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ON CONNECTING ORBITS FOR COMPETING SPECIES EQUATIONS WITH LARGE INTERACTIONS

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Abstract. We use homotopy index and monotonicity techniques to study the connecting orbits of systems of two competing species equations with diffusion and large interaction. We also use earlier work of Zhitao Zhang and the author on the dynamics of this system.

1. Introduction

In this paper, we show that the techniques in my preceding paper [15] with Zhitao Zhang can be refined to obtain a good deal of information about connecting orbits. Here we use blow-up techniques and Conley index ideas. We show that we can obtain information about connecting orbits from two of the limiting equations in [15]. The most difficult connecting orbits to study are those which involve both limiting equations. We obtain partial information on these but it seems difficult to obtain complete information. Note that the third limiting equation in [15] also plays a role in this last problem and indeed a better understanding of this equation seems necessary to improve our results further. We will restrict ourselves to Dirichlet boundary conditions though our methods are also applicable in the case of Neumann boundary conditions.

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More formally, we intend to study the following problem on a smooth bounded domain Ω in $\mathbb{R}^n$.

$$(1.1)\qquad \begin{cases} \dot{u} = \Delta u + u(a-u) - kuv & \text{in } \Omega, \\ \dot{v} = \Delta v + v(d-v) - \alpha kuv & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

for k large positive (and $\alpha > 0$). We are only interested in non-negative solutions. We always assume $a, d > \lambda_1$ where λ_1 is the principal eigenvalue of $-\Delta$ on Ω for Dirichlet boundary conditions. (Otherwise the problem is trivial.) It turns out there are three limiting equations

$$(1.2)\qquad \begin{cases} \dot{u} = \Delta u + au - uv & \text{in } \Omega, \\ \dot{v} = \Delta v + dv - \alpha uv & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

$$(1.3)\qquad \begin{cases} \dot{w} = \Delta w + aw^+ + dw^- - \alpha^{-1}(w^+)^2 + (w^-)^2 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

$$(1.4)\qquad \begin{cases} \dot{w} = \Delta w + aw^+ + dw^- & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $w = w^+ + w^-$. The results in [15] imply that under certain assumptions non-negative connecting orbits for (1.1) can have one of three forms:

(I) be close to a finite number of connecting orbits of (1.3) (in the sense that u is close to $\alpha^{-1}w^+$ and v is close to $-w^-$),

(II) be (after rescaling) close to a finite number of connected orbits of (1.2) for all time (and always uniformly small) or

(III) be close to a finite number of solutions of (1.2) on $(-\infty, t_0)$ after rescaling and then close to connecting orbits of (1.3) for later time (where the first orbit of (1.3) will come from zero).

Equation (1.4) is involved in the transition in these last orbits. Note that all three types can occur. We will use Conley index calculations to give sufficient conditions for orbits of types (I) and (II) to occur. (The case of type (II) is rather trivial). We prove that suitable connecting orbits of the limiting equations generate corresponding connecting orbits of the original system (for large k). Our results for connecting orbits of type (III) are somewhat weaker. We also obtain a number of connecting orbits from monotonicity ideas for types (I) and (III) and from a result in [13].

The main emphasis in the present work is to show that a good deal of the connection structure for (1.1) is inherited from that of (1.2) and (1.3). Thus we do not try to make a systematic study of (1.2) and (1.3). Clearly more needs to be done on (1.2) and (1.3), though a good deal is known on (1.3). Note also that the lack of smoothness of the nonlinearity in (1.3) causes us continuing technical difficulties.

Usually we assume the conditions in [15] ensuring the dynamics are simple though many of our results hold without this.

One of our basic tools is to extend some of the degree calculations in [11] to calculate the corresponding homotopy indices (or at least the cohomology of the homotopy index).

Finally, I should like to point out that nearly all the techniques in [15] and here still apply when the non-interaction nonlinear terms $u(a-u)$ and $v(d-v)$ in the two equations of (1.1) are replaced by much more general nonlinearities. Most of the proofs are essentially the same.

The reader will find it necessary to have a copy of [15] in reading this paper.

2. Behaviour of connecting orbits

In this section, we improve some arguments in [11] to obtain homotopy index calculations. These are done by combining the homotopies in [11] with the blow up estimates of [15]. These immediately imply the existence of connecting orbits.

We first consider the key case of isolating neighbourhoods for (1.1) for large k which are generated by isolating neighbourhoods for (1.3).

However, we need two technical lemmas showing that the homotopy index in Rybakowski [27] is little affected by a change of space or by a homeomorphism. Note that in our case we will be considering a continuous (nonlinear) semiflow $T(t)x$ on a closed convex set $\widehat{T}$ in a Banach space X with the strong admissibility property on a closed subset S of $\widehat{T}$ that, if $t_n \to \infty$ and $T(s_n)x_n \subseteq S$ for $0 \le s_n \le t_n$, then $\{T(t_n)x_n\}$ has a convergent subsequence in X. Moreover, as part of this assumption we also assume that solutions starting in S and staying in S for $t \ge 0$ do not blow up in S.

Lemma 2.1. *Suppose that Y is a Banach space continuously embedded in X with norm $\|\cdot\|'$ such that if K is bounded in X and $\varepsilon > 0$, $\{T(t)x : t \ge \varepsilon, x\varepsilon K\}$ is a bounded subset of Y and $T(t)$ maps $\widehat{T}$ continuously into Y if $t > 0$. If U is an isolating neighbourhood for the semiflow $T(t)$ on $\widehat{T}$ and $T(t)$ is strongly admissible on U, then $U \cap Y$ is an isolating neighbourhood for the flow on $Y \cap U$ and $C(h_{\widehat{T}}(T,U)) = C(h_{\widehat{T}\cap Y}(T|_Y, U\cap Y))$ where $C(Y)$ denotes the cohomology of Y. Here $h_{\widehat{T}}(T,U)$ denotes the homotopy index of the flow $T(t)$ on the isolating neighbourhood U in $\widehat{T}$.*

Remark 2.2. We suspect that the homotopy indices are in fact the same. In most applications, this would not help. Secondly, the continuity of $T(t)$ on Y frequently comes from the boundedness assumption on Y by interpolation.

