Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 23, 2004, 213–238

NON-AUTONOMOUS QUASILINEAR ELLIPTIC EQUATIONS AND WAŻEWSKI'S PRINCIPLE

Matteo Franca

Abstract. In this paper we investigate positive radial solutions of the following equation

$$\Delta_p u + K(r)u|u|^{\sigma-2} = 0$$

where $r = |x|$, $x \in \mathbb{R}^n$, $n > p > 1$, $\sigma = np/(n-p)$ is the Sobolev critical exponent and $K(r)$ is a function strictly positive and bounded.

This paper can be seen as a completion of the work started in [9], where structure theorems for positive solutions are obtained for potentials $K(r)$ making a finite number of oscillations. Just as in [9], the starting point is to introduce a dynamical system using a Fowler transform. In [9] the results are obtained using invariant manifold theory and a dynamical interpretation of the Pohozaev identity; but the restriction $2n/(n+2) \le p \le 2$ is necessary in order to ensure local uniqueness of the trajectories of the system. In this paper we remove this restriction, repeating the proof using a modification of Ważewski's principle; we prove for the cases $p > 2$ and $1 < p < 2n/(n+2)$ results similar to the ones obtained in the case $2n/(n+2) \le p \le 2$.

We also introduce a method to prove the existence of Ground States with fast decay for potentials $K(r)$ which oscillates indefinitely. This new tool also shed some light on the role played by regular and singular perturbations in this problem, see [10].

2000 *Mathematics Subject Classification*. 34C37, 35H30, 37B55.

Key words and phrases. p-laplace equations, radial solution, regular/singular ground state, Fowler transform, Ważewski's principle.

Research supported by the Doctoral Program in Mathematics at the Università di Firenze and the G.N.A.M.P.A.

1. Introduction

The aim of this paper is to complete the analysis of radial solutions of the following equation

$$\Delta_p u + K(|x|)u|u|^{\sigma-2} = 0$$

which we began in [9]. Here $\Delta_p u = div(|Du|^{p-2}Du)$, $p > 1$, denotes the degenerate p-Laplace operator, $x \in \mathbb{R}^n$, $n > p$ and $\sigma = np/(n-p)$ is the Sobolev critical exponent. We will focus our attention on radial solutions, so we shall in fact consider the following O.D.E.

$$(u'|u'|^{p-2})' + \frac{n-1}{r}u'|u'|^{p-2} + K(r)u|u|^{\sigma-2} = 0 \tag{1.1}$$

where $r = |x|$ and we commit the following abuse of notation. We write $u(r)$ for $u(x)$ where $|x| = r$; here and below $'$ denotes derivation with respect to r.

We are mainly interested in positive solutions and in particular in Ground States (G.S.), which are positive solutions $u(x)$ such that $\lim_{|x|\to\infty} u(x) = 0$, and Crossing Solutions, which are solutions $u(x)$ such that $u(x) > 0$ for $|x| < R$ and $u(x) = 0$ for $|x| = R$, so that they can also be regarded as Dirichlet solutions in the ball.

We will use the following notation: we will call "regular" the positive solutions of (1.1) satisfying the following initial condition

$$u(0) = u_0 > 0, \quad u'(0) = 0. \tag{1.2}$$

We will call "singular" the positive solutions $u(r)$ which are singular in the origin, that is $\lim_{r\to 0} u(r) = \infty$. We are also interested in classifying the Singular Ground States (SGS), which are positive singular solutions satisfying the condition

$$\lim_{r\to 0} u(r) = \infty, \qquad \lim_{r\to\infty} u(r) = 0.$$

The study of this problem has relevance in many applied fields, such as plasma physics, quantum mechanics and in Riemannian geometry. The autonomous equation, where $K(r) > 0$ is a constant, is well understood, see for example [12], [7], as well as the non autonomous equation where $K(r)$ is monotone, see [17], [7]. Much less is known for oscillating potentials $K(r)$. In this case the structure of positive solutions becomes richer and more complicated. A first result concerning this problem was given in [18]. In [9] the case in which $K(r)$ makes a finite number of oscillations was considered, but with the technical restriction that $2n/(n+2) \le p \le 2$. In particular the following hypotheses were introduced.

Hypotheses.

(α^-) There exists $\rho > 0$ such that $K(r)$ is monotone decreasing, for $0 \le r \le \rho$, but $\liminf_{r\to 0} |K'(r)| r^{-n/(p-1)+1} > 0$.

(α^+) There exists $\rho > 0$ such that $K(r)$ is monotone increasing, for $0 \le r \le \rho$, but $\liminf_{r\to 0} |K'(r)| r^{-n/(p-1)+1} > 0$.
(Ω^-) There exists $R > 0$ such that $K(r)$ is monotone decreasing, any $r \ge R$, but $\liminf_{r\to\infty} |K'(r)| r^{n+1} > 0$.
(Ω^+) There exists $R > 0$ such that $K(r)$ is monotone increasing, any $r \ge R$, but $\liminf_{r\to\infty} |K'(r)| r^{n+1} > 0$.

The starting point in [9], as well as in the present paper, is a Fowler inversion which transforms (1.1) into a dynamical system. The proofs in [9] used dynamical system theory and the restriction $2n/(n+2) \le p \le 2$ was essential to give enough regularity to the problem, in order to ensure local uniqueness of the solutions. On the other hand, for $p > 2$ and $1 < p < 2n/(n+2)$, the system is not even Lipschitz, so that we cannot anymore apply invariant manifold theory, which was necessary for one of the main steps of the proofs.

One of the main contribution of this paper is the construction of a stable and an unstable set, made exploiting Ważewski's principle, that allows to remove the technical restriction $2n/(n+2) \le p \le 2$.

In [10] it was proved that, when $K(r)$ is a regular or singular perturbation of a constant, that is respectively $K(r) = 1 + \varepsilon k(r)$, $K(r) = k(r^\varepsilon)$, $\varepsilon > 0$ small, we can prove the existence of G.S. with fast decay and even classify all the positive solutions. This can be done even when $K(r)$ oscillates infinitely many times (and in the singular perturbation case $K(r)$ may also change sign). In the singular perturbation case, a G.S. with fast decay bifurcates from a homoclinic trajectory of the system frozen at a non degenerate positive critical point.

It is rather amazing to find that positive solutions have a similar behaviour with these two different kinds of perturbations: in fact in the regular case $K(r)$ is assumed to have small amplitude of variation, while in the singular case it is just assumed to vary slowly. Here we present a constructive method to prove the existence of G.S. with fast decay for the non perturbative problem, that sheds some light on the role played by these two different kinds of perturbations. These techniques enable one to prove existence and even structure results for some potentials which makes an infinite number of oscillations. The conditions required in the proofs are that the potentials $K(r)$ do not vary "too quickly" and that they have "a not too large range of variation". Therefore they extend and, in some sense, connect the perturbative results. Furthermore they give, in some precise examples, an estimate of how large the perturbation could be.

The paper is organized as follows. In Section 2 we restate for the general case the theorems obtained for $2n/(n+2) \le p \le 2$ in [9]; thus we consider potentials $K(r)$ which make a finite number of oscillations. In Section 3 we show how Ważewski's Principle can be used to construct a stable and an unstable manifold for our system, even when the equations are not everywhere Lipschitz.

In Section 4 we give a sufficient condition to prove the existence of G.S. with fast decay. This condition is less explicit than the one considered in section 2 but a bit more general: it can work also for functions $K(r)$ which are flat for r small or for r large, and it can work also for potentials which oscillates indefinitely.

2. Potentials making a finite number of oscillations

We begin by recalling a known result concerning the asymptotic behaviour of positive solutions. We introduce the following notation: when we write that $u(r) \sim r^{-\alpha}$ as $r \to c$ we mean that the limits $\liminf_{r\to c} u(r)r^{\alpha}$ and $\limsup_{r\to c} u(r)r^{\alpha}$ are both finite and positive.

Proposition 2.1. *Consider a solution $u(r)$ of* (1.1) *which is well defined and positive for r small. Assume that there exists $\rho > 0$ such that $K(r)$ is monotone on $(0, \rho)$ and define $K(0) = A > 0$. Then only two asymptotic behaviour are possible as $r \to 0$.*

$$\begin{aligned} &0 < u(0) < \infty && (\textit{regular solution}),\\ &u(r) \sim r^{-\alpha} && (\textit{singular solution}). \end{aligned}$$

Furthermore for each singular solution $u(r)$ there exists a S.G.S. *$v(r)$ of the frozen equation* (1.1) *where $K(r) \equiv A$ such that $\lim_{r\to 0}(u(r) - v(r))r^{\alpha} = 0$. Analogously consider a solution $u(r)$ of* (1.1) *which is well defined and positive for r large. Assume that $K(r)$ is monotone for r large and define $\lim_{r\to\infty} K(r) = B > 0$. Then only two asymptotic behaviour are possible as $r \to \infty$.*

$$\begin{aligned} &u(r) \sim r^{-(n-p)/(p-1)} && (\textit{fast decay}),\\ &u(r) \sim r^{-\alpha} && (\textit{slow decay}). \end{aligned}$$

Furthermore for each slowly decaying solution $u(r)$ there exists a S.G.S. *$v(r)$ of the frozen equation* (1.1) *where $K(r) \equiv B$ such that*

$$\lim_{r\to\infty}(u(r) - v(r))r^{\alpha} = 0.$$

Therefore positive solutions can only exhibit two types of asymptotic behaviour as $r \to 0$ (regular and singular) and as $r \to \infty$ (slow decay and fast decay). If the monotonicity assumptions are not satisfied the estimates are not as sharp.

Corollary 2.2. *Assume that $K(r)$ is strictly positive and bounded. Consider a solution $u(r)$ of* (1.1) *which is well defined and positive for r small. Then only two asymptotic behaviour are possible as $r \to 0$.*

$$\begin{aligned} &0 < u(0) < \infty && && (\textit{regular solution}),\\ &0 < \liminf_{r\to\infty} u(r) && \textit{and} \quad \limsup_{r\to\infty} u(r)r^{\alpha} < \infty && (\textit{singular solution}). \end{aligned}$$

Analogously consider a solution $u(r)$ of (1.1) *which is well defined and positive for r large. Then only two asymptotic behaviour are possible as $r \to \infty$.*

$$\begin{aligned} &u(r) \sim r^{-(n-p)/(p-1)} && (\textit{fast decay}),\\ &0 < \liminf_{r\to\infty} u(r) \quad \textit{and} \quad \limsup_{r\to\infty} u(r) r^{\alpha} < \infty && (\textit{slow decay}). \end{aligned}$$

The proof of Proposition 2.1 can be found in [7] and [9], the proof of the corollary is given in [10].

