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UNIQUE GLOBAL SOLVABILITY OF THE FRIED–GURTIN MODEL FOR PHASE TRANSITIONS IN SOLIDS

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Abstract. The paper is concerned with the existence and uniqueness of solutions to the Allen–Cahn equation coupled with elasticity. The system represents a particular, simple version of the Fried–Gurtin model for solid-solid transitions with phase characterized by an order parameter.

The system is studied with the help of the Leray–Schauder fixed point theorem. The main tool applied in the existence proof is the solvability theory of parabolic problems in anisotropic Sobolev spaces with mixed time-space norms.

1. Introduction

In this paper we are concerned with the unique global solvability of a simple version of the Fried–Gurtin model for isothermal phase transitions in solids. The model results from a thermodynamical phase-field (diffused-interface) theory of solid-solid phase transitions developed by Fried and Gurtin (see [7]), and Fried and Grach (see [6]).

The problem which is considered here has the form of a coupled system of three-dimensional (3D) elasticity and the parabolic equation, known as the Allen–Cahn or Landau–Ginzburg equation, for a scalar order parameter. Under some physically justified assumptions on the elastic energy and the data we have proved the existence and uniqueness of a solution of the problem. The solution

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satisfies the elasticity system in a weak sense and the parabolic problem in the classical one.

The existence proof is based on the Leray–Schauder fixed point theorem. The main tool applied in the proof is the theory of maximal regularity for parabolic problems with inhomogeneous part belonging to $L_q(0,T;L_p(\Omega))$, due to Weidemaier (see [13]–[15]). This theory provides the solvability of parabolic problems in the anisotropic Sobolev space $W^{2,1}_{p,q}(\Omega^T)$.

We begin with a brief outline of the Fried–Gurtin model and the place of our study in the present theory of phase transition models. The Fried–Gurtin–Grach theory is based on balance laws of linear momentum and microforce with underlying free energy depending on deformation gradient, multicomponent order parameter and its gradient. The constitutive dependence on the order parameter and its gradient is in contrast to other well-known phase-field theories of solid-solid transitions due to Falk and Frémond (for references see e.g. the monograph [2] and the review [10]). In these theories the order parameter is identified with strain tensor, and the free energy is postulated to be a function of strain, strain gradient, and in nonisothermal situation also on temperature.

In Fried–Gurtin's theory the order parameter is not identified with the strain tensor but represents a new quantity which can have different physical status. In case of diffusive transitions it describes atomic arrangements within unit cells of crystal lattice. For pure martensitic transitions, in which the crystal lattice undergoes a machanical strain but there are no rearrangements of atoms within cells, the order parameter might be viewed as an artifice that yields a regularization of mechanical equations.

Such regularization turns out to the model the interfacial structure of phase boundaries. In [6], [7] it has been shown that, granted appriopriate scaling, the governing equations of the order-parameter based theory are asymptotic to governing equations in sharp-interface theory.

From the mathematical point of view the important difference between the theories is that in the case of Falk's and Frémond's theories stress tensor is a nonlinear function of strain whereas in Fried–Gurtin's theory this dependence is linear, the nonlinear effects are contained in the order parameter equation.

It should be pointed out that this order parameter equation generalizes to the case of deformable continua one of the central equations in materials science, namely the Allen–Cahn equation, referred also to as the Landau–Ginzburg equation, describing the ordering of atoms within unit cells on a crystal lattice.

There exists an extensive literature concerned with the mathematical analysis of Falk's and Frémond's models in 1D and 3D cases (see references in [2], [10], and [11]).

According to authors' knowledge the well-posedness of the Fried–Gurtin model has so far not been examined. In a special 1D case the model and its equilibrium solutions have been analyzed numerically in Sikora et al [12].

We present now the formulation of the Fried–Gurtin model in the special case of small strain approximation with strain represented by the linearized strain tensor $\varepsilon = \varepsilon(\mathbf{u})$ and an unconstrained scalar order parameter φ distinguishing between two phases, a and b, characterized by $\varphi = 0$ and $\varphi = 1$.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary S, occupied by a body in a fixed reference configuration. Let $\mathbf{n}$ denotes the unit outward normal to S.

Moreover, let $T > 0$ be an arbitrary fixed time, $\Omega^T = \Omega \times (0,T)$ and $S^T = S \times (0,T)$ denote the space-time cylinder and its lateral boundary.

The mechanical evolution of the body is described by the displacement field $\mathbf{u}\colon \Omega^T \to \mathbb{R}^3$ and the scalar order-parameter field $\varphi\colon \Omega^T \to \mathbb{R}$.

The free energy density f underlying the evolution of the body is assumed to be given as a function of the strain tensor $\varepsilon(\mathbf{u})$, order parameter φ, and its spatial gradient $\nabla\varphi$:

$$f = f(\varepsilon(\mathbf{u}), \varphi, \nabla\varphi).$$

The relevant Landau–Ginzburg separable form of f which is quadratic in $\varepsilon(\mathbf{u})$ and $\nabla\varphi$, and a nonlinear double-well function in φ, is specifield below.

The model has the form of a nonlinear coupled system of partial diffential equations representing linear momentum balance for the displacement (at constant mass density) and the relaxation law for the order parameter, with some prescribed initial and boundary conditions:

$$\begin{aligned} \mathbf{u}_{tt} - \nabla \cdot f_{/\varepsilon}(\varepsilon(\mathbf{u}), \varphi, \nabla\varphi) &= \mathbf{b} && \text{in } \Omega^T, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \mathbf{u}_t|_{t=0} &= \mathbf{u}_1 && \text{in } \Omega, \\ \mathbf{u} &= 0 && \text{on } S^T, \end{aligned} \tag{1.1}$$

$$\begin{aligned} \beta\varphi_t + f_{/\varphi}(\varepsilon(\mathbf{u}), \varphi, \nabla\varphi) - \nabla \cdot f_{/\nabla\varphi}(\varepsilon(\mathbf{u}), \varphi, \nabla\varphi) &= 0 && \text{in } \Omega^T, \\ \varphi|_{t=0} &= \varphi_0 && \text{in } \Omega, \\ \varphi &= 0 && \text{on } S^T. \end{aligned} \tag{1.2}$$

Here

$$\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$$

is the linearized strain tensor (upper index T denotes transposition), $\mathbf{b}\colon \Omega^T \to \mathbb{R}^3$ is an external body force, and β is a positive constant called dumping modulus (in general, β can depend on ε, φ, $\nabla\varphi$, φ_t). The functions $\mathbf{u}_0$, $\mathbf{u}_1$, φ_0 represent initial conditions for the displacement, the velocity and the order parameter.

For the sake of simplicity we consider the homogeneous Dirichlet boundary condition $(1.1)_3$ for the displacement assuming that the body is fixed at the boundary S. The results can be extended to other boundary conditions (see Section 5).

The homogeneous Dirichlet boundary condition $(1.2)_3$ for the order parameter means that on the boundary S the material remains all the time in the phase characterized by $\varphi = 0$.

We assume such condition in order to be able to apply directly the maximal regularity theory for parabolic equations due to Weidemaier ([14], [15]). Alternatively, we could consider the homogeneous Neumann boundary condition

$$\mathbf{n} \cdot \nabla\varphi = 0 \quad \text{on } S^T$$

which is the typical condition in phase field models. The maximal regularity results due to Weidemaier in [14] can be also, however indirectly, applied in this case (see Section 5).

The typical Landau–Ginzburg form of the free energy density is given by

$$f(\varepsilon(\mathbf{u}), \varphi, \nabla\varphi) = W(\varepsilon(\mathbf{u}), \varphi) + \Psi(\varphi) + \frac{\gamma}{2}|\nabla\varphi|^2, \tag{1.3}$$

with the three terms representing respectively the elastic energy, the exchange energy and the gradient energy with constant coefficient $\gamma > 0$.

The exchange energy $\Psi(\varphi)$ is a double-well potential with equal minima at $\varphi = 0$ and $\varphi = 1$, assumed in the standard form

$$\Psi(\varphi) = \frac{1}{2}\varphi^2(1-\varphi)^2. \tag{1.4}$$

The sum of the last two terms in (1.3) represents the energy of diffused phase interfaces. The relevant expressions for the elastic energy $W(\varepsilon, \varphi)$ are given by the following two examples (see [7], [6]).

Example 1.1.

$$W(\varepsilon, \varphi) = (1 - z(\varphi))W_a(\varepsilon) + z(\varphi)W_b(\varepsilon), \tag{1.5}$$

where

$$W_i(\varepsilon) = w_i + (1/2)(\varepsilon - \overline{\varepsilon}_i) \cdot \mathbf{A}_i(\varepsilon - \overline{\varepsilon}_i), \quad i = a, b,$$

is the strain energy of phase i, $\overline{\varepsilon}_i$ is the natural strain, and $w_i > 0$ is the energy at the natural state; $\overline{\varepsilon}_i$, w_i are assumed constant. Furthermore, $z(\,\cdot\,)$ is a smooth scalar interpolation function satisfying:

$$z(0) = 0, \quad z(1) = 1, \quad \text{and} \quad 0 \le z(\varphi) \le 1 \quad \text{for all } \varphi \in \mathbb{R}. \tag{1.6}$$

The inequality constraint is imposed to assure the physical sense of (1.5).

The tensor $\mathbf{A}_i = ((A_i)_{pqrs})_{p,q,r,s=1,2,3}$ is the fourth order elasticity tensor of phase i, in isotropic elasticity given by

$$\mathbf{A}_i\varepsilon(\mathbf{u}) = \lambda_i \operatorname{tr}\varepsilon(\mathbf{u})\mathbf{I} + 2\mu_i\varepsilon(\mathbf{u}), \quad i = a, b, \tag{1.7}$$

where $\mathbf{I} = (\delta_{pq})_{p,q=1,2,3}$ and λ_i, μ_i are Lamé constants of phase i, within elasticity range, i.e. satisfying conditions $\mu_i > 0$, $3\lambda_i + 2\mu_i > 0$.

For the sake of mathematical analysis in the present paper we shall confine ourselves to the case of homogeneous elasticity, i.e. we assume that the elasticity tensors are equal in both phases:

$$\mathbf{A}_a = \mathbf{A}_b = \mathbf{A}. \tag{1.8}$$

The second example is characteristic for diffusive phase transitions in elastic solids (see [7], [5]).

EXAMPLE 1.2.

$$W(\varepsilon,\varphi) = w(\varphi) + \frac{1}{2}(\varepsilon - \overline{\varepsilon}(\varphi)) \cdot \mathbf{A}(\varphi)(\varepsilon - \overline{\varepsilon}(\varphi)), \tag{1.9}$$

where

$$\overline{\varepsilon}(\varphi) = z(\varphi)\overline{\varepsilon},$$

is the natural stress-free strain depending on the order parameter, $\overline{\varepsilon}$ is the constant misfit tensor, and $z(\,\cdot\,)$ is a smooth scalar interpolation function such that

$$\begin{aligned} z(0) &= 0,\\ z(1) &= 1, \end{aligned} \tag{1.10}$$

but in this example not necessarily constrained by the inequality $(1.6)_3$.

For diffusive processes Vegard's law is commonly postulated, i.e. that the stress-free strain is isotropic and varies linearly with the order parameter (see [5, Section 4.2])

$$\overline{\varepsilon}(\varphi) = e(\varphi - \overline{\varphi})\mathbf{I}$$

with constants e and $\overline{\varphi}$. Furthermore, $w(\varphi)$ is the energy of the natural state and $\mathbf{A}(\varphi)$ is the elasticity tensor, in general depending on the order parameter.

As in Example 1.1, for mathematical analysis, we assume that

$$\mathbf{A}(\varphi) = \mathbf{A} \tag{1.11}$$

is a constant tensor.

