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BIFURCATION OF SOLUTIONS OF ELLIPTIC PROBLEMS: LOCAL AND GLOBAL BEHAVIOUR

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Abstract. Here we study the local behavior of the continua in the case of Neumann boundary conditions, pointing out some qualitative differences with the Dirichlet case. We also combine local and global behavior of the bifurcating sets to obtain existence of solutions and study of their sign for some related problems.

1. Introduction

In this paper we describe the local and global behavior of the continua of solutions of some nonlinear elliptic problems when bifurcation occurs. Let Ω be a smooth bounded domain in $\mathbb{R}^N$, let $g\colon \Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function (i.e. measurable in $x\in\Omega$ and continuous in $s\in\mathbb{R}$) and consider the boundary value problem:

$$\begin{cases} -\Delta u = \lambda u + g(x,u) & \text{if } x\in\Omega, \\ \text{B.C.} \end{cases} \tag{1.1}$$

where B.C. means either Dirichlet boundary conditions or Neumann boundary conditions.

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In order to apply bifurcation from infinity, we will asume the hypothesis:

(H1) $\lim_{|s|\to\infty} g(x,s)/s = 0$ uniformly for $x \in \Omega$.

Under the hypothesis

(H2) there exists $r > N$ and $h \in L^r(\Omega)$ such that $|g(x,s)| \leq h(x)(1+|s|)$, for all $(x,s) \in \Omega \times \mathbb{R}$,

the weak solutions of (1.1) lie in the space $W^{2,r}(\Omega)$ continuously embedded in $X = C^1(\overline{\Omega})$. We will study bifurcation of solutions $(\lambda, u) \in \mathbb{R} \times X$. P will be the cone of positive functions of X and $\pi: \mathbb{R} \times X \to \mathbb{R}$ will be the projection of pairs (λ, u) over their first variable, $\pi(\lambda, u) = \lambda$.

The Global Bifurcation Theorem of Rabinowitz (see [5]) ensures that:

Under hypotheses (H1) *and* (H2)*, every eigenvalue* λ_k *of odd algebraic multiplicity is a bifurcation point from infinity, i.e. there exists a sequence* (λ_n, u_n) *of solutions of* (1.1) *such that* $\lambda_n \to \lambda_k$ *and* $\|u_n\| \to \infty$.

Consider the set of nontrivial solutions of (1.1) *along with the bifurcation point* (λ_k, ∞)*. If* Σ *is the closed connected component of previous set containing* (λ_k, ∞)*, then either:*

(a) $\pi(\Sigma)$ *is unbounded, or*
(b) Σ *meets a bifurcation point from zero, or*
(c) Σ *meets a bifurcation point from infinity different from* (λ_k, ∞)*.*

It is well known that for any bifurcating sequence $(\lambda_n, u_n) \to (\lambda_k, \infty)$ there exists a subsequence (λ_{n_j}, u_{n_j}), such that

$$\frac{u_{n_j}}{\|u_{n_j}\|} \to \phi \quad \text{in } X,$$

where ϕ is an eigenfunction associated to λ_k with $\|\phi\| = 1$.

In the particular case of bifurcation from infinity at the principal eigenvalue λ_1, both $\phi \equiv \phi_1$ and $\phi \equiv -\phi_1$ have sequences as above. Since ϕ_1 lies in the interior of the C^1-cone of positive functions P, we deduce that, near the bifurcation points, the solutions have constant sign. Consequently, we denote Σ^+ the closed connected component bifurcating from (λ_1, ∞) and Σ^- the closed connected component bifurcating from $(\lambda_1, -\infty)$.

The behavior of continua bifurcating from infinity at the first eigenvalue with Dirichlet boundary condition has been widely studied by several authors. In [1] the authors prove that the bifurcation from ∞ is locally to the left provided that $\liminf_{s\to\infty} g(x,s) \geq \varepsilon > 0$ and locally to the right provided that $\limsup_{s\to\infty} g(x,s) \leq -\varepsilon < 0$. In [2] the authors consider the possibility of g approaching zero near infinity, or even changing sign. They point out that the local behavior of the bifurcation is decided by the sign of the integral $\int_\Omega A_\alpha \phi_1^{1-\alpha}$

where

$$A_\alpha(x) = \lim_{s\to\infty} g(x,s)s^\alpha, \quad \text{for some } \alpha \le 2.$$

Here we study the local behavior of the continua in the case of Neumann boundary conditions, pointing out some qualitative differences with the Dirichlet case. We also combine local and global behavior of the bifurcating sets to obtain existence of solutions and study of their sign for some related problems.