Now we are ready to state the main theorems of this section.

THEOREM 2.3. *Consider equation* (1.1) *and assume $K(r)$ is strictly positive and bounded, and it is continuous with its derivative ($K(r) \in \mathcal{C}^1$).*

(a) *Assume that hypothesis* (α^-) *is satisfied. Then there exists $R^- > 0$, such that for each $0 < \rho < R^-$ there exists a solution $u(r)$ of the Dirichlet problem in the exterior of the ball of radius ρ. Therefore we have $u(\rho) = 0$, $u(r) > 0$ for any $r > \rho$ and $u(r) \sim r^{-(n-p)/(p-1)}$ as $r \to \infty$.*
(b) *Assume that hypothesis* (Ω^+) *is satisfied. Then there exists $R^+ > 0$, such that for each $\rho > R^+$ there exists a solution $u(r)$ of the Dirichlet problem in any ball of radius ρ.*

THEOREM 2.4. *Consider equation* (1.1) *and assume $K(r) \in \mathcal{C}^1$ is strictly positive and bounded. Assume that hypotheses* (α^+) *and* (Ω^-) *are satisfied. Then we can classify the positive solutions as follows:*

(a) *There exist uncountably many crossing solutions.*
(b) *There exist uncountably many ground states with slow decay.*
(c) *There exists at least one ground state with fast decay.*
(d) *There exist uncountably many Dirichlet solutions $u(r)$ in exterior domains: for each such $u(r)$ there exists $R > 0$ such that $u(R) = 0$ and $u(r) \sim r^{-(n-p)/(p-1)}$, as $r \to \infty$.*
(e) *There exist uncountably many* S.G.S. *with fast decay.*
(f) *No other positive solutions can exist but the ones described and, possibly,* S.G.S. *with slow decay.*

We define now the following auxiliary functions:

$$J^+(r) = \int_0^r K'(s)s^n\,ds, \qquad J^-(r) = \int_r^\infty K'(s)s^n. \tag{2.1}$$

Adding an extra hypothesis we can complete the previous result as follows.

Corollary 2.5. *Consider all the (regular) solutions of* (1.1) *with initial condition* (1.2). *Maintain all the hypotheses of the previous theorem and assume that there exists $R > 0$ such that one of the following hypotheses is satisfied*

(a) *$J^+(r) \geq 0$ for any $r < R$ and $K'(r) \leq 0$ for any $r > R$,*
(b) *$J^-(r) \leq 0$ for any $r > R$ and $K'(r) \geq 0$ for any $r > R$.*

Then there exists $A > 0$ such that for any $u_0 > A$ the corresponding solution $u(r)$ is a G.S. *with slow decay, while for any $0 < u_0 < A$, $u(r)$ is a crossing solution. Finally if $u_0 = A$, $u(r)$ is a* G.S. *with fast decay. Furthermore there exist at least one* S.G.S. *with slow decay and uncountably many* S.G.S. *with fast decay. This classification covers all the positive solutions, both regular and singular.*

We have also another sufficient condition for the existence of G.S. with fast decay, but in this case we have a poorer structure result for the other families of positive solutions.

Theorem 2.6. *Assume that hypotheses (α^-) and (Ω^+) are satisfied, then there exist uncountably many crossing solutions,uncountably many solutions $u(r)$ of the Dirichlet problem in exterior domains, and at least one* G.S. *with fast decay.*

As in [9] the first step is to introduce the following change of variables

$$x_1 = u(r)r^{\alpha}, \quad x_2 = u'(r)|u'(r)|^{p-2}r^{\beta}, \quad r = e^t, \quad \phi(t) = K(e^t) \tag{2.2}$$
$$\text{where } \alpha = \frac{n-p}{p} \text{ and } \beta = \frac{n(p-1)}{p}$$

in order to deal with the following non-autonomous dynamical system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_2|x_2|^{(2-p)/(p-1)} \\ -\phi(t)x_1|x_1|^{\sigma-2} \end{pmatrix}. \tag{2.3}$$

Here and below the dot "$\cdot$" indicates a derivative with respect to t. We wish to stress the fact that the system is $\mathcal{C}^1$ if and only if $2n/(n+2) \leq p \leq 2$. If this condition is not satisfied the equations are not even Lipschitz on the coordinate axes. Thus local uniqueness of the solutions is not a-priori ensured; therefore the term dynamical system is not quite rigorous. In this section we try to overcome this problem; so we do not want to impose any restriction on p. It will be useful to embed system (2.3) in the following one parameter family of systems, as done in [9]:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_2|x_2|^{(2-p)/(p-1)} \\ -\phi(t+\tau)x_1|x_1|^{\sigma-2} \end{pmatrix}. \tag{2.4}$$

We will also need the following extended autonomous system obtained from system (2.7) by adding the extra variable $x_3 = \tau + t$:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} x_2|x_2|^{(2-p)/(p-1)} \\ -\phi(x_3)x_1|x_1|^{\sigma-2} \\ 1 \end{pmatrix}. \tag{2.5}$$

We will make use also of the following system, where we set $x_3 = e^{\xi t}$.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} x_2|x_2|^{(2-p)/(p-1)} \\ -\phi(\xi; x_3)x_1|x_1|^{\sigma-2} \\ 0 \end{pmatrix} \tag{2.6}$$

where $\phi(\xi; x_3) = \phi((\log(x_3))/\xi)$.

We observe now that there are some elementary correspondences between positive solutions $u(r)$ of (1.1) and trajectories $x(t)$ of (2.3). We define $\mathbb{R}^2_+ = \{(x_1, x_2) \mid x_1 \geq 0\}$.

REMARK 2.7. A trajectory $x(t)$ which lies in $\mathbb{R}^2_+$ for all $t \in \mathbb{R}$, corresponds to a positive solution $u(r)$; furthermore $x_2(t) < 0$ implies that $u(r)$ is monotone decreasing.

A solution $u(r)$ is a regular solution if and only if the corresponding trajectory of system (2.4) has the origin as α-limit point (here we use Proposition 2.1).

A solution $u(r)$ can have fast decay if and only if the corresponding trajectory of system (2.4) has the origin as ω-limit point (here again we use Proposition 2.1).

The main tool of investigation in [9], was the following function

$$H(x(t); t) := \alpha x_1 x_2 + \frac{p-1}{p}|x_2|^{p/(p-1)} + \phi(\tau + t)\frac{|x_1|^\sigma}{\sigma}.$$

This function is obtained adapting to this dynamical setting the Pohozaev function $P(u, u', r)$, defined e.g. in [18]. Observe that by differentiating we get

$$\frac{d}{dt}H(x(t), t) = \frac{d}{dt}\phi(t + \tau)\frac{|x_1|^\sigma}{\sigma}. \tag{2.7}$$

This way we obtain a "dynamical interpretation" of the Pohozaev identity. Let $x(\widetilde{\tau}, x^0; t)$ be the trajectory of system (2.4) departing at $t = 0$ from $x = x^0$, for $\tau = \widetilde{\tau}$. In [9] it was shown that there exist a stable manifold $W^s(\tau)$ and an unstable manifold $W^u(\tau)$, satisfying the following properties

$$\begin{aligned} W^u(\tau) &= \{x^0 \in \mathbb{R}^2_+ \mid \lim_{t \to -\infty} x(\tau, x^0; t) = 0\} \quad \text{and} \\ W^s(\tau) &= \{x^0 \in \mathbb{R}^2_+ \mid \lim_{t \to \infty} x(\tau, x^0; t) = 0\}. \end{aligned}$$

These existence results are achieved using invariant manifold theory for non-autonomous system, see [10], [9] and [15]. Here we use a different technique, relying on Ważewski's principle, in order to overcome the lack of regularity in the case $1 < p < 2n/(n + 2)$ or $p > 2$.

We introduce now some notation:

$$
\begin{aligned}
U^+ &:= \{(x_1, x_2) \mid x_1 \geq 0,\ x_2 \leq 0 \text{ and } \dot{x}_1 > 0\},\\
U^- &:= \{(x_1, x_2) \mid x_1 \geq 0,\ x_2 \leq 0 \text{ and } \dot{x}_1 < 0\},\\
c &:= \{(x_1, x_2) \mid x_1 \geq 0,\ x_2 \leq 0 \text{ and } \dot{x}_1 = 0\}.
\end{aligned}
$$

We will use analogous definitions for the three dimensional systems (2.5) and (2.6). We need to recall some known results concerning the autonomous system (2.4) where $\phi \equiv N > 0$ and N is a positive constant, see [7]. It is easily observed that H is a first integral for this system, therefore it is possible to draw each trajectory, see Figure 2.1.

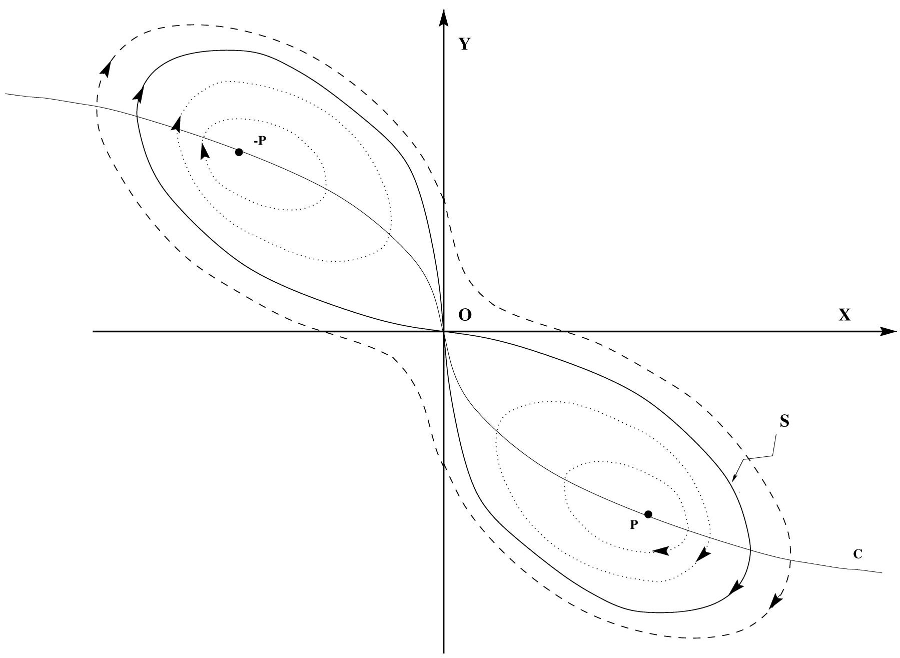

Figure 2.1. Sketch of the level sets of the function $H(\,\cdot\,, t)$. We use the solid line for the level set $H = 0$, the dotted line for $H = c_1 < 0$ and the dashed for $H = c_2 > 0$.