For further purpose we note here that in case of homogeneous elasticity (i.e. under assumptions (1.8), (1.11)) the expressions for the elastic energy and its derivatives with respect to ε and φ are given by:

in Example 1.1:

$$
\begin{aligned}
W(\varepsilon,\varphi) &= \frac{1}{2}\varepsilon\cdot A\varepsilon - \varepsilon\cdot[(1-z(\varphi))\mathbf{A}\overline{\varepsilon}_a + z(\varphi)\mathbf{A}\overline{\varepsilon}_b] \\
&\quad + (1-z(\varphi))w_a + z(\varphi)w_b + \frac{1}{2}(1-z(\varphi))\overline{\varepsilon}_a\cdot\mathbf{A}\overline{\varepsilon}_a \\
&\quad + \frac{1}{2}z(\varphi)\overline{\varepsilon}_b\cdot\mathbf{A}\overline{\varepsilon}_b, \\
W_{/\varepsilon}(\varepsilon,\varphi) &= \mathbf{A}\varepsilon - [(1-z(\varphi))\mathbf{A}\overline{\varepsilon}_a + z(\varphi)\mathbf{A}\overline{\varepsilon}_b], \\
W_{/\varphi}(\varepsilon,\varphi) &= z'(\varphi)\bigg[-\varepsilon\cdot\mathbf{A}(\overline{\varepsilon}_b-\overline{\varepsilon}_a) + \frac{1}{2}\overline{\varepsilon}_b\cdot\mathbf{A}\overline{\varepsilon}_b \\
&\quad - \frac{1}{2}\overline{\varepsilon}_a\cdot\mathbf{A}\overline{\varepsilon}_a + w_b - w_a\bigg],
\end{aligned}
\tag{1.12}
$$

in Example 1.2:

$$
\begin{aligned}
W(\varepsilon,\varphi) &= \frac{1}{2}\varepsilon\cdot A\varepsilon - z(\varphi)\varepsilon\cdot\mathbf{A}\overline{\varepsilon} + w(\varphi) + \frac{1}{2}z(\varphi)^2\overline{\varepsilon}\cdot\mathbf{A}\overline{\varepsilon}, \\
W_{/\varepsilon}(\varepsilon,\varphi) &= \mathbf{A}\varepsilon - z(\varphi)\mathbf{A}\overline{\varepsilon}, \\
W_{/\varphi}(\varepsilon,\varphi) &= w'(\varphi) + z'(\varphi)[-\varepsilon\cdot\mathbf{A}\overline{\varepsilon} + z(\varphi)\overline{\varepsilon}\cdot\mathbf{A}\overline{\varepsilon}].
\end{aligned}
\tag{1.13}
$$

We point out that in case of homogeneous elasticity the functions $W_{/\varepsilon}(\varepsilon,\varphi)$ and $W_{/\varphi}(\varepsilon,\varphi)$ are linear in respect to ε what essentially simplifies the analysis.

The problem (1.1), (1.2) corresponding to free energy (1.3), with $\Psi(\varphi)$ given by (1.4), and $W(\varepsilon,\varphi)$ as in Examples 1.1, 1.2 (homogeneous elasticity) takes the form:

$$
\begin{aligned}
\mathbf{u}_{tt} - \nabla\cdot(\mathbf{A}\varepsilon(\mathbf{u})) &= z'(\varphi)\mathbf{B}\nabla\varphi + \mathbf{b} && \text{in } \Omega^T, \\
\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \mathbf{u}_t|_{t=0} &= \mathbf{u}_1 && \text{in } \Omega, \\
\mathbf{u} &= 0 && \text{on } S^T,
\end{aligned}
\tag{1.14}
$$

$$
\begin{aligned}
\beta\varphi_t - \gamma\triangle\varphi &= -[\Psi'(\varphi) + W_{/\varphi}(\varepsilon(\mathbf{u}),\varphi)] \\
&= -[\Psi'(\varphi) + w'(\varphi) + z'(\varphi)(\mathbf{B}\cdot\varepsilon(\mathbf{u}) + h(\varphi))] && \text{in } \Omega^T, \\
\varphi|_{t=0} &= \varphi_0 && \text{in } \Omega, \\
\varphi &= 0 && \text{on } S^T,
\end{aligned}
\tag{1.15}
$$

where in Example 1.1:

$$
\begin{aligned}
\mathbf{B} &= -\mathbf{A}(\overline{\varepsilon}_b-\overline{\varepsilon}_a) \quad \text{constant tensor}, \\
h(\varphi) &= \frac{1}{2}\overline{\varepsilon}_b\cdot\mathbf{A}\overline{\varepsilon}_b - \frac{1}{2}\overline{\varepsilon}_a\cdot\mathbf{A}\overline{\varepsilon}_a + w_b - w_a = \text{const}, \\
w(\varphi) &= w_a + \frac{1}{2}\overline{\varepsilon}_a\cdot\mathbf{A}\overline{\varepsilon}_a = \text{const},
\end{aligned}
\tag{1.16}
$$

and in Example 1.2:

$$\begin{gathered}
\mathbf{B} = -\mathbf{A}\overline{\varepsilon} \quad \text{constant tensor,} \\
h(\varphi) = -z(\varphi)\overline{\varepsilon} \cdot \mathbf{B}, \\
w(\varphi) \quad \text{given function.}
\end{gathered}$$

We study the above problem under the following assumptions:

(A1) $\Omega \subset \mathbb{R}^3$, is a bounded domain with the boundary of class $C^{2+\alpha}$ for some $\alpha > 0$. Such regularity is needed in the application of the results on maximal regularity for parabolic equations (see Lemma 2.2).

(A2) The elasticity tensor $\mathbf{A} = (A_{ijkl})_{i,j,k,l=1,2,3}$ is a constant, symmetric, positive definite linear mapping from the set $\mathbf{S}^2$ of symmetric second order tensors in $\mathbb{R}^3$ onto itself, i.e. $\mathbf{A}$ satisfies the following conditions:

$$\begin{gathered}
A_{ijkl} = A_{jikl} = A_{klij}, \\
\underline{c}|\varepsilon|^2 \leq \varepsilon \cdot A\varepsilon \leq \overline{c}|\varepsilon|^2 \quad \text{for all } \varepsilon \in \mathbf{S}^2,
\end{gathered} \tag{1.17}$$

with some constants $\overline{c} > \underline{c} > 0$.

We do not require that $\mathbf{A}$ is isotropic.

We note that the isotropic tensor $\mathbf{A}$ given by (1.7) satisfies condition (1.17).

(A3) The free energy density $f(\varepsilon, \varphi, \nabla\varphi): \mathbf{S}^2 \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ has the form (1.3), with $\Psi: \mathbb{R} \to \mathbb{R}$ given by (1.4), and $W(\varepsilon, \varphi): \mathbf{S}^2 \times \mathbb{R} \to \mathbb{R}$ given in Examples 1.1 or 1.2.

We assume that the functions $z(\varphi)$, $h(\varphi)$ and $w(\varphi)$ in these examples satisfy the following conditions:
in Example 1.1:

- $z(\,\cdot\,) \in C^1(\mathbb{R})$ with $z'(\,\cdot\,)$ Lipschitz continuous, satisfies constraints (1.6), and

$$|z'(\varphi)| \leq c \quad \text{for all} \quad \varphi \in \mathbb{R}. \tag{1.18}$$

- Tensors $\overline{\varepsilon}_i \in \mathbf{S}^2$, and scalars $w_i \in \mathbb{R}$, $i = a, b$, are constant, hence by definition, $\mathbf{B} \in \mathbf{S}^2$ is constant, and functions $h(\varphi)$, $w(\varphi)$ are constant.

in Example 1.2:

- $z(\,\cdot\,) \in C^1(\mathbb{R})$ with $z'(\,\cdot\,)$ Lipschitz continuous, is subject to the growth conditions

$$|z(\varphi)| \leq c(|\varphi| + 1) \quad \text{and} \quad |z'(\varphi)| \leq c \quad \text{for all } \varphi \in \mathbb{R}. \tag{1.19}$$

- Tensor $\overline{\varepsilon} \in \mathbf{S}^2$ is constant, hence $\mathbf{B} \in \mathbf{S}^2$ is constant and function $h(\varphi) = -z(\varphi)\overline{\varepsilon} \cdot \mathbf{B}$.
- The function $w(\,\cdot\,) \in C^1(\mathbb{R})$ with $w'(\,\cdot\,)$ Lipschitz continuous, satisfies

$$w(\varphi) \geq -c \quad \text{and} \quad |w'(\varphi)| \leq c(|\varphi|^3 + 1) \quad \text{for all } \varphi \in \mathbb{R}.$$

We point on some consequences of the above assumptions. By virtue of the coercivity condition in (1.17) the following lower bounds for the elastic energy hold true:
in Example 1.1

$$W(\varepsilon,\varphi) \geq \min_{i\in\{a,b\}} \left\{ w_i + \frac{1}{2}\underline{c}\,|\varepsilon - \overline{\varepsilon}_i|^2 \right\},$$

and in Example 1.2

$$W(\varepsilon,\varphi) \geq w(\varphi) + \frac{1}{2}\underline{c}\,|\varepsilon - z(\varphi)\overline{\varepsilon}|^2.$$

Hence, in view of the growth conditions on $z(\,\cdot\,)$, taking into account that

$$\Psi(\varphi) \geq \frac{1}{8}\varphi^4 - \frac{1}{2},$$

we can see that the homogeneous part $W(\varepsilon,\varphi) + \Psi(\varphi)$ of $f(\varepsilon,\varphi,\nabla\varphi)$ satisfies the lower bound

$$W(\varepsilon,\varphi) + \Psi(\varphi) \geq c(|\varepsilon|^2 + |\varphi|^4) - c \quad \text{for all } (\varepsilon,\varphi) \in \mathbf{S}^2 \times \mathbb{R}.$$

Consequently,

$$f(\varepsilon,\varphi,\nabla\varphi) \geq c(|\varepsilon|^2 + |\varphi|^4 + |\nabla\varphi|^2) - c \quad \text{for all } (\varepsilon,\varphi,\nabla\varphi) \in \mathbf{S}^2 \times \mathbb{R} \times \mathbb{R}^3. \tag{1.20}$$

This is the main structure assumption that we use in derivation of energy estimates (see Section 2).

We note also that (A3) assures the following growth conditions (in Examples 1.1 and 1.2):

$$\begin{aligned} |W_{/\varphi}(\varepsilon,\varphi)| + |\Psi'(\varphi)| &\leq c(|\varepsilon| + |\varphi|^3 + 1), \\ |W_{/\varepsilon}(\varepsilon,\varphi)| &\leq c(|\varepsilon| + |\varphi| + 1) \end{aligned} \tag{1.21}$$

for all $(\varepsilon,\varphi) \in \mathbf{S}^2 \times \mathbb{R}$, which are applied in improving energy estimates.

Concerning the data of the problem (1.14), (1.15) we assume:

(A4) The external body force $\mathbf{b} \in L_1(0,T;\mathbf{L}_2(\Omega))$, and the initial conditions are such that:

$$\mathbf{u}_0 \in \mathbf{H}_0^1(\Omega), \quad \mathbf{u}_1 \in \mathbf{L}_2(\Omega), \quad \text{and} \quad \varphi_0 \in B_{2,q}^{2(1-1/q)}(\Omega) \cap H_0^1(\Omega), \quad 4 < q < \infty,$$

satisfies compatibility condition

$$\varphi_0 = 0 \quad \text{on } S.$$

Above $B_{2,q}^{2(1-1/q)}(\Omega)$ denotes the Besov space with the following norm equivalent to the usual norm in terms of second order differences (see [13]):

$$\|ft\|^{(T)}_{B^s_{p,q}(\Omega)} = \|f\|_{L_p(\Omega)} + |f|^{(T)}_{b^s_{p,q}(\Omega)} \quad \text{for } s \in (0,2),$$

where

$$
\begin{aligned}
|f|^{(T)}_{b^s_{p,q}(\Omega)} &= \sum_{j=1}^{3} \bigg(\int_0^{T^{1/2}} h^{-1-qs} \|\triangle^2_{j,h} f\|^q_{L_p(\Omega_{2h})}\, dh \bigg)^{1/q}, \\
\triangle_{j,h} f(\mathbf{x}) &= f(\mathbf{x} + h\mathbf{e}_j) - f(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^3,\ j = 1,2,3, \\
\triangle^k_{j,h} &= \triangle_{j,h} \triangle^{k-1}_{j,h}, \\
\Omega_\delta &= \{\mathbf{x} \in \Omega \mid \operatorname{dist}(\mathbf{x}, S) > \delta\} \quad \text{for } \delta > 0.
\end{aligned}
$$

The upper index (T) indicates the dependence of the correspodning norms and seminorms on time horizon T.

