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# SOME PROPERTIES OF INFINITE DIMENSIONAL DISCRETE OPERATORS

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Abstract. The paper is devoted to infinite dimensional discrete operators that can be considered as a difference analog of differential equations on the whole axis. We obtain a necessary and sufficient condition in order for the linear operator to be normally solvable. Topological degree for nonlinear operators is constructed.

## 1. Introduction

In this work we consider linear

$$(Lu)_j = a^j_{-m}u_{j-m} + \ldots + a^j_0 u_j + \ldots + a^j_m u_{j+m}, \quad j \in \mathbb{Z} \tag{1.1}$$

and semilinear discrete operators

$$(Au)_j = a^j_{-m}u_{j-m} + \ldots + a^j_0 u_j + \ldots + a^j_m u_{j+m} + F(u_j), \quad j \in \mathbb{Z}, \tag{1.2}$$

in the Banach space $E$ of sequences $\{u_j\}_{j\in\mathbb{Z}}$ with the supremum norm. Here $m \geq 0$ is a given integer. These operators generalize the operators in the left-hand side of the linear

$$a_j(u_{j+1} - 2u_j + u_{j-1}) + b_j(u_{j+1} - u_{j-1}) + c_j u_j = f_j, \quad j \in \mathbb{Z} \tag{1.3}$$

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and semilinear

$$a_j(u_{j+1} - 2u_j + u_{j-1}) + b_j(u_{j+1} - u_{j-1}) + F(u_j) = 0, \quad j \in \mathbb{Z}$$

problems arising as discretization of second order ordinary differential equations on the whole axis. Problems of this type are intensively studied in relation with discrete travelling waves (see [4]) or with convergence of finite difference schemes (see [1]). Properties of solutions of infinite algebraic systems are studied in [2], [3], [6], [9] in some particular cases.

If we introduce an infinite matrix $A$ and the vectors $U = (\dots, u_{-1}, u_0, u_1, \dots)$ and $f = (\dots, f_{-1}, f_0, f_1, \dots)$ we can write (1.3) in the matrix form $AU = f$. Contrary to the finite dimensional case where the solvability condition for this equation is given by the Fredholm alternative, the solvability condition in the infinite dimensional case to our knowledge is not known. This situation is to some extent similar to the case of elliptic problems where we cannot directly apply the results and the methods developed for the case of bounded domains to the case of unbounded domains. The theory of elliptic operators in unbounded domains is now well developed [11], [10]. We cannot use it directly to study discretized operators. It appears however that many of the approaches developed for elliptic problems in unbounded domains can be adapted for infinite dimensional algebraic systems. This concerns not only linear but also nonlinear problems.

In this work we prove normal solvability for a class of discrete operators. It should be noted that the condition of normal solvability is not always fulfilled and the operator may not satisfy the Fredholm property. We find conditions when $L$ is a Fredholm operator with the index zero and $A$ is proper. The topological degree for a class of nonlinear operators is constructed.

In Section 2 we define the spaces and the linear operators we study in this work. Section 3 is devoted to the linear difference equations with constant coefficients

$$a_{-m}u_{j-m} + \dots + a_0 u_j + \dots + a_m u_{j+m} = 0, \quad j \in \mathbb{Z}.$$

Necessary and sufficient conditions are established in order for this equation to have a bounded nonzero solution, and the form of the bounded solution is given (Proposition 3.3). This is an auxiliary result which is important for the theory we develop in the sequel. In Section 4 one proves that the linear operator given by (1.1) is normally solvable with a finite dimensional kernel if and only if the limiting equations

$$a_{-m}^{\pm}u_{j-m} + \dots + a_0^{\pm} u_j + \dots + a_m^{\pm} u_{j+m} = 0, \quad j \in \mathbb{Z}$$

do not have nonzero bounded solutions (Theorem 4.6). Here and everywhere below in this work we suppose that the coefficients $a_l^j$, $-m \le l \le m$, have limits

$a_l^{\pm}$ as $j \to \pm\infty$. Under some additional conditions, $L$ is a Fredholm operator and its index is zero (Theorem 4.10).

Next section deals with the spectrum of $L$. We introduce the notion of NS-spectrum as a set of points $\lambda$ in the complex plane where the operator $L - \lambda I$ is not normally solvable with a finite dimensional kernel. We give some examples and show that the NS-spectrum of the discrete operator

$$(\widetilde{L}u)_j = a_j(u_{j+1} - 2u_j + u_{j-1}) + b_j(u_{j+1} - u_j) + c_j u_j, \quad j \in \mathbb{Z}$$

approaches the NS-spectrum of the differential operator

$$\overline{L}u = au'' + bu' + cu.$$

The properties of the linear operator $L$ are used to study the properness of the semilinear operator $A$ in (1.2) (Section 6). To this end, one supposes that $F$ is smooth enough and the equations

$$[a_{-m}^{\pm} u_{j-m} + \ldots + a_0^{\pm} u_j + \ldots + a_m^{\pm} u_{j+m}] + c^{\pm} u_j = 0, \quad j \in \mathbb{Z}$$

do not have nonzero bounded solutions. Here we denote

$$c^{\pm} = \lim_{j \to \pm\infty} F'(u_j), \quad \text{if } u_j \to 0. \tag{1.4}$$

One works in a weighted space $E_\mu$, where $\mu_j \geq 1$, $j \in \mathbb{Z}$ and $\mu_j \to \infty$ as $j \to \pm\infty$. Then operator $A$ is proper in $E_\mu$. Moreover, the result is true also for semilinear difference operators $A_\tau \colon E_\mu \to E_\mu$ depending on parameter $\tau \in [0, 1]$,

$$(A_\tau u)_j = a_{-m}^j(\tau) u_{j-m} + \ldots + a_0^j(\tau) u_j + \ldots + a_m^j(\tau) u_{j+m} + F(u_j, \tau),$$

where $a_l^j(\tau)$, $-m \leq l \leq m$ and $F$ are smooth enough (Theorem 6.3).

The last section is devoted to the topological degree for the class of operators $A$ given in (1.2) and a class of homotopies $A_\tau$ of the form (1.4). One uses the topological degree constructed in [10] for Fredholm and proper operators in a more general case.

## 2. Operators and spaces

Let $E$ be the space of sequences $u = \{u_j\}$, $j \in \mathbb{Z}$ of real numbers, with the norm

$$||u|| = \sup_{j \in \mathbb{Z}} |u_j|.$$

Obviously, $E$ is a real Banach space.

Let $L \colon E \to E$ be an operator defined as follows. For any $u = \{u_j\} \in E$, $Lu$ is a sequence $f = \{f_j\}$, $j \in \mathbb{Z}$ such that $(Lu)_j \equiv A_j U_j = f_j$, where $A_j$ and $U_j$

are the $(2m+1)-$vectors,

$$U_j = (u_{j-m}, \dots, u_j, \dots, u_{j+m}), \quad A_j = (a^j_{-m}, \dots, a^j_0, \dots, a^j_m),$$

$$A_j U_j = a^j_{-m} u_{j-m} + \dots + a^j_0 u_j + \dots + a^j_m u_{j+m}, \tag{2.1}$$

the vectors $A_j$ being given.

Example 2.1. The definition is motivated by applications to finite difference approximations of differential equations on the whole axis. A typical example is

$$(Lu)_j = a_j(u_{j+1} - 2u_j + u_{j-1}) + b_j(u_{j+1} - u_{j-1}) + c_j u_j.$$

In this case, we have $A_j = (a_j - b_j, -2a_j + c_j, a_j + b_j)$, $U_j = (u_{j-1}, u_j, u_{j+1})$.

Lemma 2.2. *If*

$$\sup_{j\in\mathbb{Z}|l\in\mathbb{Z}\cap[-m,m]} |a^j_l| \le M,$$

*for some constant $M$ then the operator $L$ is bounded.*

The proof is obvious.

## 3. Equations with constant coefficients

In this section we present some auxiliary results that will be used below (see also [5]). Suppose that the vectors $A_j$ in (2.1) do not depend on $j$, hence

$$A_j = (a_{-m}, \dots, a_0, \dots, a_m) = A_*.$$

Consider the equation

$$A_* U_j = 0, \quad j \in \mathbb{Z}. \tag{3.1}$$

We will find conditions on $A_*$ such that it has nonzero solutions.

Example 3.1. If $\sum_{j\in\mathbb{Z}} a_j = 0$, then there exists a constant solution $u_j = c$, (for all) $j \in \mathbb{Z}$.

First we establish the number of linearly independent solutions of the equation (3.1).