Note that there exists a homoclinic trajectory $Z^N(t)$, corresponding to the level set $H(x) = 0$, lying in $\mathbb{R}^2_+$. Since in the autonomous case the trajectories are invariant for translations in t, there is in fact a one-parameter family of homoclinic trajectories $Z^N_\tau(t) = Z^N(t + \tau)$. Note that they have all the same graph in $\mathbb{R}^2_+$; we will denote by $Z^N(t)$ the unique trajectory satisfying $Z^N(0) \in c$. All these trajectories correspond to regular solutions of (1.1), thus we find that they are all G.S. with fast decay. In the set enclosed by $Z^N(t)$ we have $H(x) < 0$ and outside we have $H(x) > 0$. Note that, if we consider the autonomous system (2.4) where $\phi \equiv M > N > 0$, we have that $Z^M(t)$ lies in the interior of the bounded set enclosed by $Z^N(t)$ and the origin. Let us consider (2.5): we define the curve

$$S(\tau) := \{(x_1, x_2, x_3) \mid x_1 \geq 0,\ x_3 = \tau \text{ and } H(x_1, x_2, \tau) = 0\}$$

and the topological surface $S = \bigcup_{\tau \in \mathbb{R}} S(\tau)$. $S(\tau)$ will be regarded also as a curve for (2.4).

We will always assume that there exist $b > a > 0$ such that $a \leq K(r) \leq b$. We consider the homoclinic trajectories $Z^a(t)$, $Z^b(t)$ of the respective frozen systems. We identify the trajectories and the solution curves and we consider the set E delimited by the origin, the curves $Z^a(t)$ and $Z^b(t)$. Let us define $\partial E = \{Z^a(t) \mid t \in \mathbb{R}\} \cup \{Z^b(t) \mid t \in \mathbb{R}\}$. Denote now $E^{\pm} = E \cap (U^{\pm} \cup c)$ and $\partial E^{\pm} = \partial E \cap U^{\pm}$.

In the following section we will prove that there exist compact connected sets $\widetilde{W}^s(\tau) \subset U^-$ and $\widetilde{W}^u(\tau) \subset U^+$ joining c and the origin with the following properties

$$\begin{aligned} \widetilde{W}^s(\tau) &:= \{P \in E^- \mid x(\tau, P; t) \in E^- \text{ for any } t > 0\}, \\ \widetilde{W}^u(\tau) &:= \{P \in E^+ \mid x(\tau, P; t) \in E^+ \text{ for any } t < 0\}. \end{aligned} \tag{2.8}$$

Note that if $P \in \widetilde{W}^s(\tau)$ then $\lim_{t\to\infty} x(\tau, P; t) = (0,0)$, while if $P \in \widetilde{W}^u(\tau)$ then $\lim_{t\to-\infty} x(\tau, P; t) = (0,0)$. In Section 3 we prove also that the sets $\xi^{\pm}(\tau) := \widetilde{W}^{u,s}(\tau) \cap c$ are compact and connected as well. Furthermore we consider system (2.5) and define the sets

$$\widetilde{W}^{u,s} := \{(x^0, \tau) \in \mathbb{R}^3 \mid x^0 \in \widetilde{W}^{u,s}(\tau) \text{ for } \tau \in \mathbb{R}\}.$$

In the regular setting we have that both $\widetilde{W}^u$ and $\widetilde{W}^s$ are surfaces; here we will prove that they are compact and connected. Furthermore the sets

$$\xi^{\pm} := \{(x^0, \tau) \in \mathbb{R}^3 \mid x^0 \in \xi^{\pm}(\tau) \text{ for } \tau \in \mathbb{R}\}$$

are compact and connected as well.

In order to prove existence results for G.S. with fast decay for (1.1) and to classify positive solutions, we can exploit the ideas already used in [9] for the regular setting, that is when $2n/(2+n) \leq p \leq 2$ so that system (2.4) is $\mathcal{C}^1$. We sketch them here for completeness, referring the interested reader to [9]. Consider a point $P \in \xi^+(\tau)$ (respectively $P \in \xi^-(\tau)$); we indicate by $x^u(\tau, P; t)$ (respectively $x^s(P, \tau; t)$) the trajectories of system (2.4) departing at $t = 0$ from P, or generically $x^u(\tau; t)$ (respectively $x^s(\tau, t)$). First of all note that $x^u(\tau; t) \in U^+$ for any $t < 0$ and $x^s(\tau; t) \in U^-$ for any $t > 0$. These trajectories correspond respectively to solutions $u(r)$ of (1.1) which are regular in the origin or have fast decay.

We recall that we know explicitly which are the regular solutions of the autonomous problem, see [12] for example. If we denote $Z^K(t) = (U^K(t), V^K(t))$, we have

$$(2.9)\qquad U^K(t) = \left[\frac{1}{D(e^{-t}/p - 1 + e^{t/(p-1)})}\right]^{(n-p)/p} K^{-(n-p)/p^2}$$

where $D = [(n-p)^{p-1}n(p-1)]^{1/p}$ is a constant. Note that the flow on ∂E^+ points towards the interior of E^+ while on ∂E^- the flow points towards the exterior of E^-. Therefore we have that $x^u(\tau;t) \in E^+$ for any $t < 0$ and $x^s(\tau;t) \in E^-$ for any $t > 0$. Then, by elementary reasoning on the phase portrait we can prove the following estimates

$$(2.10)\qquad \begin{aligned} U^a(t) &\le x_1^u(\tau;t) \le U^b(t) \quad \text{for } t < 0, \\ U^a(t) &\le x_1^s(\tau;t) \le U^b(t) \quad \text{for } t > 0. \end{aligned}$$

Now we can obtain estimate on $H(x^u(\tau,P;t),t)$ and $H(x^s(\tau,P;t),t)$ for any $P \in \xi^{\pm}(\tau)$. We will also consider the multivalued functions $H(x^u(\tau,P;t),t) = H(x^u(\tau;t),t)$ and $H(x^s(\tau,P;t),t) = H(x^s(\tau;t),t)$ defined respectively for $P \in \xi^+(\tau)$ and for $P \in \xi^-(\tau)$. We define also $\overline{G}(\tau,t) = H(x^u(\tau;t),t) - H(x^s(\tau;t),t)$. A key role is played by the multivalued function

$$G(\tau) = \overline{G}(\tau,0) = H(\xi^+(\tau),0) - H(\xi^-(\tau),0)$$

which measures the distance between the set $\xi^+(\tau)$ and $\xi^-(\tau)$ in c. In particular if $G(\tau) > 0$ then $\xi^+(\tau)$ is on the right with respect to $\xi^-(\tau)$, while if $G(\tau) < 0$ it is on the left; here a nd later we think of the x_1 axis as horizontal and the x_2 axis as vertical. Furthermore if $0 \in G(\tau)$ then $\xi^+(\tau) \cap \xi^-(\tau) \neq \emptyset$. Note that integrating (2.7) we get the following

$$\begin{aligned} H(P,0) &= \int_{-\infty}^{0} \dot\phi(\tau+t)\frac{|x^u(\tau,P;t)|^\sigma}{\sigma}\,dt \qquad \text{where } P \in \xi^+(\tau), \\ H(P,0) &= -\int_{0}^{\infty} \dot\phi(\tau+t)\frac{|x^s(\tau,P;t)|^\sigma}{\sigma}\,dt \quad \text{where } P \in \xi^-(\tau). \end{aligned}$$

Now we can repeat step to step the analysis developed in [9]. Using the estimate (2.10) we can deduce the following.

Proposition 2.8. *Consider equation* (1.1) *and assume that* $K(r) \in \mathcal{C}^1$ *is strictly positive and bounded, and that* $K(r)$ *changes its sign a finite number of times.*

(a) *Assume that hypothesis* (α^+) *is satisfied. Then there exists* $\widetilde{T}$ *such that* $H(\xi^+(\tau),0) > 0$ *and* $H(\xi^-(\tau),0) < 0$*, for any* $\tau < \widetilde{T}$*. Assume that hypothesis* (α^-) *is satisfied, then there exists* $\widetilde{T}$ *such that we have* $H(\xi^+(\tau),0) > 0$ *and* $H(\xi^-(\tau),0) < 0$*, for any* $\tau < \widetilde{T}$*.*

(b) *Assume that hypothesis* (Ω^+) *is satisfied, then there exists* $\widehat{T}$ *such that* $H(\xi^+(\tau),0) > 0$ *and* $H(\xi^-(\tau),0) < 0$*, for any* $\tau > \widehat{T}$*. Assume that hypothesis* (Ω^-) *is satisfied, then there exists* $\widehat{T}$ *such that* $H(\xi^+(\tau),0) > 0$ *and* $H(\xi^-(\tau),0) < 0$*, for any* $\tau > \widehat{T}$*.*

A detailed proof can be found [7, Lemma 3.1]; here we just sketch the main ideas. Assume that hypothesis (Ω^+) is satisfied and choose $\tau > \lambda = \log(R)$, where $R > 0$ is such that $K'(r) > 0$ for any $r > R$. Then we have that $\dot{\phi}(\tau + t) > 0$ for $t > 0$, thus, for any $P \in \xi^-(\tau)$, we have

$$H(P,0) = -\int_0^\infty \dot{\phi}(\tau+t)\frac{|x^s(\tau,P;t)|^\sigma}{\sigma}\,dt < 0.$$

It can also be proved that there exists $\widetilde{T}$ such that $H(\xi^+(\tau),0) > 0$; the idea is that the sign of H depends on $\dot{\phi}(t)$, and the term $|x_1^u(\tau;t)|^\sigma$ and $|x_1^s(\tau;t)|^\sigma$ can be regarded as weights. Fix $P \in \xi^*(\tau)$, where $\tau > \widehat{T}$. Exploiting the exponential behaviour of these weight terms near the origin, we manage to give more weight to the term

$$\int_{\lambda-\tau}^0 \dot{\phi}(\tau+t)\frac{|x_1^u(\tau;t)|^\sigma}{\sigma}\,dt > 0,$$

with respect to the term

$$\int_{-\infty}^{\lambda-\tau} \dot{\phi}(\tau+t)\frac{|x_1^u(\tau;t)|^\sigma}{\sigma}\,dt,$$

which can be either positive or negative. The other claims can be proved in the same way.

