0$ is the first positive eigenvalue of $x \mapsto -x'' - \widehat{A}x$, then $\eta(t) \leq \lambda_m$ a.e. on T with strict inequality on a set of positive measure and

$$\limsup_{\|x\| \to 0} \frac{2j(t, x)}{\|x\|^2} \leq \eta(t) \quad \text{uniformly for almost all } t \in T,$$

(f) there exists $\delta > 0$ such that for almost all $t \in T$ and all $\|x\| \leq \delta$, we have $j(t, x) \geq 0$.

Example 3.10. The following nonsmooth locally Lipschitz function satisfies hypotheses $\mathrm{H(j)}_4$:

$$j(t, x) = \begin{cases} \dfrac{\eta(t)}{r}\,\|x\|^r & \text{if } \|x\| \leq 1, \\[2mm] \dfrac{\theta(t)}{2}\,\|x\|^2 - \dfrac{\theta(t)}{2} + \dfrac{\eta(t)}{r} & \text{if } \|x\| > 1, \end{cases}$$

with $2 \leq r < \infty$ and with $\theta, \eta \in L^\infty(T)$ as in hypotheses $\mathrm{H(j)}_4$(d) and (e), respectively.

Due to hypothesis $\mathrm{H(j)}_4$(d), the next multiplicity result applies to quadratic or superquadratic systems.

Theorem 3.11. *If hypotheses* H(A) *and* $\mathrm{H(j)}_4$ *hold and* $\dim H_- = 0$, *then problem* (1.1) *has at least two nontrivial solutions* $x_1, x_2 \in C^1(T, \mathbb{R}^N)$.

Proof. As in the proof of Theorem 3.9 we can check that φ is coercive. Hence it is bounded below and satisfies the nonsmooth PS-condition. Also because of hypothesis $\mathrm{H(j)}_4$(e), given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon)$ such that

$$j(t, x) \le \frac{1}{2}\,(\eta(t) + \varepsilon)\|x\|^2 \tag{3.39}$$

for almost all $t \in T$ and all $\|x\| \le \delta$. In addition from hypothesis $\mathrm{H(j)}_4$(c) and the mean value theorem for locally Lipschitz functions, we have

$$|j(t, x)| \le \beta_\varepsilon(t)\|x\|^\tau \tag{3.40}$$

for almost all $t \in T$, all $\|x\| \ge \delta$ and with $\beta_\varepsilon \in L^1(T)_+$, $\tau > 2$. So from (3.39) and (3.40) it follows that

$$j(t, x) \le \frac{1}{2}\,(\eta(t) + \varepsilon)\|x\|^2 + \beta_\varepsilon(t)\|x\|^\tau \quad \text{for a.a. } t \in T \text{ and all } x \in \mathbb{R}^N. \tag{3.41}$$

In view of (4.41), for $x \in H_+$ we have

$$\begin{aligned} \varphi(x) &= \frac{1}{2}\,\|x'\|_2^2 - \frac{1}{2}\int_0^b (A(t)x(t), x(t))_{\mathbb{R}^N}\,dt - \int_0^b j(t, x(t))\,dt \\ &\ge \frac{1}{2}\,\|x'\|^2 - \frac{1}{2}\int_0^b (A(t)x(t), x(t))_{\mathbb{R}^N}\,dt \\ &\quad - \frac{1}{2}\int_0^b \eta(t)\|x(t)\|^2\,dt - \frac{\varepsilon}{2}\,\|x\|_2^2 - c_1\|x\|^\tau \end{aligned} \tag{3.42}$$

for some $c_1 > 0$ (see also (3.1)). As we did for (3.23), we can show that there is $c_2 > 0$ such that

$$\|x'\|_2^2 - \int_0^b (A(t)x(t), x(t))_{\mathbb{R}^N}\,dt - \int_0^b \eta(t)\|x(t)\|^2\,dt \ge c_2\|x\|^2$$

for all $x \in H_+$. Using this in (3.42), we obtain

$$\varphi(x) \ge \frac{1}{2}\,(c_2 - \varepsilon)\|x\|^2 - c_1\|x\|^\tau \quad \text{for all } x \in H_+.$$