The Besov space specified in (A4) results from the solvability theory of parabolic problems in the anisotropic Sobolev space $W^{2,1}_{p,q}(\Omega^T)$ (see [13]–[15]). The initial trace of a function from $W^{2,1}_{p,q}(\Omega^T)$ belongs to the Besov space $B^{2(1-1/q)}_{p,q}(\Omega)$.

We state now the main results of the paper.

THEOREM 1.3 (The global existence). *Let us consider problem* (1.14), (1.15). *Let the assumptions* (A1)–(A4) *be satisfied. Then for any $T > 0$ there exists a pair $(\mathbf{u}, \varphi)$ such that:*

$$
\begin{gathered}
\mathbf{u} \in L_\infty(0,T;\mathbf{V}_0) \cap W^1_\infty(0,T;\mathbf{L}_2(\Omega)) \cap W^2_2(0,T;(\mathbf{V}_0)'), \quad \mathbf{V}_0 = \mathbf{H}^1_0(\Omega), \\
\mathbf{u}(\,\cdot\,,0) = \mathbf{u}_0, \quad \mathbf{u}_t(\,\cdot\,,0) = \mathbf{u}_1, \\
\varphi \in W^{2,1}_{2,q}(\Omega^T) = L_q(0,T;W^2_2(\Omega)) \cap W^1_q(0,T;L_2(\Omega)), \\
4 + \delta < q < \infty \quad \textit{for some } \delta > 0,
\end{gathered}
$$

which solves (1.14), (1.15) *in the following sense:*

(a) *the elasticity system* (1.14) *is satisfied in the weak sense*

$$
\begin{aligned}
(1.22)\qquad \int_0^T \langle \mathbf{u}_{tt}, \boldsymbol{\eta} \rangle_{(\mathbf{V}_0)',\mathbf{V}_0}\, dt &+ \int_{\Omega^T} \mathbf{A}\varepsilon(\mathbf{u}) \cdot \varepsilon(\boldsymbol{\eta})\, dx\, dt \\
&= \int_{\Omega^T} (z'(\varphi)\mathbf{B}\nabla\varphi + \mathbf{b}) \cdot \boldsymbol{\eta}\, dx\, dt
\end{aligned}
$$

for any $\boldsymbol{\eta} \in L_2(0,T;\mathbf{V}_0)$, where $\langle\,\cdot\,,\,\cdot\,\rangle_{(\mathbf{V}_0)',\mathbf{V}_0}$ denotes the duality pairing between $\mathbf{V}_0$ and $\mathbf{V}'_0$,

(b) *the parabolic equation* $(1.15)_1$ *is satisfied a.e. in Ω^T and initial and boundary conditions* $(1.15)_{2,3}$ *hold in the sense of appriopriate traces. Moreover, $(\mathbf{u}, \varphi)$ satisfy a priori estimates*

$$
\begin{aligned}
(1.23)\qquad \|\mathbf{u}\|_{L_\infty(0,T;\mathbf{V}_0)} + \|\mathbf{u}_t\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} &+ \|\mathbf{u}_{tt}\|_{L_2(0,T;(\mathbf{V}_0)')} \\
&+ \|\varphi\|_{W^{2,1}_{2,q}(\Omega^T)} \le c(T)
\end{aligned}
$$

with constant $c(T)$ depending only on the data $\mathbf{u}_0$, u_1, φ_0, $\mathbf{b}$, parameters $p = 2$, q, and time horizon T.

We note that by virtue of the imbedding $W^{2,1}_{2,q}(\Omega^T) \subset L_\infty(\Omega^T)$ for $q > 4$, the function φ in the solution $(\mathbf{u}, \varphi)$ satisfies

$$\|\varphi\|_{L_\infty(\Omega^T)} \le c\|\varphi\|_{W^{2,1}_{2,q}(\Omega^T)} \le c(T). \tag{1.24}$$

Theorem 1.4 (Uniqueness). *Let the assumptions* (A1)–(A4) *be satisfied. Then the solution* $(\mathbf{u}, \varphi)$ *mentioned in Theorem* 1.3 *is unique.*

We comment now on the idea of the existence proof. It is based on the Leray–Schauder fixed point theorem. The main part of the prof consitute a priori estimates for a solution of the problem. They comprise energy estimates derived on the basis of the physical form (1.1), (1.2) of the problem and the structure assumption (1.20) on the underlying free energy.

The key estimate for the order parameter φ is obtained with the help of the results on maximal regularity for the second order linear parabolic equations with inhomogeneous part belonging to the mixed space $L_q(0, T; L_p(\Omega))$, which are due to Weidemaier (see [14], [15]).

We explain the reason we apply such theory. From energy estimates we know that the right-hand side of the equation for the order parameter belongs to $L_\infty(0, T; L_2(\Omega))$ (see estimate (2.8)). Hence, by virtue of the above mentioned maximal regularity results we can conclude that the order parameter φ belongs to the anisotropic Sobolev space $W^{2,1}_{2,q}(\Omega^T)$ for some $2 \le q < \infty$ (see estimate (2.11)).

Thanks to such a regularity result we have a wide range of possibilities in the choice of the time-integration parameter q so that desired properties of the solution are ensured.

Firstly, we choose $4 + \delta < q < \infty$ for some $\delta > 0$, in order to guarantee that φ is $L_\infty(\Omega^T)$-function, i.e.

$$W^{2,1}_{2,q}(\Omega^T) \subset L_\infty(\Omega^T). \tag{1.25}$$

Secondly, we take the Besov space (see Section 2)

$$\mathcal{B} = B^{2-\delta', 1-\delta'/2}_{2,q;\theta}(\Omega^T)$$

with the parameters

$$4 + \delta < q < \infty, \quad 0 < \delta, \quad 1 \le \theta \le \infty, \quad \text{and} \quad 0 < \delta' = \frac{\delta}{2(4+\delta)} < \frac{1}{2},$$

as the working space in the Leray–Schauder fixed point theorem.

The choice of this Besov space is motivated by the following two requirements arising in the proof: $\mathcal{B}$ is the smallest space such that $W^{2,1}_{2,q}(\Omega^T)$ is compactly imbedded into $\mathcal{B}$, and at the same time $\mathcal{B}$ preserves the property (1.25) of $W^{2,1}_{2,q}(\Omega^T)$. More precisely, we require that the imbedding

$$W^{2,1}_{2,q}(\Omega^T) \subset \mathcal{B} \quad \text{s compact,} \tag{1.26}$$

and

$$\mathcal{B} \subset L_\infty(\Omega^T). \tag{1.27}$$

The imbeddings (1.26) and (1.27) are of key importance to assure the properties of the solution map in the Leray–Schauder fixed point theorem.

Finally, we underline that our results are restricted to the case of homogeneous elasticity, i.e. equal elasticity tensors of the phases. In nonhomogeneous elasticity there are additional nonlinearities in the system (1.1), (1.2) that can be handled by regularizing the elasticity equation (1.1). Such problem will be considered by the authors in a separate paper.

The paper is organized as follows: In Section 2 we derive a priori estimates for solutions of the problem (1.14), (1.15) which include the energy estimates and the maximal regularity estimates for φ.

In Section 3 we present the proof of the existence result in Theorem 1.3. It consists in constructing a solution map and checking the assumptions of the Leray–Schauder fixed point theorem. The estimates derived in Section 2 provide a priori bounds for a fixed point of the solution map.

In Section 4 we present the proof of the uniqueness of solution. It is based on direct comparison of two solutions by means of deriving appropriate energy estimates and applying Gronwall's inequality.

In Section 5 we show how the existence result can be extended to other boundary conditions.

We use following notations:

$$f_{/i} = \frac{\partial f}{\partial x_i} \quad i = 1,2,3, \qquad f_t = \frac{df}{dt}, \qquad \varepsilon(\mathbf{u}) = (\varepsilon(\mathbf{u})_{ij})_{i,j=1,2,3},$$
$$W_{/\varepsilon}(\varepsilon,\varphi) = \left(\frac{\partial W(\varepsilon,\varphi)}{\partial \varepsilon_{ij}}\right)_{i,j=1,2,3}, \qquad W_{/\varphi}(\varepsilon,\varphi) = \frac{\partial W(\varepsilon,\varphi)}{\partial \varphi},$$
$$z'(\varphi) = \frac{dz}{d\varphi},$$

where space and time derivatives are material. For simplicity, whenever there is no danger of confusion, we omit the arguments (ε,φ). Also the specification of tensor indices is omitted.

Vector and tensor valued mappings are denoted by bold letters.

The summation convention over repeated indices is used, and the following notation:

For vectors $\mathbf{a} = (a_i)$, $\widetilde{\mathbf{a}} = (\widetilde{a}_i)$, and tensors $\mathbf{B} = (B_{ij})$, $\widetilde{\mathbf{B}} = (\widetilde{B}_{ij})$, $\mathbf{A} = (A_{ijkl})$, we write $\mathbf{a}\cdot\widetilde{\mathbf{a}} = a_i\widetilde{a}_i$, $\mathbf{B}\cdot\widetilde{\mathbf{B}} = B_{ij}\widetilde{B}_{ij}$, $\mathbf{AB} = (A_{ijkl}B_{kl})$, $\mathbf{BA} = (B_{ij}A_{ijkl})$.

∇ and $\nabla\cdot$ denote the gradient and the divergence operators with respect to the material point $\mathbf{x}\in\mathbb{R}^3$.

For the divergence of the tensor fields $\varepsilon(\mathbf{x})=(\varepsilon_{ij}(\mathbf{x}))$ we use the convention of the contraction over the last index, i.e.

$$\nabla\cdot\varepsilon(\mathbf{x})=(\varepsilon_{ij/j}(\mathbf{x})).$$

We use the Sobolev spaces notation of the monograph [9]. For simplicity we write

$$H^m(\Omega)=W_2^m(\Omega)\quad\text{for } m\in N,$$
$$\mathbf{L}_2(\Omega)=(L_2(\Omega))^3,\quad \mathbf{V}=\mathbf{H}^1(\Omega)=(H^1(\Omega))^3,\quad \mathbf{V}_0=\mathbf{H}_0^1(\Omega)=(H_0^1(\Omega))^3.$$

Furthermore, $L_{p,q}(\Omega^T)=L_q(0,T;L_p(\Omega))$ with the norm

$$\|f\|_{L_{p,q}(\Omega^T)}=\bigg(\int_0^T\bigg(\int_\Omega|f|^p\,dx\bigg)^{q/p}dt\bigg)^{1/q}.$$

Throughout the paper c and $c(T)$ denote generic constants different in various instances, depending on the data of the problem and domain Ω. T he argument T indicates time horizon dependence which is always such that constant $c(T)$ stays bounded for $T\searrow 0$.

2. A priori estimates

In this section we derive a priori estimates for solutions of the problem (1.14), (1.15). These estimates will be used in the proof of Theorem 1.3 to show a fixed point property of the solution map in the Leray–Schauder theorem.

We start with establishing energy estimates. To this purpose it is convenient to consider problem (1.14), (1.15) in its original form (1.1), (1.2).

Throughout we assume that (A1)–(A4) are satisfied.

Lemma 2.1. *Assume $f(\varepsilon,\varphi,\nabla\varphi)$ satisfies structure condition* (1.20)*, and the data are such that*

$$W(\varepsilon_0,\varphi_0),\Psi(\varphi_0)\in L_1(\Omega),\quad \nabla\varphi_0\in\mathbf{L}_2(\Omega),\quad \mathbf{u}_1\in\mathbf{L}_2(\Omega),\quad \mathbf{b}\in L_1(0,T;\mathbf{L}_2(\Omega)).$$

Then for a solution $(\mathbf{u},\varphi)$ of problem (1.1), (1.2) *the following estimate holds:*

$$\begin{aligned}(2.1)\quad \|\mathbf{u}_t\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))}&+\|\varepsilon(\mathbf{u})\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))}\\ &+\|\varphi\|_{L_\infty(0,T;L_4(\Omega))}+\|\nabla\varphi\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))}+\|\varphi_t\|_{L_2(\Omega^T)}\le c_0\end{aligned}$$

with constant c_0 depending only on the data.