Now we can deduce the following

Theorem 2.9. *Consider equation* (1.1)*. Assume that* $K(r) \in \mathcal{C}^1$ *is strictly positive and bounded, and that* $K(r)$ *changes its sign a finite number of times.*

(a) *Assume that hypothesis* (α^+) *is satisfied. Then there are uncountably many* S.G.S. *with fast decay.*
(b) *Assume that hypothesis* (α^-) *is satisfied. Then there are uncountably many Dirichlet solutions* $u(r)$ *in exterior domains: for each such* $u(r)$ *there exists* $R > 0$ *such that* $u(R) = 0$ *and* $u(r) \sim r^{-(n-p)/(p-1)}$*, as* $r \to \infty$*.*
(c) *Assume that hypothesis* (Ω^+) *is satisfied. Then there are uncountable many crossing solutions.*
(d) *Assume that hypothesis* (Ω^-) *is satisfied. Then there are uncountable many ground states with slow decay.*

Proof. The proof can be found in [10, Theorem 3.2], here we sketch the main ideas for convenience of the reader. We prove just the first claim since the others are analogous. Let us consider a generic trajectory $x^s(\tau;t)$ of (2.4) whenever $\tau < \widetilde{T}$. Using Proposition 2.8 we deduce that $H(\xi^-(\tau), 0) < 0$. Now observe that $dH(x^s(\tau; t), t)/dt > 0$ for any $t < 0$, therefore $H(x^s(\tau; t), t) < 0$ for $t < 0$. So $x^s(\tau; t)$ is forced to stay in the interior of the set $S(\tau + t)$ for any $t < 0$ and in E^- for any $t > 0$. Thus it represents a S.G.S. with fast decay. □

It is rather easy now to prove Theorems 2.4 and 2.6.

Proof of Theorems 2.4 and 2.6. The only thing that remains to be proved is the existence of G.S. with fast decay: we prove it for the former case, the latter is completely analogous. Consider system (2.5); from Proposition 2.8 we deduce that $G(\tau) > 0$ as $\tau \to -\infty$ and $G(\tau) < 0$ as $\tau \to \infty$. We recall now that both $\xi^+ \subset \mathbb{R}^3$ and $\xi^- \subset \mathbb{R}^3$ are compact and connected. Therefore there is $\overline{\tau}$ such that $0 \in G(\overline{\tau})$. Thus there is at least one homoclinic trajectory $x^u(t; \overline{\tau})$ which corresponds to a G.S. with fast decay. □

We wish to stress the fact that in the regular setting $2n/(n + 2) \leq p \leq 2$, we can prove the existence of infinitely many S.G.S. with slow decay, whenever the hypotheses of Theorem 2.4 or 2.6 are satisfied. When $p > 2$, we can still prove a result of this type exploiting Ważewski's principle.

Corollary 2.10. *Consider equation* (1.1) *and assume* $K(r) \in \mathcal{C}^1$ *is strictly positive and bounded, and that* $K'(r)$ *changes sign exactly once. Then there exists at least one monotone decreasing* S.G.S. *with slow decay.*

Proof. We have assumed that there are $a, b > 0$ such that $a \leq K(r) \leq b$. We call $m = (1/2) \max_{t \in \mathbb{R}} \min_{x \in \mathbb{R}^2} H(x_1, x_2, t) < 0$; consider now system (2.5). As can be deduced from Figure 2.1 we have that the set

$$S_m := \{(x_1, x_2, x_3) \mid H(x_1, x_2, x_3) = c \text{ and } x_1 > 0\}$$

is a smooth surface homeomorphic to a cylinder. As usual we denote by $S_m(\tau) = S_m \cap \{x_3 = \tau\}$. We want to prove that there exists a trajectory $\overline{X}(t) = (\overline{x}_1(t), \overline{x}_2(t), \overline{x}_3(t))$ that is forced to stay for any t in the set enclosed by S_m: then it follows that the corresponding solution $v(r)$ of (1.1) is a monotone decreasing S.G.S. with slow decay.

We assume that there exists τ such that $\dot{\phi}(x_3) \geq 0$ for any $x_3 < \tau$, and $\dot{\phi}(x_3) \leq 0$ for any $x_3 > \tau$; the other case is completely analogous. Note that the flow on $S_m(x_3)$ is always going towards the exterior of S_m for any $x_3 < \tau$. Let us call $X(P; t)$ the trajectory departing from P at $t = 0$. Applying Ważewski's principle we deduce that there exists a point Q, belonging to the set enclosed by $S_m(\tau)$, such that $X(Q; t)$ is forced to stay inside S_m for any $t \leq 0$. Now observe that the flow on $S_m(x_3)$ is always going towards the interior of S_m for any $x_3 > \tau$, thus $X(Q; t)$ is forced to stay inside S_m also for any $t > 0$. □

Recall the definition of $J^{\pm}(r)$ given in (2.1); we can easily observe the following

$$H(x_1^u(t), x_2^u(t), t) = J^+(e^t)\frac{|u(t)|^\sigma}{\sigma} - \int_{-\infty}^t J^+(e^s)\frac{d}{ds}|u(s)|^\sigma\, ds. \tag{2.11}$$

Now we can give a slight generalization of the previous proposition.

Corollary 2.11. *Consider equation* (1.1) *and assume $K(r) \in \mathcal{C}^1$ is strictly positive and bounded, and that one of the following hypotheses is satisfied.*

(a) *There is $R > 0$ such that $J^+(r) \geq 0$ for any $0 < r < R$, and $K'(r) \leq 0$ for any $r > R$.*
(b) *There is $R > 0$ such that $K'(r) \geq 0$ for any $0 < r < R$, and $J^-(r) \leq 0$ for any $r > R$.*

Then there exists at least one S.G.S. *with slow decay.*

Proof. We begin from the first claim; observe that from (2.11) it follows that the flow on $S_m(\tau)$ points towards the exterior for $\tau \leq \log(R)$ and towards the interior for $\tau \geq \log(R)$. Then we can conclude as in Corollary 2.10 through Ważewski's principle. The second claim is analogous. □

We are ready now to prove Corollary 2.5, one of the principal results of the paper. If the first hypothesis concerning $J^+(r)$ is satisfied it is enough to put together Theorem 2.9, Corollary 2.11 and to observe that the uniqueness of the G.S. with fast decay follows from [18]. Assume that the second hypothesis concerning $J^-(r)$ is satisfied. To prove the uniqueness of the G.S. with fast decay, we have to consider the solutions $v(r)$ with fast decay of (1.1), and repeat all the reasonings developed in [18]. This way we obtain the same structure result as if the first hypothesis is satisfied, therefore the uniqueness of the G.S. with fast decay is proved.

We give now some results concerning radial solutions of the Dirichlet problem in the interior and in the exterior of a ball.

Lemma 2.12. *Consider equation* (2.4) *and assume that $G(\tau_1) > 0$ for some $\tau_1 \in \mathbb{R}$, then $x^u(\tau_1; t)$ corresponds to a crossing solution $u(r)$ of* (1.1). *Assume that there is τ_2 such that $G(\tau_2) < 0$, then $x^s(\tau_2; t)$ corresponds to a Dirichlet solution $u(r)$ of* (1.1) *in the exterior of a ball.*

Proof. The proof is analogous to the one given in Observation 3.5 in [9], but we give it here for convenience of the reader. Consider a trajectory $x^u(\tau_1; t)$: it lies on the right with respect to $\widetilde{W}^s(\tau_1)$. Assume for contradiction that $x^u(\tau_1; t) \in \mathbb{R}^2_+$ for any t, then it must cross the isocline c for some finite t. In fact otherwise we would have $\dot{x}_1^u(\tau_1; t) < 0$ for any $t > 0$; furthermore we have that $\dot{x}_1^u(\tau_1; t)$ is strictly negative for t large since $x^u(\tau_1; t)$ cannot converge to the origin, otherwise we would have $x^u(\tau_1;0)\in\widetilde{W}^s(\tau_1)$ and $0\in G(\tau_1)$. Noting that $x_1^u(\tau_1;0)<\infty$ we find that there is $t_1>0$ such that $x_1^u(\tau_1;T_2)=0$, but this is a contradiction, therefore $x^u(\tau_1;t)$ reaches the isocline c for some $T_1>0$. Now consider system (2.5); recall that the trajectory $X^u_{\tau_1}(t)$ of system (2.5), corresponding to the trajectory $x^u(\tau_1;t)$ of system (2.4) lies on the right of $\widetilde{W}^s$ for $t=0$. But from elementary reasonings on the phase portrait it follows that $X^u_{\tau_1}(t)$ must cross $\widetilde{W}^s$ for some T_0. Therefore $x^u(\tau_1;T_0)\in\widetilde{W}^s(\tau_1+T_0)\cap\widetilde{W}^u(\tau_1+T_0)$. Thus $x^u(\tau_1;0)\in\xi^+(\tau_1)\cap\xi^-(\tau_1)$. But this contradicts the hypothesis $G(\tau_1)>0$, thus $x^u(\tau_1;t)$ must represent a crossing solution.

Analogously it can be proved that $x^s(\tau_2,t)$ has to cross the positive x_1 semi-axis for some $t_2<0$. The reader can directly verify from the structure of equation (2.3) that $x^s(\tau_2,t)$ crosses the positive x_2 semi-axis for some $t_1<t_2$. □

The following proposition is a straightforward consequence of the previous lemma and of Proposition 2.8.

Proposition 2.13. *Consider equation* (1.1) *and assume $K(r)\in\mathcal{C}^1$ is strictly positive and bounded, and that $K'(r)$ changes sign a finite number of times. Assume that either hypothesis* (α^+) *or* (Ω^+) *are satisfied, then there are uncountably many crossing solutions. Assume that either hypothesis* (α^-) *or* (Ω^-) *are satisfied, then there are uncountably many Dirichlet solutions in exterior domains.*

Now reasoning as in [10, Theorem (3.6)] we can prove Theorem 2.3.

3. Construction of stable and unstable set through Ważewski's principle

We assume throughout all the paper that $a\le K(r)\le b$ for any r. We recall some definitions already used in the previous section and we introduce some new ones.