Lemma 3.2. *Let $A_* = (a_0, \dots, a_k)$, for $a_0, a_k \ne 0$, $k \ge 1$, $u = (\dots, u_{-1}, u_0, u_1, \dots)$, and $U_j = (u_{j-k}, u_{j-k+1}, \dots, u_{j-1}, u_j)$, $j = 0, \pm 1, \dots$ be $(k+1)$-component subvectors of the infinite vector $u$. Then the system of equations*

$$A_* U_j \equiv a_0 u_{j-k} + a_1 u_{j-k+1} + \dots + a_{k-1} u_{j-1} + a_k u_j = 0, \quad j = 0, \pm 1, \dots$$

*has $k$ linearly independent solutions $u^m = (\dots, u^m_{-1}, u^m_0, u^m_1, \dots)$, $m = 1, \dots, k$.*

Proof. Suppose that $u_{-k}, \dots, u_{-1}$ are given. Then from the equation $A_* U_0 = 0$ we find a uniquely defined $u_0$. The equation $A_* U_1 = 0$ allows us to find $u_1$, from $A_*U_2 = 0$ we find $u_2$, and so on. Similarly we determine $u_{-k-1}, u_{-k-2}, \ldots$

Thus the dimension of the subspace of solutions is less or equal to $k$. On the other hand, each vector

$$\begin{gathered} u_{-k} = 1, u_{-k+1} = 0, \ldots, u_{-1} = 0, \\ u_{-k} = 0, u_{-k+1} = 1, \ldots, u_{-1} = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{gathered}$$

generates a solution. There are $k$ of them, and they are linearly independent. Therefore there are exactly $k$ linearly independent solutions. The lemma is proved. $\square$

Now we are going to find some necessary and sufficient conditions in order that equation (3.1) has a bounded nonzero solution.

Assuming that $a_0, a_k \neq 0$, we can rewrite (3.1) as

$$a_0 u_j + a_1 u_{j+1} + \ldots + a_k u_{j+k} = 0, \quad j \in \mathbb{Z} \tag{3.2}$$

and look for the solution of this equation in the form $u_j = e^{\mu j}$, $j \in \mathbb{Z}$. Then,

$$a_0 + a_1 e^{\mu} + \ldots + a_k e^{\mu k} = 0.$$

If we denote $\sigma = e^{\mu}$, this equation can be written as

$$a_k \sigma^k + \ldots + a_1 \sigma + a_0 = 0, \tag{3.3}$$

therefore there are $k$ solutions (possibly multiple) of this equation, $\sigma_1, \ldots, \sigma_k$. We note that if the values of $\mu$ differ by $2\pi n$ with an integer $n$, then they give the same value of $\sigma$. To have a one to one correspondence we will assume that $\mu \in [0, 2\pi)$.

If all solutions $\sigma_1, \ldots, \sigma_k$ are different, then we have $k$ linearly independent solutions of the equation (3.2), $u_j^{(l)} = (\sigma_l)^j$, $l = 1, \ldots, k$. From Lemma 3.2 it follows that all solutions of the equation (3.2) have the form

$$u_j = c_1 \sigma_1^j + \ldots + c_k \sigma_k^j, \quad j \in \mathbb{Z}.$$

Suppose now that not all solutions of (3.3) are different. Consider the function

$$G(\sigma) = a_k \sigma^k + \ldots + a_1 \sigma + a_0.$$

If for example $\sigma = \sigma_1$ is a double root, then $G(\sigma_1) = 0$, $G'(\sigma_1) = 0$. The last equality gives

$$a_k k \sigma_1^{k-1} + a_{k-1}(k-1)\sigma_1^{k-2} + \ldots + 2a_2\sigma_1 + a_1 = 0. \tag{3.4}$$

Put

$$u_j = j\sigma_1^j, \quad j \in \mathbb{Z}. \tag{3.5}$$

Substituting in (3.2), we have

$$a_0 j\sigma_1^j + a_1(j+1)\sigma_1^{j+1} + \ldots + a_k(j+k)\sigma_1^{j+k} = 0.$$

From this equality and $G(\sigma_1) = 0$ we obtain $a_1 + \ldots + a_k k\sigma_1^{k-1} = 0$. By virtue of (3.4), $u_j$ given by (3.5) is a solution. It is linearly independent with respect to other solutions.

Hence in the case of multiple roots, if for example $\sigma$ is a root of order $q$ of (3.3), then solutions are given by the formula

$$u_j = P(j)\sigma^j, \quad j \in \mathbb{Z},$$

where $P(j)$ is a polynomial of degree $q-1$. Thus we have proved the following result (cf. [8]).

Proposition 3.3. *Equation*

$$A_* U_k = 0, \quad k \in \mathbb{Z}$$

*has a bounded for all $j$ nonzero solution $u_j$, $j \in \mathbb{Z}$ if and only if equation* (3.3) *has a root $\sigma = e^{i\xi}$, for some real $\xi$. The bounded nonzero solution has the form*

$$u_j = e^{i\xi j}.$$

We note that if $\xi$ is not in a rational relation with $\pi$, $u_j$ is not periodic but quasi-periodic.

## 4. Normal solvability and Fredholm property

In this section we give a condition in order for the operator $L$ to be normally solvable with a finite dimensional kernel. Next we prove that, under some additional assumptions, $L$ is a Fredholm operator and its index is zero.

We recall that a bounded linear operator $L\colon E \to E$ is *normally solvable* if its image $\operatorname{Im} L$ is closed. A bounded linear operator $L\colon E \to E$ is called a *Fredholm operator* if $L$ is normally solvable, it has a finite dimensional kernel and the codimension of its image is finite. The *index* $\kappa(L)$ of the Fredholm operator $L$ is $\kappa(L) = \alpha(L) - \beta(L)$, where $\alpha(L) = \dim(\ker L)$ and $\beta(L) = co\dim(\operatorname{Im} L)$. The index does not change under deformation in the class of Fredholm operators.

Consider linear difference operators of the form

$$(Lu)_j \equiv A_j U_j = a_{-m}^j u_{j-m} + \ldots + a_0^j u_j + \ldots + a_m^j u_{j+m}, \tag{4.1}$$

where $A_j = (a_{-m}^j, \ldots, a_0^j, \ldots, a_m^j)$ is given. Suppose that there exist limits

$$\lim_{j\to\pm\infty} A_j = A^{\pm}. \tag{4.2}$$

CONDITION 4.1. Equations $A^{\pm}U_j = 0$ for $j \in \mathbb{Z}$ do not have nonzero bounded solutions.

LEMMA 4.2. *If Condition* 4.1 *is satisfied, then the image of the operator* $L$ *is closed.*

PROOF. Let $\{f_n\}$ be a sequence in $\operatorname{Im} L$, $f_n \to f_0$. We show that $f_0 \in \operatorname{Im} L$. Consider a sequence $\{u_n\}$, such that $Lu_n = f_n$.

*Case 1.* Suppose that $\{u_n\}$ is bounded in $E$, $||u_n|| \leq M$, (for all) $n \geq 1$. We prove that it has a convergent subsequence. Observe that for every integer $N > 0$, we can choose a subsequence $u_{n_k} = \{u_j^{n_k}\}$ and some $u_0 = \{u_j^0\}$ for $-N \leq j \leq N$, such that

$$\sup_{-N \leq j \leq N} |u_j^{n_k} - u_j^0| \to 0.$$

Using a diagonalization process we can extend $u_j^0$ for all $j \in \mathbb{Z}$. Obviously, $\sup_j |u_j^0| \leq M$. Passing to the limit in $A_j U_j^{n_k} = f_j^{n_k}$, i.e. in

$$a_{-m}^j u_{j-m}^{n_k} + \ldots + a_0^j u_j^{n_k} + \ldots + a_m^j u_{j+m}^{n_k} = f_j^{n_k}, \quad j \in \mathbb{Z},$$

one obtains

$$a_{-m}^j u_{j-m}^0 + \ldots + a_0^j u_j^0 + \ldots + a_m^j u_{j+m}^0 = f_j^0, \quad j \in \mathbb{Z}.$$