Choose $\varepsilon < c_2$. Because $\tau > 2$, if we choose $\rho_1 > 0$ small we will have

$$\varphi(x) \ge 0 \quad \text{for all } x \in H_+, \quad \|x\| \le \rho_1.$$

Since H_0 is finite dimensional, all norms are equivalent and so we can find $0 < \rho \le \rho_1$ such that if $x^0 \in H_0$ with $\|x^0\| \le \rho$, then $\|x^0\|_\infty \le \delta$. Thus by virtue of hypothesis H(j)$_4$(f) we have $j(t,x^0(t)) \geq 0$ for a.e. $t \in T$. So we have

$$\begin{aligned}\varphi(x^0) &= \frac{1}{2}\,\|(x^0)'\|_2^2 - \frac{1}{2}\int_0^b (A(t)x^0(t),x^0(t))_{\mathbb{R}^N}\,dt - \int_0^b j(t,x^0(t))\,dt \\ &= -\int_0^b j(t,x^0(t))\,dt \leq 0 \quad \text{for } x^0 \in H_0 \text{ with } \|x^0\| \leq \rho.\end{aligned}$$

If $\inf\varphi = 0$, then all $x^0 \in H_0 \setminus \{0\}$ with $\|x^0\| \leq \rho$ are critical points of φ, hence solutions of problem (1.1). If $\inf\varphi < 0 = \varphi(0)$, then we can apply Theorem 2.3 and obtain $x_1, x_2 \in W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N)$ with $x_1, x_2 \neq 0$ such that $0 \in \partial\varphi(x_i)$, $i = 1, 2$. These are two nontrivial solutions of problem (1.1). $\square$

As for the existence theory, if we assume that $\dim(H_- \oplus H_0) = 0$ and we impose a symmetry condition on the potential $j(t,\cdot)$, we can strengthen our conclusion and produce a whole infinite sequence of solutions. The precise hypotheses on the nonsmooth potential $j(t,x)$ are the following:

H(j)$_5$ $j\colon T \times \mathbb{R}^N \to \mathbb{R}$ is a function such that $j(t,0) = 0$ a.e. on T and

(a) for all $x \in \mathbb{R}^N$, $t \mapsto j(t,x)$ is measurable,
(b) for almost all $t \in T$, $x \mapsto j(t,x)$ is locally Lipschitz and even,
(c) for almost all $t \in T$, all $x \in \mathbb{R}^N$ and all $u \in \partial j(t,x)$, we have

$$\|u\| \leq a(t) + c(t)\|x\|^{r-1} \quad \text{with } a, c \in L^1(T)_+,\ 1 \leq r < \infty,$$

(d) there exist $\mu > 2$, $M > 0$ and $\beta \in L^\infty(T)_+$, $\beta \neq 0$ a.e. on T such that

$$\beta(t)\|x\|^\mu \leq \mu j(t,x) \leq -j^0(t,x;-x) \quad \text{for a.a. } t \in T \text{ and all } \|x\| \geq M,$$

(e) there exists $\theta \in L^\infty(T)_+$ such that if $\lambda_1 > 0$ is the first eigenvalue of $x \mapsto -x'' - \widehat{A}x$, we have $\theta(t) \leq \lambda_1$ a.e. on T with strict inequality on a set of positive measure and

$$\limsup_{\|x\|\to 0} \frac{2j(t,x)}{\|x\|^2} \leq \theta(t) \quad \text{uniformly for almost all } t \in T.$$

Example 3.12. Because of hypothesis H(j)$_5$(d), this multiplicity result applies to strictly superquadratic systems. The following nonsmooth locally Lipschitz function $j(t,x)$ satisfies hypotheses H(j)$_5$:

$$j(t,x) = \begin{cases} \dfrac{\theta(t)}{2}\,\|x\|^2 & \text{if } \|x\| \leq 1, \\[2mm] \dfrac{1}{\mu}\,\|x\|^\mu + \dfrac{\theta(t)}{2} - \dfrac{1}{\mu} & \text{if } \|x\| > 1, \end{cases}$$

where $\mu > 2$ and $\theta \in L^\infty(T)_+$ as in hypotheses H(j)$_5$(e) with $\theta(t) \leq 2/\mu$.

We have the following multiplicity result.