We call E the bounded set enclosed by $Z^a(t)$, $Z^b(t)$ and the origin. We call $E_a=\{Z^a(t)\mid t\in\mathbb{R}\}$, $E_b=\{Z^b(t)\mid t\in\mathbb{R}\}$ and $\partial E=E_a\cup E_b$. Denote now $E^\pm=E\cap(U^\pm\cup c)$ and $\partial E^\pm=\partial E\cap U^\pm$. We call A the point of intersection $\{A\}=E_a\cap c$ and $\{B\}=E_b\cap c$. We use the subscript 0 to mean that we consider the union of that set and the origin.

Consider now the non-autonomous system (2.4), and observe that the flow of the solutions on ∂E^- is always going towards the exterior of E, while on ∂E^+ it is always going towards the interior of E. We will construct a stable and unstable set $\widetilde{W}^{s,u}(\tau)$ satisfying (2.8). In [9] it is proved that these sets are manifolds, due to a generalization to non-autonomous systems of invariant manifold theory. Here we can just prove that they are compact connected sets joining the origin and c.

Here we will follow the ideas of the Ważewski principle, explained in [13]. Consider the set E^-. We say that Q is a point of strict egress, if there exists $\delta > 0$ such that the trajectory $x(\tau, Q; t)$ lies outside E^- for any $t \in (0, \delta)$. Note that the set of the point of strict egress of E^- is ∂E^- (we recall that the origin does not belong to ∂E^-). Consider now a point $Q \in E^-$ and assume that there exists $\widehat{T} > 0$ such that $x(\tau, Q; \widehat{T})$ does not belong to E^-. Then there exists $T > 0$ such that $x(\tau, Q; T) \in \partial E^-$ and $x(\tau, Q; t) \in E^-$ for any $t \in [0, T]$. In this case we say that $x(\tau, Q; T) = C(Q)$ is the consequent point of Q.

We want to prove now that there exists a point $\overline{Q} \in c$ such that the trajectory $x(\tau, \overline{Q}; t)$ is in E^- for any t. Then observing that in E^- we have that $\dot{x}_1 \leq 0$ for any t we can conclude that $\lim_{t\to\infty} x(\tau, \overline{Q}; t) = (0, 0)$. The existence of $\overline{Q}$ is proved by contradiction; thus we assume that for any point in c there exists a consequent point belonging to ∂E^-. We can define the function $C: c \to \partial E^-$ which to a point $Q \in c$ associates its consequent point $C(Q)$. First of all note that C is continuous. In fact, let us consider a point Q and its image $C(Q) = x(\tau, Q; T)$, then there exists $\delta > 0$ such that $x(\tau, Q; T + \delta)$ does not lie in E^-. Then, exploiting the continuity of the flow of the solution, we can affirm that, for any $\varepsilon > 0$ there is $\eta > 0$ such that if $|Q - P| < \eta$ then $|x(\tau, Q; t) - x(\tau, P; t)| < \varepsilon$, for any $t \in (T - \delta, T + \delta)$. This implies that $C(P) \to C(Q)$ if $P \to Q$ and this proves the claim.

Now observe that $C(A) = A$ and $C(B) = B$, therefore the sets $C^{-1}(\partial E_A^-)$ and $C^{-1}(\partial E_B^-)$ are both nonempty. Furthermore, exploiting once again the continuity of C we can conclude that they are open in c. Now, recalling that c is connected we can conclude that there exists $\overline{Q} \in c - C^{-1}(\partial E^-)$, thus we have the following.

PROPOSITION 3.1. *Consider system* (2.4) *and assume that ϕ is strictly positive and bounded. Then for any τ there exists a point $\overline{Q} \in c$ such that*

$$\lim_{t\to\infty} x(t, \overline{Q}; \tau) = (0, 0).$$

We define now $\widetilde{W}^s(\tau)$ and $\widetilde{W}^u(\tau)$ to be the sets satisfying (2.8). Now we want to prove that, for any τ, $\widetilde{W}^s(\tau)$ contains a continuum connecting the origin and c. Therefore we define the map $\overline{C}: E_0^- \to \partial E_0^-$ that associates to a point $P \in (E^- - W^s(\tau))$ its consequent point $\overline{C}(P)$ and to a point in $W^s(\tau)$ the origin. The function $\overline{C}$ is clearly an extension of the function C defined above. Repeating the proof given for C we can prove that $\overline{C}$ is continuous in a neighborhood of any point $P \in (E^- - W^s(\tau))$. Now we prove that $\overline{C}$ is continuous in $W^s(\tau)$ as well, thus we get that $\overline{C}$ is continuous. Let us consider $Q \in W^s(\tau)$. We recall that, for any $\varepsilon > 0$ there exists $T > 0$ such that $|x(\tau, Q; t)| < \varepsilon$ for any $t > T$. We fix $\delta > 0$ small and consider the points P such that $|P - Q| < \delta$. There exists a $T(\delta) > T$ for which $P_1 = x(\tau, P; T(\delta))$ is such that $|P_1| < \varepsilon$. Note that $\overline{C}(P) = \overline{C}(P_1)$ and that $|C(P_1)| < |P_1| < \varepsilon$ since $x(\tau, P; t) \in E^-$ until the trajectory crosses ∂E^-. Thus we have proved the continuity of the map $\overline{C}$.

Observe that $\widetilde{W}^s(\tau) = \overline{C}^{-1}((0,0))$, therefore it is closed and bounded, thus it is compact. Furthermore it contains the origin and a point belonging to c. Using the flow of the solution it is now possible to define a global stable set

$$W^s(\tau) := \{x^0 \in \mathbb{R}^2 \mid x(\tau, x^0; t) \in \widetilde{W}^s(\tau + t)\}.$$

We want to prove now that $\widetilde{W}^s(\tau)$ contains a continuum, say $\widehat{W}^s(\tau)$, joining the origin and c; this fact follows by the following topological Lemma which is a slight variant of [21, Lemma 4].

Lemma 3.2 ([13]). *Let $\mathcal{R}$ be a closed set homeomorphic to a full triangle. We call the vertex O, A and B and o, a, b the edges which are opposite to the respective vertex. Let $\mathcal{S} \subset \mathcal{R}$ be a closed set such that $\sigma \cap \mathcal{S} \neq \emptyset$, for any path $\sigma \subset \mathcal{R}$ joining a with b. Then $\mathcal{S}$ contains a closed connected set which contains O and at least one point of o.*

In our case we set $\mathcal{R} = E_0^-$, $\mathcal{S} = \widetilde{W}^s(\tau)$, $a = \partial E_a \cup U^-$ and $b = \partial E_b \cup U^-$. Note that, applying Proposition 3.1 substituting to c a generic path σ joining a and b, we find that σ intersects $\widetilde{W}^s(\tau)$. Thus we can apply the lemma and prove the existence of $\widehat{W}^s(\tau)$.

Reasoning in the same way we can prove the existence of $\widehat{W}^u(\tau)$.

We want to prove now that the sets $\widetilde{W}^s(\tau)$ and $\widetilde{W}^u(\tau)$ vary continuously with respect to τ. Let us now denote by $d(x,y)$ the Euclidean distance between $x, y \in \mathbb{R}^n$. Given two compact sets $X, Y \subset \mathbb{R}^n$ we define the Hausdorff distance

$$D(X,Y) := \max_{x \in X} \min_{y \in Y} d(x,y).$$

We claim that $D(\widetilde{W}^u(\tau), \widetilde{W}^u(\tau + \varepsilon)) \to 0$ as $\varepsilon \to 0$. Let us denote by $\Phi_t(\tau, x^0)$ the morphism $x^0 \to x(\tau, x^0; t)$, then we have $\Phi_t(\tau, \widetilde{W}^u(\tau)) \supset \widetilde{W}^u(\tau + t)$ for any $t > 0$. Thus the claim follows easily by a continuity argument.

We recall now that for any τ a priori we could have more than one set satisfying the property of the set $\widehat{W}^u(\tau)$ described above. We fix a value τ_0 and choose a set $\widehat{W}^u(\tau_0)$. Note that for $\varepsilon > 0$ small we have that $\Phi_\varepsilon(\tau_0, \widehat{W}^u(\tau_0)) \cap (U^+ \cup c)$ is a compact connected subset of $\widetilde{W}^u(\tau_0 + \varepsilon)$ to which the origin and a point in c belong: we call $\widehat{W}^u(\tau_0 + \varepsilon) = \Phi_\varepsilon(\tau_0, \widehat{W}^u(\tau_0)) \cap (U^+ \cup c)$. This way we can construct a family of compacts $\widehat{W}^u(\tau)$ satisfying the desired property for any $\tau > \tau_0$. Then we can choose a sequence of values $\tau_n \to -\infty$ and, iterating the reasoning, we can construct a family of connected compacts $\widehat{W}^u(\tau) \subset \widetilde{W}^u(\tau)$ to which the origin and a point of c belong. The compacts of this family vary continuously with respect to τ, in the sense of the distance defined above.

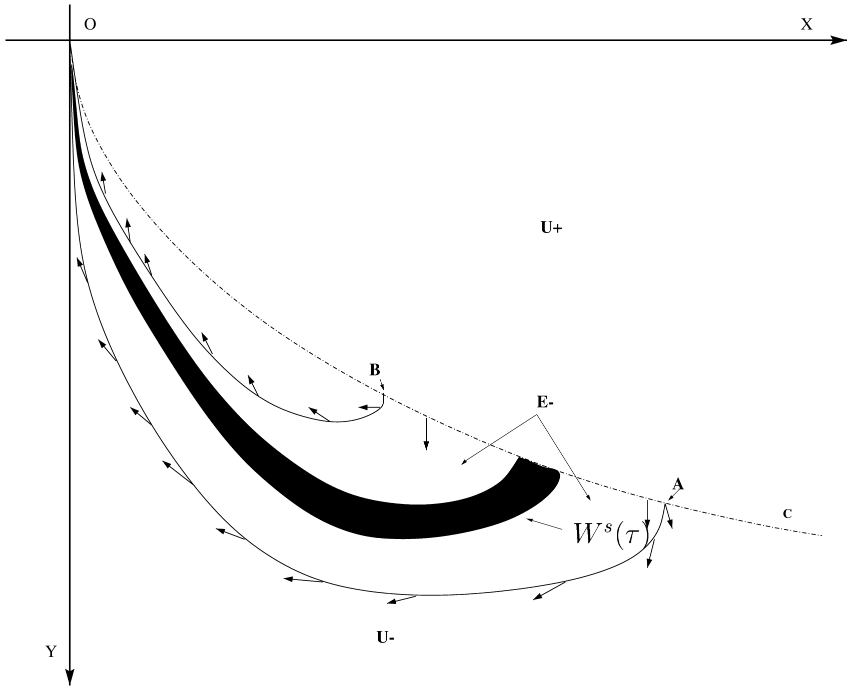

FIGURE 3.1. Construction of the set $\widetilde{W}^s(\tau)$.