The paper is organized as follows: in Section 2 the local behavior of bifurcation from infinity at the first eigenvalue with Neumann boundary condition is studied. We give an example to point out the differences in the local behavior from infinity when different boundary conditions are considered.

In Section 3 we study the global behavior of the continua bifurcating from infinity, when $g(x,0)$ has constant sign.

2. Local behavior of the continua

Local behavior of the bifurcation from infinity with Dirichlet boundary condition has been studied in [1] and [2]. In this section, we will center our attention in equation (1.1) with Neumann boundary condition and we will find some differences between two cases.

Consider the problem

$$(2.1)\qquad \begin{cases} -\Delta u(x) = \lambda u(x) + g(x,u(x)) & \text{if } x\in\Omega, \\ \dfrac{\partial u}{\partial n}(x) = 0 & \text{if } x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded regular domain.

The eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

has a sequence of positive eigenvalues $0 = \lambda_1 \le \ldots \le \lambda_n \le \ldots \to \infty$. It is well-known that $\lambda_1 = 0$ is simple and its eigenspace is spanned by the positive function $\phi_1 \equiv 1$.

In [2] the authors prove that, with Dirichlet boundary conditions, the sign of $g(x,s)$ near $s = \infty$ does not decide the local behavior of the set of solutions near the bifurcation. In particular, they prove that if, as s goes to infinity, g tends to zero (roughly) like the function a/s^α ($a\in\mathbb{R}$), then the sign of a decides the behavior of the bifurcation from infinity only if $\alpha \le 2$.

Here we use similar ideas to prove that, in the Neumann case, the sign of $g(x,s)$ near $s = \pm\infty$ *does decide* the local behavior of the set of solutions near the bifurcation point.

Theorem 2.1. *Assume one of the following two conditions, either*

(a) *there exists* $g(x,\infty) = \lim_{s\to\infty} g(x,s)$, *such that* $\int_\Omega g(x,\infty) > 0$ (*resp.* < 0), *or*

(b) *there exists* $a(x)$ *such that* $\int_\Omega a(x) = 0$, *and there exists* $M > 0$ *such that if* $s \geq M$ *then* $g(x,s) > a(x)$ (*resp.* $< a(x)$).

Then the bifurcation of solutions of (2.1) *from* $(\lambda_1,\infty) = (0,\infty)$ *is locally to the left* (*resp. to the right*).

Proof. Multiplying in (2.1) by $\phi_1 \equiv 1$ and integrating by parts,

$$-\lambda \int_\Omega u = \int_\Omega g(x,u). \tag{2.2}$$

Moreover, there exists a sequence (μ_n, u_n) of solutions of (2.1) such that $\mu_n \to \lambda_1 = 0$, $\|u_n\| \to \infty$, and a subsequence still denoted by (μ_n, u_n) such that

$$v_n = \frac{u_n}{\|u_n\|} \to \phi_1 \equiv 1 \quad \text{in } X.$$

In particular, $u_n \to \infty$ uniformly in Ω. Consequently, for n sufficiently large, $\int_\Omega u_n > 0$, and from (2.1) we conclude that for all (λ, u) near the bifurcation from $(0,\infty)$

$$\operatorname{sgn}(\lambda) = -\operatorname{sgn}\bigg(\int_\Omega g(x,u)\bigg).$$

Using the uniform convergence of u_n to ∞ and the hypotheses of this theorem, we conclude that $\int_\Omega g(x,u) > 0$ (resp. < 0), and the proof is concluded. □

Remarks 2.2. (1) Observe that if $\int_\Omega g(x,\infty) = 0$, then (a) cannot be applied, but one can take $a(x) = g(x,\infty)$ and apply (b), provided that g approaches such limit "from one side".