Proof. Multiplying equation $(1.1)_1$ by $\mathbf{u}_t$, integrating over Ω and by parts, in view of boundary condition $(1.1)_3$, we get

$$\frac12\frac{d}{dt}\int_\Omega|\mathbf{u}_t|^2\,dx+\int_\Omega f_{/\varepsilon}(\varepsilon,\varphi,\nabla\varphi)\cdot\varepsilon_t\,dx=\int_\Omega\mathbf{b}\cdot\mathbf{u}_t\,dx.\tag{2.2}$$

Further, multiplying $(1.2)_1$ by φ_t, integrating over Ω and by parts, in view of $(1.2)_3$, we deduce that

$$\beta\int_\Omega \varphi_t^2\,dx+\int_\Omega f_{/\varphi}(\varepsilon,\varphi,\nabla\varphi)\varphi_t\,dx+\int_\Omega f_{/\nabla\varphi}(\varepsilon,\varphi,\nabla\varphi)\cdot\nabla\varphi_t\,dx=0. \tag{2.3}$$

Adding (2.2) and (2.3) we arrive at the energy identity

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{u}_t|^2\,dx+\frac{d}{dt}\int_\Omega f(\varepsilon,\varphi,\nabla\varphi)\,dx+\beta\int_\Omega \varphi_t^2\,dx=\int_\Omega \mathbf{b}\cdot\mathbf{u}_t\,dx. \tag{2.4}$$

Integrating (2.4) with respect to t, estimating the source term by

$$\begin{aligned}\int_0^t\int_\Omega \mathbf{b}\cdot u_{t'}\,dx\,dt' &\le \|\mathbf{u}_{t'}\|_{L_\infty(0,t;\mathbf{L}_2(\Omega))}\|\mathbf{b}\|_{L_1(0,t;\mathbf{L}_2(\Omega))}\\ &\le \frac{1}{4}\|\mathbf{u}_{t'}\|^2_{L_\infty(0,t;\mathbf{L}_2(\Omega))}+\|\mathbf{b}\|^2_{L_1(0,t;\mathbf{L}_2(\Omega))},\end{aligned}$$

and using structure condition (1.20), we conclude the assertion. □

By standard duality argument we can get estimate for the second time derivative $\mathbf{u}_{tt}$. In fact, from $(1.1)_1$ it follows that

$$\int_{\Omega^T}\mathbf{u}_{tt}\cdot\boldsymbol{\eta}\,dx\,dt=-\int_{\Omega^T}W_{/\varepsilon}(\varepsilon(\mathbf{u}),\varphi)\cdot\varepsilon(\boldsymbol{\eta})\,dx\,dt+\int_{\Omega^T}\mathbf{b}\cdot\boldsymbol{\eta}\,dx\,dt$$

for any $\boldsymbol{\eta}\in L_2(0,T;\mathbf{V}_0)$. Hence, using growth condition $(1.21)_2$ and estimate (2.1), we get

$$\begin{aligned}\left|\int_{\Omega^T}\mathbf{u}_{tt}\cdot\boldsymbol{\eta}\,dx\,dt\right| &\le \|W_{/\varepsilon}(\varepsilon(\mathbf{u}),\varphi)\|_{\mathbf{L}_2(\Omega^T)}\|\nabla\boldsymbol{\eta}\|_{\mathbf{L}_2(\Omega^T)}+\|\ \mathbf{b}\|_{\mathbf{L}_2(\Omega^T)}\|\boldsymbol{\eta}\|_{\mathbf{L}_2(\Omega^T)}\\ &\le c(\|\varepsilon(\mathbf{u})\|_{\mathbf{L}_2(\Omega^T)}+\|\varphi\|_{L_2(\Omega^T)}+\|\mathbf{b}\|_{\mathbf{L}_2(\Omega^T)}+1)\|\boldsymbol{\eta}\|_{L_2(0,T;\mathbf{V}_0)}\\ &\le c\|\boldsymbol{\eta}\|_{L_2(0,T;\mathbf{V}_0)}.\end{aligned}$$

This shows that

$$\|\mathbf{u}_{tt}\|_{L_2(0,T;(\mathbf{V}_0)')}\le c. \tag{2.5}$$

Our goal now is to improve energy estimates (2.1). Firstly, we shall note that from (2.1) it follows that

$$\|\varphi\|_{L_\infty(0,T;H^1(\Omega))}\le c_0,$$

so that, by Sobolev's imbedding,

$$\|\varphi\|_{L_\infty(0,T;L_6(\Omega))}\le c_0. \tag{2.6}$$

Let us consider parabolic problem (1.15). For simplicity, let us denote

$$H\equiv-[\Psi'(\varphi)+w'(\varphi)+z'(\varphi)(\mathbf{B}\cdot\varepsilon(\mathbf{u})+h(\varphi))]. \tag{2.7}$$

In view of growth conditions $(1.21)_1$, estimates (2.1) and (2.6) imply that

$$\|H\|_{L_\infty(0,T;L_2(\Omega))}\le c_0. \tag{2.8}$$

To conclude from (2.8) the regularity of the solution to the problem (1.15) we apply the results on maximal regularity for second order linear parabolic equations with the inhomogeneous part belonging to the space $L_q(0,T;L_p(\Omega))$, obtained recently by Weidemaier [15].

A direct application of [15, Theorem 3.1] yields:

Lemma 2.2. *Consider problem* (1.15). *Assume that Ω is a bounded domain in $\mathbb{R}^3$ of class $C^{2+\alpha}$ for some $\alpha > 0$ and that*

$$3/2 < p = 2 \le q < \infty, \quad H \in L_q(0,T;L_2(\Omega)), \quad \varphi_0 \in B^{2(1-1/q)}_{2,q}(\Omega) \tag{2.9}$$

and φ_0 satisfies compatibility condition $\varphi_0 = 0$ on S. Then the problem (1.15) *has a unique solution $\varphi \in W^{2,1}_{2,q}(\Omega^T)$ and there is a constant $c^*(p,q,T)$ (which stays bounded for $T \searrow 0$), such that*

$$\|\varphi\|_{W^{2,1}_{2,q}(\Omega^T)} \le c^*(\|H\|_{L_q(0,T;L_2(\Omega))} + \|\varphi_0\|_{B^{2(1-1/q)}_{2,q}(\Omega)}). \tag{2.10}$$

By virtue of Lemma 2.2 and estimate (2.8) we conclude that

$$\|\varphi\|_{W^{2,1}_{2,q}(\Omega^T)} \le c(T), \quad 2 \le q < \infty, \tag{2.11}$$

where c denotes a constant depending only on the data, $p = 2$, q and T.

To prove the existence of solutions we shall restrict ourselves to the range

$$4 < 4 + \delta < q < \infty \quad \text{for some } \delta > 0.$$

In this range, by virtue of Sobolev's imbedding,

$$W^{2,1}_{2,q}(\Omega^T) \subset L_\infty(\Omega^T) \quad \text{for } 4 < 4 + \delta < q < \infty. \tag{2.12}$$

We introduce now the Besov space $B^{2-\delta',1-\delta'/2}_{2,q;\theta}(\Omega^T)$ which will be used later as the working space in the Leray-Schauder fixed point theorem. We recall that the norm of this space is defined by (see [1, Definition 18.1])

$$\begin{aligned}\|\varphi\|_{B^{2-\delta',1-\delta'/2}_{2,q;\theta}(\Omega^T)} &= \|\varphi\|_{L_q(0,T;L_2(\Omega))} \\ &\quad + \sum_{i=1}^{3}\bigg\{\int_0^{h_0}\bigg[\frac{\|\triangle_{x_i}(h,\Omega^T)\partial_{x_i}\varphi\|_{L_q(0,T;L_2(\Omega))}}{h^{1-\delta'}}\bigg]^\theta \frac{dh}{h}\bigg\}^{1/\theta} \\ &\quad + \bigg\{\int_0^{h_0}\bigg[\frac{\|\triangle_t(h,\Omega^T)\varphi\|_{L_q(0,T;L_2(\Omega))}}{h^{1-\delta'/2}}\bigg]^\theta \frac{dh}{h}\bigg\}^{1/\theta},\end{aligned}$$

where the parameter $0 < h_0 < \infty$, and

$$\triangle_{x_i}(h,\Omega^T)\varphi(\mathbf{x},t) = \begin{cases} \triangle_{x_i}(h)\varphi(\mathbf{x},t) & \text{if } [(\mathbf{x},t),(\mathbf{x}+h\mathbf{e}_i,t)] \subset \Omega^T, \\ 0 & \text{if } [(\mathbf{x},t),(\mathbf{x}+h\mathbf{e}_i,t)] \not\subset \Omega^T, \end{cases}$$

$$\triangle_{t}(h,\Omega^T)\varphi(\mathbf{x},t) = \begin{cases} \triangle_{t}(h)\varphi(\mathbf{x},t) & \text{if } [(\mathbf{x},t),(\mathbf{x},t+h)] \subset \Omega^T, \\ 0 & \text{if } [(\mathbf{x},t),(\mathbf{x},t+h)] \not\subset \Omega^T, \end{cases}$$

$$\begin{aligned} \triangle_{x_i}(h)\varphi(\mathbf{x},t) &= \varphi(\mathbf{x}+h\mathbf{e}_i,t) - \varphi(\mathbf{x},t), \\ \triangle_{t}(h)\varphi(\mathbf{x},t) &= \varphi(\mathbf{x},t+h) - \varphi(\mathbf{x},t). \end{aligned}$$

For further purposes we recall here that, by virtue of [1, Theorem 18.13], the imbedding

$$W^{2,1}_{2,q}(\Omega^T) \subset B^{2-\delta',1-\delta'/2}_{2,q;\theta}(\Omega^T) \quad \text{for any } \delta' > 0 \text{ and } 1 \le \theta \le \infty \tag{2.13}$$

is compact. At the same time, by virtue of [1, Theorem 18.10],

$$B^{2-\delta',1-\delta'/2}_{2,q;\theta}(\Omega^T) \subset L_\infty(\Omega^T) \quad \text{for } \frac{3}{2} + \frac{2}{q} < 2 - \delta'. \tag{2.14}$$

The later inequality implies that

$$q > \frac{4}{1-2\delta'},$$

so, setting

$$\frac{4}{1-2\delta'} = 4 + \delta,$$

we get the condition on δ'

$$\delta' = \frac{\delta}{2(4+\delta)} < \frac{1}{2}.$$

The above condition will be used in the definition of the working space $\mathcal{B} = B^{2-\delta',1-\delta'/2}_{2,q;\theta}(\Omega^T)$.

3. Proof of Theorem 1.3 (global existence)

3.1. Preparation of the Leray–Schauder fixed point theorem. The proof of Theorem 1.3 is based on the classical Leray–Schauder fixed point theorem which we recall here in one of its equivalent formulations for reader's convenience (see e.g. [3]).

Theorem 3.1. *Let $\mathcal{B}$ be a Banach space. Assume that $\mathcal{T}\colon [0,1] \times \mathcal{B} \to \mathcal{B}$ is a map with the following properties:*

(a) *For any fixed $\tau \in [0,1]$ the map $\mathcal{T}(\tau,\,\cdot\,)\colon \mathcal{B} \to \mathcal{B}$ is completely continuous.*

(b) *For every bounded subset $\mathcal{C}$ of $\mathcal{B}$, the family of maps $\mathcal{T}(\,\cdot\,,\chi)\colon [0,1] \to \mathcal{B}$, $\chi \in \mathcal{C}$, is uniformly equicontinuous.*

(c) *There is a bounded subset $\mathcal{C}$ of $\mathcal{B}$ such that any fixed point in $\mathcal{B}$ of $\mathcal{T}(\tau,\,\cdot\,)$, $0 \le \tau \le 1$, is contained in $\mathcal{C}$.*

(d) $\mathcal{T}(0,\,\cdot\,)$ *has precisely one fixed point in $\mathcal{B}$.*

Then $\mathcal{T}(1,\,\cdot\,)$ has at least one fixed point in $\mathcal{B}$.

For our purposes, let $\mathcal{B}$ be the Besov space of functions φ on Ω^T, given by

$$\mathcal{B} \equiv B^{2-\delta',1-\delta'/2}_{2,q;\theta}(\Omega^T),$$

with the parameters

$$4+\delta < q < \infty, \quad 0 < \delta, \quad 1 \le \theta \le \infty, \quad \text{and} \quad 0 < \delta' = \frac{\delta}{2(4+\delta)} < \frac{1}{2}.$$

We denote by $\|\cdot\|_{\mathcal{B}}$ the norm induced by the space $B^{2-\delta',1-\delta'/2}_{2,q;\theta}(\Omega^T)$.