We can also construct the sets $\widehat{W}^u(\tau)$ in such a way that $\widehat{W}^u(\tau) - \{O\}$ is connected. Furthermore for any τ, $\widetilde{W}^u(\tau)$ contains only one maximal set $\widehat{W}^u(\tau)$ satisfying all these properties. We just sketch the proof of this uniqueness result since we do not need it in this paper. The proof is for contradiction. If there are two disjoint sets $\widehat{W}^u(\tau)$, we can choose a point P in the set enclosed by these two sets and the isocline c, which does not belong to $\widetilde{W}^u(\tau)$. It might be proved that the trajectory departing from P must stay in E^+ for any $t < 0$ and that it has the origin as α-limit point. Then it follows that $P \in \widetilde{W}^u(\tau)$, but this is a contradiction.

Repeating the same reasoning we can get the same conclusions for $\widetilde{W}^s(\tau)$ and $\widehat{W}^s(\tau)$.

THEOREM 3.3. *Consider system* (2.4) *and assume that ϕ is strictly positive and bounded. Then for any τ the sets $\widehat{W}^{s,u}(\tau)$ are compact, connected and vary continuously with respect to τ. Each set $\widehat{W}^{s,u}(\tau)$ contains a continuum joining c and the origin.*

We denote by $\xi^{\pm}(\tau) = \widehat{W}^{u,s}(\tau) \cap c$. Note that both $\xi^+(\tau)$ and $\xi^-(\tau)$ are compact connected subsets and vary continuously with respect to τ. Furthermore consider (2.5); note that the sets

$$\xi^{\pm} = \bigcup_{\tau\in\mathbb{R}} \{(x_1, x_2, \tau) \mid (x_1, x_2) \in \xi^{\pm}(\tau)\}$$

are connected.

4. Indefinitely oscillating potentials

In this section we focus our attention on equation (1.1), where $K(r)$ can oscillate indefinitely. Furthermore we give an explanation of the existence of G.S. with fast decay in the perturbative case and we try to find a law which unifies the results obtained in [10] for the regular and the singular perturbation settings.

We continue to assume that there are $b > a > 0$ such that $a \le K(r) \le b$. As done in Section 2, we use (2.10) to approximate the solutions $x^{u,s}(\tau; t)$ of the non-autonomous problem with the homoclinic trajectories of the autonomous system, whose exact expression is known, see (2.9). Note that, if $P \in \xi^+(\tau)$ then

$$H(P, 0) = \int_{-\infty}^{0} \dot\phi(t+\tau) \frac{|x^u(\tau, P; t)|^\sigma}{\sigma}\, dt = \int_{-\infty}^{\tau} \dot\phi(s) \frac{|x^u(\tau, P; s-\tau)|^\sigma}{\sigma}\, ds.$$

Let us denote by $\dot\phi^+(t) = \max\{\dot\phi(t), 0\}$ and $\dot\phi^-(t) = \max\{-\dot\phi(t), 0\}$.

We can define now some important auxiliary functions which can be explicitly computed:

$$\begin{aligned}
F^u_-(\tau) &:= \int_{-\infty}^{0} \dot\phi^+(s+\tau)\frac{|U^b(s)|^\sigma}{\sigma}\, ds - \int_{-\infty}^{0} \dot\phi^-(s+\tau)\frac{|U^a(s)|^\sigma}{\sigma}\, ds,\\
F^u_+(\tau) &:= \int_{-\infty}^{0} \dot\phi^+(s+\tau)\frac{|U^a(s)|^\sigma}{\sigma}\, ds - \int_{-\infty}^{0} \dot\phi^-(s+\tau)\frac{|U^b(s)|^\sigma}{\sigma}\, ds,\\
F^s_-(\tau) &:= \int_{0}^{\infty} \dot\phi^-(s+\tau)\frac{|U^b(s)|^\sigma}{\sigma}\, ds - \int_{0}^{\infty} \dot\phi^+(s+\tau)\frac{|U^a(s)|^\sigma}{\sigma}\, ds,\\
F^s_+(\tau) &:= \int_{0}^{\infty} \dot\phi^-(s+\tau)\frac{|U^a(s)|^\sigma}{\sigma}\, ds - \int_{0}^{\infty} \dot\phi^+(s+\tau)\frac{|U^b(s)|^\sigma}{\sigma}\, ds.
\end{aligned}$$

Now exploiting estimate (2.10) we can deduce the following:

$$F^u_-(\tau) \le H(\xi^+(\tau), 0) \le F^u_+(\tau), \qquad F^s_-(\tau) \le H(\xi^-(\tau), 0) \le F^s_+(\tau).$$

Thus we can easily prove the following theorem.

Theorem 4.1. *Assume that there are τ_1 and τ_2 such that $F^s_-(\tau_1) \ge F^u_+(\tau_1)$ and $F^u_-(\tau_2) \ge F^s_+(\tau_2)$. Then there exists at least one homoclinic trajectory for system* (2.4)*, which corresponds to a* G.S. *with fast decay of* (1.1).

Proof. The theorem easily follows if we observe that

$$H(\xi^+(\tau_1), 0) - H(\xi^-(\tau_1), 0) < 0 \quad \text{while} \quad H(\xi^+(\tau_2), 0) - H(\xi^-(\tau_2), 0) > 0$$

and reasoning as in Theorem 2.4. $\square$

REMARK 4.2. Note that applying the previous theorem it is possible to prove the existence of multiple G.S. with fast decay. In fact, if we have an increasing sequence of values τ_k, $1 \leq j \leq k$, for which $F^s_-(\tau_j) \geq F^u_+(\tau_j)$ and $F^u_-(\tau_{j+1}) \geq F^s_+(\tau_{j+1})$, then there are at least $k-1$ G.S. with fast decay.

Recall that in the oscillatory case we lose something in the precision of the estimate of the asymptotic behaviour of positive solutions. However we can still apply Corollary 2.2. Observe that we can reformulate Theorems 2.3, 2.4, 2.6 and 2.9, replacing hypotheses $(\alpha^{\pm})$ and $(\Omega^{\pm})$ with the following which are more general but less explicit.

HYPOTHESES.

(α^-_O) There exists $\tau_1 \in \mathbb{R}$ such that $F^s_-(\tau) - F^u_+(\tau) \geq 0$, for any $\tau < \tau_1$.
(α^+_O) There exists $\tau_1 \in \mathbb{R}$ such that $F^u_-(\tau) - F^s_+(\tau) \geq 0$, for any $\tau < \tau_1$.
(Ω^-_O) There exists $\tau_2 \in \mathbb{R}$ such that $F^s_-(\tau) - F^u_+(\tau) \geq 0$, for any $\tau > \tau_2$.
(Ω^+_O) There exists $\tau_2 \in \mathbb{R}$ such that $F^u_-(\tau) - F^s_+(\tau) \geq 0$, for any $\tau > \tau_2$.

This construction find further applications also to the case where the sequence τ_k defined in Remark 4.2 is infinite. But in this case we have to restrict the range of the parameter p. We assume that $2n/(2+n) \leq p \leq 2$, so that, following [10], we find that $W^u(\tau)$ and $W^s(\tau)$ are manifolds. We consider the following hypotheses:

(O_1) There exists an infinite sequence τ_k, $\tau_k > \tau_{k+1}$ and $\tau_k \to -\infty$, such that $F^s_-(\tau_{2k}) - F^u_+(\tau_{2k}) > 0$ and $F^u_-(\tau_{2k+1}) - F^s_+(\tau_{2k+1}) < 0$.
(O_2) There exists an infinite sequence τ_k, $\tau_k < \tau_{k+1}$ and $\tau_k \to \infty$, such that $F^s_-(\tau_{2k}) - F^u_+(\tau_{2k}) > 0$ and $F^u_-(\tau_{2k+1}) - F^s_+(\tau_{2k+1}) < 0$.

Now we state the following structure result analogous to Theorem 4.1:

THEOREM 4.3. *Consider equation* (1.1) *and assume that* $2n/(2+n) \leq p \leq 2$. *Furthermore assume that hypothesis* (O_1) *is satisfied. Then there exist infinitely many crossing solutions, infinitely many* G.S. *with fast decay, infinitely many* S.G.S. *with fast decay and infinitely many solutions of Dirichlet problem in the exterior of balls. Assume that hypothesis* (O_2) *is satisfied. Then there exist infinitely many crossing solutions, infinitely many* G.S. *with fast decay, infinitely many* G.S. *with slow decay and infinitely many solutions of the Dirichlet problem in the exterior of balls. Assume that either hypothesis* (O_1) *or* (α^+_O) *is satisfied. Moreover, assume that either hypothesis* (Ω^-_O) *or* (O_2) *is satisfied. Then we can completely classify positive solutions as follows:*

(a) *There exist uncountably many crossing solutions.*
(b) *There exist uncountably many* G.S. *with slow decay.*
(c) *There exist uncountably many* S.G.S. *with fast decay*
(d) *There exist uncountably many* S.G.S. *with slow decay.*

(e) *There exists a non empty set of* G.S. *with fast decay.*

No other positive solutions, regular or singular, can exist.

This theorem can be proved exploiting the geometrical construction developed in Section 4 in [10]; we just sketch the main ideas for convenience of the reader. It can be proved that, for any τ, there exists a closed curve $L(\tau)$ which is made up of arcs of $\widetilde{W}^u(\tau)$ and $\widetilde{W}^s(\tau)$, which is contained in the 4th quadrant and to which the origin belongs, see Figure 4.1.

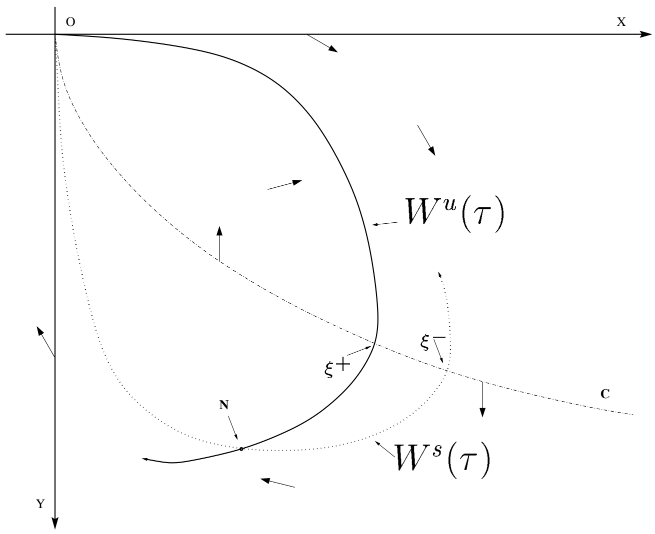

Figure 4.1. Sketch of the curve $L(\tau)$ when hypotheses (α^+) and (Ω^-) are satisfied.