Hence $Lu_0 = f_0$. We now prove that $u_{n_k} \to u_0$ in $E$. To do this, denote $v_k = u_{n_k} - u_0$ and observe $v_k \to 0$, as $k \to \infty$, uniformly on each bounded interval of $j$. We show this convergence is uniform with respect to all $j \in \mathbb{Z}$. Assume that it is not true. Then there is an unbounded subsequence $j_k$ such that $|v_{j_k}^k| \geq \varepsilon > 0$. Without loss of generality we can suppose that $j_k \to \infty$. Denote

$$w_j^k = v_{j+j_k}^k = u_{j+j_k}^k - u_{j+j_k}^0.$$

Then $w_j^k$ satisfies the difference equation

$$a_{-m}^{j+j_k} w_{j-m}^k + \ldots + a_0^{j+j_k} w_j^k + \ldots + a_m^{j+j_k} w_{j+m}^k = f_{j+j_k}^{n_k} - f_{j+j_k}^0, \quad j \in \mathbb{Z} \tag{4.3}$$

and the inequality

$$|w_0^k| = |v_{j_k}^k| \geq \varepsilon. \tag{4.4}$$

As above, we can choose a subsequence $\{w_j^{k_l}\}$ converging to some $\{w_j^0\}$ uniformly on every bounded interval of $j$. Letting $k_l \to \infty$ in (4.3) and (4.4) one obtains with the aid of (4.2),

$$a_{-m}^+ w_{j-m}^0 + \ldots + a_0^+ w_j^0 + \ldots + a_m^+ w_{j+m}^0 = 0, \quad j \in \mathbb{Z}, \quad |w_0^0| \geq \varepsilon.$$

Thus the limiting equation $A^+U_j = 0$, $j \in \mathbb{Z}$ has a nonzero bounded solution. This contradiction proves the existence of a subsequence $u_{n_k} \to u_0$ (in the supremum norm), such that $Lu_0 = f_0$. Therefore $\operatorname{Im} L$ is closed.

*Case 2.* Suppose $\{u_n\}$ is unbounded in $E$. Writing $u_n = v_n + w_n$, where $v_n \in \ker L$ and $w_n$ belongs to the supplement of $\ker L$, we get $Lu_n = Lw_n = f_n$. If $\{w_n\}$ is bounded, we repeat the reasoning of Case 1, for $\{w_n\}$ instead of $\{u_n\}$ and find that $\operatorname{Im} L$ is closed. If $\{w_n\}$ is unbounded, we take

$$\widetilde{w}_n = \frac{w_n}{\|w_n\|}, \quad \widetilde{f}_n = \frac{f_n}{\|w_n\|}.$$

Then $L\widetilde{w}_n = \widetilde{f}_n$ and $\widetilde{f}_n \to 0$. Using Case 1 for $\widetilde{w}_n$, we can choose a convergent subsequence $\{\widetilde{w}_{n_k}\}$, say $\widetilde{w}_{n_k} \to \widetilde{w}_0$. Passing to the limit in $L\widetilde{w}_{n_k} = \widetilde{f}_{n_k}$, we arrive at $L\,\widetilde{w}_0 = 0$, i.e. $\widetilde{w}_0 \in \ker L$. This contradiction shows that this case is not possible, and therefore $\operatorname{Im} L$ is closed. The lemma is proved. $\square$

We now prove that Condition 4.1 implies that $\ker L$ is finite dimensional.

Lemma 4.3. *If the operator $L: E \to E$ satisfies Condition* 4.1 *then the kernel of $L$ has a finite dimension.*

Proof. It is sufficient to prove that if the sequence $u_n = \{u_j^n\}$, $n = 1, 2, \ldots$ belongs to the unit ball $B$ in $\ker L$, then it has a converging sequence. Using the same method as in the previous lemma for $f_n \equiv 0$, we can choose a subsequence $u_{n_k} \to u_0$, for some $u_0$, such that $Lu_0 = 0$. As above we can show that the convergence $u_j^{n_k} \to u_j^0$ is uniform with respect to all integers $j$. The lemma is proved. $\square$

Definition 4.4. Operator $L: E \to E$ is called *proper* if the inverse image $L^{-1}(G)$ of any compact set $G \subset E$ is compact in any bounded ball $B$.

It is known that an operator $L$ is proper if and only if it is normally solvable with a finite dimensional kernel.

Example 4.5. Let $L$ be defined by $(Lu)_j = u_{j+1} - 2u_j + u_{j-1}$. Condition 4.1 is not satisfied since $u_j \equiv const$ is a solution of the equation

$$u_{j+1} - 2u_j + u_{j-1} = 0, \quad j \in \mathbb{Z}.$$

We can construct a sequence $\{u_n\}$, such that $Lu_n = f_n \to 0$ and $\{u_n\}$ is not compact. Indeed, we put $u_j^n = \sin(\varepsilon_n j)$, $j \in \mathbb{Z}$ where $\varepsilon_n \to 0$ as $n \to \infty$. This means that $L$ is not proper.

We are ready to establish one of the main results of the section.

Theorem 4.6. *The operator $L$ is proper if and only if Condition* 4.1 *is satisfied.*

Proof. The sufficiency follows from Lemmas 4.2 and 4.3.

To prove the necessity, suppose that Condition 4.1 is not satisfied, i.e. there exists a nontrivial bounded solution $u$ to one of the equations $L^{\pm} u = 0$. Let it be the first one.

From Proposition 3.3 it follows that such a solution has the form $u = \{u_j\}$, $j \in \mathbb{Z}$, where for some $\xi \in \mathbb{R}$

$$u_j = e^{i\xi j}, \quad j \in \mathbb{Z}. \tag{4.5}$$

Let $\alpha = \{\alpha_j\}$, $\beta^N = \{\beta_j^N\}$, $\gamma^N = \{\gamma_j^N\}$ be a partition of unity ($\alpha_j + \beta_j^N + \gamma_j^N = 1$), such that

$$\alpha_j = \begin{cases} 1 & \text{for } j \le 0, \\ 0 & \text{for } j \ge 1, \end{cases}$$

$$\beta_j^N = \begin{cases} 1 & \text{for } 1 \le j \le N, \\ 0 & \text{for } j \le 0,\ j \ge N+1, \end{cases}$$

$$\gamma_j^N = \begin{cases} 1 & \text{for } j \ge N+1, \\ 0 & \text{for } j \le N, \end{cases}$$

respectively. Consider the sequences $u_n = \{u_j^n\}$, $v_n = \{v_j^n\}$, where $u_j^n = e^{i(\xi+\varepsilon_n)j}$, with $\varepsilon_n \to 0$, as $n \to \infty$ and $v_j^n = (1-\alpha_j)(u_j^n - u_j)$, $j \in \mathbb{Z}$. Obviously, $u_j^n \to u_j$ $(n \to \infty)$ uniformly on every bounded set of integers $j$.

In what follows we will denote by $vw$ the component-wise product of sequences $v$ and $w$ such that $(vw)_j = v_j w_j$.

Denoting $f_n = Lv_n$, we are going to prove that $f_n \to 0$. To this end, we write $f_n = \{f_j^n\}$ under the form

$$\begin{aligned} f_j^n &= (\alpha_j + \beta_j^N + \gamma_j^N)(L[(\beta^N + \gamma^N)(u^n - u)])_j \\ &= \alpha_j (L[(\beta^N + \gamma^N)(u^n - u)])_j + \beta_j^N (L\left[(\beta^N + \gamma^N)(u^n - u)\right])_j \\ &\quad + \gamma_j^N (L[\beta^N (u^n - u)])_j + \gamma_j^N ((L - L^+)[\gamma^N (u^n - u)])_j \\ &\quad + \gamma_j^N (L^+[\gamma^N (u^n - u)])_j. \end{aligned} \tag{4.6}$$

Denote by $T_1, \dots, T_5$ the terms in the right-hand side of (4.6). We estimate each term and show that it goes to 0 as $n \to \infty$. Observe that $T_1 = 0$, $j \le -m$, $j \ge 1$, where $m$ is defined in (4.1),

$$T_2 = \begin{cases} (L(u^n - u))_j & \text{for } 1 \le j \le N, \\ 0 & \text{for } j \le 0,\ j \ge N+1, \end{cases}$$

$$T_3 = 0, \quad j \le N,\ j \ge N+m+1,$$

$$T_5 = \begin{cases} (L^+(u^n - u))_j & \text{for } j \ge N+m+1, \\ 0 & \text{for } j \le N. \end{cases}$$

Using the uniform convergence $u_j^n \to u_j$ on every bounded interval of $j$, we get, for $N$ fixed, $T_1 \to 0$, $T_2 \to 0$, $T_3 \to 0$ as $n \to \infty$, uniformly with respect to all integers $j$. By hypotheses (4.2) and the boundedness $||u^n|| = \|u\| = 1$, one deduces that

$$|T_4| \leq \sharp\gamma^N(L - L^+)\sharp \cdot \|\gamma^N(u^n - u)\| \to 0, \quad N \to \infty.$$