(2) In the case (b), one can also consider the weaker condition for all $s \geq M$, $g(x,s) \geq a(x)$. Then the same argument concludes that the bifurcation is locally to the left, but "not strictly". Explicitly, every bifurcating sequence $(\lambda_n, u_n) \to (0,\infty)$, will satisfy that $\lambda_n \leq 0$ for n large enough.

(2) In [1] the authors prove that the bifurcation is locally to the left provided that $\liminf_{s\to+\infty} g(x,s) \geq \varepsilon > 0$. Here we improve that condition by allowing $g(x,\infty)$ to change sign, or even g to approach zero.

(3) Of special interest are the problems with mixed-type boundary conditions, and nonlinearities satisfying (b). As one can see in previous proof, as well as in the corresponding result in [2], the qualitative behavior of the bifurcation is determined by the type of convergence of solutions to ∞. This convergence will be uniform if and only if the eigenfunction associated to the first eigenvalue is strictly positive in $\overline{\Omega}$. In such a case, previous theorem applies. On the other hand, if the eigenfunction vanishes somewhere in the boundary, the results and counterexample of [2] can be adapted to this special case.

EXAMPLES 2.3 (Difference between Neumann and Dirichlet boundary conditions). Note that usually the hypotheses to decide the behavior of local bifurcation are referred to the non-linearity g function. However, these two examples emphasize that, using the same g, the boundary conditions can determine the local behavior of the continua.

(1) Let us denote by ϕ_1 the first eigenfunction of the Laplace operator with Dirichlet boundary conditions. Now take a nonlinearity $g(x,s)$ such that

$$\int_\Omega g(x,\infty)\phi_1 < 0 < \int_\Omega g(x,\infty).$$

Applying the results in [2], the bifurcation from (λ_1,∞) is locally to the right for the Dirichlet case. However, for Neumann boundary condition previous theorem ensures that bifurcation from infinity at the first eigenvalue is to the left.

(2) Now we avoid the use of spatial dependence and fix our attention on how function g approaches zero as s goes to infinity. Consider

$$\begin{cases} -\Delta u = \lambda u + g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where $\Omega = (0,\pi)$, $K > 0$ and

$$g(s) = \begin{cases} -K & \text{for } -\infty < s \le 1, \\ K(s-2) & \text{for } 1 \le s \le 2, \\ (s-2)/s^4 & \text{for } 2 \le s. \end{cases}$$

In [2] the authors prove that if K large enough (for example $K > 64$) the bifurcation of (2.3) from (λ_1,∞) is locally to the right.

On the other hand, consider (2.3) with Neumann boundary condition,

$$\begin{cases} -\Delta u = \lambda u + g(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Using Theorem 2.1 we obtain that the bifurcation of (2.4) is locally to the left against the Dirichlet case.

3. Global behavior of the continua

Consider the problem

$$\begin{cases} -\Delta u(x) = \lambda u(x) + g(x,u(x)), & \text{if } x \in \Omega, \\ \text{B.C.} \end{cases} \tag{3.1}$$

where Ω is a regular, bounded domain.

In order to decide the global behavior of the continua in problem (3.1) we will add the next hypotheses:

(G1) g is L-Lipschitz in the second variable.

(G2) $g(x,0)$ does not change its sign and is not identically 0 in Ω.

We denote the possibilities by:

(G2+) $g(x,0) \geq 0$, for all $x \in \Omega$ and $g(x,0)$ is not identically 0 in Ω.
(G2$-$) $g(x,0) \leq 0$, for all $x \in \Omega$ and $g(x,0)$ is not identically 0 in Ω.

Theorem 3.1. *Assume* (H1), (H2) *and* (G1). *Then we have*

(a) *If* (G2+) *holds, then*
 (i) $\Sigma^+ \subset \mathbb{R}\times$ int(P)
 (ii) $\Sigma^- \cap (\mathbb{R}\times P) = \emptyset$,
(b) *If* (G2$-$) *holds, then*
 (i') $\Sigma^- \subset \mathbb{R}\times$ int$(-P)$,
 (ii') $\Sigma^+ \cap (\mathbb{R}\times(-P)) = \emptyset$.