Proof. We first need to note that the flow $T(t)$ on $\widehat{T} \cap Y$ is strongly admissible for the neighbourhood $U \cap Y$ in $\widehat{T} \cap Y$. If $t_n \to \infty, x_n \in U \cap Y$ and if $T(s_n)x_n \in U \cap Y$ for $0 \le s_n \le t_n$, then by our assumption on $U, T(t_n - \alpha)x_n$ has a subsequence converging in X. Thus $T(t_n)x_n = T(\alpha)(T(t_n - \alpha)x_n)$ has a subsequence convergent on Y (since $T(\alpha)$ maps X continuously into Y). The no blow up condition is proved similarly. Thus both homotopy indices are defined. It suffices to prove the result for the case of an isolating block U. □

To prove the theorem, we use the formula for the homotopy index in Theorem 4.6 in Rybakowski (see [28])

$$H^n(h_{\widehat{T}}(T, U)) = H^n(A \cup B, \widetilde{C})$$

where B is the set of points of U which lie on bounded solutions (in $\mathbb{R}$) of our semiflow which lie in U for all t while A is the set of points of U which lie on solutions $u(t)$ which are defined on $(-\infty, \gamma)$ (where γ depends on the solution) such that $u(t) \in U$ if $t < \gamma, u(\gamma) \in \partial_{\widehat{T}} U$ and $u(t)$ then leaves U. Finally $\widetilde{C} = A \cap \partial_{\widehat{T}} U$. By our regularity, we see that all these solutions lie in $Y \cap \widehat{T}$. By the strong admissibility, it is easy to see that $A \cup B$ is compact in X and hence by the regularity is also compact in Y. Hence we see that $A, B, \widetilde{C}$ are unchanged if we work in X or Y and our claim follows.

We need the following very simple lemma.

Lemma 2.3. *Assume that $\widehat{T}$ is a closed convex subset of $X, T(t)$ is a semi flow on $\widehat{T}$ and U is an isolating neighbourhood for $T(t)$ such that $T(t)$ is strongly admissible on U. If Z is a Banach space and $q: X \to Z$ is a homeomorphism, then $q(U)$ is an isolating neighbourhood for the semiflow $qT(t)q^{-1}$ on $q(\widehat{T})$ and $h_{\widehat{T}}(T, U) = h_{q(\widehat{T})}(qT(t)q^{-1}, q(U))$.*

Proof. We can reduce to the case where U is an isolating block. It is easy to prove that the new semiflow is strongly admissible on $q(U)$ and that q is a homeomorphism of U onto $q(U)$ and of the corresponding exit sets of U and $q(U)$, respectively. Hence the result follows. □

The main theorem of this section is the following. We consider $T(t), S(t)$ the semi flows for (1.1) and (1.3) respectively. For spaces, we use $X_1 = L^p(\Omega) \oplus L^p(\Omega)$ for (1.2) and $X_2 = L^p(\Omega)$ for (1.3). Here $p > n$ but is fixed. In fact $p > 2$ would suffice with care. In this case standard theory as in Henry [18] shows that the semiflows are defined for all $t \ge 0$ where in the first case the convex set $\widehat{T}$ is the set $K = \{(u, v) \in X_1 : u, v \ge 0 \text{ on } \Omega\}$ while in the second $\widehat{T}$ is the whole space. Moreover, by using fractional power spaces $(X_1)^\theta$ with θ close to but less than 1, the regularity theory in Henry ensures that the assumptions of Lemma 2.1 are satisfied if we take $Y = \{(u, v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) : u = v = 0 \text{ on } \partial\Omega\}$ with the corresponding choice in the second case. This will be important below in our index calculations.

Theorem 2.4. *Assume that U is a bounded isolating neighbourhood for the semi flow $S(t)$ of* (1.3) *and $w^+ \neq 0$ and $w^- \neq 0$ for every stationary solution w of* (1.3) *in U and that $0 \notin U$. If M is large, let $\mathcal{U} = \{(u,v) \in X_1 : u, v \geq 0$, $\alpha u - v \in U$, $\|u\|_p, \|v\|_p < M\}$. Then $\mathcal{U} \cap K$ is an isolating neighbourhood for the flow of* (1.1) *on $X_1 \cap K$ for large k and $C(h_{X_1 \cap K}(T, \mathcal{U} \cap K)) = C(h_{X2}(S, U))$ for large k.*

Proof. Note that similar results were proved in [11] but with homotopy indices replaced by degrees. The idea here is to combine the blow up methods of [15] and the homotopies in [11]. The key point is that the two components of $T(t)w$ are close to $(\alpha^{-1}S(t)w)^+$ and $-(S(t)w)^-$ for appropriate w if k is large. The proof is by a series of steps, partly following [11].

Step 1. We first consider the homotopy

$$(2.1) \qquad \begin{cases} \dot{u} = \Delta u + sf(u) + (1-s)f((u - \alpha^{-1}v)^+) - kuv & \text{in } \Omega, \\ \dot{v} = \Delta v + sg(v) + (1-s)g((v - \alpha u)^+) - \alpha kuv & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

for $0 \leq s \leq 1$. Here $f(y) = ay - y^2, g(y) = dy - y^2$. We consider bounded solution on $\mathbb{R}$ such that $\alpha u - v \in U$ for all t (and $u, v \geq 0$). Since it is easy to check that $(u \dot{+} v) - \Delta(u+v) \leq -1$ if $u + v$ is large, we easily see that there is an a priori sup bound independent of s and large k for solutions of (2.1) defined and bounded on $\mathbb{R}$. This determines the M in the definition of $\mathcal{U}$. Now if (u,v) is such a solution and if k is large, the argument used to derive (8) and (9) in [15] ensure that $w = \alpha u - v$ satisfies

$$\dot{w} = \Delta w + h(w) + O_k(1)$$

where $O_k(1)$ tends to zero uniformly on $\overline{\Omega}$ as $k \to \infty$ and $h(y) = ay^+ + dy^- - \alpha^{-1}(w^+)^2 + (w^-)^2$. Since $w(t)$ is bounded in $L^\infty(\Omega)$ uniformly in t (by above), standard local results imply $w(t)$ is bounded in $\dot{W}^{1,2}(\Omega)$ uniformly in t and large k. Hence by the Sobolev embedding theorem and interpolation, we see that $w(t)$ lies in a compact subset of $L^p(\Omega)$ uniformly in t and large k. We now prove that $\mathcal{U}$ is an isolating neighbourhood for all $s \in [0,1]$ and all large k. Suppose by way of contradiction that $(u_i(t), v_i(t))$ are solutions of (2.1) for $s = s_i$ and $k = k_i \to \infty$ which are in $\mathcal{U}$ for all t such that $d((u_i(t_i), v_i(t_i)), \partial\mathcal{U}) \to 0$ as $i \to \infty$ for some t_i. By a translation, we can assume $t_i = 0$ for all i. By our earlier estimates, the only possibility is that, if $w_i(t) = \alpha u_i(t) - v_i(t), w_i(0) \to \partial U$ as $i \to \infty$. Note that w_i satisfies $\dot{w}_i = \Delta w_i + h(w_i) + \widetilde{O}_i$ where $\widetilde{O}_i$ tends to zero uniformly on Ω as $i \to \infty$. Since $w_i(t)$ all lie in a compact subset of Ω, we can easily use a variation of constants formulation of the equation to show that w_i converges to w in $L^p(\Omega)$ uniformly on compact t intervals to w where $w(t) \in \overline{U}$ for all $t, w(0) \in \partial U, \dot{w} = \Delta w + h(w)$. (Note that standard local estimates ensure that $w_i(t)$ are uniformly continuous in t uniformly in i). This contradicts that U is an isolating neighbourhood for (1.3). Hence we see that for large k the homotopy index of (1.1) on $\mathcal{U}$ is the same as that for the flow of

$$(2.2) \qquad \begin{cases} \dot{u} = \Delta u + f((u - \alpha^{-1}v)^+) - kuv & \text{in } \Omega, \\ \dot{v} = \Delta v + g((v - \alpha u)^+) - \alpha kav & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that this argument also ensures that the homotopy index of $T(t)$ on $\mathcal{U}$ is defined for large k.