Here $\sharp\cdot\sharp$ is the norm of the operator. We now estimate $(L^+(u^n - u))_j$ for a fixed $N$. Since $u$ in (4.5) is a solution of the equation $L^+u = 0$, we have

$$\begin{aligned}(L^+(u^n - u))_j &= (L^+u^n)_j = (L^+u^n)_j - e^{i\varepsilon_n j}(L^+u)_j \\ &= e^{i(\xi+\varepsilon_n)j}[a^+_{-m}e^{-i\xi m}(e^{-i\varepsilon_n m} - 1) + \ldots + a^+_{-1}e^{-i\xi}(e^{-i\varepsilon_n} - 1) \\ &\quad + a^+_1 e^{i\xi}(e^{i\varepsilon_n} - 1) + \ldots + a^+_m e^{i\xi m}(e^{i\varepsilon_n m} - 1)].\end{aligned}$$

Therefore

$$\begin{aligned}(L^+(u^n - u))_j &= i\varepsilon_n e^{i(\xi+\varepsilon_n)j}[a^+_{-m}(-m)e^{-i\xi m}e^{ic_{-m}} + \ldots - a^+_{-1}e^{-i\xi}e^{ic_{-1}} \\ &\quad + a^+_1 e^{i\xi}.e^{ic_1} + \ldots + a^+_m m e^{i\xi m}e^{ic_m}], \quad j \in \mathbb{Z},\end{aligned}$$

where $c_{-m}, \ldots c_{-1}, c_1, \ldots, c_m$ are intermediate points. Since $\varepsilon_n \to 0$, then

$$(L^+(u^n - u))_j \to 0, \quad \text{as } n \to \infty,$$

uniformly with respect to all $j \in \mathbb{Z}$. Hence $T_5 \to 0$ as $n \to \infty$. From the estimates for $T_1, \ldots, T_5$ one deduces that $f^n_j \to 0$, uniformly with respect to all $j \in \mathbb{Z}$. Indeed, for any given positive $\varepsilon$ we first choose $N$ sufficiently large to get an estimate for $T_4$ (independent of $n$), and then for the fixed $N$ we choose $n$ sufficiently large to estimate other terms in such a way that $|f^n_j| \leq \varepsilon$ for all $j$.

Since $L$ is proper and $f_n \to 0$, it follows that $\{v_n\} \to 0$. On the other hand,

$$\|v_n\| = \sup_{j>0} |e^{i(\xi+\varepsilon_n)j} - e^{i\xi j}| \geq \sigma > 0$$

for some $\sigma$. The contradiction proves the theorem. □

Condition 4.7. Equations $L^{\pm}u_j - \lambda u_j = 0$ for $j \in \mathbb{Z}$ do not have nonzero bounded solutions for any $\lambda \geq 0$.

We begin with some auxiliary results.

Lemma 4.8. *Consider the linear difference operators* $(L_0u)_j = A_jU_j - \rho u_j$*¡* $(L_1u)_j = u_{j+1} - (2+\rho)u_j + u_{j-1}$ *and the homotopy* $L_\tau = (1-\tau)L_0 + \tau L_1$, $\tau \in [0,1]$. *Then there exists* $\rho \geq 0$ *such that the equations*

$$L^{\pm}_\tau u = 0 \tag{4.7}$$

*do not have nonzero bounded solutions for any* $\tau \in [0,1]$.

Proof. Suppose that at least one of the equations (4.7) admits nonzero bounded solutions. By Proposition 3.3 such a solution has the form $u_k =$

$\exp(i\xi k)$, $k \in \mathbb{Z}$, for some $\xi \in \mathbb{R}$. Writing $L_0$ under the form

$$(L_0 u)_k = a^k_{-m} u_{k-m} + \ldots + a^k_0 u_k + \ldots + a^k_m u_{k+m} - \rho u_k, \quad k \in \mathbb{Z},$$

equation (4.7) becomes

$$\begin{aligned}(1-\tau)a^{\pm}_{-m}u_{k-m} &+ \ldots + (1-\tau)a^{\pm}_{-2}u_{k-2} + [(1-\tau)a^{\pm}_{-1} + \tau]u_{k-1} \\ &+ [(1-\tau)(a^{\pm}_0 - \rho) - \tau(2+\rho)]u_k + [(1-\tau)a^{\pm}_1 + \tau]u_{k+1} \\ &+ (1-\tau)a^{\pm}_2 u_{k+2} + \ldots + (1-\tau)a^{\pm}_m u_{k+m} = 0,\end{aligned}$$

for $k \in \mathbb{Z}$. Since $u_k = \exp(i\xi k)$, $k \in \mathbb{Z}$ is a solution of this equation, it follows that

$$\begin{aligned}(1-\tau)a^{\pm}_{-m}e^{-im\xi} &+ \ldots + (1-\tau)a^{\pm}_{-2}e^{-i2\xi} + [(1-\tau)a^{\pm}_{-1} + \tau]e^{-i\xi} \\ &+ [(1-\tau)a^{\pm}_0 - 2\tau - \rho] + [(1-\tau)a^{\pm}_1 + \tau]e^{i\xi} \\ &+ (1-\tau)a^{\pm}_2 e^{i2\xi} + \ldots + (1-\tau)a^{\pm}_m e^{im\xi} = 0.\end{aligned}$$

Taking the real part of the last equality, we obtain

$$\begin{aligned}(1-\tau)a^{\pm}_{-m}\cos m\xi + \ldots + (1-\tau)a^{\pm}_{-2}\cos 2\xi + [(1-\tau)a^{\pm}_{-1} + \tau]\cos\xi + (1-\tau)a^{\pm}_0 - 2\tau \\ + [(1-\tau)a^{\pm}_1 + \tau]\cos\xi + (1-\tau)a^{\pm}_2\cos 2\xi + \ldots + (1-\tau)a^{\pm}_m \cos m\xi = \rho.\end{aligned}$$

Since the coefficients $a^{\pm}_j$ are bounded, this equality does not hold for $\rho$ sufficiently large. This contradiction proves the lemma. □

Lemma 4.9. *If* $(L_1 u)_j = u_{j+1} - (2+\rho)u_j + u_{j-1}$, *then for sufficiently large* $\rho$, *we have* $\operatorname{Im} L_1 = E$.

Proof. The operator $B$ given by

$$(Bu)_j = (1/\rho)(u_{j+1} - 2u_j + u_{j-1}), \quad j \in \mathbb{Z},$$

has a small norm, for each $\rho > 0$ large enough. Therefore the operator $T = B - I$ is invertible. The lemma is proved. □

Theorem 4.10. *If the operator* $(Lu)_j = A_j U_j$ *verifies Condition* 4.7, *then* $L$ *is a Fredholm operator with the zero index.*

Proof. Denote by $\alpha(L)$ and $\beta(L)$ the dimension of $\ker L$ and the codimension of $\operatorname{Im} L$, respectively. Condition 4.7 for $L$ implies that Condition 4.1 for $L - \lambda I$ is satisfied for any $\lambda \geq 0$. Therefore, by Lemmas 4.2 and 4.3, it follows that $\ker(L - \lambda I)$ is finite dimensional and $\operatorname{Im}(L - \lambda I)$ is closed. In order to obtain the Fredholm property for $L - \lambda I$, it is enough to prove that $\beta(L - \lambda I)$ is finite for all $\lambda \geq 0$.

To do this, we put $L_0 = L - \lambda I$, $(L_1 u)_j = u_{j+1} - 2u_j + u_{j-1} - \lambda u_j$ and $L_\tau = (1-\tau)L_0 + \tau L_1$, $\tau \in [0,1]$. By Lemma 4.8, $L_\tau$ verifies Condition 4.1 for sufficiently large $\lambda \geq 0$ and consequently $L_\tau$ is normally solvable with a finite dimensional kernel.

Thus we have a continuous deformation of the operator $L$ to the invertible operator $L_1$ in the class of normally solvable operators with finite dimensional kernels. Therefore the index of $L$ equals zero. The theorem is proved. $\square$

## 5. NS-spectrum

By essential spectrum of a linear operator $L$ we understand the set $\sigma_{\text{ess}}$ of all values $\lambda$ such that the operator $L - \lambda I$ is not Fredholm. Here $I$ is the identity operator. We introduce also the notion of NS-spectrum $\sigma_{\text{NS}}$ as the set of $\lambda$ for which the operator $L - \lambda I$ is not normally solvable with a finite dimensional kernel. It is clear that $\sigma_{\text{NS}} \subset \sigma_{\text{ess}}$.