For $\tau \in [0,1]$ we define $\mathcal{T}(\tau,\,\cdot\,)$ as the map that carries $\overline{\varphi} \in \mathcal{B}$ into $\varphi \in \mathcal{B}$ by the following procedure:

First we construct $\mathbf{u}(\mathbf{x},t)$ by solving the system

$$\begin{aligned} \mathbf{u}_{tt} - \nabla\cdot\mathbf{A}\varepsilon(\mathbf{u}) = \tau[z'(\overline{\varphi})\mathbf{B}\nabla\overline{\varphi} + \ \mathbf{b}] &\equiv \overline{\mathbf{G}} && \text{in } \Omega^T,\\ \mathbf{u}|_{t=0} = \tau\mathbf{u}_0, \quad \mathbf{u}_t|_{t=0} &= \tau\mathbf{u}_1 && \text{in } \Omega,\\ \mathbf{u} &= 0 && \text{on } S^T, \end{aligned} \tag{3.1}$$

and then compute $\varphi(\mathbf{x},t)$ through the problem

$$\begin{aligned} \beta\varphi_t - \gamma\triangle\varphi &= -\tau[\Psi'(\overline{\varphi}) + w'(\overline{\varphi}) + z'(\overline{\varphi})(\mathbf{B}\cdot\varepsilon(\mathbf{u}) + h(\overline{\varphi}))] \equiv \overline{H} && \text{in } \Omega^T,\\ \varphi|_{t=0} &= \tau\varphi_0 && \text{in } \Omega,\\ \varphi &= 0 && \text{on } S^T. \end{aligned} \tag{3.2}$$

Clearly, $(\mathbf{u},\varphi)$ defined as a fixed point of $\mathcal{T}(1,\,\cdot\,)$ is a solution to problem (1.14), (1.15).

Our goal is to show that the map $\mathcal{T}(\tau,)$ satisfies assumptions of Theorem 3.1. We consider the first step of the construction.

3.2. The elasticity system. We show that the map

$$\mathcal{T}_1(\tau,\,\cdot\,)\colon \mathcal{B} \ni \overline{\varphi} \mapsto \mathbf{u} \in \mathcal{V} \equiv L_\infty(0,T;\mathbf{V}_0) \tag{3.3}$$

that gives a solution $\mathbf{u}$ of (3.1) for a given $\overline{\varphi}$ is well-defined, i.e. the solution exists and is unique, and that $\mathcal{T}_1(\tau,\,\cdot\,)$ is continuous. Firstly, we note that (see [1, Theorem 18.4])

$$\nabla\overline{\varphi} \in B^{1-\delta',(1-\delta')/2}_{2,q;\theta}(\Omega^T).$$

By virtue of [1, Theorem. 18.10], we have the imbedding

$$B^{1-\delta',(1-\delta')/2}_{2,q;\theta}(\Omega^T) \subset L_2(\Omega^T) \quad \text{for } \frac{3}{2} + \frac{2}{q} - \frac{5}{2} \le 1 - \delta', \tag{3.4}$$

which clearly holds true for $q > 4$. Hence, on account of growth condition on $z'(\,\cdot\,)$ in (A3), we can see that

$$
\begin{aligned}
(3.5)\quad \|\overline{\mathbf{G}}\|_{L_1(0,T;\mathbf{L}_2(\Omega))} &\le c(\|\nabla\overline{\varphi}\|_{L_1(0,T;\mathbf{L}_2(\Omega))} + \|\mathbf{b}\|_{L_1(0,T;\mathbf{L}_2(\Omega))})\\
&\le c(T)(\|\nabla\overline{\varphi}\|_{B^{1-\delta',(1-\delta')/2}_{2,q;\theta}(\Omega^T)} + \|\mathbf{b}\|_{L_1(0,T;\mathbf{L}_2(\Omega))})\\
&\le c(T)(\|\overline{\varphi}\|_{\mathcal{B}} + \|\mathbf{b}\|_{L_1(0,T;\mathbf{L}_2(\Omega))}).
\end{aligned}
$$

By virtue of the known results (see [4, Chapter III, Theorem 4.1]) it follows that if $\overline{\mathbf{G}} \in L_1(0,T;\mathbf{L}_2(\Omega))$, $\mathbf{u}_0 \in \mathbf{H}^1_0(\Omega)$ and $\mathbf{u}_1 \in \mathbf{L}_2(\Omega)$ then there exists a unique $\mathbf{u}$ such that

$$
\begin{gathered}
\mathbf{u} \in L_\infty(0,T;\mathbf{V}_0) \cap W^1_\infty(0,T;\mathbf{L}_2(\Omega)) \cap W^2_2(0,T;(\mathbf{V}_0)'),\\
\mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}_t(0) = \mathbf{u}_1,
\end{gathered}
$$

and satisfying (3.1) in the following weak sense

$$
(3.7)\qquad \int_0^T \langle \mathbf{u}_{tt}, \boldsymbol{\eta}\rangle_{(\mathbf{V}_0)',\mathbf{V}_0}\,dt + \int_{\Omega^T} \mathbf{A}\varepsilon(\mathbf{u})\cdot\varepsilon(\boldsymbol{\eta})\,dx\,dt = \int_{\Omega^T} \overline{\mathbf{G}}\cdot\boldsymbol{\eta}\,dx\,dt
$$

for any $\boldsymbol{\eta} \in L_2(0,T;\mathbf{V}_0)$. The basic a priori estimate for system (3.1) follows by testing $(3.1)_1$ by $\mathbf{u}_t$, integrating over Ω and by parts to get

$$
(3.8)\qquad \frac{1}{2}\frac{d}{dt}\int_\Omega \Big(|\mathbf{u}_t|^2 + |\mathbf{A}\varepsilon(\mathbf{u})|^2\Big)\,dx = \int_\Omega \overline{\mathbf{G}}\cdot\mathbf{u}_t\,dx.
$$

Hence, with the help of the Cauchy-Schwartz inequality, it follows that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\bigg\{\bigg[\int_\Omega \Big(|\mathbf{u}_t|^2 + |\mathbf{A}\varepsilon(\mathbf{u})|^2\Big)\,dx\bigg]^{1/2}\bigg\}^2\\
&\qquad\qquad \le \bigg(\int_\Omega |\overline{\mathbf{G}}|^2\,dx\bigg)^{1/2}\bigg(\int_\Omega (|\mathbf{u}_t|^2 + |\mathbf{A}\varepsilon(\mathbf{u})|^2)\,dx\bigg)^{1/2},
\end{aligned}
$$

thus

$$
(3.9)\qquad \frac{d}{dt}\bigg[\int_\Omega \Big(|\mathbf{u}_t|^2 + |\,\mathbf{A}\varepsilon(\mathbf{u})|^2\Big)\,dx\bigg]^{1/2} \le \bigg(\int_\Omega |\overline{\mathbf{G}}|^2\,dx\bigg)^{1/2}.
$$

Integrating (3.9) with respect to t yields

$$
\begin{aligned}
&\|\mathbf{u}_t\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + \|\mathbf{A}\varepsilon(\mathbf{u})\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))}\\
&\qquad\qquad \le \|\overline{\mathbf{G}}\|_{L_1(0,T;\,\mathbf{L}_2(\Omega))} + \|\mathbf{A}\varepsilon(\mathbf{u}_0)\|_{\mathbf{L}_2(\Omega)} + \|\mathbf{u}_1\|_{\mathbf{L}_2(\Omega)}.
\end{aligned}
$$

Hence, using coercivity and bounedness (1.17) of $\mathbf{A}$, by virtue of Korn's inequality (see e.g. [4, Chapter II, Theorem 3.3]) we get

$$
\begin{aligned}
(3.10)\quad &\|\mathbf{u}_t\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + \|\mathbf{u}\|_{L_\infty(0,T;\mathbf{V}_0)}\\
&\le c(\|\overline{\mathbf{G}}\|_{L_1(0,T;\mathbf{L}_2(\Omega))} + \|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_1\|_{\mathbf{L}_2(\Omega)})\\
&\le c(T)(\|\overline{\varphi}\|_{\mathcal{B}} + \|\mathbf{b}\|_{L_1(0,T;\mathbf{L}_2(\Omega))} + \|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_1\|_{\mathbf{L}_2(\Omega)}) \equiv c_1.
\end{aligned}
$$

In the next step we examine the continuity of $\mathcal{T}_1(\tau,\,\cdot\,)$. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be the solutions of (3.1) coresponding to $\overline{\varphi}_1 \in \mathcal{B}$ and $\overline{\varphi}_2 \in \mathcal{B}$, respectively. Subtracting the corresponding equations and denoting

$$\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2, \quad \overline{\psi} = \overline{\varphi}_1 - \overline{\varphi}_2,$$

it follows that

$$\tag{3.11} \begin{aligned} \mathbf{v}_{tt} - \nabla\cdot\mathbf{A}\varepsilon(\mathbf{v}) &= \tau[(z'(\overline{\varphi}_1)) - z'(\overline{\varphi}_2))\mathbf{B}\nabla\overline{\varphi}_1 + z'(\overline{\varphi}_2)\mathbf{B}\nabla\overline{\psi}] \\ &\equiv \overline{\overline{G}} && \text{in } \Omega^T, \\ \mathbf{v}|_{t=0} &= 0, \qquad \mathbf{v}_t|_{t=0} = 0 && \text{in } \Omega, \\ \mathbf{v} &= 0 && \text{on } S^T. \end{aligned}$$

We prove the following

Lemma 3.2. *Under assumptions* (A1)–(A4) *the map* $\mathcal{T}_1(\tau,\,\cdot\,)$ *defined by* (3.3) *is continuous.*

Proof. We proceed similarly as in derivation of (3.10). Multiplying $(3.11)_1$ by $\mathbf{v}_t$, integrating over Ω and by parts, and using the Cauchy-Schwartz inequality, it follows that

$$\frac{d}{dt}\bigg[\int_\Omega \Big(|\mathbf{v}_t|^2 + |\ \mathbf{A}\varepsilon(\mathbf{v})|^2\Big)\,dx\bigg]^{1/2} \le \bigg(\int_\Omega |\overline{\overline{\mathbf{G}}}|^2\,dx\bigg)^{1/2}.$$

Integrating the above inequality with respect to t , using the boundedness assumption on $z'(\,\cdot\,)$, we get

$$\tag{3.12} \begin{aligned} \bigg[\int_\Omega \Big(|\mathbf{v}_t|^2 + |\mathbf{A}\varepsilon(\mathbf{v})|^2\Big)\,dx\bigg]^{1/2} &\le c\int_0^t \bigg(\int_\Omega |\overline{\psi}|^2|\nabla\overline{\varphi}_1|^2\,dx\bigg)^{1/2} dt' \\ &\quad + c\int_0^t \bigg(\int_\Omega |\nabla\overline{\psi}|^2\,dx\bigg)^{1/2} dt' \equiv R_1 + R_2 \end{aligned}$$

for a.a. $t \in [0,T]$. By Hölder's inequality,

$$R_1 \le cT^{1/2}\bigg(\int_{\Omega^T} |\overline{\psi}|^2|\nabla\overline{\varphi}_1|^2\,dx\,dt\bigg)^{1/2} \le cT^{1/2}\|\overline{\psi}\|_{L_{2\lambda_1}(\Omega^T)}\|\nabla\overline{\varphi}_1\|_{\mathbf{L}_{2\lambda_2}(\Omega^T)},$$

where $1/\lambda_1 + 1/\lambda_2 = 1$.