Step 2. We use Lemma 2.1 to replace the underlying space X_1 by $\{(u,v) \in C^1(\Omega) : u, v \geq 0,\ u = v = 0 \text{ on } \partial\Omega\}$. Note that it is because of this step we can only prove the final result for the cohomological part of the homotopy index.

For the technical details, we note that since $p > n$, we can choose $\theta < 1$ but close to 1 so that the fractional power space $(X_2)^\theta$ is continuously embedded in $C^1(\Omega)$. (For fractional power spaces, see [18]). Now the theory in [18] ensures that the semigroup $\widetilde{T}(t)$ for (1.1) on $L^p(\Omega) \oplus L^p(\Omega)$ maps continuously into $\{(u,v) : (X_2)^\theta \oplus (X_2)^\theta : u = v = 0 \text{ on } \partial\Omega\}$ if $t > 0$ and hence into $Z = \{(u,v) \in C^1(\overline{\Omega}) \oplus C^1(\overline{\Omega}) : u = v = 0 \text{ on } \partial\Omega\}$ by the Sobolev embedding theorem. Hence our claim follows.

We can do a little more here and this is the point of choosing our space. Every bounded solution of (1.1) with $u, v \geq 0$ and $\alpha u - v \in U$ for all t lies in the interior of the natural cone $\widetilde{K}$ in Z and lies at a positive distance from $\partial\widetilde{K}$ (where the distance depends on k). This follows easily since the set of bounded solutions in our neighbourhood lies in a compact set of Z (by our earlier remarks and standard estimates) and does not contain a point where $u(t) = 0$ (or $v(t) = 0$). (This follows because $u(t) = 0$ would imply by uniqueness and backward uniqueness, as in [18], that $u(t) = 0$ for all t and hence $w(t)$ is negative for all t. Now the w equation is easily seen to have a unique negative stationary solution and hence $w(t)$ must approach 0 as $t \to -\infty$ or $t \to \infty$. This contradicts that $0 \notin U$.) Hence, by the parabolic maximum principle, $c\partial u/\partial n$ has a positive lower bound on $\partial\Omega$ and the result follows easily. Hence we can choose a closed neighbourhood Y_k of our set of positive solutions so that $Y_k \subseteq \operatorname{int} \widetilde{K}$. Hence we see that our homotopy index is the same in $\widetilde{K}$ or Z.

Step 3. We consider the homeomorphism $q(u,v) = (\alpha u - v, u)$. By Lemma 2.3, we see that our homotopy index is the same as that of the system

$$(2.3) \qquad \begin{cases} \dot{w} = \Delta w + h(w), \\ \dot{u} = \Delta u + f(\alpha^{-1}w^+) - ku(\alpha u - w), \end{cases}$$

on $q^{-1}(Y_k)$ in Z. Our new map is essentially a product so we are close to applying the product theorem for the homotopy index. We choose $\widetilde{M} > 0$ such that

$$f(\alpha^{-1}t) - k\widetilde{M}(\alpha\widetilde{M} - t) < 0$$

if k is large and $t \in [-b, a]$. We can also shrink U so that $w(t) \in [-b, a]$ if $w \in U$. To prove this, it suffices to show that each bounded solution of $\dot{w} = \Delta w + h(w)$ lies in $(-b, a)$ because a compactness argument shows that each such solution lies is $(-b + \delta, a - \delta)$ where $\delta > 0$ is independent of w and then we can choose U a small neighbourhood of these bounded solutions. Now bounded solutions of the w equation on $\mathbb{R}$ are either stationary solutions or heteroclinic solutions joining stationary solutions. It is easy to use the elliptic maximum principle to show that the stationary solutions have the required property and then apply the parabolic maximum principle to $a - w$ and $b + w$ to prove our claim.

We now consider bounded solutions of (2.3) such that $w(t) \in U$ for all t and $0 \leq u(t) \leq M$ for all t. For each such $w(t)$, M is an upper solution of the second equation and zero is a lower solution. Hence, by using the method of sub and super solution on the time interval $[-n, n]$ and letting n tend to infinity, we find that there is a non-negative solution $u(t)$ of the second equation of (2.3) (including the boundary condition) such that $u \leq M$. We will prove below that any such solution u of the second equation satisfies $u \geq \alpha^{-1}w^+$ for large k and u converges uniformly to $\alpha^{-1}w^+$ as $k \to \infty$. Moreover, what is meant by large is uniform in w. Assuming this for the moment, we complete the proof.

Since $\alpha^{-1}w^+ \leq u \leq M$ for any non-negative solution u of the second equation, $v = \alpha u - w \geq 0$. Thus, for large k, the only bounded solution of (2.3) in $U \times [0, M]$ must lie in $q^{-1}(Y_k)$ (because αu is close to w^+ and v is close to $-w^-$ in $L^p(\Omega)$ and u, v are non-negative). Thus the homotopy index on $q^{-1}(Y_k)$ and $U \times [0, M]$ is the same. We calculate the index on this last space. Since 0 and M are sub and supersolutions of the second equation and not solutions (since $0 < w(x, t) < a$ for some x in Ω for each t), the u component never exits $[0, M]$. Hence the homotopy index is the homotopy type of $(U \times [0, M])/(\widehat{U} \times [0, M])$ where $\widehat{U}$ is the exit set for the w flow on U. Hence the homotopy index is the homotopy type of $U/\widehat{U}$, which is the homotopy type of the flow of the w equation on U as required.

Thus, to complete the proof, we need only prove our claims on the second equation of (2.3) above. First, we prove the convergence assuming $u \geq \alpha^{-1}w^+$. It suffices to prove uniform convergence on compact sets. To prove the convergence is a blow up argument rather similar to the proof of Theorem 1 in [15]. We have to consider the solutions on $\mathbb{R}^n \times \mathbb{R}$ of

$$\begin{cases} \dot{w} = \Delta w, \\ \dot{u} = \Delta u - u(\alpha u - w), \end{cases}$$

where w is bounded u is non-negative and bounded, and $u \geq \alpha^{-1}w$. We need to prove that $u \equiv \alpha w^+$ for this limit equation. As in [15], the first equation ensures that w is constant, $w \equiv C$, and then the second equation becomes u bounded and non-negative $u \geq \alpha^{-1}C$ and

$$\dot{u} = \Delta u - u(\alpha u - C)$$

As in [15], we find $u \equiv 0$ if $C \leq 0$ and $u \equiv \alpha^{-1}C$ if $C > 0$. Hence $u = \alpha w^+$ and our claim follows. (If $C > 0$, we use the equation for $\widetilde{v} = \alpha u - C$).