From the results of the previous section, it follows that the NS-spectrum of the operator $L$ given by (4.1) is the set of all $\lambda$ with the property that one of the equations

$$a^{\pm}_{-m} u_{j-m} + \ldots + a^{\pm}_0 u_j + \ldots + a^{\pm}_m u_{j+m} = \lambda u_j, \quad j \in \mathbb{Z}$$

has nonzero bounded solutions. If $u_j = e^{i\xi j}$, $j \in \mathbb{Z}$ ($\xi \in \mathbb{R}$) is a solution (see Proposition 3.3), then

$$\lambda = a^{\pm}_{-m} e^{-i\xi m} + \ldots + a^{\pm}_0 + \ldots + a^{\pm}_m e^{i\xi m}, \quad \xi \in \mathbb{R},$$

that is

$$\begin{aligned}\lambda = {}& a^{\pm}_{-m} \cos \xi m + a^{\pm}_{-m+1} \cos \xi(-m+1) + \ldots + a^{\pm}_m \cos \xi m \\ & + i(a^{\pm}_{-m} \sin(-\xi m) + a^{\pm}_{-m+1} \sin \xi(-m+1) + \ldots + a^{\pm}_m \sin \xi m).\end{aligned}$$

Now we apply this result to some particular difference operators, arising as discrete variants of some differential operators.

Example 5.1. Consider the linear difference operator

$$(Lu)_j = a_j(u_{j+1} - 2u_j + u_{j-1}) + b_j(u_{j+1} - u_j) + c_j u_j, \quad j \in \mathbb{Z}.$$

The NS-spectrum is given by

$$a^{\pm}(u_{j+1} - 2u_j + u_{j-1}) + b^{\pm}(u_{j+1} - u_j) + c^{\pm} u_j = \lambda u_j.$$

Then

$$\lambda = a^{\pm}(e^{i\xi} - 2 + e^{-i\xi}) + b^{\pm}(e^{i\xi} - 1) + c^{\pm},$$

or

$$\lambda = (2a^{\pm} + b^{\pm}) \cos \xi + i b^{\pm} \sin \xi - 2a^{\pm} - b^{\pm} + c^{\pm}. \tag{5.1}$$

Consider $\lambda$ as a function of $\xi$. Let us take the case $a^+$, $b^+$, $c^+$. Put

$$a^+ = \frac{a}{h^2}, \quad b^+ = \frac{b}{h}, \quad c^+ = c.$$

Then $\lambda$ in (5.1) becomes

$$\lambda = (2\frac{a}{h^2} + \frac{b}{h})\cos\xi + i\frac{b}{h}\sin\xi - 2\frac{a}{h^2} - \frac{b}{h} + c.$$

Let $\lambda = \mu + i\nu$. Then

$$\mu = \left(2\frac{a}{h^2} + \frac{b}{h}\right)\cos\xi - 2\frac{a}{h^2} - \frac{b}{h} + c, \quad \nu = \frac{b}{h}\sin\xi.$$

This implies that

$$\nu = \frac{b}{h}\sqrt{-\frac{(c-\mu)^2}{(2a/h^2 + b/h)^2} + \frac{2(c-\mu)}{2a/h^2 + b/h}}.$$

Passing to the limit as $h \to 0$, one obtains

$$\nu = b\sqrt{\frac{c-\mu}{a}}.$$

From this we deduce $\mu = c - a\nu^2/b^2$, therefore

$$\lambda = \mu + i\nu = c - a\frac{\nu^2}{b^2} + i\nu, \quad \nu \in \mathbb{R}.$$

Since $\nu$ takes all real values, we can replace it by $b\xi$. Hence

$$\lambda = -a\xi^2 + ib\xi + c, \quad \xi \in \mathbb{R}.$$

This is the NS-spectrum of the differential operator $Lu = au'' + bu' + cu$. Thus the spectrum of the difference operator converges to the spectrum of the differential operator.

Example 5.2. We put

$$(Lu)_j = a_j(u_{j+1} - 2u_j + u_{j-1}) + b_j(u_{j+1} - u_{j-1}) + c_j u_j, \quad j \in \mathbb{Z}.$$

In this case, the NS-spectrum of $L$ is given by

$$\begin{aligned}\lambda &= a^\pm(e^{i\xi} - 2 + e^{-i\xi}) + b^\pm(e^{i\xi} - e^{-i\xi}) + c^\pm \\ &= 2a^\pm\cos\xi + 2ib^\pm\sin\xi - 2a^\pm + c^\pm.\end{aligned}$$

We note that since the operator is bounded, its spectrum lies in a circle of radius $||L||$. Outside of it, the operator $L - \lambda I$ is invertible.

If $\lambda = 0$ is not inside the domain bounded by the NS-spectrum, then the index of $L$ equals zero. To show this, we construct a homotopy to invertible operators.

## 6. Properness of semilinear difference operators

In this section, we consider semilinear difference operators $A\colon E \to E$, of the form $(Au)_j = A_jU_j + F(u_j)$, $j \in \mathbb{Z}$, that is

$$(Au)_j = a^j_{-m}u_{j-m} + \ldots + a^j_0 u_j + \ldots + a^j_m u_{j+m} + F(u_j), \tag{6.1}$$

where the coefficients $a^j_l$, $-m \le l \le m$, have limits

$$\lim_{j\to\pm\infty} a^j_l = a^\pm_l, \quad -m \le l \le m, \tag{6.2}$$

the function $F$ is continuous, and its derivative satisfies the Lipschitz condition. It is easy to verify that the operator $A$ is bounded and continuous.

In this section we study its properness. Recall that an operator $A\colon E \to E$ is *proper* if the intersection of an inverse image $A^{-1}(D)$ of a compact set $D$ with any bounded closed set is compact in $E$.

Example 6.1. Let $u = \{u_j\}$ be a symmetric sequence ($u_j = u_{-j}$, for all $j \in \mathbb{Z}$), such that $u_j \to 0$ as $j \to \pm\infty$. We put $F(u_j) = -(u_{j+1} - 2u_j + u_{j-1})$. Then $u$ is a solution of the equation

$$u_{j+1} - 2u_j + u_{j-1} + F(u_j) = 0.$$

For any integer $k$, the sequence $v^{(k)} = \{v^{(k)}_j\}$, where $v^{(k)}_j = u_{j+k}$, is also a solution of this equation. Hence an inverse image of 0 for the operator

$$(Au)_j = u_{j+1} - 2u_j + u_{j-1} + F(u_j)$$

is not compact.

This example shows that discrete operators similar to differential operators in unbounded domains may be not proper. To obtain properness, we introduce weighted spaces.

Denote $E_\mu$ the space of the sequences $u = \{u_j\}$ with the norm

$$\|u\|_\mu = \sup_j |\mu_j u_j|,$$

where $\mu_j \ge 1$, $j \in \mathbb{Z}$ and $\mu_j \to \infty$, as $j \to \pm\infty$. Denote

$$\nu_{j+l} = \frac{\mu_j - \mu_{j+l}}{\mu_{j+l}}, \quad -m \le l \le m$$

and assume that, for $-m \le l \le m$,

$$\nu_{j+l} \to 0, \quad \text{as } j \to \pm\infty. \tag{6.3}$$

As an example we can take $\mu_j = 1 + j^2$. Then $\nu_{j+l} = (-2jl - l^2)/[1 + (j+l)^2]$ and the condition above is satisfied.

To study the topological degree in the next section, we need the properness of the operator $A$ in the case where the coefficients $a_l^j$ $(-m \le l \le m)$ and the function $F$ depend also on a parameter $\tau \in [0,1]$.

In what follows we consider semilinear $\tau$-dependent operators $A_\tau: E_\mu \to E_\mu$, $\tau \in [0,1]$,

$$(A_\tau u)_j = a_{-m}^j(\tau)u_{j-m} + \ldots + a_0^j(\tau)u_j + \ldots + a_m^j(\tau)u_{j+m} + F(u_j,\tau),$$

satisfying the following conditions:

(H1) The function $F(u,\tau)$ is continuous together with its first derivative with respect to $u$, and $F(0,\tau) = 0$ for all $\tau$,

(H2) There exists limiting functions $a_l^\pm(\tau)$ $(-m \le l \le m)$ and $c^\pm(\tau)$ such that, for all $\tau \in [0,1]$,

$$\begin{aligned} \lim_{j\to\pm\infty} a_l^j(\tau) &= a_l^\pm(\tau), \quad -m \le l \le m, \\ \lim_{j\to\pm\infty} F_u'(u_j,\tau) &= c^\pm(\tau), \qquad \text{if } u_j \to 0, \end{aligned}$$

(H3) For every $\tau, \tau_0 \in [0,1]$, we have

$$\|a_l(\tau) - a_l(\tau_0)\| \le c|\tau-\tau_0|, \quad \|F(u,\tau) - F(u,\tau_0)\|_\mu \le k|\tau-\tau_0|,$$

for all $u$ from a bounded set in $E$ and some $c$ and $k$.

