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SOME APPLICATIONS OF GROUPS OF ESSENTIAL VALUES OF COCYCLES IN TOPOLOGICAL DYNAMICS

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Abstract. A class of examples showing that a measure-theoretical characterization of regular cocycles in terms of essential values is not valid in topological dynamics is constructed. An example that in topological dynamics for the case of non-abelian groups, the groups of essential values of cohomologous cocycles need not be conjugate is given. A class of base preserving equivariant isomorphisms of Rokhlin cocycle extensions of topologically transitive flows is described. In particular, the topological centralizer of Rokhlin cocycle extension of minimal rotation defined by an action of the group $\mathbb{R}^m$ is determined.

1. Introduction

Some notions and theorems in topological dynamics imitate their analogues from measure-theoretic ergodic theory (see [5]). However the structure of objects in topological dynamics is sometimes more complicated than in ergodic theory. In particular the theorem saying that each measure-theoretic dynamical system is built up from ergodic components has no appropriate version in topological dynamics. The two most similar counterparts in topological dynamics of

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measure-theoretic ergodicity are minimality and topological transitivity (topological ergodicity). Both notions have some properties similar to ergodicity, unfortunately not all of them.

In this paper we will compare some properties of special cocycle extensions (see (1.2) below) in measure-theoretic ergodic theory and in topological dynamics. It is known that each measure-theoretic extension is a cocycle extension (see [1]), however the cocycle takes its values in a big Polish group, namely in the group of all automorphisms of a fixed Lebesgue space. In topological dynamics there are extensions that cannot be represented as cocycle extensions (see Example 3.1). The special cocycle extensions considered below will strongly depend on cocycles taking values in locally compact groups.

To study them, the main tool we will use is the notion of the group of essential values of a cocycle. This notion was introduced by Klaus Schmidt ([14]) in the measure-theoretic context. A topological version of the notion of group of essential values inherits many properties and consequences of the original Schmidt's definition (see [2], [8]). However some theorems valid in ergodic theory are no longer valid in topological dynamics and we will present relevant examples for these phenomena, for instance we will show that groups of essential values of cohomologous cocycles need not be conjugate – this shows that a relevant measure-theoretic theorem [3, Proposition 1.1] can not be proved in topological dynamics. On the other hand, for some constructions and strong theorems in ergodic theory there is a topological counterpart. In this paper we compare descriptions of isomorphisms of Rokhlin cocycle extensions in ergodic theory and topological dynamics.

Let us now define more precisely the objects that will appear in this paper. As mentioned above an ergodic extension $\widetilde{T}\colon (Z,\mathcal{A},m) \to (Z,\mathcal{A},m)$ of an automorphism $T\colon (X,\mathcal{B},\mu) \to (X,\mathcal{B},\mu)$ is of the form

$$\begin{gathered}\widetilde{T}\colon (X\times Y,\mathcal{B}\otimes\mathcal{C},\mu\otimes\nu) \to (X\times Y,\mathcal{B}\otimes\mathcal{C},\mu\otimes\nu),\\ \widetilde{T}(x,y) = (Tx,\psi(x)(y)),\end{gathered}\tag{1.1}$$

where $\psi\colon X \to \mathrm{Aut}(Y,\nu)$ is a measurable map (and ψ is often a *Rokhlin cocycle*). Some examples of Rokhlin cocycles can be obtained in the following way. First take G a locally compact second countable group and let $\varphi\colon X \to G$ be a cocycle. Then suppose that G acts measurably on $(Y,\mathcal{C},\nu)$ as $G \ni g \mapsto \gamma_g \in \Gamma = \{\gamma_g : g \in G\}$. Then let

$$T_{\varphi,\Gamma}\colon (X\times Y,\mathcal{B}\otimes\mathcal{C},\mu\otimes\nu) \to (X\times Y,\mathcal{B}\otimes\mathcal{C},\mu\otimes\nu)$$

be given by

$$T_{\varphi,\Gamma}(x,y) = (Tx,\gamma_{\varphi(x)}(y)).\tag{1.2}$$

The extensions of the form (1.2) seem to be a very particular case of the general situation (1.1). However, quite surprisingly, as noticed in [4], each Rokhlin extension (1.1) is isomorphic, as an extension, to (1.2); moreover G may be taken countable and amenable.

In the topological context we will study only extensions of the form (1.2) and here Γ is assumed to be a continuous action of a locally compact second countable group G on a compact metric space Y. In the study of extensions of the form (1.2) an important role is played by associated, so named, cylindrical transformations $T_\varphi\colon X\times G\to X\times G$, $T_\varphi(x,g)=(Tx,\varphi(x)g)$. Similarly to the measure-theoretic situation central object is the set $E_\infty(\varphi)$ of essential values of φ. We will give (Section 3) examples that some important properties of $E_\infty(\varphi)$ that hold in ergodic theory are not inherited by topological dynamics. In this paper we also describe (Section 4) base preserving equivariant homeomorphisms of Rokhlin cocycle extensions of minimal flows, that means, equivariant homeomorphisms of the form $\widehat{S}\colon(X\times Y_1,\widetilde{T})\to(X\times Y_2,\overline{T})$, where both $\widetilde{T}$ and $\overline{T}$ are Rokhlin cocycle extensions of a given topologically transitive flows (X,T), and both these extensions are defined by the same cocycle $\varphi\colon X\to G$. The results of this paper refer to [7, Proposition 5], [4, Theorem 7.3], [10, Proposition 2.1].

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2. Preliminaries

2.1. Measure-theoretic context. We start with measure-theoretic definitions. Let $(X,\mathcal{B},\mu)$ be a standard probability space, i.e. X is a Polish space equipped with the σ-algebra $\mathcal{B}$ of Borel sets and μ is a probability measure. Assume $T\colon(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ is an ergodic automorphism. Let $C(T)$ be the set of all automorphisms $S\colon(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ that commutes with T: $ST=TS$. The set $C(T)$ is called the *centralizer* of T.

Let G be a locally compact group with the unit element e. We consider here G with the natural Borel structure $\mathcal{G}$ and a left-invariant Haar measure $m=m_G$. Suppose $\varphi\colon X\to G$ is a measurable map i.e. $\varphi^{-1}(A)\in\mathcal{B}$ up to set of μ-measure zero for each Borel set $A\subset G$. Then φ defines a measurable $\mathbb{Z}$-cocycle $\varphi^{(\cdot)}\colon\mathbb{Z}\times X\to G$ by the formula

$$\varphi^{(n)}(x)=\begin{cases}\varphi(T^{n-1}x)\varphi(T^{n-2}x)\ldots\varphi(Tx)\varphi(x) & \text{for } n\geq 1,\\ e & \text{for } n=0,\\ \varphi(T^nx)^{-1}\varphi(T^{n+1}x)^{-1}\ldots\varphi(T^{-1}x)^{-1} & \text{for } n\leq -1.\end{cases}\tag{2.1}$$

Then the cocycle identity $\varphi^{(n+k)}(x)=\varphi^{(n)}(T^kx)\varphi^{(k)}(x)$ is fulfilled. Note that each (measurable) $\mathbb{Z}$-cocycle $\Phi=\Phi(n,x)$ is of the form (2.1): simply define

$\varphi(x) = \Phi(1, x)$. In what follows we will shortly call measurable $\varphi: X \to G$ a *cocycle*. Such a cocycle allows us to define a group extension $T_\varphi: X \times G \to X \times G$ by

$$T_\varphi(x, g) = (Tx, \varphi(x)g). \tag{2.2}$$

Then

$$(T_\varphi)^n(x, g) = (T^n x, \varphi^{(n)}(x)g), \quad n \in \mathbb{Z}. \tag{2.3}$$

The map T_φ preserves the infinite measure $\mu \otimes m_G$. We say that the cocycle φ is *ergodic* if the corresponding group extension T_φ is ergodic, i.e. if for each T_φ-invariant set $A \in \mathcal{B} \otimes \mathcal{G}$, either $(\mu \times m)(A) = 0$ or $(\mu \times m)(A^c) = 0$.