Thus it remains to prove that $u \geq \alpha^{-1}w^+$ for any non-negative solution of the second equation of (2.3). To prove this we use sweeping families of subsolutions. It suffices to prove this for a fixed k. First note that $\alpha^{-1}w^+$ is easily seen to be a subsolution of our problem. (This is easily seen most easily by using Kato's inequality [20, Section 16.8] to prove that $d(w^+)/dt \leq \Delta(w^+) + h(w^+)$ in the sense of distributions.) It follows easily by concavity that sw^+ is a subsolution of the second equation of (2.3) when $0 \leq s < \alpha^{-1}$ (and a strict subsolution when $0 < s < \alpha^{-1}$ and $w(x,t) > 0$). Now let s^* be the maximal s in $[0, \alpha^{-1}]$ for which $u \geq s^*w^+$ on $\Omega \times \mathbb{R}$. We are finished if $s^* \geq \alpha^{-1}$ and we do not exclude that $s^* = 0$. Then by a simple calculation $y = u - s^*w^+$ is a non-negative weak solution of

$$\dot{y} \geq \Delta y - \widehat{K}y + (\alpha^{-1} - s^*)f(\alpha^{-1}w^+) + s^*k(1 - \alpha s^*)(w^+)^2 \tag{2.4}$$

on $\Omega \times \mathbb{R}$ where $\widehat{K}$ is uniformly bounded (independent of u). Now it is not difficult to prove that there exists $\mu, B > 0$ independent of t such that $w(x,t) \geq \mu$ on a set B_t with $m(B_t) \geq B$. (Remember that $w(x,t) \leq 0$ on Ω for some $\widetilde{t}$ implies $w(x,t) \leq 0$ for $t > \widetilde{t}$ and $x \in \Omega$). This ensures that the forcing term on the right-hand side of (2.4) (that is the terms independent of y) is non-negative for each t and has a positive lower bound on a set not of small measure (since $s^* < \alpha^{-1}$). Hence by applying the Green's function for $\dot{y} - \Delta y + \widehat{K}y$ on $[t-1, t]$, we deduce that $y(t)$ has a positive lower bound on compact subsets of Ω and $\partial y(\,\cdot\,, t)/\partial n$ has a positive lower bound on $\partial\Omega$ where the bounds are independent of t. Hence we see that if δ is small and positive $y - \delta w^+ \geq 0$ on $\Omega \times \mathbb{R}$. Thus $u \geq (s^* + \delta)w^+$ on $\Omega \times \mathbb{R}$. This contradicts the maximality of s^* and hence $u \geq \alpha^{-1}w^+$ as required. This completes the proof. □

Remark 2.5. The proof of Theorem 2.4 is made much more difficult because some of the homotopy indices are in cones and some are whole space homotopy indices. This seems to preclude using the abstract theory in Huang [21] though he informs me that it can be used in some awkward (non-normable) spaces.

Remark 2.5. It is possible to prove a variant of Theorem 2.4 for the case where U contains a positive solution of (1.3). We do not consider this case further here since it only seems of use in proving the existence of infinitely many connections. Otherwise Proposition 2.9 below gives better results.

The main use of the theorem for our present purposes is to obtain connections for (1.1). Assume that w_1 and w_2 are non-zero changing sign hyperbolic solutions of (1.3). Note that as in [12], the linearization is defined. For large k there are locally unique positive solutions (u_i, v_i) near $(\alpha^{-1}w_i^+, -w_i^-)$ in X for $i = 1, 2$ (see [12]). By [15, Remark 4, p. 482], (u_i, v_i) are hyperbolic for large k. By the theory in [15], it is necessary for there to be a connection from w_1 to w_2 for there to be a connection from (u_1, v_1) to (u_2, v_2). We prove a converse to this.

Corollary 2.7.

(a) *Assume that* $w_i, i = 1, 2$, *are changing sign hyperbolic solutions of* (1.3) *and* U *is an isolating neighbourhood for the semiflow of* (1.3) *such that* $w_i \in U$ *for* $i = 1, 2$, *no other stationary solution of* (1.3) *belongs to* $\overline{U}$ *and* $C(h(S(t), U)) \neq C(h(S(t), w_1)) \oplus C(h(S(t), w_2))$. *Then for large* k *there is a connection joining* (u_1, v_1) *and* (u_2, v_2).

(b) *Assume that* w_i, $i = 1, 2$ *are hyperbolic changing sign stationary solutions of* (1.3) *such that the stable manifold of* w_1 *intersects the unstable manifold of* w_2 *transversally and Morse index of* $w_1 -$ *Morse index* $w_2 = -1$. *Then there is a connection joining* (u_1, v_1) *and* (u_2, v_2) *for large* k.

(c) *Assume that* $n = 1$, w_i, $i = 1, 2$ *are changing sign stationary solutions of* (1.3) *such that Morse index* $w_1 -$ *Morse index* $w_2 = -1$ *and there is a connection from* w_2 *to* w_1. *Then for large* k *there is a connection joining* (u_1, v_1) *and* (u_2, v_2).

Proof. (a) If we define $\mathcal{U}$ as in the proof of Theorem 2.4, then the theory in Theorems 2 and 3 of [15] implies that for large k the only bounded solutions of (1.1) in $\overline{\mathcal{U}}$ for all t are $(u_1, v_1), (u_2, v_2)$ and connections joining (u_1, v_1) and (u_2, v_2) which lie in $\mathcal{U}$ for all t. Moreover, the connections must be uniformly close to $(\alpha^{-1}w^+(t), -w^-(t))$, where $w(t)$ is a connection of (1.3) joining w_1 and w_2. Hence it suffices to prove that we have contradiction if there is no non-constant solution of (1.1) which lies in $\mathcal{U}$ for all t. This follows since otherwise, Theorem 10.4 in [27] and elementary cohomology theory ensures that $C(h_K(T(t), \mathcal{U})) = C(h_K(T(t), (u_1, v_1)) \oplus C(h_K(T(t), (u_2, v_2))$. By Theorem 2.4 applied to each of these homotopy indices, we see that for large k

$$C(h(S(t), \mathcal{U}) = C(h(S(t), w_1)) \oplus C(h(S(t), w_2)).$$

This contradicts our assumptions, completing the proof.

(b) To prove this, choose a connection $z(t)$ joining w_1 to w_2. By the transversality assumption, the same argument as in [23] (which is the finite dimensional case) ensures that this connection is isolated among connecting orbits joining w_1 and w_2. We choose U to be a small neighbourhood of $\{w_1\}\cup\{w_2\}\cup\{z(t) : -\infty < t < \infty\}$ in $L^p(\Omega)$. Our construction ensures that the only bounded solutions of (1.3) defined on $\mathbb{R}$ and in $\overline{U}$ for all t are the constant in time solutions of w_1, w_2 and $\{z(t) : t \in \mathbb{R}\}$. Thus it suffices to prove that U satisfies the cohomology condition in (a). We in fact prove that $C(h(S(t), U)$ is $\overline{O}$, from which our claim follows. For finite dimensional smooth mappings, this follows immediately from Theorem 3.1 in [23]. The general case can be reduced to this by smoothing and finite dimensional approximations. Details will appear elsewhere.