Denote $E' = E_\mu \times [0,1]$. Assume that the following hypothesis holds:

CONDITION 6.2. For each $\tau \in [0,1]$ the system

$$\begin{aligned} a_{-m}^\pm(\tau)u_{j-m} + \ldots + a_{-1}^\pm(\tau)u_{j-1} + [a_0^\pm(\tau) + c^\pm(\tau)]u_j \\ + a_1^\pm(\tau)u_{j+1} + \ldots + a_m^\pm(\tau)u_{j+m} = 0, \end{aligned}$$

for $j = 0, \pm1, \pm2, \ldots$, does not have nonzero bounded solutions.

We can state now the main result of this section.

THEOREM 6.3. *If* (6.3)*,* (H1)–(H3) *and Condition* 6.2 *hold, then the operator $A_\tau(u)$ is proper with respect to both $u$ and $\tau$ (in $E'$).*

PROOF. Let $B$ be a ball of radius $M$ in $E$, $f^{(n)} \in E_\mu$ a converging sequence, $f^{(n)} \to f^{(0)}$ in $E_\mu$. Suppose that $(u^{(n)},\tau_n)$ satisfies

$$A_{\tau_n}(u^{(n)}) = f^{(n)}, \quad \|u^{(n)}\|_\mu \le M, \quad n = 1,2,\ldots \tag{6.4}$$

Without loss of generality, we may assume $\tau_n \to \tau_0$ as $n \to \infty$. We show that there exists a converging subsequence of the sequence $u^{(n)}$.

In order to do this, we write equation (6.4) in the form

$$a_{-m}^j(\tau_n)u_{j-m}^{(n)} + \ldots + a_0^j(\tau_n)u_j^{(n)} + \ldots + a_m^j(\tau_n)u_{j+m}^{(n)} + F(u_j^{(n)},\tau_n) = f_j^{(n)} \tag{6.5}$$

and multiply it by $\mu_j$. Denoting $v_j^{(n)} = \mu_j u_j^{(n)}$ and $g_j^{(n)} = \mu_j f_j^{(n)}$ and using the equality

$$\mu_j u_{j+l}^{(n)} = \mu_{j+l} u_{j+l}^{(n)} + (\mu_j - \mu_{j+l}) u_{j+l}^{(n)} = v_{j+l}^{(n)} + \nu_{j+l} v_{j+l}^{(n)},$$

for $-m \le l \le m$, we obtain from equation (6.5)

$$\begin{aligned}(6.6)\quad (1+\nu_{j-m}) a_{-m}^j(\tau_n) v_{j-m}^{(n)} + \ldots + (1+\nu_j) a_0^j(\tau_n) v_j^{(n)} + \ldots \\ + (1+\nu_{j+m}) a_m^j(\tau_n) v_{j+m}^{(n)} + \mu_j F(u_j^{(n)}, \tau_n) = g_j^{(n)}.\end{aligned}$$

We note that the sequence $v^{(n)} = \{v_j^{(n)}\}$, $v_j^{(n)} = \mu_j u_j^{(n)}$ is uniformly bounded in $E$. Hence, for any interval $[-N, N]$ of $j$, we can choose a subsequence (denoted also $v^{(n)}$) converging to some limiting element $v^{(0)} = \{v_j^{(0)}\}$ uniformly in $j \in [-N, N]$. Using a diagonalization process, we can extend it for all $j \in \mathbb{Z}$. Let $u^{(0)}$ be the limit of the corresponding subsequence of $u^{(n)}$. It is easy to verify that $\|v^{(0)}\| \le M$. Since $F$ and $a_l^j$ satisfy the hypothesis (H3), passing to the limit as $n \to \infty$ in (6.5), we find

$$a_{-m}^j(\tau_0) u_{j-m}^{(0)} + \ldots + a_0^j(\tau_0) u_j^{(0)} + \ldots + a_m^j(\tau_0) u_{j+m}^{(0)} + F(u_j^{(0)}, \tau_0) = f_j^{(0)},$$

i.e. $A_{\tau_0}(u^{(0)}) = f^{(0)}$. We have used the convergence

$$\begin{aligned}&F(u_j^{(n)}, \tau_n) - F(u_j^{(0)}, \tau_0) \\ &\quad = [F(u_j^{(n)}, \tau_n) - F(u_j^{(n)}, \tau_0)] + [F(u_j^{(n)}, \tau_0) - F(u_j^{(0)}, \tau_0)] \to 0, \quad n \to \infty,\end{aligned}$$

by virtue of (H3) and the continuity of $F$ with respect to the first variable.

On the other hand, letting $n \to \infty$ in (6.6) and using (H3) again, together with the uniform boundedness of $v_j^{(n)}$, one obtains

$$\begin{aligned}(6.7)\quad (1+\nu_{j-m}) a_{-m}^j(\tau_0) v_{j-m}^{(0)} + \ldots + (1+\nu_j) a_0^j(\tau_0) v_j^{(0)} + \ldots \\ + (1+\nu_{j+m}) a_m^j(\tau_0) v_{j+m}^{(0)} + \mu_j F(u_j^{(0)}, \tau_0) = g_j^{(0)},\end{aligned}$$

where $g_j^{(0)} = \mu_j f_j^{(0)}$. Subtracting (6.6) and (6.7), we get

$$\begin{aligned}(6.8)\qquad (1+\nu_{j-m})[a_{-m}^j(\tau_n) v_{j-m}^{(n)} - a_{-m}^j(\tau_0) v_{j-m}^{(0)}] + \ldots \\ + (1+\nu_j)[a_0^j(\tau_n) v_j^{(n)} - a_0^j(\tau_0) v_j^{(0)}] + \ldots \\ + (1+\nu_{j+m})[a_m^j(\tau_n) v_{j+m}^{(n)} - a_m^j(\tau_0) v_{j+m}^{(0)}] \\ + \mu_j [F(u_j^{(n)}, \tau_n) - F(u_j^{(0)}, \tau_0)] = g_j^{(n)} - g_j^{(0)}.\end{aligned}$$

Denote $w^{(n)} = v^{(n)} - v^{(0)}$ and observe that for each $l \in \mathbb{Z}$, $-m \le l \le m$,

$$a_l^j(\tau_n) v_{j+l}^{(n)} - a_l^j(\tau_0) v_{j+l}^{(0)} = a_l^j(\tau_n) w_{j+l}^{(n)} + [a_l^j(\tau_n) - a_l^j(\tau_0)] v_{j+l}^{(0)}.$$

Then (6.8) becomes

$$\begin{aligned}(6.9)\quad &(1+\nu_{j-m})a^j_{-m}(\tau_n)w^{(n)}_{j-m}+\ldots+(1+\nu_j)a^j_0(\tau_n)w^{(n)}_j+\ldots\\ &+(1+\nu_{j+m})a^j_m(\tau_n)w^{(n)}_{j+m}\\ &+\mu_j[F(u^{(n)}_j,\tau_n)-F(u^{(0)}_j,\tau_0)]\\ &+(1+\nu_{j-m})[a^j_{-m}(\tau_n)-a^j_{-m}(\tau_0)]v^{(0)}_{j-m}+\ldots\\ &+(1+\nu_j)[a^j_0(\tau_n)-a^j_0(\tau_0)]v^{(0)}_j+\ldots\\ &+(1+\nu_{j+m})[a^j_m(\tau_n)-a^j_m(\tau_0)]v^{(0)}_{j+m}=g^{(n)}_j-g^{(0)}_j.\end{aligned}$$

Recall that $w^{(n)}_j\to 0$ as $n\to\infty$ uniformly on each bounded interval of $j$. Suppose that this convergence is not uniform for all $j\in\mathbb{Z}$. Then, without loss of generality, we may assume that there exists a sequence $j_n\to\infty$ such that $|w^{(n)}_{j_n}|\geq\varepsilon>0$. Denote

$$(6.10)\qquad \widetilde{w}^{(n)}_j=w^{(n)}_{j+j_n}=\mu_{j+j_n}(u^{(n)}_{j+j_n}-u^{(0)}_{j+j_n}).$$