We say that a cocycle φ is *coboundary* if $\varphi(x) = (f(Tx))^{-1} f(x)$ μ-a.e. for some measurable $f: X \to G$. Two cocycles φ and ψ are said to be *cohomologous*, if there exists a measurable $f: X \to G$ such that $\psi(x) = (f(Tx))^{-1}\varphi(x)f(x)$ μ-a.e. If G is Abelian and the cocycles φ, ψ are cohomologous, then the corresponding group extensions are isomorphic; an isomorphism is of the form $(x, g) \mapsto (x, f(x)^{-1}g)$, where $\psi(x) = (f(Tx))^{-1}\varphi(x)f(x)$.

Denote by G_∞ the one-point compactification of G: $G_\infty = G \cup \{\infty\}$.

DEFINITION 2.1. We say that $g \in G_\infty$ is an *essential value of* φ if for any positive measure $A \in \mathcal{B}$ and for each open neighbourhood $G_\infty \supset V \ni g$ there exists an integer n such that the set

$$A \cap T^{-n}A \cap \{x \in X : \varphi^{(n)}(x) \in V\}$$

has positive measure. The set of all essential values of φ will be denoted by $E_\infty(\varphi)$. Moreover, set $E(\varphi) = E_\infty(\varphi) \cap G$.

It turns out that $E(\varphi)$ is always a closed subgroup of G. Also the following are true ([14]).

FACT 2.2. *The group extension T_φ is ergodic if and only if $E(\varphi) = G$.*

FACT 2.3. *Suppose G is Abelian. Then the cocycle φ is a coboundary if and only if $E_\infty(\varphi) = \{0\}$.*

Both facts above are valid in topological dynamics – see [8, Propositions 3.2 and 3.4].

It is easy to observe that if G is Abelian and φ is cohomologous to ψ, then $E(\varphi) = E(\psi)$. This fails when G is not Abelian, nevertheless the following theorem holds (see [3, Proposition 1.1]).

FACT 2.4. *If the cocycles φ and ψ are cohomologous, then the groups $E(\varphi)$ and $E(\psi)$ are conjugate in G, i.e. $E(\psi) = g^{-1}E(\varphi)g$ for some $g \in G$.*

In Section 3 we will give an example that in topological dynamics Fact 2.4 is not true (Example 3.3).

Definition 2.5. Assume that the group G is Abelian. We say that φ is *regular* if it is cohomologous to a an ergodic cocycle ψ taking all values in $E(\varphi)$ i.e. if there exists a measurable function $f\colon X\to G$ such that all values of the cocycle $\psi(x)=(f(Tx))^{-1}\varphi(x)f(x)$ are in $E(\varphi)$.

Fact 2.6 ([14]). *Assume that the group G is Abelian. Given a cocycle $\varphi\colon X\to G$ define a cocycle $\widetilde{\varphi}\colon X\to G/E(\varphi)$ by $\widetilde{\varphi}(x)=\varphi(x)E(\varphi)$. Then φ is regular if and only if $E_\infty(\widetilde{\varphi})=\{0\}$.*

Clearly $E_\infty(\widetilde{\varphi})\subset\{0,\infty\}$. The equivalence in Fact 2.6 is shown making use of Fact 2.3 and of the existence of a measurable selector for the quotient map $G\to G/E(\varphi)$. In the topological case continuous selectors may not exist. We will show that Fact 2.6 is not true in topological dynamics – see Proposition 3.2.

Assume now that $(Y,\mathcal{C},\nu)$ is a standard probability space. Consider the set $\mathrm{Aut}(Y,\mathcal{C},\nu)$ of all automorphisms of $(Y,\mathcal{C},\nu)$. Then considering the map

$$\mathrm{Aut}(Y,\mathcal{C},\nu)\ni S\mapsto U_S: L^2(Y,\mathcal{C},\nu)\to L^2(Y,\mathcal{C},\nu),\quad U_S(f)=f\circ S$$

we may consider $\mathrm{Aut}(Y,\mathcal{C},\nu)$ as a closed subset of the group $U(L^2(Y,\mathcal{C},\nu))$ of unitary operators on $L^2(Y,\mathcal{C},\nu)$ in the strong operator topology. With this topology the set $\mathrm{Aut}(Y,\mathcal{C},\nu)$ is a Polish space. Given a locally compact group G, its representation $\Gamma=\{\gamma_g : g\in G\}$ and a measurable cocycle $\varphi\colon X\to G$ we may consider the Rokhlin cocycle extension $T_{\varphi,\Gamma}$ defined by (1.2). For cohomologous cocycles φ and ψ, $\psi(x)=(f(Tx))^{-1}\varphi(x)f(x)$, the corresponding skew products $T_{\varphi,\Gamma}$ and $T_{\psi,\Gamma}$ are isomorphic via the map $(x,y)\mapsto(x,f(x)^{-1}(y))$. A complete description of all invertible elements from the centralizer of the automorphism $T_{\varphi,\Gamma}$ for a locally compact second countable group G is given in [7, Proposition 5].

Fact 2.7. *Let $T\colon(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ be an ergodic automorphism of a probability standard space, $(Y,\mathcal{C},\nu)$ a probability standard space, Γ a closed locally compact second countable subgroup of $\mathrm{Aut}(Y,\mathcal{C},\nu)$. Let $\varphi\colon X\to G$ be an ergodic cocycle. Then each invertible element $\widetilde{R}$ of $C(T_{\varphi,\Gamma})$ is of the form*

$$\widetilde{R}(x,y)=(Rx,f(x)\circ W(y)),$$

where $R\in C(T)$, $f\colon X\to G$ is measurable and $W\in\mathrm{Aut}(Y,\mathcal{C},\nu)$ normalizes the group Γ in $\mathrm{Aut}(Y,\mathcal{C},\nu)$.

We will give analogous characterization of invertible elements of $C(T_{\varphi,\Gamma})$ in topological dynamics context (see Theorem 4.10).

2.2. Topological dynamics context. Assume that G is a locally compact group with the unit element e, X a compact Hausdorff space and let $\Gamma=\{\gamma_g : g\in G\}$ be a left continuous action of G on X, i.e. there is a continuous map

$\gamma\colon G\times X\to X$ satisfying the following conditions:

$$\gamma(e,x)=x, \qquad \text{for all } x\in X, \tag{2.4}$$

$$\gamma(g_1g_2,x)=\gamma(g_1,\gamma(g_2,x)) \quad \text{for all } g_1,g_2\in G,\ x\in X. \tag{2.5}$$

As usual we denote $\gamma(g,\,\cdot\,)=\gamma_g$. In what follows we will assume that all actions of topological group we consider are effective, i.e. $\gamma_g=\mathrm{Id}_X$ implies $g=e$. The pair (X,Γ) will be called a *compact G-flow*, or shortly a *G-flow.* If $G=\mathbb{Z}$, the group of integers, then the action $\Gamma=\{\gamma_n:n\in\mathbb{Z}\}$ of $\mathbb{Z}$ is determined by γ_1 – a single homeomorphism. Conversely, given a homeomorphism $T\colon X\to X$, we may define an action $\Gamma=\{\gamma_n:n\in\mathbb{Z}\}$ of $\mathbb{Z}$ on X by $\gamma_n(x)=T^nx$. In the sequel in case of actions of the group of integers we will denote the flow (X,Γ) by (X,T), where $T=\gamma_1$.

Let (X,Γ) be a G-flow. For non-empty sets $U,V\subset X$ the *dwelling set* $D(U,V)\subset G$ is defined by

$$D(U,V)=\{g\in G:\gamma_g(U)\cap V\neq\emptyset\}.$$

Several notions in topological dynamics can be defined by choosing some properties of the dwelling sets. A point $x\in X$ is *almost periodic* if for each non-empty open neighborhood $U\ni x$ the dwelling set $D(x,U)$ is syndetic (a set $A\subset G$ is *syndetic* whenever there exists a compact subset C of G such that $G=CA$; see e.g. [15, IV(1.2)]). The flow (X,Γ) is *topologically ergodic* if for any non-empty open sets $U,V\subset X$, $D(U,V)\neq\emptyset$. Each point transitive flow, i.e. flow with a point that has dense orbit, is topologically ergodic, not vice versa. However, if X is metrizable, then topological ergodicity is equivalent to point transitivity.