(c) This follows from (b) if we prove that the w_i are hyperbolic and that the transversality condition holds. The hyperbolicity follows from a straightforward adaption of the arguments in Laetsch [22]. (One uses that w' satisfies the linearized equation and the Sturm comparison theorem.) The jump in the derivative at zero does not affect the argument much at all. Note that for (1.3) we need only worry about whether zero is an eigenvalue of the linearization. This is also proved in [30]. The transversality is a variant of the main result in Argument [1] (or [19]) where we assume less smoothness. We need to modify the proof in [1] by using the results in [2] and by proving a variant of Theorem 3.4.4 in [18] where we assume strict differentiability rather than C^1. □

Remarks 2.8. (a) If $n = 1$, arguments in Section 4 of [14] can be easily adapted to prove that (1.3) has at most 1 solution with a given number of zeros and a given sign of $u'(a)$ (where $\Omega = (a, b)$. This has also been noted by a number of other mathematicians. (See [30].)

(b) Assume that w_1 and w_2 are both hyperbolic changing sign solutions of (1.3) and U is an isolating neighbourhood for the semiflow of (1.3) such that w_1 and w_2 are the only stationary solutions of (1.3) in U. Then the condition $C(h(S(t), U)) \neq C(h(S(t), w_1)) \oplus C(h(S(t), w_2))$ is equivalent to the connection index δ (in the sense of [24, p. 156]) is non-trivial. (This is sometimes called the connection map.) This follows easily from the exact sequence for the attractor-repeller pair (as in [29]). Moreover, the same exact sequence shows that this can only occur if the Morse indices of w_1 and w_2 differ by 1. Moreover, if $\delta \neq 0$, δ is an injection in the dimension where it is non-zero (since it there is a mapping from Z to Z). This is useful for Remark (b) below. There have been many results on which solutions of (1.3) are joined by connecting orbits and how many connecting orbits, especially when $n = 1$. Rather more complete information can be found for $n = 1$ in [3]. Another way to prove that the connection index δ is non-zero by continuation (usually starting from a bifurcation). Lastly, it is possible with a good deal of care to prove the local uniqueness of the connection obtained in Corollary 2.7(b).

(c) We suspect the transversality is true in higher dimensions for generic Ω. For a smoother nonlinearity a related but different result is proved in [4].

(d) It is often possible to use the ideas in [17] with our remark above that the connection index δ is injective under certain conditions to prove the existence of connections of (1.1) for large k between hyperbolic stationary solutions of (1.1) whose Morse indices differ by more than 1. A much simpler proof can be given if the Morse indices differ by 2.

We now consider two other ways of obtaining connections. First we use the order structure. It is well known and easy to prove that the semiflow $S(t)$ of (1.1) preserves the order $\leq_S$ the cone of non-negative functions in X where $(u_1, v_1) \leq_S (u_2, v_2)$ if $u_1 \leq u_2$ and $v_1 \geq v_2$. Then the following result follows immediately from Remark 1.3 in Dancer and Hess [13]. Note it does not use that k is large and is easy to use when it applies.

Proposition 2.9. *Assume that (u_1, v_1), (u_2, v_2) are non-negative stationary solutions of* (1.1) *such that $(u_1, v_1) \geq_S (u_2, v_2)$ and there is no other stationary solution of* (1.1) *(u, v) with $(u_1, v_1) \geq_S (u, v) \geq_S (u_2, v_2)$. Then there is a monotone connection* (*in the order $\leq_S$*) *joining (u_1, v_1) and (u_2, v_2).*

Remark 2.10. Conversely, if there is a stationary solution (u, v) other than (u_1, v_1) and (u_2, v_2) in the order interval between (u_1, v_1) and (u_2, v_2) then it is easy to use strong monotonicity to deduce that there is no connection joining (u_1, v_1) and (u_2, v_2). Note that the solution $(\overline{u}, 0)$ where $\overline{u}$ is positive is a maximal solution of (1.1) in our order and so there is always a connection joining $(\overline{u}, 0)$ to a maximal element of the set of positive solutions of (1.1) with $(\overline{u}, 0)$ excluded. If k is large, we can say more. Now suppose k is large, $(u_3, v_3) \leq_S (u_4, v_4)$, (u_3, v_3) is uniformly close to $(\alpha^{-1}a^+, -a^-)$ and (u_4, v_4) is uniformly close to $(\alpha^{-1}b^+, -b^-)$ where a and b are non-trivial hyperbolic solutions of (1.3). (Note that for k large, any stationary solution of (1.1) which is not small must be of this form). It is easy to see that $(u_3, v_3) \leq_S (u_4, v_4)$ forces that $a \leq b$. If $a \neq b$ and $a \leq b$, and a and b are hyperbolic changing sign solutions of (1.3), it is possible to prove with care that conversely $(u_3, v_3) \leq_S (u_4, v_4)$ if k is large. If $n = 1$, it is not difficult to prove using the first integral that such ordered a, b never exist with a and b changing sign, but it can be shown that this is not always true in higher dimensions (for example on dumbbells, by the use of domain variation techniques, as in [10]).

Proposition 2.11. *Let D be the set of points of X_1 which lie on bounded non-negative solutions of* (1.2). *Then D is a compact acyclic set.*

Proof. This follows immediately from Theorem 4.6 in [28] if we note that we can choose an isolating neighbourhood $\widetilde{Z}$ for the flow of (1.1) in X_1 which is the product of two balls in X_1, has no exit set and contains all the bounded solutions. (Thus the homotopy index on $\widetilde{Z}$ is the two point space $\overline{1}$). Note that the compactness comes from the strong admissibility of the semiflow.

Once again this does not use that k is large. In particular, if we know that the only bounded solutions are stationary solutions or connecting orbits (which is proved in [15] under certain assumptions), it follows that every non-negative stationary solution of (1.1) must be joined by a connecting orbit to another non-negative stationary solution. □

3. Small-small connections

From the theory in [15], there is another type of positive stationary solution of (1.1) for large k. These are of order k^{-1}. Thus, it is natural to look for connecting orbits of the equation

$$\begin{aligned} \dot{\mathcal{U}} &= \Delta\mathcal{U} + a\mathcal{U} - \alpha^{-1}k^{-1}\mathcal{U}^2 - \mathcal{U}V, \\ \dot{V} &= \Delta V + dV - k^{-1}V^2 - \mathcal{U}V, \\ \mathcal{U} &= V = 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

where $\mathcal{U} = \alpha ku, V = kv$. We consider this briefly. This is a small perturbation of (1.2) (for $\alpha = 1$) and hence it is natural to look for connecting orbits of this as perturbations of connecting orbits of (1.2). This is a regular perturbation so the theory is much simpler than that in Section 2.