Consider now system (2.5); setting $x_3 = \tau$ and letting τ takes value in the whole of $\mathbb{R}$ we obtain a topological surface L. Note that $L(\tau)$ can also be regarded as the intersection of L with the plane $x_3 = \tau$. Then exploiting the fact that the trajectories $x^s(\tau;t)$ and $x^u(\tau;t)$ cannot cross $L(\tau)$ we use the surface L to "entrap" the solutions. This way we can classify all positive solutions, see [10] for more details.

Now assume that $2n/(2+n) \leq p \leq 2$ and that ϕ is periodic of period T: we want to show that system (2.4) exhibits chaotic-like behaviour. We will follow almost step to step the construction developed for the perturbative case in [15] and [10]. Assume that hypotheses (O_1) and (O_2) are satisfied, then we know that $W^u(\tau)$ and $W^s(\tau)$ have infinitely many topological crossings, which corresponds to homoclinic points for (2.4). Let us call $\Phi_t(x)$ the flow associated to (2.4): applying the construction developed in [5] by Burns and Weiss we can prove the existence of a horseshoe factor. To be more precise there exists an integer

$N \in \mathbb{N}$ and a Cantor-like set $\acute{\Lambda}$ such that $\acute{\Lambda}$ is invariant under the action of the map $\Phi_{NT}(\,\cdot\,)$. Furthermore the discrete dynamical system made up of $\acute{\Lambda}$ and the map $\Phi_{NT}(\,\cdot\,)$ is topologically semiconjugated to the Bernoulli shift on a sequence of k symbols, where $k \in \mathbb{N}$. In particular there exists a subset $\Lambda \subset \acute{\Lambda}$, which is homeomorphic to a Cantor set, made up of periodic points.

We will show that the trajectories of (2.4) passing through these points are periodic and are forced to stay in a open subset of the 4th quadrant. Thus they correspond to S.G.S. with slow decay $v(r)$ of (1.1), such that $v(r) \sim r^{-\alpha}$ both as $r \to 0$ and as $r \to \infty$. Thus, even if $K(r)$ is oscillatory these S.G.S. satisfy the asymptotic estimates given for the monotone case in Proposition 2.1.

THEOREM 4.4. *Consider equation* (1.1) *and assume that* $2n/(2+n) \leq p \leq 2$. *Furthermore assume that* $\phi(t)$ *is periodic and that hypothesis* (O_1) *and* (O_2) *are satisfied. Then there exists a Cantor like set of* S.G.S. *with slow decay* $v(r)$ *of* (1.1), *such that* $v(r) \sim r^{-\alpha}$ *both as* $r \to 0$ *and as* $r \to \infty$.

PROOF. We have already seen how [5] can be used to show that system (2.4) exhibits a chaotic behaviour and admits a Cantor set of periodic orbits. Now we can repeat step to step the construction developed in [15] or the construction of [10] for the perturbative problem, to show that these trajectories are periodic. We just give a sketch of the ideas used in [10] for convenience of the reader. As mentioned above there exists a closed curve $L(\tau)$, contained in the 4th quadrant, which is made up of arcs of $\widetilde{W}^u(\tau)$ and $\widetilde{W}^s(\tau)$. We can arrange to find Λ inside the set enclosed by $L(\tau)$.

Observe that the periodic trajectories $x(\tau,t)$ of (2.4) departing at $t=0$ from a point in Λ cannot have the origin as α or Ω limit point. Therefore $x(\tau,t)$ cannot cross $L(t+\tau)$ for any $t \in \mathbb{R}$. Thus the corresponding trajectories $X(\tau,t)$ of (2.5) are forced to stay in the interior of L for any t. Therefore the corresponding solutions $v(r)$ of (1.1) are positive and monotone decreasing. The asymptotic estimates are easily deduced recalling that $x(\tau,t)$ is periodic in t. □

Now we try to clarify the meaning of the hypotheses (α_O^+), (α_O^-), (ω_O^+), (ω_O^-), (O_1) and (O_2) and we will show that both the regular and the singular perturbation settings will be included in these hypotheses. First of all observe that for $t<0$ we have

$$e^{(n-p)/pt}\left(\frac{p-1}{pD\sqrt[p]{a}}\right)^{(n-p)/p} \leq U^a(t) \leq e^{(n-p)t/p}\left(\frac{p-1}{D\sqrt[p]{a}}\right)^{(n-p)/p}$$

and for $t>0$ we have

$$e^{-(n-p)t/(p-1)}\left(\frac{p-1}{pD\sqrt[p]{a}}\right)^{(n-p)/p} \leq U^a(t) \leq e^{-(n-p)t/(p-1)}\left(\frac{p-1}{D\sqrt[p]{a}}\right)^{(n-p)/p}.$$

Therefore we can obtain the following estimates:

$$
\begin{aligned}
F^u_-(\tau) &\geq \frac{(p-1)^n}{\sigma D^n} \int_{-\infty}^0 \bigg(\frac{\dot\phi^+(s+\tau)}{[p\sqrt[p]{b}]^n} - \frac{\dot\phi^-(s+\tau)}{\sqrt[p]{a^n}} \bigg) e^{ns}\, ds, \\
F^u_+(\tau) &\leq \frac{(p-1)^n}{\sigma D^n} \int_{-\infty}^0 \bigg(\frac{\dot\phi^+(s+\tau)}{\sqrt[p]{a^n}} - \frac{\dot\phi^-(s+\tau)}{[p\sqrt[p]{b}]^n} \bigg) e^{ns}\, ds, \\
F^s_-(\tau) &\geq \frac{(p-1)^n}{\sigma D^n} \int_0^{\infty} \bigg(\frac{\dot\phi^-(s+\tau)}{[p\sqrt[p]{b}]^n} - \frac{\dot\phi^+(s+\tau)}{\sqrt[p]{a^n}} \bigg) e^{-nps/(p-1)}\, ds, \\
F^s_+(\tau) &\leq \frac{(p-1)^n}{\sigma D^n} \int_0^{\infty} \bigg(\frac{\dot\phi^-(s+\tau)}{\sqrt[p]{a^n}} - \frac{\dot\phi^+(s+\tau)}{[p\sqrt[p]{b}]^n} \bigg) e^{-nps/(p-1)}\, ds.
\end{aligned}
$$

Now consider functions $K(r)$ which can be written as $K(r) = 1 + \delta k(r)$, where the parameter δ is positive and small. In [10] a Melnikov function, which is closely related to $G(\tau)$, is introduced. The computation of this function involves the integration of the homoclinic solution of the autonomous system. Through this function one obtains a sufficient condition for the existence of G.S. with fast decay, when δ is small enough. Using Theorem 4.3 we can quantify the "smallness" of δ. In fact it can be shown using explicit computations (and below this will be done, in some examples) that, if $K(r)$ is oscillatory, we can find τ_1 and τ_2 in such a way that $F^u_-(\tau_1) - F^s_+(\tau_1) > 0$, and $F^u_+(\tau_2) - F^s_-(\tau_2) < 0$. Hence we can apply Theorem 4.1. We will see that, in many cases, it will be sufficient to take $\delta < 1/2$. The underlying idea is that, when we choose δ small, we are shrinking the sets E^+ and E^-, so that the estimates (2.10) become good ones. Therefore we may say that the Melnikov function works because, if δ is small, the sets E^+ and E^- become so thin that we can approximate $x^u(\tau)$ and $x^s(\tau)$ with the homoclinic trajectory of the autonomous system.

The computation becomes clearer in the singular perturbation case, that is $K(r) = k(r^\varepsilon)$, which gives $\phi = \phi(\varepsilon t)$. In fact in [10] it is proved that if ε is small enough, there exists a G.S. with fast decay for each positive non degenerate critical point of ϕ. Once again using Theorem 4.3 and evaluating $F^{u,s}_\pm$ we can quantify the "smallness" of ε and give an interpretation of the results obtained in [10]. The Melnikov function used in [10] measures the value of $G(\tau)$; in that paper it was proved that $G(\tau)$ has the same sign as $\dot\phi(\tau)$. Let us call I^+_j and I^-_j the intervals in which respectively $\dot\phi(\tau + t) \geq 0$ and $\dot\phi(\tau + t) \leq 0$. Note that if the system is singularly perturbed these intervals become larger. Thus the main contribution in $F^u(\tau)$ and $F^s(\tau)$ is given by the integration in the interval I_j to which $t = 0$ belongs, so that we can neglect the others. This justifies the fact that in the perturbative case the sign of $G(\tau)$ and of $\dot\phi(\tau)$ are equal. It is worthwhile to point out that, in our examples, it is enough that $\log\varepsilon$ is small to make our Theorems work.

To illustrate these claims we give an example with a potential $K(r)$ which can be easily computed. Let $K(r) = 1 + \delta\sin(\varepsilon\log r)$ that gives $\phi(t) = 1 + \delta\sin(\varepsilon t)$. We assume that $\pi/2\varepsilon \le \tau \le 3\pi/2\varepsilon$; we skip the computations even if they can be made explicitly, since they are too technical.

$$\begin{aligned}\frac{\sigma D^n(\varepsilon^2+n^2\delta)}{(p-1)^n\varepsilon\delta}F^u_+(\tau) &< \left(\frac{1}{[\sqrt[p]{a}]^n} - \frac{1}{[p\sqrt[p]{b}]^n}\right)\frac{e^{n(\pi/(2\varepsilon)-\tau)}}{1-e^{-n\pi/\varepsilon}} \\ &\quad + \frac{1}{[p\sqrt[p]{b}]^n}\left(\sin(\varepsilon\tau) + \frac{n\cos(\varepsilon\tau)}{\varepsilon}\right), \\ -\frac{\sigma D^n(\varepsilon^2(p-1)^2+n^2\delta)}{(p-1)^n\varepsilon\delta(p-1)^2}F^s_-(\tau) &> \left(\frac{1}{[\sqrt[p]{a}]^n} - \frac{1}{[p\sqrt[p]{b}]^n}\right)\frac{e^{-(n/p-1)(\pi/(2\varepsilon)-\tau)}}{1-e^{-n\pi/((p-1)\varepsilon)}} \\ &\quad + \frac{1}{[p\sqrt[p]{b}]^n}\left(\sin(\varepsilon\tau) - \frac{n}{(p-1)\varepsilon}\cos(\varepsilon\tau)\right).\end{aligned}$$

Recall that $a = 1-\delta$ and $b = 1+\delta$. If we set $\tau = \pi/\varepsilon$ and $p = 2$, we obtain the following

$$F^u_+\left(\frac{\pi}{\varepsilon}\right) - F^s_-\left(\frac{\pi}{\varepsilon}\right) < C,$$

where

$$C = \frac{2\varepsilon\delta}{\sigma D^n(\varepsilon^2+n^2\delta)}\left[\left(\frac{1}{[\sqrt[p]{a}]^n} - \frac{1}{[p\sqrt[p]{b}]^n}\right)\frac{e^{-n\pi/(2\varepsilon)}}{1-e^{-n\pi/\varepsilon}} - \frac{n}{\varepsilon}\frac{1}{[p\sqrt[p]{b}]^n}\right].$$

Observe that, if for example we take $\delta < 1$ and $\varepsilon < n$, we find that C is negative. Furthermore note that the positive contribution to C behaves like $e^{-n\pi/(2\varepsilon)}$.