Now, we use the imbeddings (see [1, Theorem 18.10])

$$B^{2-\delta',1-\delta'/2}_{2,q;\theta}(\Omega^T) \subset L_{2\lambda_1}(\Omega^T) \quad\text{and}\quad B^{1-\delta',(1-\delta')/2}_{2,q;\theta}(\Omega^T) \subset L_{2\lambda_2}(\Omega^T),$$

which hold true under the following conditions:

$$\frac{3}{2} + \frac{2}{q} - \frac{5}{2\lambda_1} \le 2 - \delta' \quad\text{and}\quad \frac{3}{2} + \frac{2}{q} - \frac{5}{2\lambda_2} \le 1 - \delta'.$$

The above conditions imply that

$$q \ge \frac{8}{5 - 4\delta'} = \frac{8(4+\delta)}{20 + 3\delta}$$

which is obviously satisfied since

$$q > 4 + \delta > \frac{8(4+\delta)}{20+3\delta}.$$

Therefore,

$$R_1 \le c(T)\|\overline{\psi}\|_{\mathcal{B}}\|\overline{\varphi}_1\|_{\mathcal{B}} \le c(T)\|\overline{\psi}\|_{\mathcal{B}}. \tag{3.13}$$

Similarly, by virtue of the imbedding (3.4),

$$R_2 \le cT^{1/2}\|\nabla\overline{\psi}\|_{\mathbf{L}_2(\Omega^T)} \le c(T)\|\overline{\psi}\|_{\mathcal{B}}. \tag{3.14}$$

Combining estimates (3.12)–(3.14) and using Korn's inequality, we conclude that

$$\|\mathbf{v}_t\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + \|\mathbf{v}\|_{L_\infty(0,T;\mathbf{V}_0)} \le c(T)\|\overline{\psi}\|_{\mathcal{B}}. \tag{3.15}$$

This shows the assertion. □

3.3. The parabolic problem. Here we study the second step of the construction of the solution map, i.e. the map

$$\mathcal{T}_2(\tau, \cdot\,, \cdot): \mathcal{B} \times \mathcal{V} \ni (\overline{\varphi}, \mathbf{u}) \mapsto \varphi \in \mathcal{B} \tag{3.16}$$

that gives a solution φ of (3.2) for a given $\overline{\varphi}$ and $\mathbf{u} = \mathcal{T}_1(\tau, \overline{\varphi})$. In view of growth conditions $(1.21)_1$,

$$\|\overline{H}\|_{L_\infty(0,T;L_2(\Omega))} \le c(T)(\|\overline{\varphi}\|_{\mathcal{B}} + \|\varepsilon(\mathbf{u})\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))}). \tag{3.17}$$

Hence, applying the regularity result [15, Theorem 3.1], we conclude that the parabolic problem (3.2) has a unique solution $\varphi \in W^{2,1}_{2,q}(\Omega^T)$ for $p = 2 \le q < \infty$, and there is a constant $c(T)$ such that

$$\begin{aligned} \|\varphi\|_{W^{2,1}_{2,q}(\Omega^T)} &\le c^*(\|\overline{H}\|_{L_q(0,T;L_2(\Omega))} + \|\varphi_0\|_{B^{2(1-1/q)}_{2,q}(\Omega)}) \\ &\le c(T)(\|\overline{\varphi}\|_{\mathcal{B}} + \|\varepsilon(\mathbf{u})\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + \|\varphi_0\|_{B^{2(1-1/q)}_{2,q}(\Omega)}). \end{aligned} \tag{3.18}$$

In view of the compactness of the imbedding of the space $W^{2,1}_{2,q}(\Omega^T)$ into $\mathcal{B}$ (see (2.13)) estimate (3.18) for $4 + \delta < q < \infty$, shows that the map $\mathcal{T}_2(\tau, \cdot\,, \cdot)$ is well-defined and compact.

We proceed to show that $\mathcal{T}_2(\tau, \cdot\,, \cdot)$ is continuous. Let φ_1 and φ_2 be the solutions of (3.2) corresponding respectively to $(\overline{\varphi}_1, \mathbf{u}_1) \in \mathcal{B} \times \mathcal{V}$ and $(\overline{\varphi}_2, \mathbf{u}_2) \in \mathcal{B} \times \mathcal{V}$, where $\mathbf{u}_1 = \mathcal{T}_1(\tau, \overline{\varphi}_1)$ and $\mathbf{u}_2 = \mathcal{T}_1(\tau, \overline{\varphi}_2)$. Subtracting the corresponding equations, and denoting

$$\psi = \varphi_1 - \varphi_2, \quad \overline{\psi} = \overline{\varphi}_1 - \overline{\varphi}_2, \quad \mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2,$$

we get the following problem

$$
(3.19)\qquad
\begin{aligned}
\beta\psi_t - \gamma \triangle \psi &= -\tau[(\Psi'(\overline{\varphi}_1) - \Psi'(\overline{\varphi}_2)) + (w'(\overline{\varphi}_1) - w'(\overline{\varphi}_2))]\\
&\quad - \tau[(z'(\overline{\varphi}_1) - z'(\overline{\varphi}_2))(\mathbf{B}\cdot\varepsilon(\mathbf{u}_1) + h(\overline{\varphi}_1))]\\
&\quad - \tau[z'(\overline{\varphi}_2)(\mathbf{B}\cdot\varepsilon(\mathbf{v}) + h(\overline{\varphi}_1) - h(\overline{\varphi}_2))]\\
&\equiv \overline{\overline{H}}_1 + \overline{\overline{H}}_2 + \overline{\overline{H}}_3 = \overline{\overline{H}} && \text{in } \Omega^T,\\
\psi|_{t=0} &= 0 && \text{in } \Omega,\\
\psi &= 0 && \text{on } S^T.
\end{aligned}
$$

Lemma 3.3. *Under assumptions* (A1)–(A4) *the map* $\mathcal{T}_2(\tau,\,\cdot\,,\,\cdot\,)$ *defined by* (3.16) *is continuous.*

Proof. By conditions on $\Psi(\,\cdot\,)$, $z(\,\cdot\,)$, $h(\,\cdot\,)$ and $w(\,\cdot\,)$ specified in (A3) it follows that

$$
\begin{aligned}
|\overline{\overline{H}}_1| &\le c(\overline{\varphi}_1^2 + \overline{\varphi}_2^2 + 1)|\overline{\psi}|,\\
|\overline{\overline{H}}_2| &\le c(|\varepsilon(\mathbf{u}_1)| + |\overline{\varphi}_1| + 1)|\overline{\psi}|,\\
|\overline{\overline{H}}_3| &\le c(|\varepsilon(\mathbf{v})| + |\overline{\psi}|).
\end{aligned}
$$

Hence, in view of the imbedding (2.14),

$$
\begin{aligned}
\|\overline{\overline{H}}_1\|_{L_\infty(0,T;L_2(\Omega))} &\le c(T)(\|\overline{\varphi}_1\|_{\mathcal{B}}^2 + \|\overline{\varphi}_2\|_{\mathcal{B}}^2 + 1)\|\overline{\psi}\|_{\mathcal{B}},\\
\|\overline{\overline{H}}_2\|_{L_\infty(0,T;L_2(\Omega))} &\le c(T)(\|\varepsilon(\mathbf{u}_1)\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + \|\overline{\varphi}_1\|_{\mathcal{B}} + 1)\|\overline{\psi}\|_{\mathcal{B}},\\
\|\overline{\overline{H}}_3\|_{L_\infty(0,T;L_2(\Omega))} &\le c(T)(\|\varepsilon(\mathbf{v})\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + \|\overline{\psi}\|_{\mathcal{B}}).
\end{aligned}
$$

Consequently, using a priori estimate (3.10), we get

$$
(3.20)\qquad \|\overline{\overline{H}}\|_{L_\infty(0,T;L_2(\Omega))} \le c(T)(\|\overline{\psi}\|_{\mathcal{B}} + \|\varepsilon(\mathbf{v})\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))}).
$$

Applying the regularity result [15, Theorem 3.1] to problem (3.19), it follows that there exists a constant $c(T)$ such that

$$
(3.21)\quad \|\psi\|_{W^{2,1}_{2,q}(\Omega^T)} \le c^*\|\overline{\overline{H}}\|_{L_q(0,T;L_2(\Omega))} \le c(T)(\|\overline{\psi}\|_{\mathcal{B}} + \|\varepsilon(\mathbf{v})\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))})
$$

for $p = 2 \le q < \infty$. This shows the assertion. $\square$

3.4. Properties of the solution map. In the view of the properties of $\mathcal{T}_1(\tau,\,\cdot\,)$ and $\mathcal{T}_2(\tau,\,\cdot\,,\,\cdot\,)$ we can conclude that for any $\tau \in [0,1]$ the composed map

$$
(3.22)\qquad \mathcal{T}(\tau,\cdot) = \mathcal{T}_2(\tau,\,\cdot\,,\mathcal{T}_1(\tau,\,\cdot\,)) : \mathcal{B} \ni \overline{\varphi} \mapsto \varphi \in \mathcal{B}
$$

is well-defined, continuous and compact, i.e., completely continuous. In particular, by virtue of (3.10) and (3.18) the following a priori estimate is satisfied

$$
(3.23)\qquad \|\varphi\|_{W^{2,1}_{2,q}(\Omega^T)} \le c(T)(\|\overline{\varphi}\|_{\mathcal{B}} + \|\varphi_0\|_{B^{2(1-1/q)}_{2,q}(\Omega)} + c_1) \equiv c_2.
$$

This shows property (a) of the Leray–Schauder theorem.

The property (b) follows by direct comparison of two solutions $(\mathbf{u},\varphi)$ and $(\widetilde{\mathbf{u}},\widetilde{\varphi})$ to problem (3.1), (3.2) corresponding respectively to parameters τ and $\widetilde{\tau}$. The differences

$$\mathbf{v}=\mathbf{u}-\widetilde{\mathbf{u}},\qquad \psi=\varphi-\widetilde{\varphi}$$

satisfy

$$\begin{aligned} \mathbf{v}_{tt}-\nabla\cdot\mathbf{A}\varepsilon(\mathbf{v}) &= (\tau-\widetilde{\tau})[z'(\overline{\varphi})\mathbf{B}\nabla\overline{\varphi}+\mathbf{b}] && \text{in } \Omega^T,\\ \mathbf{v}|_{t=0} &= (\tau-\widetilde{\tau})\mathbf{u}_0,\quad \mathbf{v}_t|_{t=0}=(\tau-\widetilde{\tau})\mathbf{u}_1 && \text{in } \Omega,\\ \mathbf{v} &= 0 && \text{on } S^T, \end{aligned} \tag{3.24}$$

$$\begin{aligned} \beta\psi_t-\gamma\,\triangle\,\psi =& -(\tau-\widetilde{\tau})[\Psi'(\overline{\varphi})+w'(\overline{\varphi})+z'(\overline{\varphi})(\mathbf{B}\cdot\varepsilon(\mathbf{u})+h(\overline{\varphi}))]\\ &-\widetilde{\tau}z'(\overline{\varphi})\mathbf{B}\cdot\varepsilon(\mathbf{v})\equiv\widetilde{H}\quad \text{in } \Omega^T,\\ \psi|_{t=0} =& (\tau-\widetilde{\tau})\varphi_0 \qquad \text{in } \Omega,\\ \psi =& 0 \qquad \text{on } S^T. \end{aligned} \tag{3.25}$$

Analogously to (3.10) and (3.23) we get the following estimates on $\mathbf{v}$ and ψ:

$$\|\mathbf{v}_t\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))}+\|\mathbf{v}\|_{L_\infty(0,T;\mathbf{V}_0)}\le c_1|\tau-\widetilde{\tau}|, \tag{3.26}$$

$$\begin{aligned} \|\psi\|_{W^{2,1}_{2,q}(\Omega^T)} &\le c^*(\|\widetilde{H}\|_{L_q(0,T;L_2(\Omega))}+\|(\tau-\widetilde{\tau})\varphi_0\|_{B^{2(1-1/q)}_{2,q}(\Omega)})\\ &\le (c_1+c_2)|\tau-\widetilde{\tau}|, \end{aligned} \tag{3.27}$$

where in the last inequality we have used (3.26). This means that for $\overline{\varphi}$ in a bounded subset of $\mathcal{B}$ the map $\mathcal{T}(\,\cdot\,,\overline{\varphi})\colon[0,1]\to\mathcal{B}$ is equicontinuous and the property (b) is satisfied.

The property (c) for $\tau=1$ results from a priori estimate (2.11). It is easy to see that the same holds also true for $0<\tau<1$.

For $\tau=0$ problem (3.1), (3.2) has the unique solution $\mathbf{u}=0$, $\varphi=0$, so that property (d) is satisfied.