THEOREM 3.1. *Suppose that $\mathcal{W}$ is a bounded isolating neighbourhood for the semiflow of* (1.2) (*for* $\alpha = 1$) *on the natural cone in* $L^p(\Omega) \oplus L^p(\Omega)$. *Then for large k, $\mathcal{W}$ is also an isolating neighbourhood for the semiflow of* (3.1) *on $\mathcal{W}$ and two homotopy indices are the same.* (*Here p is as before*).

PROOF. This is trivial from the homotopy invariance of the homotopy index. □

REMARKS 3.2. By a simple rescaling we see that the homotopy index of (1.2) is independent of α (provided $\mathcal{W}$ is also sealed appropriately). Theorem 2 can be used much as Theorem 1. The comments here are very similar to those after Theorem 2.4. We simply point out the differences. The first major difference is that it is unclear if the only bounded positive solutions of (1.2) are stationary solutions or connecting orbits. (This is unclear even if $n = 1$). Thus we tend to need to make extra assumptions on (1.2). (The one easy case is when $a = d$ which we discuss later.) The second change is that we do not know hyperbolicity and transversality results if $n = 1$. The third is that the stationary solution $(0, 0)$ is unstable as a positive solution for small perturbations (by adding a small positive constant to each equation). Thus $(0, 0)$ has trivial homotopy index and there is no hope of using the homotopy index to obtain connections from $(0, 0)$

to another positive solution. It is easy to see that this connection always exists if $a = d$ (see below) or for Neuman boundary condition but it is unclear if it exists in general. Another change is that (1.2) can not have two positive stationary solutions ordered in the order $\leq_S$. In fact we cannot even have two solutions with $u_1 \geq u_2$. This follows from comparison theory for the eigenvalues of positive linear operators if we note that if (u, v) is a positive stationary solution of (1.2), then for large enough C, 1 is the principal eigenvalue of the eigenvalue problem

$$\begin{cases} -\Delta h + Ch = \lambda(C + d - u)h & \text{in } \Omega, \\ h = 0 & \text{on } \partial\Omega, \end{cases}$$

(with eigenfunction v). Thus the theorem in [13] is not applicable in this case. Lastly, note that the stationary solutions and the connecting orbit structure for (1.2) are not always simple. This can be proved easily by bifurcations from the constant positive solutions for the case of Neumann boundary conditions and by rather more effort by bifurcation from manifolds of solutions and then domain variation [9] in the case of Dirichlet boundary conditions. It seems likely that other examples can be created from the results in [8].

Note that if $a = d$ and a is not an eigenvalue of $-\Delta$ (for Dirichlet boundary conditions), it is easy to prove as in [15], §2, that there is a unique positive stationary solution $(\overline{w}, \overline{w})$ of (1.2) (where purely for simplicity we are assuming $\alpha = 1$) and the only non-constant bounded positive solution is the connecting orbits joining $(0, 0)$ to $(\overline{w}, \overline{w})$. The structure of the stationary points persists if we perturb a and d slightly and the only possible connecting orbits are those joining $(0, 0)$ to the unique positive stationary solution. We could also obtain a good deal of information when $a = d$ is an eigenvalue. (Here the set of stationary positive solutions is an unbounded manifold). We will briefly consider another case below.

Provided we assume that (1.4) has only the trivial stationary solution, the argument in [15, pp. 483–484] (which is a blow up argument) shows that, for large k, there cannot be any other connecting orbits of (1.1) approaching stationary solutions of (1.1) of order k^{-1} as $t \to \pm\infty$. Moreover, the arguments in §1 of [15] show that there cannot be connecting orbits $\widetilde{w}(t) = (u(t), v(t))$, of (1.1) such that $\lim_{t\to-\infty} \widetilde{w}(t)$ is not small but $\widetilde{w}(t)$ is order k^{-1} as $t \to \infty$. (The key point is that, as in [15], any non-trivial stationary solution of (1.3) has negative energy). We will explore in the next section the connecting orbits $\widetilde{w}(t)$ such that $\lim_{t\to-\infty} \widetilde{w}(t)$ is order k^{-1} while $\lim_{t\to\infty} \widetilde{w}(t)$ is not small.

Theorem 3.3. *Suppose that* (1.4) *has only the trivial stationary solution and* $\widehat{K} > 0$ *such that all the bounded positive solutions of* (1.2) *in* $L^p(\Omega) \oplus L^p(\Omega)$ *lie in the ball of radius* $\widehat{K}$. *Then, for k large,*

$$h_K(\widetilde{T}(t), K \cap B_{k^{-1}\widehat{K}}) = h_K(T_1(t), K \cap B_{\widehat{K}})$$

where $\widetilde{T}(t)$ is the semiflow for (1.2) *for* $\alpha = 1$ *and* $T_1(t)$ *is the semiflow for* (3.1). *Part of the result is that the homotopy indices are defined. Moreover,*

$$C(h_K(\widetilde{T}(t), K \cap B_{k^{-1}\widehat{K}})) = C(h_{E_1}(T_2(t), \widetilde{B}_1)$$

where $\widetilde{B}_1$ *is the ball of radius* 1 *in* $E_1 = L^p(\Omega)$ *and* $T_2(t)$ *is the semiflow of* (1.4).

REMARK 3.4. The arguments in [24, Section 2], ensure that $\widehat{K}$ exists.

PROOF. The first part is simply a rescaling of (1.1) to (3.1) and then using the homotopy invariance of the homotopy index. To prove the second result we use a very similar argument to that in the proof of Theorem 2.4. As in Step 1 there, we first deform (1.2) to the equation

$$\begin{aligned} \dot{u} &= \Delta u + a(u-v)^+ - uv, \\ \dot{v} &= \Delta v + d(u-v)^- - uv, \end{aligned}$$

on a large enough ball in the cone by the obvious homotopy. That we have a uniform bound $\widehat{K}$ for the positive solutions in the deformation is very much the same as the proofs on in [15, pp. 483–484]. We now fix $\widehat{K}$. By adding a small positive constant ε_0 to each equation of (1.2), we do not change the homotopy index on $B_{\widehat{K}}$ but ensure that every bounded non-negative solution of (1.2) defined on $\mathbb{R}$ has both components strictly positive on $\Omega \times \mathbb{R}$. As in the proof of Theorem 2.4, the cohomology of the homotopy index is unchanged if we replace $L^p(\Omega)$ by $C_0^1(\Omega)$ as the underlying space. If we note that every non-trivial non-negative bounded solution of (1.2) on $\mathbb{R}$ has both components in the interior of the natural cone in $C_0^1(\Omega)$ by the strong parabolic maximum principle, we can complete the proof by a similar but rather easier argument to that in the proof of Theorem 2.4. □

REMARK 3.5. The cohomology of the homotopy index for the semiflow of (1.4) on a ball centre zero is not easy to calculate though there are many partial results in [6]–[8], [16] and [25] where many further references can be found. (Note that the cohomology of the homotopy index is the same as the critical groups discussed in these papers.) The results in [8] suggest that it usually changes if (a, d) crosses a curve in $A_0 = \{(a, d) : -\Delta u = au^+ + du^-, u = 0 \text{ on } \partial\Omega$ has a non-trivial solution$\}$.