Then,

$$(6.11)\qquad |\widetilde{w}^{(n)}_0|=|w^{(n)}_{j_n}|\geq\varepsilon>0$$

and from (6.9) we get

$$\begin{aligned}(6.12)\quad &(1+\nu_{j+j_n-m})a^{j+j_n}_{-m}(\tau_n)\widetilde{w}^{(n)}_{j-m}+\ldots+(1+\nu_{j+j_n})a^{j+j_n}_0(\tau_n)\widetilde{w}^{(n)}_j+\ldots\\ &+(1+\nu_{j+j_n+m})a^{j+j_n}_m(\tau_n)\widetilde{w}^{(n)}_{j+m}+\mu_{j+j_n}[F(u^{(n)}_{j+j_n},\tau_n)-F(u^{(0)}_{j+j_n},\tau_0)]\\ &+(1+\nu_{j+j_n-m})[a^{j+j_n}_{-m}(\tau_n)-a^{j+j_n}_{-m}(\tau_0)]v^{(0)}_{j+j_n-m}+\ldots\\ &+(1+\nu_{j+j_n})[a^{j+j_n}_0(\tau_n)-a^{j+j_n}_0(\tau_0)]v^{(0)}_{j+j_n}+\ldots\\ &+(1+\nu_{j+j_n+m})[a^{j+j_n}_m(\tau_n)-a^{j+j_n}_m(\tau_0)]v^{(0)}_{j+j_n+m}=g^{(n)}_{j+j_n}-g^{(0)}_{j+j_n}.\end{aligned}$$

We intend to take the limit as $n\to\infty$ in (6.12). To this end, first observe that (H3) and the boundedness $\|v^{(0)}\|\leq M$ imply

$$(6.13)\qquad [a^{j+j_n}_l(\tau_n)-a^{j+j_n}_l(\tau_0)]v^{(0)}_{j+j_n+l}\to 0,$$

for $-m\leq l\leq m$. By (H2), we also have

$$(6.14)\qquad a^{j+j_n}_l(\tau_n)=[a^{j+j_n}_l(\tau_n)-a^{j+j_n}_l(\tau_0)]+a^{j+j_n}_l(\tau_0)\to a^+_l(\tau_0),$$

for every integer $l$, with $-m\leq l\leq m$. Next, we write

$$\begin{aligned}(6.15)\quad &\mu_{j+j_n}[F(u^{(n)}_{j+j_n},\tau_n)-F(u^{(0)}_{j+j_n},\tau_0)]\\ &=\mu_{j+j_n}[F(u^{(n)}_{j+j_n},\tau_n)-F(u^{(n)}_{j+j_n},\tau_0)]+\mu_{j+j_n}[F(u^{(n)}_{j+j_n},\tau_0)-F(u^{(0)}_{j+j_n},\tau_0)]\end{aligned}$$

and denote by $T^n_1$, $T^n_2$ the two terms in the right-hand side of (6.15). Hypothesis (H3) for $F$ leads us to the convergence $T^n_1\to 0$ as $n\to\infty$.

Using (6.10), we get

$$T_2^n = \widetilde{w}_j^{(n)} \cdot \frac{F(u_{j+j_n}^{(n)}, \tau_0) - F(u_{j+j_n}^{(0)}, \tau_0)}{u_{j+j_n}^{(n)} - u_{j+j_n}^{(0)}} = \widetilde{w}_j^{(n)} F'_u(tu_{j+j_n}^{(n)} + (1-t)u_{j+j_n}^{(0)}, \tau_0),$$

for some $t \in [0,1]$. By (6.4) we have

$$|u_{j+j_n}^{(n)}| \le \frac{M}{\mu_{j+j_n}}, \quad |u_{j+j_n}^{(0)}| \le \frac{M}{\mu_{j+j_n}},$$

hence

$$tu_{j+j_n}^{(n)} + (1-t)u_{j+j_n}^{(0)} \to 0, \quad \text{as } n \to \infty,$$

uniformly in $j$ on every bounded interval. The equality (6.10) shows that

$$\widetilde{w}_j^{(n)} \to \widetilde{w}_j^{(0)}, \tag{6.16}$$

for some $\widetilde{w}^{(0)} \in E$, also uniformly in $j$ on bounded intervals. Thus, using (H2) one obtains

$$T_2^n \to c^+(\tau_0)\widetilde{w}_j^{(0)}.$$

Introducing this, together with $T_1^n \to 0$ in (6.15), we find

$$\mu_{j+j_n}[F(u_{j+j_n}^{(n)}, \tau_n) - F(u_{j+j_n}^{(0)}, \tau_0)] \to c^+(\tau_0)\widetilde{w}_j^{(0)}. \tag{6.17}$$

Now we may pass to the limit in (6.12). Taking into account (6.3), (6.13), (6.14), (6.16), (6.17), and the convergence $g_{j+j_n}^{(n)} - g_{j+j_n}^{(0)} \to 0$, we get

$$a_{-m}^+(\tau_0)\widetilde{w}_{j-m}^{(0)} + \ldots + a_0^+(\tau_0)\widetilde{w}_j^{(0)} + \ldots + a_m^+(\tau_0)\widetilde{w}_{j+m}^{(0)} + c^+(\tau_0)\widetilde{w}_j^{(0)} = 0.$$

From (6.11) it is obvious that $\widetilde{w}^{(0)} \neq 0$. Thus Condition 6.2 is not satisfied. This contradiction shows that the convergence $w_j^{(n)} \to 0$ as $n \to \infty$ is uniform with respect to all $j \in \mathbb{Z}$ and thus the theorem is completely proved. □

We note that the sequences from $E_\mu$ converge to 0 as $j \to \pm\infty$. If we want to have different limits for $j \to \infty$ and $j \to -\infty$ in the definition of $c^\pm(\tau)$ (see (H2)), we can consider operators of the form

$$(Au)_j = A_jU_j + A_j\Psi_j + F(\psi_j + u_j),$$

where $\Psi_j = (\psi_{j-m}, \ldots, \psi_j, \ldots, \psi_{j+m})$ and $\psi_j \to u^\pm$ as $j \to \pm\infty$. In this case, in Condition 6.2 we take $c^\pm(\tau) = F'_u(u^\pm, \tau)$.

Finally, taking $\tau = \text{const}$, we obtain the properness of $A: E \to E$ given by (6.1). In this case, Condition 6.2 becomes:

Condition 6.2'. The equation

$$a_{-m}^\pm u_{j-m} + \ldots + a_{-1}^\pm u_{j-1} + (a_0^\pm + c^\pm)u_j + a_1^\pm u_{j+1} + \ldots + a_m^\pm u_{j+m} = 0,$$

where $c^\pm = F'(0)$, does not have nonzero bounded solutions.

Then we have the following consequence of Theorem 6.3.

Corollary 6.4. *Suppose that $F$ is bounded and continuous on every bounded interval together with its first derivative. If* (6.2), (6.3) *and Condition* 6.2' *are satisfied, then the operator $A$ given by* (6.1) *is proper.*

## 7. Topological degree

We are going to prove that a topological degree can be constructed for the class of the difference operators $A_\tau: E \to E$, $\tau \in [0,1]$ which have two Fréchet derivatives with respect to $u$ and $\tau$, $A_\tau$ satisfying Condition 7.1 below for every $\tau \in [0,1]$.

We begin with the definition of the topological degree (see e.g. [7]).

Consider $E_1$, $E_2$ two Banach spaces, a class $\Phi$ of operators acting from $E_1$ to $E_2$ and a class of homotopies

$$H = \{A_\tau(u): E_1 \to E_2,\ \tau \in [0,1],\ u \in E_1 \\ \text{such that } A_\tau(u) \in \Phi \text{ for all } \tau \in [0,1]\}.$$

Let $D \subset E_1$ be an open bounded set and $A \in \Phi$ such that $A(u) \neq 0$, $u \in \partial D$, where $\partial D$ is the boundary of $D$. Suppose that for such a pair $(D, A)$, there exists an integer $\gamma(A, D)$ with the following properties:

(i) (Homotopy invariance) If $A_\tau(u) \in H$ and $A_\tau(u) \neq 0$, for $u \in \partial D$, $\tau \in [0,1]$, then $\gamma(A_0, D) = \gamma(A_1, D)$.

(ii) (Additivity) If $A \in \Phi$ and $A(u) \neq 0$, $u \in \overline{D} \setminus (D_1 \cup D_2)$, where $D_1, D_2 \subset D$ are open sets, $D_1 \cap D_2 = \Phi$, then

$$\gamma(A, D) = \gamma(A, D_1) + \gamma(A, D_2).$$

(Here $\overline{D}$ denotes the closure of $D$.)