Summarizing, we have shown that the solution map (3.22) satisfies assumptions (a)–(d) of the Leray–Schauder fixed point theorem. Thus

$$\mathcal{T}(1,\,\cdot\,)=\mathcal{T}_2(1,\,\cdot\,,\,\cdot\,)\circ\mathcal{T}_1(1,\,\cdot\,)$$

has at least one fixed point $\varphi\in\mathcal{B}$. At the same time the pair $(\mathbf{u},\varphi)$, with $\mathbf{u}=\mathcal{T}_1(1,\varphi)$, is a solution of problem (1.14), (1.15).

Recalling a priori bounds (2.1), (2.5) and (2.11) the proof of Theorem 1.3 is completed. □

4. Proof of Theorem 1.4 (uniqueness)

Let $(\mathbf{u},\varphi)$ and $(\widetilde{\mathbf{u}},\widetilde{\varphi})$ be two solutions of (1.14), (1.15) corresponding to the same data. Subtracting the corresponding equations and denoting:

$$\mathbf{v}=\mathbf{u}-\widetilde{\mathbf{u}},\quad \psi=\varphi-\widetilde{\varphi},$$

we see that $\mathbf{v}$, ψ satisfy the following problems

$$\begin{aligned}
&\mathbf{v}_{tt}-\nabla\cdot(\mathbf{A}\varepsilon(\mathbf{v}))=(z'(\varphi)-z'(\widetilde{\varphi}))\mathbf{B}\nabla\varphi+z'(\widetilde{\varphi})\mathbf{B}\nabla\psi\equiv\mathbf{G} && \text{in } \Omega^T,\\
&\mathbf{v}|_{t=0}=0,\quad \mathbf{v}_t|_{t=0}=0 && \text{in } \Omega,\\
&\mathbf{v}=0 && \text{on } S^T,
\end{aligned}\tag{4.1}$$

$$\begin{aligned}
\beta\psi_t-\gamma\,\triangle\,\psi={}&-(\Psi'(\varphi)-\Psi'(\widetilde{\varphi}))-(w'(\varphi)-w'(\widetilde{\varphi}))\\
&-[(z'(\varphi)-z'(\widetilde{\varphi}))(\mathbf{B}\cdot\varepsilon(\mathbf{u})+h(\varphi))]\\
&-[z'(\widetilde{\varphi})(\mathbf{B}\cdot\varepsilon(\mathbf{v})+h(\varphi)-h(\widetilde{\varphi}))]\equiv H && \text{in } \Omega^T,\\
\psi|_{t=0}={}&0 && \text{in } \Omega,\\
\psi={}&0 && \text{on } S^T.
\end{aligned}\tag{4.2}$$

The idea of the proof is to derive energy estimates for the system (4.1), (4.2) which will allow to conclude the uniqueness by means of Gronwall's inequality.

In the first step, proceeding similarly as in Lemma 3.2, we obtain estimates on $\mathbf{v}$ in terms of ψ. Multiplying $(4.1)_1$ by $\mathbf{v}_t$, integrating over Ω and by parts, and using the Cauchy-Schwartz inequality, we deduce that

$$\frac{d}{dt}\bigg[\int_\Omega(|\mathbf{v}_t|^2+|\mathbf{A}\varepsilon(\mathbf{v})|^2)\,dx\bigg]^{1/2}\le\bigg(\int_\Omega|\mathbf{G}|^2\,dx\bigg)^{1/2}.$$

Hence, upon inegrating with respect to time,

$$\begin{aligned}
\bigg[\int_\Omega(|\mathbf{v}_t|^2+|\ \mathbf{A}\varepsilon(\mathbf{v})|^2)\,dx\bigg]^{1/2}&\le\int_0^t\bigg(\int_\Omega|\mathbf{G}(t')|^2\,dx\bigg)^{1/2}dt'\\
&=\int_0^t\|\mathbf{G}(t')\|_{\mathbf{L}_2(\Omega)}dt'\equiv R(t)
\end{aligned}\tag{4.3}$$

for all $t\in[0,T]$. By means of Hölder's inequality, recalling the boundedeness assumption on $z'(\,\cdot\,)$, we have

$$\begin{aligned}
R(t)&\le c\int_0^t(\|\psi\nabla\varphi\|_{\mathbf{L}_2(\Omega)}+\|\nabla\psi\|_{\mathbf{L}_2(\Omega)})\,dt'\\
&\le c\int_0^t(\|\psi\|_{L_4(\Omega)}\|\nabla\varphi\|_{\mathbf{L}_4(\Omega)}+\|\nabla\psi\|_{\mathbf{L}_2(\Omega)})\,dt'\\
&\le c(\|\psi\|_{L_2(0,t;L_4(\Omega))}\|\nabla\varphi\|_{L_2(0,t;\mathbf{L}_4(\Omega))}+\|\nabla\psi\|_{L_1(0,t;\mathbf{L}_2(\Omega))}).
\end{aligned}$$

In view of the imbedding $W^{1,1/2}_{2,q}(\Omega^T) \subset L_2(0,T;L_4(\Omega))$, which holds true provided

$$\frac{3}{2}+\frac{2}{q}-\frac{3}{4}-\frac{2}{2}\le 1, \quad \text{that is } q \ge \frac{8}{5},$$

we can see that

$$\|\nabla\varphi\|_{L_2(0,t;\mathbf{L}_4(\Omega))} \le c\|\nabla\varphi\|_{\mathbf{W}^{1,1/2}_{2,q}(\Omega^T)} \le c\|\varphi\|_{W^{2,1}_{2,q}(\Omega^T)} \le c(T),$$

where in the last inequality we have used a priori estimate (1.23). Hence

$$R(t) \le c(T)(\|\psi\|_{L_2(0,t;L_4(\Omega))} + \|\nabla\psi\|_{L_1(0,t;\mathbf{L}_2(\Omega))}).$$

Consequently, by Korn's inequality, (4.3) yields

$$\|\mathbf{v}_t\|_{L_\infty(0,t;\mathbf{L}_2(\Omega))} + \|\mathbf{v}\|_{L_\infty(0,t;\mathbf{V}_0)} \le c(T)(\|\psi\|_{L_2(0,t;L_4(\Omega))} + \|\nabla\psi\|_{L_1(0,t;\mathbf{L}_2(\Omega))}) \tag{4.4}$$

for $t\in[0,T]$.

In the second step we estimate the solution ψ of (4.2). We multiply $(4.2)_1$ by ψ, integrate over Ω and by parts to get:

$$\frac{\beta}{2}\frac{d}{dt}\int_\Omega |\psi|^2\,dx + \gamma\int_\Omega |\nabla\psi|^2 dx = \int_\Omega H\psi\,dx,$$

whence

$$\frac{\beta}{2}\int_\Omega |\psi|^2\,dx + \gamma\int_{\Omega^t} |\nabla\psi|^2\,dx\,dt' = \int_{\Omega^t} H\psi\,dx\,dt' \quad \text{for all } t\in[0,T]. \tag{4.5}$$

We estimate the right-hand side of (4.5)

$$\begin{aligned}\int_{\Omega^t} H\psi\,dx\,dt' \le& \int_{\Omega^t} (|\Psi'(\varphi)-\Psi'(\widetilde{\varphi})| + |w'(\varphi)-w'(\widetilde{\varphi})|)|\psi|\,dx\,dt'\\ &+ \int_{\Omega^t} |z'(\varphi)-z'(\widetilde{\varphi})|\,|\mathbf{B}\cdot\varepsilon(\mathbf{u})+h(\varphi)|\,|\psi|\,dx\,dt'\\ &+ \int_{\Omega^t} |z'(\widetilde{\varphi})|\,|\mathbf{B}\cdot\varepsilon(\mathbf{v})+h(\varphi)-h(\widetilde{\varphi})|\,|\psi|\,dx\,dt' \equiv \mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}$$

Due to $L_\infty(\Omega^T)$-norm estimate (1.24) on φ,

$$I \le c(T)\int_{\Omega^t} |\psi|^2\,dx\,dt'.$$

Further, in view of assumptions on $z(\,\cdot\,)$ and $h(\,\cdot\,)$,

$$\mathrm{II} \le c\int_{\Omega^t} |\varepsilon(\mathbf{u})|\,|\psi|^2 dxdt' + c\int_{\Omega^t} |\psi|^2\,dx\,dt' \equiv \mathrm{II}_1+\mathrm{II}_2.$$

For the term II_1, using the Cauchy–Schwartz inequality and the interpolation inequality

$$\|\psi\|_{L_4(\Omega)} \le c\|\psi\|^{1/4}_{L_2(\Omega)}\|\nabla\psi\|^{3/4}_{\mathbf{L}_2(\Omega)} + c\|\psi\|_{L_2(\Omega)}, \tag{4.6}$$

we have

$$
\begin{aligned}
\mathrm{II}_1 &\le c\int_0^t \|\varepsilon(\mathbf{u})\|_{\mathbf{L}_2(\Omega)}\|\psi\|^2_{L_4(\Omega)}\,dt' \\
&\le c\int_0^t \|\varepsilon(\mathbf{u})\|_{\mathbf{L}_2(\Omega)}\|\psi\|^{1/2}_{L_2(\Omega)}\|\nabla\psi\|^{3/2}_{\mathbf{L}_2(\Omega)}\,dt' + c\int_0^t \|\varepsilon(\mathbf{u})\|_{\mathbf{L}_2(\Omega)}\|\psi\|^2_{L_2(\Omega)}\,dt' \\
&\equiv \mathrm{II}_{11} + \mathrm{II}_{12}.
\end{aligned}
$$

By virtue of a priori estimate (1.23),

$$
\|\varepsilon(\mathbf{u})\|_{L_\infty(0,t;\mathbf{L}_2(\Omega))} \le c(T), \tag{4.7}
$$

hence

$$
\mathrm{II}_{12} \le c(T)\int_{\Omega^t} |\psi|^2\,dx\,dt'.
$$

Turning to II_{11}, with the help of Young's inequality we obtain

$$
\begin{aligned}
\mathrm{II}_{11} &\le c\int_0^t \left[\frac{\delta^{\lambda_1}}{\lambda_1}\|\nabla\psi\|^{3\lambda_1/2}_{\mathbf{L}_2(\Omega)} + \frac{1}{\lambda_2\delta^{\lambda_2}}(\|\varepsilon(\mathbf{u})\|_{\mathbf{L}_2(\Omega)}\|\psi\|^{1/2}_{L_2(\Omega)})^{\lambda_2}\right] dt' \\
&\le \int_0^t [\delta\|\nabla\psi\|^2_{\mathbf{L}_2(\Omega)} + c_\delta\|\varepsilon(\mathbf{u})\|^4_{\mathbf{L}_2(\Omega)}\|\psi\|^2_{L_2(\Omega)}]\,dt',
\end{aligned}
$$

where $\lambda_1 = 4/3$, $\lambda_2 = 4$, $\delta > 0$. Hence, using (4.7) again, it follows that

$$
\mathrm{II}_{11} \le \delta\int_{\Omega^t} |\nabla\psi|^2\,dx\,dt' + c(T)\int_{\Omega^t} |\psi|^2\,dx\,dt'.
$$

Next, in view of the assumptions on $z(\,\cdot\,)$ and $h(\,\cdot\,)$,

$$
\mathrm{III} \le c\int_{\Omega^t} |\varepsilon(\mathbf{v})|\,|\psi|\,dxdt' + c\int_{\Omega^t} |\psi|^2\,dx\,dt' \equiv \mathrm{III}_1 + \mathrm{III}_2.
$$

By virtue of estimate (4.4),

$$
\begin{aligned}
\mathrm{III}_1 &\le c\|\varepsilon(\mathbf{v})\|_{L_2(0,t;\mathbf{L}_2(\Omega))}\|\psi\|_{L_2(0,t;L_2(\Omega))} \\
&\le c(T)(\|\psi\|_{L_2(0,t;L_4(\Omega))} + \|\nabla\psi\|_{L_1(0,t;\mathbf{L}_2(\Omega))})\|\psi\|_{L_2(0,t;L_2(\Omega))} \\
&\le c(T)\|\psi\|^2_{L_2(0,t;L_4(\Omega))} + c(T)\|\nabla\psi\|_{L_2(0,t;\mathbf{L}_2(\Omega))}\|\psi\|_{L_2(0,t;L_2(\Omega))} \\
&\equiv \mathrm{III}_{11} + \mathrm{III}_{12}.
\end{aligned}
$$

Applying interpolation inequality (4.6) and next Young's inequality we infer that

$$
\begin{aligned}
\mathrm{III}_{11} &\le c(T)\int_0^t \|\psi\|^{1/2}_{L_2(\Omega)}\|\nabla\psi\|^{3/2}_{\mathbf{L}_2(\Omega)}\,dt' + c(T)\int_0^t \|\psi\|^2_{L_2(\Omega)}\,dt' \\
&\le c(T)\int_0^t \left(\frac{\delta^{\lambda_1}}{\lambda_1}\|\nabla\psi\|^{3\lambda_1/2}_{\mathbf{L}_2(\Omega)} + \frac{1}{\lambda_2\delta^{\lambda_2}}\|\psi\|^{\lambda_2/2}_{L_2(\Omega)}\right) dt' + c(T)\int_0^t \|\psi\|^2_{L_2(\Omega)}\,dt' \\
&\le \delta\int_{\Omega^t} |\nabla\psi|^2\,dx\,dt' + c(\delta,T)\int_{\Omega^t} |\psi|^2\,dx\,dt',
\end{aligned}
$$

where $\lambda_1 = 4/3$, $\lambda_2 = 4$, $\delta > 0$.