Note that the homotopy index is constant on components of $\mathbb{R}^2 \setminus A_0$ and the structure of A_0 is far from understood. Lastly, since $(0, 0)$ has trivial homotopy index for (1.2) and is easily seen to be a repeller, we can easily use the exact sequence for an attractor repeller pair to see that the cohomology of the homotopy index of (1.2) on large balls in the cone is unaffected by $(0, 0)$ or connections joining $(0, 0)$ to any other stationary point.

Lastly for this section, we consider very briefly another case where we can understand (1.2) very well. We assume $d > \lambda_1$, is fixed and assume $a > \lambda_1$ but a is close to λ_1. We first look at the stationary solutions of (1.2). We write $u = \|u\|_\infty \widehat{u}, v = \|v\|_\infty \widehat{v}$. We first assume that $\|u\|_\infty$ and $\|v\|_\infty$ are uniformly bounded. Passing to the limit as $a \to \lambda_1$, we easily see that $\|\widehat{u}\|_\infty = 1$,

$$\begin{cases} -\Delta \widehat{u} = (\lambda_1 - B\widehat{v})\widehat{u} & \text{on } \Omega, \\ -\Delta \widehat{v} = (d - \alpha \widehat{u})\widehat{v} & \text{on } \Omega, \\ \widehat{u} = \widehat{v} = 0 & \text{on } \Omega. \end{cases}$$

Here B is the limit of $\|v\|_\infty$ and α is the limit of $\|u\|_\infty$ (through subsequences). By scalar multiplying the first equation by $\widehat{u}$, we easily deduce that $\widehat{u}$ is the principal eigenfunction ϕ_1 of $-\Delta$ for Dirichlet boundary condition on Ω (normalized so that $\|\phi_1\|_\infty = 1$) and $B\widehat{v} = 0$ (and so $B = 0$ since $\|\widehat{v}\|_\infty = 1$). Now the least eigenvalue $\lambda_1(\alpha)$ of $-\Delta - (d - \alpha\phi_1)I$ on Ω (with the boundary condition) is easily seen to strictly increase with α (by the variational characterization of eigenvalues), $\lambda(0) = \lambda_1 - d < 0$ and $\lambda(\alpha) > 0$ for large positive α (comp. [9]). Hence there is a unique positive α_0 for which $\lambda(\alpha_0) = 0$. This uniquely determines α and hence $\widehat{v}$ (in the limit).

We use this to prove the uniqueness of the positive stationary solution of (1.2) in this case (assuming the bounds). By the above, if $(u_1, v_1), (u_2, v_2)$ are positive solutions of (1.2) and a tends to λ_1, then $u_i \to \alpha_0\phi_1$ and $v_i \to 0$ in $C_0^1(\Omega)$ for $i = 1, 2$. Hence by a standard argument $(u_1 - u_2, v_1 - v_2)$ normalized converges to a non-trivial solution of the linearization of (1.2) at $(\alpha_0\phi_1, 0)$ for $a = \lambda_1$, that is, to a non-trivial solution $(h, \widetilde{k})$ of

$$\begin{cases} -\Delta h = \lambda_1 h - \alpha_0 \phi_1 \widetilde{k}, \\ -\Delta \widetilde{k} = (d - \alpha_0\phi_1)\widetilde{k}, \end{cases} \tag{3.2}$$

(with the boundary condition). By the second equation, $\widetilde{k} = 0$ or $\widetilde{k}$ has fixed sign. By scalar multiplying the first equation by ϕ_1, we deduce that $\widetilde{k} = 0$. Hence the only non-trivial solutions of (3.2) are $(r\phi_1, 0)$. Because of the C^1 convergence (and that ϕ_1 is an interior element of the usual cone in $C_0^1(\Omega)$), we deduce that $\pm(u_1 - u_2)$ is positive if a is close to λ_1. This contradicts earlier remarks of ours on (1.2) and so our claim follows. Thus, it suffices to establish the bounds. If the bounds do not hold we can argue very similarly to that in [15, p. 484] to deduce that $\widehat{u}\widehat{v} \to 0$ uniformly on $\overline{\Omega}$ as $a \to \lambda_1$. On the other hand, by the first equation of (1.2), $\int |\nabla \widehat{u}|^2 \le a \int \widehat{u}^2$. Thus $\widehat{u}$ is bounded in $\dot{W}^{1,2}(\Omega)$. Since $a \to \lambda_1$, it follows that $\widehat{u}$ converges in $L^2(\Omega)$ to a non-negative multiple of ϕ_1 Now $\widehat{u} \to 0$ in $L^2(\Omega)$ is impossible because, we could then bootstrap from the equation $-\Delta\widehat{u} \le a\widehat{u}$ to deduce that $\widehat{u} \to 0$ in $L^\infty(\Omega)$. Thus $\widetilde{u} = B\phi_1$ where $B > 0$. (Here $\widetilde{u}$ is the limit of the $\widehat{u}$). Similarly $\widehat{v}$ converges to $\widetilde{v}$ where $\widetilde{v}$ is non-negative and non-trivial. But $\widehat{u}\widehat{v} \to 0$ in $L^2(\Omega)$ as $a \to \lambda_1$. Hence, taking the limit, $B\phi_1\widetilde{v} = 0$. Since $B > 0, \phi_1(x) > 0$ on Ω and $\widetilde{v}$ is non-trivial, this is impossible and our bound follows. Thus uniqueness holds. In fact, using similar ideas but with a great deal more care, it is possible to prove that if a is close to λ_1 (and d is fixed), the only possible non stationary bounded solutions of (1.2) are connections joining $(0,0)$ to the unique positive stationary solution. In proving this, one obtains an asymptotic estimate for B. In fact

$$B \sim (a - \lambda_1) < \phi_1, \quad \phi_1 > \left(\int \widehat{v}\phi_1^2 \right)^{-1}$$

as $a \to \lambda_1$. Moreover, if d is fixed and a is close to λ_1, it is not difficult to show that there are no sign changing solutions of (1.3) and so we can obtain a complete understanding of the dynamics of (1.1) in this case (for k large).

4. Connections from small solutions to non-small solutions

We understand these rather less well but in this short section we obtain a few results. The difficulty with these connections that they involve all three limiting equations. Note that we can easily find connections from $(0,0)$ to $(\overline{u},0)$ and $(0,\overline{v})$ by looking in the subspaces $v = 0$ and $u = 0$, respectively.

THEOREM 4.1. *Assume that* (1.4) *has only the trivial stationary solution and* $C > 0$. *There is a* k_0 *such that if* $k \geq k_0$ *and if* (u,v) *is a non-trivial positive solution of* (1.1) *with* $\|u\|_\infty + \|v\|_\infty \leq Ck^{-1}$, *then there is a connection joining* (u,v) *with* $(\overline{u},0)$ (*and a connection joining* (u,v) *with* $(0,\overline{v})$).