Analogously $F^u_-(0) - F^s_+(0) > -C$; furthermore note that in this case the functions ϕ and $F^{u,s}_\mp$ are periodic. Therefore, applying Theorem 4.1 we find that there exist infinitely many G.S. with fast decay and applying Theorem 4.3 we can classify positive solutions of (1.1). Furthermore applying Theorem 4.4 we deduce the existence of a horseshoe factor for (2.4) and of a Cantor like set of S.G.S. $v(r)$ of (1.1), such that $v(r)r^\alpha$ is strictly positive and bounded.

However, working directly on the definition of $F^{u,s}_\pm$, with the help of numerical computation, we can get better estimate and work also with generic functions. Consider the function $K(r) = 1 - \delta\cos(\varepsilon\log r)$. If we set $\delta = 0.5$, $n = 5$, $p = 4$, $\varepsilon = 1$ we obtain that

- if we set $\tau = -\pi/(2\varepsilon)$:

$$\begin{aligned}-0.086 = F^u_-(\tau) &< H^u(\tau) < F^u_+(\tau) = -0.021,\\ 0.014 = F^s_-(\tau) &< H^s(\tau) < F^s_+(\tau) = 0.091,\end{aligned}$$

- if we set $\tau = \pi/(2\varepsilon)$:

$$\begin{aligned}0.021 = F^u_-(\tau) &< H^u(\tau) < F^u_+(\tau) = 0.086,\\ -0.091 = F^s_-(\tau) &< H^s(\tau) < F^s_+(\tau) = -0.014.\end{aligned}$$

Recalling that system (2.3) is periodic of period $T = 2\pi/\varepsilon$ and applying Theorem 4.1 we prove the existence of infinitely many G.S. with fast decay. We cannot apply Theorem 4.3 since $p > 2$.

If we set $\delta = 0.5$, $n = 3$, $p = 1.5$, $\varepsilon = 1$ we obtain that

- if we set $\tau = -\pi/(2\varepsilon)$:

$$-7.3 \times 10^{-4} = F^u_-(\tau) < H^u(\tau) < F^u_+(\tau) = -0.8 \times 10^{-4},$$
$$7.0 \times 10^{-4} = F^s_-(\tau) < H^s(\tau) < F^s_+(\tau) = 6.3 \times 10^{-4},$$

- if we set $\tau = \pi/(2\varepsilon)$:

$$0.8 \times 10^{-4} = F^u_-(\tau) < H^u(\tau) < F^u_+(\tau) = 7.3 \times 10^{-4},$$
$$-6.3 \times 10^{-4} = F^s_-(\tau) < H^s(\tau) < F^s_+(\tau) = -7.0 \times 10^{-4}.$$

Thus applying Theorem 4.1 we prove the existence of infinitely many G.S. with fast decay. Furthermore, since $2n/(n+2) \leq p \leq 2$ and hypotheses (O_1) and (O_2) are satisfied, we can apply Theorems 4.3 and 4.4, and classify all the positive solutions.

Now consider $K(r) = 1 + \delta\sin(r^\varepsilon)$. Note now that the function $\phi(t)$ in this case is recurrent, that is $\phi(t + \log(1 + 2k\pi e^{-t})) = \phi(t)$ for any $k \in \mathbb{N}$. Therefore, if we set $T = \log(1 + 2k\pi e^{-t})$, $k \in \mathbb{N}$ we have $\phi(t + T) = \phi(t)$. Thus also the values of the functions $F^{u,s}_\pm(\tau)$ have some kind of oscillatory behaviour, therefore it is possible to deduce the existence of multiple G.S. with fast decay.

If we set $\delta = 0.01$, $n = 3$, $p = 2$, $\varepsilon = 1$ we obtain that

- if we set $\tau = \log(\pi/(2\varepsilon))$:

$$1.89 \times 10^{-5} = F^u_-(\tau) < H^u(\tau) < F^u_+(\tau) = 1.92 \times 10^{-5},$$
$$1.73 \times 10^{-5} = F^s_-(\tau) < H^s(\tau) < F^s_+(\tau) = 1.77 \times 10^{-5},$$

- if we set $\tau = \log(3\pi/(2\varepsilon))$:

$$-2.15 \times 10^{-5} = F^u_-(\tau) < H^u(\tau) < F^u_+(\tau) = -2.08 \times 10^{-5},$$
$$-1.04 \times 10^{-5} = F^s_-(\tau) < H^s(\tau) < F^s_+(\tau) = -0.96 \times 10^{-5},$$

- if we set $\tau = \log(3\pi/(2\varepsilon))$:

$$4.23 \times 10^{-6} = F^u_-(\tau) < H^u(\tau) < F^u_+(\tau) = 5.00 \times 10^{-6},$$
$$5.77 \times 10^{-6} = F^s_-(\tau) < H^s(\tau) < F^s_+(\tau) = 6.54 \times 10^{-6}.$$

Therefore we can apply Theorem 4.1 and prove the existence of two G.S. with fast decay. The problem is that as $\tau \to \infty$, the functions $F^{u,s}_\pm(\tau)$ tend to 0 oscillating. Therefore, when τ is too large, the errors in the approximations become too relevant and the estimates are not good enough anymore. However, recalling that hypothesis (α^+_O) is satisfied, we can still apply Theorem 2.9 to prove the existence of uncountably many S.G.S. with fast decay, and Lemma 2.13 to prove the existence of uncountably many solutions of the Dirichlet problem in the interior and in the exterior of a ball.

Acknowledgements. The author wishes to thank his supervisor, Professor R. A. Johnson, for the helpful discussions as well as for having introduced him to the study of this topic.

References

[1] F. Battelli and R. Johnson, *Singular ground states of the scalar curvature equation in* $\mathbb{R}^n$, Differential Integral Equations **14** (2000), 123–139.

[2] ———, *On positive solutions of the scalar curvature equation when the curvature has variable sign*, Nonlinear Anal. **47** (2001), 1029–1037.

[3] ———, *On transversal smoothness of invariant manifold*, Comm. Appl. Anal. **5** (2001), 383–401.

[4] G. Bianchi, H. Egnell, *An ODE approach to the equation* $\Delta u + Ku^{(n+2)/(n-2)} = 0$ *in* $\mathbb{R}^n$, Math. Z. **210** (1992), 137–166.

[5] K. Burns and H. Weiss, *A geometric criterion for positive topological entropy*, Comm. Math. Phys. **172** (1995), 95–118.

[6] L. Damascelli, F. Pacella and M. Ramaswamy, *Symmetry of ground states of p-Laplace equations via the Moving Plane Method*, Arch. Rational Mech. Anal. **148** (1999), 291–308.

[7] M. Franca, *Classification of positive solutions of p-Laplace equation with a growth term*, Arch. Math. (to appear).

[8] ———, *Some results on the m-Laplace equations with two growth terms*, preprint.

[9] ———, *Structure theorems for positive radial solutions of the generalized scalar curvature equation, when the curvature exhibits a finite number of oscillations*, preprint.

[10] M. Franca and R. A. Johnson, *Ground states and singular ground states for quasilinear partial differential equations with critical exponent in the perturbative case*, Adv. Nonlinear Studies (to appear).

[11] B. Franchi, E. Lanconelli and J. Serrin, *Existence and uniqueness of nonnegative solutions of quasilinear equations in* $\mathbb{R}^n$, Adv. Math. **118** (1996), 177–243.

[12] F. Gazzola, *Critical exponents which relate embedding inequalities with quasilinear elliptic operator*, preprint.

[13] J. Hale, *Ordinary Differential Equations*, vol. 21, Pure and Applied Mathematics, 1980.

[14] R. Johnson, *Concerning a theorem of Sell*, J. Differential Equations **30** (1978), 324–339.

[15] R. Johnson, X. B. Pan and Y. F. Yi, *The Melnikov method and elliptic equations with critical exponent*, Indiana Math. J. **43** (1994), 1045–1077.

[16] ———, *Singular ground states of semilinear elliptic equations via invariant manifold theory*, Nonlinear Anal. **20**.

[17] N. Kawano, W. M. Ni and S. Yotsutani, *A generalized Pohozaev identity and its applications*, J. Math. Soc. Japan **42** (1990), 541–564.

[18] N. Kawano, E. Yanagida and S. Yotsutani, *Structure theorems for positive radial solutions to* $\operatorname{div}(|Du|^{m-2}Du) + K(|x|)u^q = 0$ *in* $\mathbb{R}^n$, J. Math. Soc. Japan **45** (1993), 719–742.

[19] M. C. Knapp and L. A. Peletier, *Quasilinear elliptic equations with nearly critical growth*, Comm. Partial Differential Equations **14** (1989), 1351–1383.

[20] W. M. Ni and J. Serrin, *Nonexistence theorems for quasilinear partial differential equations*, Rend. Circ. Mat. Palermo (2) **8** (1985), 171–185.

[21] D. Papini and F. Zanolin, *Chaotic dynamics for non-linear Hill's equation*, Georgian Math J. **9** (2002).

Manuscript received November 6, 2003

Matteo Franca
Dipartimento di Matematica
Università di Firenze
Viale Morgagni 67a
50134 Firenze, ITALY

E-mail address: franca@math.unifi.it