Let (X,T) be a compact $\mathbb{Z}$-flow, G a locally compact group with the unit element e. For a continuous map $\varphi\colon X\to G$ one can define a $\mathbb{Z}$-cocycle $\varphi^{(\,\cdot\,)}$ by formula (2.1). As in the measure-theoretic case we will call a continuous function $\varphi\colon X\to G$ a *$\mathbb{Z}$-cocycle* and such a φ is a *coboundary*, if $\varphi(x)=\xi(Tx)^{-1}\xi(x)$ for some continuous function ξ. For such a cocycle φ define $T_\varphi\colon X\times G\to X\times G$ by formula (2.2). The flow $(X\times G,T_\varphi)$, defined on a locally compact space, is called a *cocycle group extension* of (X,T). Clearly formula (2.3) holds. We say that the cocycle φ is *ergodic* if T_φ is topologically ergodic.

Definition 2.8. Let (X,T) be a $\mathbb{Z}$-flow, $\varphi\colon X\to G$ be a cocycle. We say that $v\in G_\infty$ is an *essential value of* φ if for each nonempty open $U\subset X$ and each neighbourhood V of v there exists $N\in\mathbb{Z}$ such that

$$U\cap T^{-N}U\cap\{x\in X:\varphi^{(N)}(x)\in V\}\neq\emptyset.$$

The set of all essential values of φ will be denoted by $E_\infty(\varphi)$. Moreover, denote $E(\varphi)=E_\infty(\varphi)\cap G$.

The set $E(\varphi)$ turns out to be a closed subgroup of G (see [8, Proposition 3.1]). From [8, Proposition 3.1] we also have that if G is Abelian then $E_\infty(\varphi) = E_\infty(\psi)$ for cohomologous cocycles φ and ψ. In [8, Proposition 3.2] the following characterization of topological ergodicity in the language of essential values is given.

Fact 2.9. *Assume that (X,T) is a compact topologically ergodic flow, G a locally compact group, $\varphi: X \to G$ a continuous map. Then $(X \times G, T_\varphi)$ is topologically ergodic if and only if $E(\varphi) = G$.*

It follows from [8, Proposition 3.4] that for minimal (X,T) and Abelian G, if $E_\infty(\varphi) = \{e\}$, then φ is a coboundary. Conversely, if $\varphi = f \circ T \cdot f^{-1}$ for some continuous f, then taking an open neighbourhood $V \subset G$ of the unit element of G and an open $U \subset X$ such that $x', x'' \in U$ implies $f(x')f(x'')^{-1} \in V$ we get that whenever $U \cap T^{-n}U \neq \emptyset$, then $f(T^n x)f(x)^{-1} \in V$ for each $x \in U \cap T^{-n}U$. Therefore $E_\infty(\varphi) = \{e\}$. Thus we have the following fact.

Fact 2.10. *Let (X,T) be a compact minimal flow, G a locally compact Abelian group, $\varphi: X \to G$ a continuous map. Then $E_\infty(\varphi) = \{e\}$ if and only if φ is a coboundary.*

For an Abelian group G, cohomologous cocycles have the same essential values (see [8, Proposition 3.1(2)]):

Fact 2.11. *Let (X,T) be a compact flow, G a locally compact Abelian group. If $\varphi, \xi: X \to G$ are continuous maps, then $E_\infty(\varphi) = E_\infty((\xi \circ T)^{-1}\varphi\xi)$.*

This is not true when G is not Abelian, even for the groups of essential values, and moreover, in topological dynamics Fact 2.4 fails – see Example 3.3.

Definition 2.12 ([12]). Let (X,T) be a $\mathbb{Z}$-flow, G a locally compact Abelian group, $\varphi: X \to G$ a continuous map. We say that the cocycle φ is *regular* if there exists a continuous map $f: X \to G$ such that all values of the cocycle $\psi = (f \circ T)^{-1}\varphi f$ are in $E(\varphi)$.

For regular cocycle φ the following equality $E(\widetilde{\varphi}) = \{0\}$ holds ([12, Corollary 2.8]), where $\widetilde{\varphi}(x) = \varphi(x)E(\varphi) \in G/E(\varphi)$. In the measure-theoretic case the equality $E(\widetilde{\varphi}) = \{0\}$ is equivalent to regularity of φ. Proposition 3.2 shows that this is not true in topological dynamics.

Definition 2.13. Let (X,T) be a compact $\mathbb{Z}$-flow, (Y,Γ) a compact G-flow, where G is a locally compact Abelian group and $\Gamma = \{\gamma_g : g \in G\}$ an effective continuous left action of G on Y. Assume that $\varphi: X \to G$ is a continuous map. We define a homeomorphism $T_{\varphi,\Gamma}: X \times Y \to X \times Y$ by

$$T_{\varphi,\Gamma}(x,y) = (Tx, \gamma_{\varphi(x)}(y)), \quad x \in X,\ y \in Y.$$

The $\mathbb{Z}$-flow $(X \times Y, T_{\varphi,\Gamma})$ we will call a *Rokhlin cocycle extension* of T.

Definition 2.14. Let (Y, d) be a compact metric space. By $\mathrm{Hom}(Y, Y)$ denote the topological group of all homeomorphisms of the space Y with the topology of uniform convergence defined by the metric

$$d(p,q) = \sup_{y\in Y} d(p(y), q(y)) + \sup_{y\in Y} d(p^{-1}(y), q^{-1}(y))$$

for $p, q \in \mathrm{Hom}(Y, Y)$.

3. Counterexamples in topological dynamics

First we present a simple example of an extension $\widetilde{T} \to T$ of topological flow such that $\widetilde{T}$ is not of the form (1.1).

Example 3.1. Let $\mathbb{T}$ be the unit circle represented as the interval $[0, 1)$. Consider $\widetilde{T}: \mathbb{T} \to \mathbb{T}$, $\widetilde{T}x = x + \alpha \bmod 1$, where α is irrational. Then $T = \widetilde{T}^2: \mathbb{T} \to \mathbb{T}$, $Tx = x + 2\alpha \bmod 1$, is a factor of $\widetilde{T}$ with two-point fibers. It is easy to check that $\widetilde{T}$ and T are not isomorphic. Clearly $\widetilde{T}$ is not isomorphic to any skew product $(\mathbb{T} \times Y, T_\psi)$ with continuous $\psi: \mathbb{T} \to \mathrm{Hom}(Y, Y)$.

The following proposition defines a family of topological counterexamples for valid in ergodic theory Fact 2.6.

Proposition 3.2. *Assume that (X, T) is a compact metric minimal flow, $\varphi: X \to \mathbb{R}^m$ an ergodic cocycle. Define $\overline{T}: X \times \mathbb{R}^m/\mathbb{Z}^m \to X \times \mathbb{R}^m/\mathbb{Z}^m$ by $\overline{T}(x, g + \mathbb{Z}^m) = (Tx, \varphi(x) + g + \mathbb{Z}^m)$. Let $\psi: X \times \mathbb{R}^m/\mathbb{Z}^m \to \mathbb{R}^m$, $\psi(x, g + \mathbb{Z}^m) = \varphi(x)$. Then $E(\psi) = \mathbb{Z}^m$, $E_\infty(\widetilde{\psi}) = \{0\}$, where $\widetilde{\psi}: X \times \mathbb{R}^m/\mathbb{Z}^m \to \mathbb{R}^m/E(\psi) = \mathbb{R}^m/\mathbb{Z}^m$, $\widetilde{\psi}(x, g + \mathbb{Z}^m) = \psi(x, g + \mathbb{Z}^m) + \mathbb{Z}^m$, and ψ is not regular. If moreover (X, T) is distal, then $(X \times \mathbb{R}^m/\mathbb{Z}^m, \overline{T})$ is minimal.*

Proof. Clearly $\overline{T}$ is topologically ergodic. If moreover T is distal, $\overline{T}$ is also distal. Thus $\overline{T}$ is minimal provided T is distal. Let us compute $E(\psi)$. Let $g_0 \in E(\psi)$, $g_0 \neq 0$. Then for any nonempty open sets $U \subset X$, $W \subset \mathbb{R}^m/\mathbb{Z}^m$, $g_0 \in V \subset \mathbb{R}^m$, we can find an $n \in \mathbb{Z}$ such that

$$(U \times W) \cap \overline{T}^{-n}(U \times W) \cap \{(x, g + \mathbb{Z}^m) : \psi^{(n)}(x, g\mathbb{Z}^m) \in V\} \neq \emptyset.$$