(iii) (Normalization) There exists a bounded linear operator $J: E_1 \to E_2$ with a bounded inverse defined on all $E_2$ such that $\gamma(J, D) = 1$ for every bounded set $D \subset E_1$ with $0 \in D$.

The integer $\gamma(A, D)$ is called topological degree. In [11], [10] a topological degree is constructed for the class $\mathcal{F}$ of operators and the class $\mathcal{H}$ of homotopies given below.

Let $E_0$ and $E_1$ be Banach spaces, $E_0 \subseteq E_1$ algebraically and topologically and $G \subset E_0$ be an open bounded set. Consider a class $\Phi$ of bounded linear operators $A: E_0 \to E_1$ satisfying:

(a) The operator $A + \lambda I: E_0 \to E_1$, where $I$ is the identity operator, is Fredholm for all $\lambda \geq 0$,

(b) There is $\lambda_0 = \lambda_0(A)$ such that each operator $A + \lambda I$ has a uniformly bounded inverse for all $\lambda > \lambda_0$.

Denote by $\mathcal{F}$ the class

$$\mathcal{F} = \{f \in C^1(G, E_1),\ f \text{ proper},\ f'(x) \in \Phi,\ \text{for all } x \in G\},$$

where $f'(x)$ is the Fréchet derivative of the operator $f$.

Finally, one introduces the class $\mathcal{H}$ of homotopies given by

$$\mathcal{H} = \{f(x,\tau) \in C^1(G \times [0,1], E_1) : \\ f \text{ proper},\ f(\,\cdot\,,\tau) \in \mathcal{F},\ \text{for all } \tau \in [0,1]\}. \tag{7.1}$$

Here the properness of $f$ is understood in both variables $x \in G$ and $\tau \in [0,1]$.

In [10] the authors prove that for every $f \in \mathcal{F}$ and every open set $D$, with $\overline{D} \subset G$, there is a topological degree $\gamma(f, D)$.

Now let $E_0 = E_1 = E_2 = E_\mu$ be the weighted space of sequences defined in Section 6. We apply this result for difference operators of the form

$$(Au)_j = A_j U_j + F(u_j) = a^j_{-m} u_{j-m} + \ldots + a^j_0 u_j + \ldots + a^j_m u_{j+m} + F(u_j), \tag{7.2}$$

such that

$$\lim_{j\to\pm\infty} a^j_l = a^\pm_l, \quad -m \le l \le m. \tag{7.3}$$

To this end, assume that the following condition takes place.

Condition 7.1. The equation

$$A^\pm U_j + c^\pm u_j - \lambda u_j = 0$$

(where $c^\pm = \lim_{j\to\pm\infty} F'(u_j)$, if $u_j \to 0$) do not have nonzero bounded solutions, for all $\lambda \ge 0$.

Let $\widetilde{\mathcal{F}}$ be the class of operators $A$ defined in (7.2), where $F$ has two derivatives and such that (6.3), (7.3) and Condition 7.1 are satisfied.

Corollary 6.4 and Theorem 4.10 assure that $A$ is proper and the Fréchet derivative $A'$ of $A$ is a Fredholm operator.

Now we take the $\tau$-dependent operators $A_\tau(u)\colon E_\mu \to E_\mu$, $\tau \in [0,1]$,

$$(A_\tau u)_j = a^j_{-m}(\tau) u_{j-m} + \ldots + a^j_0(\tau) u_j + \ldots + a^j_m(\tau) u_{j+m} + F(u_j, \tau) \tag{7.4}$$

satisfying hypotheses (6.3), (H1)–(H3) and

Condition 7.2. For every $\tau \in [0,1]$, the equations

$$A^\pm(\tau) U_j + c^\pm(\tau) u_j - \lambda u_j = 0$$

do not have nonzero bounded solutions, for all $\lambda \ge 0$, where

$$c^\pm(\tau) = \lim_{j\to\pm\infty} F'_u(u_j, \tau) \quad \text{if } u_j \to 0.$$

Let $\widetilde{\mathcal{H}}$ be the class of all operators $A_\tau\colon E_\mu \to E_\mu$ of the form (7.4) satisfying the above assumptions, such that $F$ has two derivatives with respect to $u$ and $\tau$.

Using Theorems 6.3 and 4.10 for each $\tau \in [0,1]$, we deduce that $\widetilde{\mathcal{H}}$ has the form (7.1).

Now we may apply the above result to conclude that the topological degree can be constructed for difference operators of the form (7.2). More precisely, we have

Theorem 7.3. *Suppose that the function $F$ has two derivatives with respect to $u$ and $\tau$, Condition* 7.2, *hypotheses* (H1)–(H3) *and* (6.3) *hold. Then the topological degree exists for the class $\widetilde{\mathcal{F}}$ of operators and the class $\widetilde{\mathcal{H}}$ of homotopies.*

We will finish this section with one example of application of the topological degree to prove existence of solutions. Consider the system

$$u_{i+1} - 2u_i + u_{i-1} + F(u_i) = 0, \quad i = 0, \pm 1, \pm 2, \dots, \tag{7.5}$$

where $F = -au$ with $a > 0$. Put

$$F_i(u,\tau) = (1-\tau)F(u_i) + \tau F_i(u_i), \tag{7.6}$$

where $F_i$ are sufficiently smooth functions, and $F_i(u_i) = F(u_i)$ for $|i| \geq N$ with some positive $N$. Consider equation (7.5) with the function $F_i(u,\tau)$ instead of $F(u_i)$. This means that we change our problem on a finite interval of $i$. Then for $\tau = 0$ we have problem (7.5), and for $\tau = 1$ the problem

$$u_{i+1} - 2u_i + u_{i-1} + F_i(u_i) = 0, \quad i = 0, \pm 1, \pm 2, \dots \tag{7.7}$$

This homotopy satisfies conditions of Sections 6 and 7, and the topological degree can be defined for it.

For $\tau = 0$, $u_i = 0$ is a solution of this problem. Moreover this solution is unique. Indeed, suppose that there exists another bounded solution. It can be easily verified that it cannot have a positive maximum or a negative minimum. Therefore there exist limits of the sequence $u_i$ as $i \to \pm\infty$. If these limits equal zero, then the solution is identically zero. Otherwise, the sequence $u_i$ converges to some nonzero value either at $\infty$ or at $-\infty$. Substituting this value into (7.5) we obtain a contradiction.

Problem (7.5) linearized around $u_i = 0$ has all spectrum in the left-half plane. Therefore the index of the solution $u_i = 0$, that is $(-1)^\nu$ where $\nu$ is the number of positive eigenvalues, equals 1.

Suppose in addition that for some positive $r$,

$$F_i(u) < 0, \quad u \geq r, \quad F_i(u) > 0, \quad u \leq -r, \quad i = 0, \pm 1, \pm 2, \dots \tag{7.8}$$

Let us verify that there exist a priori estimates of solutions for homotopy (7.6) in the weighted space. First of all, we note that $|u_i| \le r$ for all $i$. Next, for $|j| \ge N$, $u_j = \exp(\mu j)$, where $\mu = \ln \sigma$, and $\sigma$ is one of two solutions of the equation

$$\sigma^2 - (2+a)\sigma + 1 = 0.$$

Since $a > 0$, then one of the solutions of this equation is greater than 1, and another one is less than 1. Therefore

$$|u_j| \le re^{\mu_1(j-N)}, \quad j \ge N, \quad |u_j| \le re^{\mu_2(j+N)}, \quad j \le -N,$$

where $\mu_1 < 0, \mu_2 > 0$. This estimate gives a priori estimates of solutions in the weighted space.

Thus, applying the Leray–Schauder method we conclude that problem (7.7) has a solution satisfying the estimate $|u_i| \le r$ for all $i$. This estimate does not depend on $N$. Consider a sequence $N_k \to \infty$. We have a solution $u^k$ for each $N_k$. From the sequence of solutions $u^k$ we can choose a subsequence converging to some limiting solution $u^0$ on every bounded interval of $i$. It is easy to see that $u^0$ is a solution of the limiting problem. We have proved the following theorem.

Theorem 7.4. *Suppose that the functions $F_i(u_i)$ are sufficiently smooth and satisfy condition* (7.8). *Then problem* (7.7) *has a solution satisfying the estimate $|u_i| \le r$, $i = 0, \pm 1, \pm 2, \ldots$*

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