PROOF. To prove this we can simply apply the Proposition 2.9 if we know that there is no stationary solution (u_1,v_1) with

$$(\widetilde{u},\widetilde{v}) \leq_S (u_1,v_1) \leq_S (\overline{u},0). \tag{4.1}$$

There are two possibilities, $\|u_1\|_\infty + \|v_1\|_\infty \to 0$ as $k \to \infty$ or $\|u_1\|_\infty + \|v_1\|_\infty \geq \mu > 0$ (at least through subsequences). In the second case, through a subsequence, $u_1 \to w^+$ and $v_1 \to w^-$ where w is a sign changing solution of (1.3) (by [11]). Passing to the limit in the first inequality of (4.1), we deduce that $0 \geq -w^-$ which is impossible. Thus $\|u_1\|_\infty + \|v_1\|_\infty \to 0$ as $k \to \infty$. Since (1.4) has only the trivial solution, we see from [11] that (u_1,v_1) is of the form $k^{-1}((\widetilde{u}_1,\widetilde{v}_1) + o(1))$ where $(\widetilde{u}_1,\widetilde{v}_1)$ is a positive solution of (1.2). Similarly $(\widetilde{u},\widetilde{v}) = k^{-1}((\widetilde{u}_2,\widetilde{v}_2) + o(1))$. Hence using the first inequality of (4.1) and passing to the limit we find that $(\widetilde{u}_2,\widetilde{v}_2) \leq_S (\widetilde{u}_1,\widetilde{v}_1)$. By our earlier comments on solutions of (1.2), this implies that $(\widetilde{u}_1,\widetilde{v}_1) = (\widetilde{u}_2,\widetilde{v}_2)$. We show that this cannot occur. If it occurred, we would have distinct ordered (in the order $\leq_S$) solutions of (3.1) with the same non-zero limit as $k \to \infty$. If we use the equation for their difference (normalized) and passing to the limit as $k \to \infty$, we find that we have a non-trivial solution (s, y) with $(s, y) \geq_S (0, 0)$ of

$$\begin{cases} -\Delta s = (a - \widehat{v})s - \widehat{u}y & \text{on } \Omega, \\ -\Delta y = (d - \widehat{u})y - \widehat{v}s & \text{on } \Omega, \\ s = y = 0 & \text{on } \partial\Omega, \end{cases}$$

where $(\widehat{u}, \widehat{v})$ is a non-trivial positive solution of (1.2). This is the linearized equation of (1.2) at $(\widehat{u}, \widehat{v})$.

Thus, by applying the Krein Rutman theory (comp. Lemma 2.4 in [12]) after the change of variable $y = -r$, we see that zero is an eigenvalue of smallest real part of the eigenvalue problem

$$\begin{cases} -\Delta z = (a - \widehat{v})z + \widehat{u}r + \lambda z & \text{on } \Omega, \\ -\Delta r = (d - \widehat{u})r + \overline{v}z + \lambda r & \text{on } \Omega, \\ z = r = 0 & \text{on } \partial\Omega. \end{cases}$$

(This uses essentially that $(s, y) \geq_S (0, 0)$). This contradicts Lemma 2.5 in [12] (or more strictly contradicts its proof). Hence our claim follows. Thus the claimed connections exist. By the same argument we always have connections to $(0, \overline{v})$. □

We can use a connectedness argument to show that frequently there are connections to solutions of (1.1) which are not small. *Suppose that* $(\widehat{u}, \widehat{v})$ *is a positive solution of* (1.2) *which is hyperbolic, has Morse index at least* 2 *and there is no connection of* (1.2) *joining* $(\widehat{u}, \widehat{v})$ *to another non-trivial solution of* (1.2). (Here the order is important, I mean connections starting from $(\widehat{u}, \widehat{v})$ at $t = -\infty$). The case $a = d > \lambda_2$, a is not an eigenvalue of $-\Delta$ shows that this case is non-empty. Let $(\widehat{u}_k, \widehat{v}_k)$ by the corresponding small solution of (1.1) for large k. This is hyperbolic and has an unstable manifold T_k of dimension at least 2. For large k there are no connections starting from $(\widehat{u}_k, \widehat{v}_k)$ and going to another small solution (by a limit argument). Suppose that W_k is the boundary of a small sphere centre $(\widehat{u}, \widehat{v}_k)$ in T_k. Since $\dim T_k \geq 2, W_k$ is connected. We see by continuous dependence that the set of points W_k' in W_k such that if $z \in W_k'$ the solution through z approaches $(\overline{u}, 0)$ as $t \to \infty$ is non-empty and open. (Here we use that $(\overline{u}, 0)$ is asymptotically stable and Theorem 4 implies the non-emptiness.) Similarly the set W_k^2 of points z in W_k such that the solution through z approaches $(0, \overline{v})$ as $t \to \infty$ is non-empty and open. By connectedness, there is a point in W_k not in $W_k^1 \cup W_k^2$. Hence if in addition all the assumptions of [15] hold there must be a connection for large k between $(\widehat{u}_k, \widehat{v}_k)$ and a strictly positive solution of (1.1).

It seems quite difficult to fully understand these connections. One of the problems is that they involves three limiting equations (1.2), (1.3) and (1.4).

(A blow up argument similar but more complicated than that in [15, p. 484] shows that for a connecting orbit joining a small solution at $t = -\infty$ to a non small solution at $t = \infty$, the limiting equation when $(u(t), v(t))$ is small but not of order k^{-1} is where $u(t)$ is close to $\alpha^{-1} w(t)^+, v(t)$ is close to $-(w(t))^-$, $w(t)$ solves (1.4) and blows up as $t \to \infty$. This limiting equation does not seem easy to understand in general when $a \neq d$. This argument also explains how the various limiting equations fit together.

Lastly, a variant of our connectedness arguments above can be used to obtain a few other results. For simplicity, we assume that the conditions in [15] are all satisfied in this paragraph. Here we work with the space $C_0^1(\Omega) \oplus C_0^1(\Omega)$. Since $(\overline{u}, 0)$ and $(0, \overline{v})$ are attractors and since it is easy to see that there is a connection in $v = 0$ joining $(0, 0)$ to $(\overline{u}, 0)$ and a connection joining $(0, 0)$ to $(0, \overline{v})$ in $u = 0$, we can again use a continuous dependence and connectedness argument to prove that there is always a connection joining $(0, 0)$ to a strictly positive stationary solution (u, v) but it is unclear whether (u, v) is a small solution or not. By a similar argument (and by Theorem 4), there is always a connection joining a given small positive non-degenerate solution of (1.1) of Morse index at least 2 to another positive solution of (1.1) (not $(\overline{u}, 0)$ or $(0, \overline{v})$). Here, once again, we stress that we are assuming that the conditions of [15] are satisfied. We could also obtain some extra results on connections by using Proposition 2.11.

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