Now, if $(x, g + \mathbb{Z}^m)$ belongs to the set above, then $\psi^{(n)}(x, g + \mathbb{Z}^m) \in V$, $x, T^n x \in U$, $g + \mathbb{Z}^m$, $\varphi^{(n)}(x) + g + \mathbb{Z}^m \in W$, $\varphi^{(n)}(x) \in V$. Consider the sequences

$$U = U_1 \supset U_2 \supset \dots, \quad W = W_1 \supset W_2 \supset \dots, \quad V = V_1 \supset V_2 \supset \dots$$

of open sets with $\bigcap_{n\geq 1} U_i = \{x\}$, $\bigcap_{i\geq 1} W_i = \{g + \mathbb{Z}^m\}$, $\bigcap_{i\geq 1} V_i = \{g_0\}$. Thus we can choose $x_i \in U_i$ with $x_i, T^{n_i} x_i \to x$, $g_i + \mathbb{Z}^m, \varphi^{(n_i)}(x_i) g_i + \mathbb{Z}^m \to g + \mathbb{Z}^m$, $\varphi^{(n_i)}(x_i) \to g_0$. This implies $\varphi^{(n_i)}(x_i) + \mathbb{Z}^m \to \mathbb{Z}^m$, so $g_0 \in \mathbb{Z}^m$. Suppose now that $h \in \mathbb{Z}^m$. Take nonempty open sets $U \subset X$, $W \subset \mathbb{R}^m/\mathbb{Z}^m$, and fix

$h \in V \subset \mathbb{R}^m$. Find open sets $h \in V_0 \subset V$ and $W_0 \subset W$ such that $V_0 + W_0 \subset W$. As φ is ergodic, there exists $n \in \mathbb{Z}$ such that the set $U \cap T^{-1}U \cap \{x : \varphi^{(n)}(x) \in V_0\}$ is non-empty, say z belongs to it. Let $w_0 + \mathbb{Z}^m \in W_0$. Then $z, T^n z \in U$, $w_0 + \mathbb{Z}^m \in W$, $\varphi^{(n)}(z)w_0 + \mathbb{Z}^m \in V_0 + W_0 \subset W$ and $\varphi^{(n)}(z) \in V_0 \subset V$. Thus $(z, w_0 + \mathbb{Z}^m) \in (U \times W) \cap \widetilde{T}^{-n}(U \times W) \cap \{\psi^{(n)} \in V\}$ and $h \in E(\psi)$. We have shown that $E(\psi) = \mathbb{Z}^m$.

Now, if ψ were regular, ψ would have the form

$$\varphi(x) = \psi(x, g + \mathbb{Z}^m) = F(x, g + \mathbb{Z}^m) - F \circ \overline{T}(x, g + \mathbb{Z}^m) + \chi(x, g + \mathbb{Z}^m),$$

where $\chi: X \times \mathbb{R}^m/\mathbb{Z}^m \to \mathbb{Z}^m$. Integrating both sides of the above equation over $\mathbb{R}^m/\mathbb{Z}^m$ with respect to the normalized Haar measure we get $\varphi(x) = f(x) - f(Tx) + \chi_0(x)$, which is impossible as φ is ergodic. Thus ψ is not regular.

Now observe that since $\mathbb{R}^m/\mathbb{Z}^m$ is compact, $\infty \notin E_\infty(\widetilde{\psi})$. □

It follows from [12, Theorem 4.9] that each zero mean cocycle defined on a minimal rotation on a compact monothetic metric group and with values in $\mathbb{R}^m$, is regular. Taking in Proposition 3.2 a minimal rotation on a circle as (X, T) with topologically ergodic continuous $\varphi: \mathbb{T} \to \mathbb{R}$ we get that the compact flow $(\mathbb{T} \times \mathbb{R}/\mathbb{Z}, \widetilde{T})$ is minimal and distal. Moreover, this flow admits a non-regular real cocycle ψ with $E_\infty(\widetilde{\psi}) = \{0\}$. This shows that the theory of topological cocycles is more complex than this theory in measure-theoretic context.

Below we will present an example of two cohomologous cocycles with values in (non-abelian) group $SL(2, \mathbb{R})$ such that their groups of essential values are not conjugate.

Example 3.3. Let $X = \{0,1\}^{\mathbb{Z}}$ be the set of all 0–1 bisequences with product topology. For $x \in X$ denote by $x[n]$ the nth coordinate of x and let $x[n,m] = x[n]x[n+1]\ldots x[m]$ for $m \geq 0$. The product topology on X is defined by the metric

$$d(x,y) = \big(1 + \min\{|n| : x[n] \neq y[n]\}\big)^{-1}.$$

Let $T: X \to X$ be left side shift, $Tx[n] = x[n+1]$. Then the flow (X, T) is topologically ergodic. Define $f: X \to \mathbb{Z}$, $f(x) = (-1)^{x[0]}$. Clearly f is continuous and has zero mean with respect to the Bernoulli probability measure $(1/2, 1/2)$ on X. Now we will show that f is ergodic i.e. $T_f: X \times \mathbb{Z} \to X \times \mathbb{Z}$ is topologically transitive. To do this take an arbitrary positive integer m and fix $B = a_{-m}a_{-m+1}\ldots a_0 a_1 \ldots a_m$, where all a_i are either zero or one. Let $U = \{x \in X : x[-m, m] = B\}$. Set $n = 4m + 3$. Denote $\widetilde{B} = \widetilde{a}_{-m} \ldots \widetilde{a}_m$, where $\widetilde{a} = 1 - a$ for $a \in \{0,1\}$. Choose an $x_0 \in X$ satisfying

$$x_0[-m, m] = B, \quad x_0[m+1, 3m+1] = \widetilde{B},$$
$$x_0[3m+2] = 0, \quad x_0[3m+3, 5m+3] = B.$$

Then $x_0 \in U$, $T^n x_0 \in U$. We also have

$$\begin{aligned} f^{(n)}(x_0) &= \sum_{i=0}^{4m+2} (-1)^{x_0[i]} \\ &= \sum_{i=0}^{m} (-1)^{x_0[i]} + \sum_{i=m+1}^{3m+1} (-1)^{x_0[i]} + (-1)^{x_0[3m+2]} + \sum_{i=4m+2}^{3m+1} (-1)^{x_0[i]} \\ &= \sum_{i=0}^{m} (-1)^{a_i} + \sum_{i=-m}^{m} (-1)^{1-a_i} + 1 + \sum_{i=-m}^{-1} (-1)^{a_i} = 1. \end{aligned}$$

As U was arbitrary, we conclude that $1 \in E(f)$. Since $E(f)$ is a group and f is integer-valued, $E(f) = \mathbb{Z}$. By Fact 2.9, f is ergodic.

Now we define a continuous map $\varphi: X \to SL(2, \mathbb{R})$ setting

$$\varphi(x) = \begin{bmatrix} 1 & f(x) \\ 0 & 1 \end{bmatrix}.$$

Then clearly

$$E(\varphi) = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}.$$

Define a continuous map $\xi: X \to SL(2, \mathbb{R})$ by

$$\xi(x) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } x[0] = 0, \\ \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} & \text{if } x[0] = 1. \end{cases}$$

Let $\psi: X \to SL(2, \mathbb{R})$,

$$\psi(x) = (\xi(Tx))^{-1} \varphi(x) \xi(x).$$

We will show that $E(\psi)$ is trivial, hence not conjugate to $E(\varphi)$. To prove this take $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in E(\psi)$. Then for each $x \in X$ there exists a sequence $(n_i)_{i \ge 1}$ of integers and a sequence $(x_i)_{i \ge 1}$ such that

$$x_i \to x, \quad T^{n_i} x_i \to x, \quad \psi^{(n_i)}(x_i) \to \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Suppose first that $x[0] = 0$. Then we may assume that $x_i[0] = T^{n_i}[0] = 0$ for all $i \ge 1$. Then

$$\begin{aligned} \psi^{(n_i)}(x_i) &= (\xi(T^{n_i} x_i)^{-1} \varphi(x_i) \xi(x_i) \\ &= \varphi^{(n_i)}(x_i) = \begin{bmatrix} 1 & f^{(n_i)}(x_i) \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{aligned}$$

that means $c = 0$, $a = 1$, $d = 1$ i.e. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$.