Finally, by Young's inequality,

$$\mathrm{III}_{12} \leq \delta \int_{\Omega^t} |\nabla\psi|^2 dx dt' + c(\delta, T) \int_{\Omega^t} |\psi|^2 \, dx \, dt'.$$

Combining the estimates on the terms I, II and III in (4.5) we see that

$$\frac{\beta}{2} \int_{\Omega} |\psi|^2 \, dx + \gamma \int_{\Omega^t} |\nabla\psi|^2 \, dx \, dt' \leq \delta \int_{\Omega^t} |\nabla\psi|^2 \, dx \, dt' + c(\delta, T) \int_{\Omega^t} |\psi|^2 \, dx \, dt',$$

whence, choosing $\delta = \gamma/2$,

$$\frac{\beta}{2} \|\psi\|^2_{L_2(\Omega)} + \frac{\gamma}{2} \int_0^t \|\nabla\psi\|^2_{\mathbf{L}_2(\Omega)} \, dt' \leq c(T) \int_0^t \|\psi\|^2_{L_2(\Omega)} \, dt' \tag{4.8}$$

for all $t \in [0, T]$. Consequently, due to Gronwall's inequality, it follows from (4.8) that $\psi = 0$ a.e. in Ω^T. At the same time, by virtue of (4.4), $\mathbf{v} = 0$ a.e. in Ω^T. Hence the uniqueness of $(\mathbf{u}, \varphi)$ is proved. $\square$

5. The existence of solution for other boundary conditions

As mentioned in the introduction the problem (1.1), (1.2) can be considered with other boundary conditions. We show here that the existence result of Theorem 1.3 can be extended to the following boundary conditions:

For the elasticity system we assume

$$\begin{aligned} \mathbf{u} &= 0 \quad \text{on } S_D^T = S_D \times (0, T), \\ W_{/\varepsilon}(\varepsilon, \varphi)\mathbf{n} &= 0 \quad \text{on } S_N^T = S_N \times (0, T). \end{aligned} \tag{5.1}$$

Here we have the boundary decomposition $S = \overline{S_D} \cup \overline{S_N}$, $\quad S_D \cap S_N = \varnothing$ where S_D and S_N are open subsets of S, and *meas* $S_D > 0$, i.e. S_D and S_N are disjoint parts of the boundary S on which zero displacement and zero traction boundary conditions are prescribed.

The corresponding expressions for the stress tensor $W_{/\varepsilon}(\varepsilon, \varphi)$ are given by $(1.12)_2$ or $(1.13)_2$.

The condition *meas* $S_D > 0$ is assumed in order to guarantee Korn's inequality.

For the order parameter we assume the homogeneous Neumann boundary condition (no flux condition)

$$\mathbf{n} \cdot \nabla\varphi = 0 \quad \text{on } S^T = S \times (0, T), \tag{5.2}$$

which is the typical condition in phase field models.

We have the following

Theorem 5.1. *Let us consider problem* (1.14), (1.15) *with boundary conditions* (5.1) *and* (5.2) *in place of* $(1.14)_3$ *and* $(1.15)_3$. *Let the assumptions* (A1)–(A4) *be satisfied, and meas* $S_D > 0$. *Then for any* $T > 0$ *there exists a pair* $(\mathbf{u}, \varphi)$ *which satisfies the assertion of Theorem* 1.3 *with*

$$\mathbf{V}_0 = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \mathbf{u} = 0 \textit{ on } S_D\}. \tag{5.3}$$

Proof. We proceed the same way as in Theorem 1.3. In connection with different boundary conditions there are three main points that have to be examined:

- The energy estimates.
- The applicability of the maximal regularity results for parabolic problem with homogeneous Neumann boundary condition.
- The solvability of the elasticity system with mixed boundary condition (5.1).

We consider these points one after another.

The procedure of getting energy estimates is the same as presented in Section 2. We multiply equation $(1.1)_1$ by $\mathbf{u}_t$, integrate over Ω and by parts. Further, we multiply $(1.2)_1$ by φ_t, integrate over Ω and by parts. Adding the results we arrive at

Lemma 5.2. *For the system of equations* $(1.1)_1$ *and* $(1.2)_1$ *the following energy identity is satisfied*

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{u}_t|^2 dx + \frac{d}{dt}\int_\Omega f(\varepsilon,\varphi,\nabla\varphi)\,dx + \beta\int_\Omega \varphi_t^2\,dx \\ + \int_S [-(f_{/\varepsilon}(\varepsilon,\varphi,\nabla\varphi)\mathbf{n})\cdot\mathbf{u}_t - (\mathbf{n}\cdot f_{/\nabla\varphi}(\varepsilon,\varphi,\nabla\varphi))\varphi_t]\,dS \\ = \int_\Omega \mathbf{b}\cdot\mathbf{u}_t\,dx \qquad \textit{for } t\in(0,T). \end{aligned} \tag{5.4}$$

From the above energy identity we can conclude immediately the following extension of Lemma 2.1.

Lemma 5.3. *Let us consider problem* (1.1), (1.2) *with boundary conditions on* S *consistent with the requirement*

$$(f_{/\varepsilon}(\varepsilon,\varphi,\nabla\varphi)\mathbf{n})\cdot\mathbf{u}_t = 0, \quad (\mathbf{n}\cdot f_{/\nabla\varphi}(\varepsilon,\varphi,\nabla\varphi))\varphi_t = 0 \quad \textit{on } S^T. \tag{5.5}$$

Assume $f(\varepsilon,\varphi,\nabla\varphi)$ *satisfies structure condition (1.20), and the data are such that*

$$f(\varepsilon_0,\varphi_0,\nabla\varphi_0)\in L_1(\Omega), \quad \mathbf{u}_1\in\mathbf{L}_1(\Omega), \quad \mathbf{b}\in L_1(0,T;\mathbf{L}_2\Omega)).$$

Then for a solution $(\mathbf{u}, \varphi)$ *the following energy estimates hold*

$$\|\mathbf{u}_t\|_{L_\infty(0,T:\mathbf{L}_2(\Omega))} + \|\varepsilon(\mathbf{u})\|_{L_\infty(0,T:\mathbf{L}_2(\Omega))} + \|\varphi\|_{L_\infty(0,T:L_4(\Omega))} + \|\nabla\varphi\|_{L_\infty(0,T:\mathbf{L}_2(\Omega))} + \|\varphi_t\|_{L_2(\Omega^T)} \le c_0 \tag{5.6}$$

with constant c_0 *depending only on the data.*

We note that for $f(\varepsilon, \varphi, \nabla\varphi)$ given by (1.3) the boundary conditions (5.1), (5.2) are consistent with (5.5).

In view of estimate (5.6) the conclusions (2.6)–(2.8) in Section 2 remain unchanged.

The second, most important question concerns the applicability of the maximal regularity results. First we note that the results due to Weidemaier [14], [15] originally apply to parabolic problems with inhomogeneous boundary conditions, Dirichlet or conormal boundary conditions. These results require some range restrictions on the parameters p and q in the space $L_q(0, t; L_p(\Omega))$ to which the right-hand side of the parabolic equation belongs.

In particular, in case of inhomogeneous Dirichlet boundary condition the relation is (see [15, Theorem 3.1])

$$3/2 < p \le q < \infty \tag{5.7}$$

and in case of conormal boundary condition (including Neumann condition) the relation is (see [15, Theorem 3.2])

$$3 < p \le q < \infty. \tag{5.8}$$

Such range restrictions assure that the corresponding trace operators are well-defined and onto.

We note that condition (5.8) does not allow to apply directly the maximal regularity result [15, Theorem 3.2], to problem (1.15) with homogeneous Neumann condition (5.2) because for the right-hand side H we know only that (see (2.7), (2.8))

$$H \in L_\infty(0, T; L_2(\Omega)).$$

At this point we underline that the above restrictions (5.7) and (5.8) can be relased in case of homogeneous boundary conditions. In particular, for problem $(1.15)_{1,2}$ with homogeneous Neumann condition (5.2) the following compact version of known results can be formulated

Lemma 5.4 (P. Weidemaier, personal letters). *Let us consider problem*

$$\begin{aligned} \varphi_t - \triangle\varphi &= H \quad && \text{in } \Omega^T, \\ \varphi|_{t=0} &= \varphi_0 && \text{in } \Omega, \\ \mathbf{n}\cdot\nabla\varphi &= 0 && \text{on } S^T. \end{aligned} \tag{5.9}$$

Assme that Ω is a bounded domain in $\mathbb{R}^3$ of class $C^{2+\alpha}$ for some $\alpha > 0$ and that

$$1 < p, q < \infty \quad \textit{and} \quad \frac{1}{p} + \frac{2}{q} \neq 1,$$
$$H \in L_q(0,T; L_p(\Omega)), \quad \varphi_0 \in B^{2(1-1/q)}_{p,q}(\Omega),$$

and φ_0 satisfies compatibility condition

$$\mathbf{n} \cdot \nabla \varphi_0 = 0 \quad \textit{for } \frac{1}{p} + \frac{2}{q} < 1.$$

Then problem (5.9) *has a unique solution $\varphi \in W^{2,1}_{p,q}(\Omega^T)$, and there exists a constant $c^*(p,q,T)$ such that*

$$\|\varphi\|_{W^{2,1}_{p,q}(\Omega^T)} \leq c^*(\|H\|_{L_q(0,T;L_p(\Omega))} + \|\varphi_0\|_{B^{2(1-1/q)}_{p,q}(\Omega)}). \tag{5.10}$$

By virtue of Lemma 5.3 and estimate (2.8) it follows that

$$\|\varphi\|_{W^{2,1}_{2,q}(\Omega^T)} \leq c(T) \quad \text{for } q \neq 4. \tag{5.11}$$

In particular, (5.11) holds true in the range under consideration

$$4 + \delta < q < \infty \quad \text{for some } \delta > 0.$$

Consequently the key imbeddings (2.12)–(2.14) remain unaltered.

The last point concerns the solvability of the elasticity system $(3.1)_{1,2}$ with boundary conditions (5.1). Similarly as in Section 3.2 the existence of solution follows from general results [4, Chapter III, Theorem 4.1]. The statements (3.6), (3.7) hold with the space $\mathbf{V}_0$ defined by (5.3).

Consequently, the identity (3.8) and the resulting estimation remain unaltered.

Finally, we point out that in (3.10) we make use of the following version of Korn's inequality (see [4, Chap. II, Thm 3.3])

$$\int_\Omega (\mathbf{A}\varepsilon(\mathbf{u})) \cdot \varepsilon(\mathbf{u})\, dx \geq \alpha_0 \|\mathbf{u}\|^2_{\mathbf{H}^1(\Omega)}$$
$$\text{for all } \mathbf{u} \in \mathbf{V}_0 = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \mathbf{u} = 0 \text{ on } S_D\}$$

with some constant $\alpha_0 > 0$. With the above remark the proof is completed. □

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