Theorem 3.13. *If hypotheses* H(A) *and* $\mathrm{H(j)}_5$ *hold and* $\dim(H_- \oplus H_0) = 0$, *then problem* (1.1) *has infinitely many distinct pairs* $(x, -x)$, $x \in C^1(T, \mathbb{R}^N)$ *of solutions.*

Proof. From the proof of Theorem 3.7 we know that the locally Lipschitz Euler functional φ satisfies the nonsmooth PS-condition.

Let $Y \subset W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N)$ be a finite dimensional subspace and let $\gamma > M$, where $M > 0$ is as in hypothesis $\mathrm{H(j)}_5$(d). Set $\beta_1 = \beta/mu$ and

$$\xi_0 = \inf\bigg\{\int_{\{\|y(t)\|>M\}} \beta_1(t)\|y(t)\|^{\mu}\,dt : y \in Y,\ \|y\|_\infty = \gamma\bigg\} > 0.$$

Because Y is finite dimensional, all the norms are equivalent. So for all $y \in Y$ we can write

$$\begin{aligned}
\varphi(y) &= \frac{1}{2}\,\|y'\|_2^2 - \frac{1}{2}\int_0^b (Ay(t), y(t))_{\mathbb{R}^N}\,dt - \int_0^b j(t, y(t))\,dt \\
&\leq \eta\|y\|_\infty^2 - \int_0^b j(t, y(t))\,dt \\
&= \eta\|y\|_\infty^2 - \int_{\{\|y(t)>M\|\}} j(t, y(t))\,dt - \int_{\{\|y(t)\leq M\|\}} j(t, y(t))\,dt \\
&\leq \eta\|y\|_\infty^2 - \int_{\{\|y(t)>M\|\}} \beta_1(t)\|y(t)\|^2\,dt + \eta_1
\end{aligned} \tag{3.43}$$

for some $\eta > 0$ and $\eta_1 > 0$ (see hypotheses $\mathrm{H(j)}_5$(c) and (d)). Since we will send $\|y\|$ to ∞, then $\|y\|_\infty$ will also go to ∞ and so we may assume that $\|y\|_\infty > \gamma$. So

$$\{t \in T : \|y(t)\| > M\} \supseteq \bigg\{t \in T : \frac{\gamma}{\|y\|_\infty}\|y(t)\| > M\bigg\}$$

and we have

$$\begin{aligned}
-\int_{\{\|y(t)>M\|\}} \beta_1(t)\|y(t)\|^{\mu}\,dt &\leq -\int_{\{\frac{\gamma}{\|y\|_\infty}\|y(t)\|>M\}} \beta_1(t)\|y(t)\|^{\mu}\,dt \\
&= -\frac{\|y\|_\infty^{\mu}}{\gamma^{\mu}}\int_{\{\frac{\gamma}{\|y\|_\infty}\|y(t)\|>M\}} \beta_1(t)\gamma^{\mu}\bigg(\frac{\|y(t)\|}{\|y\|_\infty}\bigg)^{\mu}\,dt \leq -\frac{\xi_0\|y\|_\infty^{\mu}}{\gamma^{\mu}}.
\end{aligned}$$

We use this estimate in (3.43). Hence we obtain

$$\varphi(y) \leq \eta\|y\|_\infty^2 - \frac{\xi_0}{\gamma^{\mu}}\|y\|_\infty^{\mu} + \eta_1.$$

Since $\mu > 2$ and the norms on Y are equivalent, it follows that

$$\varphi(y) \to -\infty \quad \text{as } \|y\| \to \infty,\ y \in Y.$$

On the other hand because of hypothesis $\mathrm{H(j)}_5$(e), as in the proof of Theorem 3.5 (Claim 2), we can find $\rho > 0$ small such that

$$\varphi(x) \geq \beta_0 > 0 \quad \text{for all } x \in W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N) \text{ with } \|x\| = \rho.$$

Therefore we can apply Theorem 2.4 and obtain a sequence $\{(x_n, -x_n)\}_{n\geq 1}$, $x_n \in W^{1,2}_{\rm per}((0,b),\mathbb{R}^N)$ of distinct critical points of φ. So $0 \in \partial\varphi(\pm x_n)$ for all $n \geq 1$ and from this we infer that $x_n \in C^1(T,\mathbb{R}^N)$ and the pairs $\{(x_n, -x_n)\}_{n\geq 1}$ are solutions of problem (1.1). $\square$

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