Assume now that $x[0]=1$. Then we may assume that $x_i[0]=T^{n_i}[0]=1$ for all $i\geq 1$. Then

$$\begin{aligned}\psi^{(n_i)}(x_i) &= \xi(T^{n_i}x_i)^{-1}\varphi(x_i)\xi(x_i)\\ &= \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & \varphi^{(n_i)}(x_i)\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}\\ &= \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\cdot\begin{bmatrix}1-\varphi^{(n_i)}(x_i) & \varphi^{(n_i)}(x_i)\\ -1 & 1\end{bmatrix}\\ &= \begin{bmatrix}1-\varphi^{(n_i)}(x_i) & \varphi^{(n_i)}(x_i)\\ -\varphi^{(n_i)}(x_i) & 1+\varphi^{(n_i)}(x_i)\end{bmatrix}\to\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix}.\end{aligned}$$

Therefore $\varphi^{(n_i)}(x_i)\to 0$ and $b=0$.

4. Isomorphisms of Rokhlin cocycle extensions of point transitive flows

The following proposition is a topological version of [11, Proposition 11].

Proposition 4.1. *Let (X,T) be a $\mathbb{Z}$-flow. Assume that G,H are locally compact Abelian groups and let $\pi\colon G\to H$ be a continuous group homomorphism. If $\varphi\colon X\to G$ is a continuous map, then*

$$\overline{\pi(E(\varphi))}\subset E(\pi\circ\varphi).$$

If additionally φ is regular, then

$$\overline{\pi(E(\varphi))}=E(\pi\circ\varphi).$$

Proof. The inclusion is clear. Assume now that φ is regular, that means $\varphi=(f\circ T)^{-1}\psi f$, where $f\colon X\to G$, $\psi\colon X\to E(\varphi)$ are continuous maps. Let $g\in E(\pi\circ\varphi)$. To prove that $g\in\overline{\pi(E(\varphi))}$, fix an open neighbourhood V of the unit element in H. We will show that $(gV)\cap\pi(E(\varphi))\neq\emptyset$. Let V_0 be an open symmetric neighbourhood of the unit element in H such that $V_0V_0\subset V$. Take an open $U\subset X$ such that $x,y\in U$ implies $\pi(f(y)^{-1})\pi(f(x))\in V_0$. Now, as $g\in E(\pi\circ\varphi)$, there exists n such that the set $U\cap T^{-n}U\cap\{\pi\circ\varphi^{(n)}\in gV_0\}$ is nonempty, say x belongs to it. Then $x\in U$, $T^nx\in U$, and, by our assumption, $\pi(f(T^nx)^{-1})\pi(f(x))\in V_0$. Moreover

$$gV_0\ni\pi\circ\varphi^{(n)}(x)=\pi(f(T^nx)^{-1})\pi(f(x))\pi\circ\psi^{(n)}(x)$$

and we get

$$\pi\circ\psi^{(n)}(x)\in gV_0V_0\subset gV\quad\text{and}\quad\pi\circ\psi^{(n)}(x)\in\pi(E(\varphi)),$$

which finishes the proof. □

Theorem 4.2. *Let (X,T) be a compact metric point transitive flow, G a locally compact second countable Abelian group, Y a compact metric space, $\Gamma = \{\gamma_g : g \in G\}$ an effective left continuous actions of G on Y, $\varphi: X \to G$ a continuous map such that T_φ is point transitive. Assume moreover that $\Gamma \subset \mathrm{Hom}(Y,Y)$ is a closed subgroup. Let $\widehat{S} \in C(T_{\varphi,\Gamma})$ be an invertible extension of some $S \in C(T)$. Then there exist $p \in \mathrm{Hom}(Y,Y)$, a topological group automorphism $v: G \to G$ and a continuous map $\psi: X \to G$ such that*

$$\widehat{S}(x,y) = \big(Sx, \gamma_{\psi(x)} \circ p(y)\big), \tag{4.1}$$

and p satisfies

$$\gamma_{v(g)} = p \circ \gamma_g \circ p^{-1}, \quad g \in G. \tag{4.2}$$

Proof. Let $\widehat{S}(x,y) = \big(Sx, \kappa(x,y)\big)$, where $\kappa: X \times Y \to Y$ is a continuous map. Because $\widehat{S}$ commutes with $T_{\varphi,\Gamma}$, we have

$$\gamma_{\varphi(Sx)}\kappa(x,y) = \kappa(Tx, \gamma_{\varphi(x)}(y)). \tag{4.3}$$

For $x \in X$ let $\kappa_x: Y \to Y$, $\kappa_x(y) = \kappa(x,y)$. Then (4.3) may be written as

$$\gamma_{\varphi(Sx)} \circ \kappa_x = \kappa_{Tx} \circ \gamma_{\varphi(x)}. \tag{4.4}$$

Consider now the map

$$X \ni x \mapsto \kappa_x \in \mathrm{Hom}(Y,Y).$$

We will show that the map above is continuous. Take $\varepsilon > 0$. Find $\delta_1 > 0$ such that $d((x,y),(x',y')) < \delta_1$ implies both $d(\widehat{S}(x,y), \widehat{S}(x',y')) < \varepsilon/2$ and $d(\widehat{S}^{-1}(x,y), \widehat{S}^{-1}(x',y')) < \varepsilon/2$. Now, find $\delta > 0$ such that $\delta < \delta_1$ and if $d(x,x') < \delta$ then $d(Sx,Sx') < \delta_1$. Now assume that $d(x',x) < \delta$. Then

$$\begin{aligned} d(\kappa_x, \kappa_{x'}) &= \sup_{y\in Y} d(\kappa_x(y), \kappa_{x'}(y)) + \sup_{y\in Y} d(\kappa_x^{-1}(y), \kappa_{x'}^{-1}(y)) \\ &\le \sup_{y\in Y} d(\widehat{S}(x,y), \widehat{S}(x',y)) + \sup_{y\in Y} d(\widehat{S}^{-1}(x,y), \widehat{S}^{-1}(x',y)) < \varepsilon. \end{aligned}$$

Define a (continuous) map $F: X \times G \to \mathrm{Hom}(Y,Y)$ by $F(x,g) = \kappa_x \circ \gamma_g$. Then, by (4.4), $F \circ T_\varphi(x,g) = \kappa_{Tx} \circ \gamma_{\varphi(x)} \circ \gamma_g = \gamma_{\varphi(Sx)} \circ F(x,g)$. Considering the identity

$$F \circ T_\varphi(X,g) = \gamma_{\varphi(Sx)} \circ F(x,g) \tag{4.5}$$

in the quotient space $\mathrm{Hom}(Y,Y) \setminus \Gamma$ of left cosets of Γ in $\mathrm{Hom}(Y,Y)$ we get $\Gamma F \circ T_\varphi(x,g) = \Gamma F(x,g)$. As T_φ is topologically ergodic, the map $\Gamma F: X \to \mathrm{Hom}(Y,Y) \setminus \Gamma$ is constant, $\Gamma F(x,y) = \Gamma p$ for some $p \in \mathrm{Hom}(Y,Y)$. This means that

$$F(x,g) = \gamma_{\overline{\psi}(x,g)} \circ p, \tag{4.6}$$

where $\overline{\psi}: X \times G \to G$ is a continuous map. By (4.5) we have

$$\gamma_{\varphi(Sx)} = \gamma_{\overline{\psi}(Tx,\varphi(x)g)\cdot\overline{\psi}(x,g)^{-1}}$$

which gives

$$\varphi(Sx) = \overline{\psi} \circ T_\varphi(x,g) \cdot \overline{\psi}(x,g)^{-1}. \tag{4.7}$$

Now, for arbitrary $h \in G$ consider a map $A_h: X \times G \to G$, $A_h(x,g) = \overline{\psi}(x,gh)\overline{\psi}(x,g)^{-1}$. Then, by (4.7),

$$\begin{aligned} A_h \circ T_\varphi(x,g) &= \overline{\psi}(Tx,\varphi(x)gh)\overline{\psi}(Tx,\varphi(x)g)^{-1} \\ &= (\overline{\psi}(Tx,\varphi(x)gh)\overline{\psi}(x,gh)^{-1})(\overline{\psi}(x,gh)\overline{\psi}(x,g)^{-1})(\overline{\psi}(x,g)\overline{\psi}(Tx,\varphi(x)g)) \\ &= \varphi(Sx)A_h(x,g)\varphi(Sx)^{-1} = A_h(x,g). \end{aligned}$$

Since T_φ is topologically ergodic, A_h is constant:

$$\overline{\psi}(x,gh)\overline{\psi}(x,g)^{-1} = v(h), \quad \text{where } v: G \to G. \tag{4.8}$$

Clearly v is a continuous group homomorphism. In particular we have $v(h) = \overline{\psi}(x,h)\overline{\psi}(x,e)^{-1}$ i.e.

$$\overline{\psi}(x,h) = v(h)\overline{\psi}(x,e) = v(h)\psi(x). \tag{4.9}$$

By (4.6), $\kappa_x = \gamma_{\psi(x)} \circ \gamma_{v(g)} \circ p \circ \gamma_g^{-1}$ and κ_x does not depend on g, so $\gamma_{v(g)}p\gamma_g^{-1}$ also does not depend on g. In particular, taking $g = e$ we get

$$\gamma_{v(g)}p\gamma_g^{-1} = p \quad \text{and} \quad \kappa_x = \gamma_{\psi(x)} \circ p. \tag{4.10}$$

Therefore $\widetilde{S}(x,y) = (Sx, \gamma_{\psi(x)} \circ p(y))$. By (4.10), $\gamma_{v(g)} = p \circ \gamma_g \circ p^{-1}$.

To finish the proof observe that as the action Γ is effective, v is a monomorphism. It remains to show that v is onto. By virtue of (4.7) and (4.9) we have $\varphi(Sx) = \overline{\psi}(Tx,\varphi(x))\overline{\psi}(x,e)^{-1} = v(\varphi(x)\psi(Tx)\psi(x)^{-1}$. Thus we have obtained

$$(\varphi, \varphi \circ S) = (\varphi, v \circ \varphi) \cdot (e, \psi \circ T \cdot \psi^{-1}), \tag{4.11}$$

an equation giving that the cocycle $\varphi \times \varphi \circ S: X \to G \times G$ is cohomologous to $\varphi \times v \circ \varphi$. In particular we have equality of the groups of essential values: $E(\varphi \times \varphi \circ S) = E(\varphi \times v \circ \varphi)$. On the other hand, it is easy to see that $E(v \circ \varphi) = \overline{v(G)}$, $E(\varphi \times v \circ \varphi) = \overline{\{(g, v(g)) : g \in G\}} = \overline{\Delta_v}$ and the cocycle $\varphi \times v \circ \varphi$ takes all values in the group Δ_v. Thus the cocycle $\varphi \times \varphi \circ S$ is regular and, by Proposition 4.1,

$$G = E(\pi_2 \circ (\varphi \times \varphi \circ S)) = E(\pi_2 \circ (\varphi \times v \circ \varphi)) = \overline{E(v \circ \varphi)} = \overline{v(G)}$$

i.e. $v(G)$ is dense in G.

If G is compact, then $v(G)$ is a closed subset of g, hence $v(G) = G$. If G is connected, then $G = \mathbb{R}^m \oplus K$, where K is compact group. In such a case, as v is a monomorphism, $v(\mathbb{R}^m) = \mathbb{R}^m$ and $v(K) = K$, so $v(G) = v(\mathbb{R}^m \oplus K) = \mathbb{R}^m \oplus K =$

G. If G is an arbitrary locally compact Abelian group, then G possesses an open subgroup of the form $\mathbb{R}^m \oplus K$ for some compact group K. Then clearly $v(\mathbb{R}^m \oplus K) = \mathbb{R}^m \oplus K$. As $G/\mathbb{R}^m \oplus K$ is discrete, $v(G/\mathbb{R}^m \oplus K) = G/\mathbb{R}^m \oplus K$ (since $\overline{v(G)} = G$) and therefore v is onto and the result follows. □

If $\Gamma \subset \mathrm{Hom}(Y,Y)$ is closed and acts effectively, then the following corollary from the proof of Theorem 4.2 holds.

Corollary 4.3. *If $S \in C(T)$ can be lifted to an invertible $\widehat{S} \in C(T_{\varphi,\Gamma})$, then the cocycle $\varphi \times \varphi \circ S$ is regular and $E(\varphi \times \varphi \circ S) = \Delta_v$ for some topological group automorphism v of G. In particular, both projections of $E(\varphi \times \varphi \circ S)$ are equal to G.*

In view of Theorem 4.2 observe, that if the actions of Γ on Y is not uniformly rigid, i.e. $\gamma_{g_n} \not\to \mathrm{Id}$ uniformly for any sequence $G \ni g_n \to \infty$ (see e.g. [6] for this and other related notions of rigidity in topological dynamics), then $\Gamma \subset \mathrm{Hom}(Y,Y)$ is closed. Indeed, if Γ is not closed in $\mathrm{Hom}(Y,Y)$, then $\gamma_{g_n} \to \gamma \notin \Gamma$. As $\gamma \notin \Gamma$, we have $g_n \to \infty$. Therefore $\gamma_{g_n}^{-1} = \gamma_{-g_n} \to \gamma^{-1} \notin \Gamma$. Taking a subsequence g_{k_n} such that $h_n = g_n - g_{k_n} \to \infty$ we get $\gamma_{h_n} = \gamma_{g_n}\gamma_{g_{k_n}}^{-1} \to \mathrm{Id}_Y$, so the action of Γ on Y is uniformly rigid.

In general, for an element S of $C(T)$ that can be lifted to an $\widehat{S} \in C(T_{\varphi,\Gamma})$, the following lemma is true.

Lemma 4.4. *If (X,T) is a $\mathbb{Z}$-flow, G a locally compact Abelian group, and let $\varphi\colon X \to G$ be a cocycle. Let $\Gamma \subset \mathrm{Hom}(Y,Y)$ be a continuous representation of G, where Y is a compact Hausdorff space. If $S \in C(T)$ can be lifted to a $\widehat{S} \in C(T_{\varphi,\Gamma})$ and the cocycle $\varphi \times \varphi \circ S$ is regular, then both projections of $E(\varphi \times \varphi \circ S)$ are dense in G.*

Proof. If $\pi_i\colon G \times G \to G$ denotes the projection onto the ith coordinate, then, by Proposition 4.1, $\overline{\pi_i(E(\varphi \times \varphi \circ S))} = E(\pi(\varphi \times \varphi \circ S)) = E(\varphi) = G$, which finishes the proof. □

The requirement of full projections of the group of essential values of the cocycle $\varphi \times \varphi \circ S$ has the following algebraic interpretation.

Lemma 4.5. *Let (G,e) be a group, $H \subset G\times G$ a subgroup. Consider the natural action of $G\times G$ on $(G\times G)/H$ given by $((\widetilde{g}_1, \widetilde{g}_2), (g_1, g_2)H) \mapsto (\widetilde{g}_1 g_1, \widetilde{g}_2 g_2)H$. Then the natural action of $\{e\} \times G$ on $(G \times G)/H$ is transitive if and only if the projection of H on the first coordinate is equal to G. Similarly, the natural action of $G \times \{e\}$ on $(G \times G)/H$ is transitive if and only if the projection of H on the second coordinate is equal to G.*

Proof. Assume that the action of $\{e\}\times G$ on $(G\times G)/H$ is transitive. Then

$$\{(e,g)H : g \in G\} = (G \times G)/H. \tag{4.12}$$

Given a $g_1 \in G$, we will find a $g_2 \in G$ such that $(g_1, g_2) \in H$. In view of (4.12), there exists a $g \in G$ such that $(e, g)H = (g_1, e)H$. In particular $(g_1, e) = (h_1, gh_2)$ for some $(h_1, h_2) \in H$, so $(g_1, gh_2) = (g_1, g_2) \in H$.

Conversely, assume that the projection of H on the first coordinate is equal to G. Fix $(g_1, g_2) \in G \times G$. By assumption, there exists an $h \in G$ such that $(g_1, h) \in H$. Let $g = g_2 h^{-1}$. Then $(e, g)H = (e, g)(g_1, h)H = (g_1, g_2)H$ and we are done. □

Motivated by Corollary 4.3, Lemmas 4.4 and 4.5, we will weaken the assumption of Theorem 4.2 by skipping the requirement that Γ is closed in $\mathrm{Hom}(Y, Y)$, and replacing it by regularity of $\varphi \times \varphi \circ S$ and full projections of $E(\varphi \times \varphi \circ S)$ in G (Theorem 4.8). These two conditions are indeed weaker than the requirement that Γ be closed. For instance, if $G = \mathbb{Z}$ and $\Gamma = \{\gamma_n : n \in \mathbb{Z}\} \subset \mathrm{Hom}(\mathbb{T}, \mathbb{T})$, where $\mathbb{T}$ denotes the unit circle, and $\gamma_n(y) = y + n\alpha \bmod 1$ for some irrational α, then clearly Γ is not closed in $\mathrm{Hom}(\mathbb{T}, \mathbb{T})$. On the other hand, for any extension $\widehat{\mathrm{Id}}_X$ of Id_X, the group $E(\varphi \times \varphi \circ \mathrm{Id}) = E(\varphi \times \varphi) = \Delta_{\mathbb{Z}}$ has full projections and the cocycle $\varphi \times \varphi \circ \mathrm{Id}$ is regular.

In our considerations we need a generalization of [8, Proposition 7.1(i)]. First, following [13] we define the notion of relatively minimal extensions of topological flows. We say, that if $\pi: X \to Y$ is a factor map of topological flows, then Y is a *relatively minimal extension* of X if for each closed and invariant $Y_0 \subset Y$ satisfying $\pi(Y_0) = X$, we have $Y_0 = Y$.

Proposition 4.6. *Let (X, T) be a point transitive flow, G a locally compact Abelian group, $\varphi: X \to G$ a continuous map such that T_φ is point transitive. Let Y be a compact Hausdorff space and $\Gamma = \{\gamma_g : g \in G\}$ a left continuous action of G on Y. If $M \subset X \times Y$ is a $T_{\varphi,\Gamma}$-invariant closed set that is a relatively minimal extension of X via the natural projection, then there exists a closed set $Y_0 \subset Y$ such that $M = X \times Y_0$. Moreover the G-flow (Y_0, Γ) is point transitive.*

Proof. By assumptions, we can find an $x_0 \in X$ such that $\overline{\mathrm{Orb}}(x_0, e) = X \times G$. Since M is an extension of X via the natural projection, there exists a $y_0 \in Y$ such that $(x_0, y_0) \in M$. Put $D = \{(x, g) : (x, \gamma_g(y_0)) \in M\}$. Clearly $(x_0, e) \in D$, D is closed and $T_{\varphi,\Gamma}$-invariant, hence $D = X \times G$. Let

$$Y_0 = \overline{\mathrm{Orb}}_\Gamma(y_0) = \overline{\{\gamma_g(y_0) : g \in G\}}.$$

Since $D = X \times G$, $X \times Y_0 \subset M$. By assumption of this proposition, the extension $\Pi_X: M \to X$ is relatively minimal, therefore $M = X \times Y_0$. □

The proposition below is a topological counterpart of [9, Theorem 3]].

Proposition 4.7. *Let (X, T) be a compact point transitive flow, G a locally compact Abelian group, Y, Z compact Hausdorff spaces, $\Gamma = \{\gamma_g : g \in G\}$,*

$\Lambda = \{\lambda_g : g \in G\}$ left effective continuous actions of G on Y and Z respectively, $\varphi: X \to G$ a continuous map such that T_φ is point transitive. Assume that $M \subset (X \times Y) \times (X \times Z)$ is a $T_{\varphi,\Gamma} \times T_{\varphi,\Lambda}$-invariant closed set that is point transitive and the extension $\pi_{X\times X}: M \to \pi_{X\times X}(M) = M_0$ is relatively minimal. Assume moreover that the restriction $(\varphi \times \varphi)_{M_0}$ of $\varphi \times \varphi$ to M_0 is regular i.e. there exist functions $f_1, f_2: M_0 \to G \times G$, $\eta_1, \eta_2: M_0 \to E((\varphi \times \varphi)_{M_0})$ such that

$$\big(\varphi(x_1), \varphi(x_2)\big) = (f_1(x_1,x_2), f_2(x_1,x_2)) - (f_1(Tx_1,Tx_2), f_2(Tx_1,Tx_2)) + (\eta_1(x_1,x_2), \eta_2(x_1,x_2))$$

for all $(x_1,x_2) \in M_0$. Then there exists a compact $E((\varphi\times\varphi)_{M_0})$-invariant set $A \subset Y \times Z$ such that the map $J: M \to M_0 \times (Y \times Z)$ given by

$$J(x_1,y,x_2,z) = (x_1, x_2, \gamma_{f_1(x_1,x_2)}(y), \lambda_{f_2(x_1,x_2)}(z))$$

is an isomorphism of the flows $(M, T_{\varphi,\Gamma} \times T_{\varphi,\Lambda})$ and $(M_0 \times A, (T\times T)_{(\theta_1,\theta_2),H})$, where $H = \{(\gamma_{g_1}, \lambda_{g_2}) : (g_1,g_2) \in E\big((\varphi,\varphi)_{M_0}\big)\}$.

Proof. Clearly $J \circ (T_{\varphi,\Gamma} \times T_{\varphi,\Lambda}) = (T\times T)_{(\theta_1,\theta_2),H} \circ J$ on M. Thus $J: M \to J(M)$ is an isomorphism and, by [13, Proposition 2.3], $J(M)$ is a relatively minimal extension of M_0. By Proposition 4.6, there exists a closed set $Y_0 \subset Y$ such that $M = X \times A$ and the $E\big((\varphi,\varphi)_{M_0}\big)$-flow (A,H) is point transitive. □

Theorem 4.8. *Let (X,T) be a compact point transitive flow, G a locally compact Abelian group, Y, Z compact Hausdorff spaces, $\Gamma = \{\gamma_g : g \in G\}$, $\Lambda = \{\lambda_g : g \in G\}$ left effective continuous actions of G on Y and Z respectively, $\varphi: X \to G$ a continuous map such that T_φ is point transitive. Assume that $\widehat{S}: X \times Y \to X \times Z$ is an isomorphism of $(X\times Y, T_{\varphi,\Gamma})$ and $(X \times Z, T_{\varphi,\Lambda})$, that is an extension of some $S \in C(T)$. Assume moreover, that the cocycle $\varphi \times \varphi \circ S$ is regular and that both projections of $E(\varphi \times \varphi \circ S)$ are equal to G. Then there exist a homeomorphism $p: Y \to Z$, a topological group automorphism $v: G \to G$ and a continuous map $\psi: X \to G$ such that*

$$\widehat{S}(x,y) = (Sx, \lambda_{\psi(x)} \circ p(y)), \tag{4.13}$$

and p satisfies

$$p \circ \gamma_g(y) = \lambda_{v(g)} \circ p(y), \quad g \in G,\ y \in Y. \tag{4.14}$$

Proof. By Proposition 4.7, $J(\Delta_{\widehat{S}}) = \Delta_S \times A$ and $A \subset Y \times Z$ is a compact, H-invariant set, where $H = \{(\gamma_{g_1}, \lambda_{g_2}) : (g_1,g_2) \in E(\varphi \times \varphi \circ S)\}$. Therefore

$$\widehat{S}(x,y) = (Sx, \kappa(x,y)) \tag{4.15}$$

for some continuous map $\gamma: X \times Y \to Z$.

First we define the topological group automorphism $v: G \to G$. To do this take $(g,g_1), (g,g_2) \in E(\varphi \times \varphi \circ S)$. Since A is H-invariant, for each $y \in Y$ we have $\lambda_{g_1} \circ \kappa(x,y) = \kappa(x, \gamma_g(y)) = \lambda_{g_2} \circ \kappa(x,y)$, and therefore $\lambda_{g_1 g_2^{-1}} = \mathrm{Id}_Z$, i.e. $g_1 = g_2$. This implies that there exists a map $v\colon G \to G$ such that

$$E(\varphi \times \varphi \circ S) = \{(g, v(g)) : g \in G\} = \Delta_v. \tag{4.16}$$

As $E(\varphi \times \varphi \circ S)$ is a group, v is a group homomorphism. By the assumption that both projections of $E(\varphi \times \varphi \circ S)$ are equal to G, v is onto. In particular v is continuous. Since $\widehat{S}$ is an isomorphism, in a similar way we show that if $(g_1, g), (g_2, g) \in E(\varphi \times \varphi \circ S)$, then $g_1 = g_2$, i.e. v is a topological group automorphism.

Because the cocycle $\varphi \times \varphi \circ S$ is regular, there exist functions $f_1, f_2, \theta\colon X \to G$ such that

$$\varphi = f_1 \cdot (f_1 \circ T)^{-1} \cdot \theta, \tag{4.17}$$

$$\varphi \circ S = f_2 \cdot (f_2 \circ T)^{-1} \cdot (v \circ \theta). \tag{4.18}$$

Now we are able to prove the existence of the map $p\colon Y \to Z$. More precisely, we will show that

$$A = \{(y, p(y)) : y \in Y\} \quad \text{and} \quad \lambda_{v(g)} \circ p = p \circ \gamma_g,\ g \in G. \tag{4.19}$$

Indeed, as $J(\Delta_{\widehat{S}}) = \Delta_S \times A$, the set A is a graph of some continuous map $p\colon Y \to Z$. As $\widehat{S}$ is an isomorphism, p is a homeomorphism. To prove that $\lambda_{v(g)} \circ p = p \circ \gamma_g$ fix $(x,y) \in X \times Y$ and denote $\overline{y} = \lambda^{-1}_{f_1(x)} y$. Then $(\gamma_{f_1(x)} \overline{y}, l_{f_2(x)} \kappa(x, \overline{y})) \in A$, hence $p(y) = \lambda_{f_2(x)} \kappa(x, \gamma^{-1}_{f_1(x)} y)$, equivalently

$$\kappa(x, \gamma^{-1}_{f_1(x)} y) = \lambda^{-1}_{f_2(x)} p(y). \tag{4.20}$$

Since A is Δ_v-invariant, for each $g \in G$ we have

$$(\gamma_g \circ \gamma_{f_1(x)} \overline{y}, \lambda_{v(g)} \circ \lambda_{f_2(x)} \circ \kappa(x, \overline{y})) \in A,$$

i.e.

$$\lambda_{v(g)} \circ \lambda_{f_2(x)} \circ \kappa(x, \gamma^{-1}_{f_1(x)} y) = p(\gamma_g \circ \gamma_{f_1(x)} \overline{y}) = p \circ \gamma_g(y).$$

By (4.20), $\lambda_{v(g)} \circ p(y) = p \circ \gamma_g(y)$ and (4.19) is proved.

To finish the proof let $(\gamma_{f_1(x)} y, \lambda_{f_2(x)} \kappa(x,y)) \in A$. By (4.20),

$$\lambda_{f_2(x)} \circ \kappa(x,y) = p \circ \gamma_{f_1(x)}(y) = \lambda_{v(f_1(x))} \circ p(y),$$

hence

$$\kappa(x,y) = \lambda^{-1}_{f_2(x)} \circ \lambda_{v(f_1(x))} \circ p(y).$$

Denote $\psi(x) = v(f_1(x)) f_2(x)^{-1}$. Then $\lambda_{\psi(x)} = \lambda^{-1}_{f_2(x)} \circ \lambda_{v(f_1(x))}$ and

$$\widehat{S}(x,y) = (Sx, \kappa(x,y)) = (Sx, \lambda_{\psi(x)} \circ p(y))$$

and the proof is complete. □

Since $\widehat{S}$ from Theorem 4.8 satisfies $\widehat{S} \circ T_{\varphi,\Gamma} = T_{\varphi,\Lambda} \circ \widehat{S}$, $\kappa(Tx, \gamma_{\varphi(x)}(y)) = \lambda_{\varphi(Sx)} \circ \kappa(x, y)$. By Theorem 4.8,

$$\lambda_{\psi(Tx)} \circ \lambda_{v(\varphi(x))} \circ p(y) = \lambda_{\psi(Tx)} \circ p \circ \gamma_{\varphi(y)}(y) = \lambda_{\psi(Sx)} \circ \lambda_{\psi(x)} \circ p(y),$$

hence $\psi \circ T \cdot v \circ \varphi = \varphi \circ S \cdot \psi$ and we have the following

COROLLARY 4.9. *Under the assumptions of Theorem* 4.8, *the cocycles* $\varphi \circ S$ *and* $v \circ \varphi$ *are cohomologous.*

Directly from Theorem 4.8 we get the following description of the elements of the centralizer $C(T_{\varphi,\Gamma})$.

THEOREM 4.10. *Let* (X, T) *be a compact minimal flow, G a locally compact Abelian group, Y a compact Hausdorff space,* $\Gamma = \{\gamma_g : g \in G\}$ *a left action of G on Y,* $\varphi: X \to G$ *a continuous map such that* T_φ *is point transitive. If* $\widehat{S} \in C(T_{\varphi,\Gamma})$ *is such an invertible extension of* $S \in C(T)$ *that the cocycle* $\varphi \times \varphi \circ S$ *is regular and that both projections of* $E(\varphi \times \varphi \circ S)$ *are equal to G, then there exist: a homeomorphism* $p: Y \to Y$, *topological group automorphism* $v: G \to G$ *and a continuous map* $\psi: X \to G$ *such that*

$$\widehat{S}(x, y) = (Sx, \gamma_{\psi(x)} \circ p(y)) \quad \textit{and} \quad p \circ \gamma_g = \gamma_{v(g)} \circ p, \ g \in G. \tag{4.21}$$

COROLLARY 4.11. *If, under assumptions of Theorem* 4.10, $\widehat{S} \in C(T_{\varphi,\Gamma})$ *is invertible, then p normalizes* Γ *in* $\mathrm{Hom}(Y, Y)$.

COROLLARY 4.12. *If* $\widehat{\mathrm{Id}} \in C(T_{\varphi,\Gamma})$ *is an extension of the identity map on X, then* $\widehat{\mathrm{Id}}(x, y) = (x, p(y))$, *where* $p \in \mathrm{Hom}(Y, Y) \cap C(K)$.

PROOF. In this case $E(\varphi \times \varphi) = \Delta_G$, hence the cocycle $\varphi \times \varphi$ is regular, $f_1 = f_2 \equiv 0$, $v = \mathrm{Id}_G$ and $\gamma_g \circ p = p \circ \gamma_g$, $g \in G$. Thus $\psi \equiv 0$ and $\widehat{\mathrm{Id}}(x, y) = (x, p(y))$. □

If $\widehat{S}: X \times Y \to X \times Z$ is a factor map of flows, where (X, T) is a minimal rotation, then the set $\Pi_{X \times X}(\Delta_{\widehat{S}}) \subset X \times X$ is minimal, so $\Pi_{X \times X}(\Delta_{\widehat{S}}) = \Delta_S$ for some $S \in C(T)$.

THEOREM 4.13. *Let T be a minimal rotation on a compact metric monothetic group X,* $\Gamma = \{\gamma_g : g \in \mathbb{R}^m\}$ *a left continuous action of* $\mathbb{R}^m$ *on a compact metric space Y,* $\varphi: X \to \mathbb{R}^m$ *a continuous map such that* T_φ *is point transitive. If* $\widehat{S} \in C(T_{\varphi,K})$ *is invertible, then there exist* $S \in C(T)$, $p \in \mathrm{Hom}(Y, Y)$, *topological group automorphism* $v: \mathbb{R}^m \to \mathbb{R}^m$ *and a continuous map* $\psi: X \to \mathbb{R}^m$ *such that*

$$\widehat{S}(x, y) = (Sx, \gamma_{\psi(x)} \circ p(y))$$

and $p \circ \gamma_g = \gamma_{v(g)} \circ p$, $g \in \mathbb{R}^m$.

PROOF. Clearly $\widehat{S}$ is an extension of some $S \in C(T)$. φ is ergodic, so by [12, Theorem 4.9], $\varphi \times \varphi \circ S$ is regular and $E(\varphi \times \varphi \circ S)$ is a linear subspace of $\mathbb{R}^m \times \mathbb{R}^m$. As in the proof of Theorem 4.8 we deduce that $E(\varphi \times \varphi \circ S)$ has dense both projections on $\mathbb{R}^m$, hence the projections are equal to $\mathbb{R}^m$. An application of Theorem 4.8 finishes the proof. $\square$

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