Consequently, if (G2) *holds, then* $\Sigma^+ \cap \Sigma^- = \emptyset$.

Proof. We will prove item (a). Proof of item (b) is similar.

From [3] using (G1) if $\underline{u}$ subsolution (no solution) and $u \in C^1(\overline{\Omega})$ solution of (3.1), then $u - \underline{u} \notin \partial P$.

In this case, $\underline{u} \equiv 0$. Hypothesis (G2+) implies $\underline{u}$ is subsolution but no solution and then $u \notin \partial P$.

Consider the projection $T\colon \mathbb{R}\times C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$ given by $T(\lambda,u) = u$.

(i) Since T is continuous, $T(\Sigma^+)$ is a connected set and $T(\Sigma^+) \cap \partial P = \emptyset$. It's known that if $(\lambda,u) \in \Sigma^+$ and (λ,u) near the bifurcation point then $u \in P$, therefore $T(\Sigma^+) \subset$ int(P).
(ii) In the same way, if $(\lambda,u) \in \Sigma^-$ and (λ,u) near $(\lambda_1,-\infty)$ then $u < 0$ $(u \notin P)$ and $T(\Sigma^-) \cap \partial P = \emptyset$. Consequently, $T(\Sigma^-) \cap P = \emptyset$. □

We have proved that the hypothesis (G2) implies the continua bifurcating from λ_1 are disjoint. Now, we will describe the global behavior of them.

Theorem 3.2 (Figure 1). *Under hypotheses* (H1), (H2), (G1) *and* (G2+), Σ^+ *does not meet another bifurcation point neither from infinity nor from zero. Moreover,* $\pi(\Sigma^+)$ *is bounded from above and unbounded from below. Explicitly,*

$$]-\infty,\lambda_1[\, \subset \pi(\Sigma^+) \subset\,]-\infty,\lambda_1+L].$$

Proof. Suppose Σ^+ meets (λ_k,∞) $(k \neq 1)$. Then there exists a sequence (λ_n,u_n) such that $\lambda_n \to \lambda_k$, $\|u_n\| \to \infty$ and $u_n/\|u_n\| \to \phi_k$. That is not posible because it is known that ϕ_k changes sign and $\Sigma^+ \subset R\times$ int(P). Therefore, Σ^+ does not bifurcate from any (λ_k,∞), $k \neq 1$.

Since $g(x,0)$ is not identically zero the function $u \equiv 0$ is not a solution of (3.1) for any value of λ, and then bifurcation from zero does not occur.

Thus, Σ^+ does not meet another bifurcation point different from (λ_1,∞).

Now consider $(\lambda, u) \in \Sigma^+$ a solution of (3.1). Multiplying by ϕ_1 and integrating by parts,

$$\int_\Omega \nabla u \nabla \phi_1 = \lambda \int_\Omega u\phi_1 + \int_\Omega g(x,u)\phi_1.$$

Multiplying $-\Delta\phi_1 = \lambda_1\phi_1$ by u and integrating by parts,

$$\int_\Omega \nabla u \nabla \phi_1 = \lambda_1 \int_\Omega u\phi_1,$$

we obtain

$$(\lambda_1 - \lambda)\int_\Omega u\phi_1 = \int_\Omega g(x,u)\phi_1.$$

From (G1) and (G2+), $g(x,s) \geq g(x,0) - Ls \geq -Ls$, for all $s > 0$, and using $\Sigma^+ \subset R \times \mathrm{int}(P)$

$$(\lambda_1 - \lambda)\int_\Omega u\phi_1 = \int_\Omega g(x,u)\phi_1 \geq -L\int_\Omega u\phi_1 \quad \text{and} \quad \int_\Omega u\phi_1 > 0,$$

then,

$$\lambda \leq \lambda_1 + L. \tag{3.2}$$

Since Σ^+ does not meet another bifurcation point different of λ_1, then, taking into account the Global Bifurcation Theorem of Rabinowitz given in the introduction of this paper, we have proved that Σ^+ does not satisfy neither (b) nor (c) options. Then Σ^+ satisfies option (a) and by (3.2) we deduce $\pi(\Sigma^+)$ is lower unbounded. □

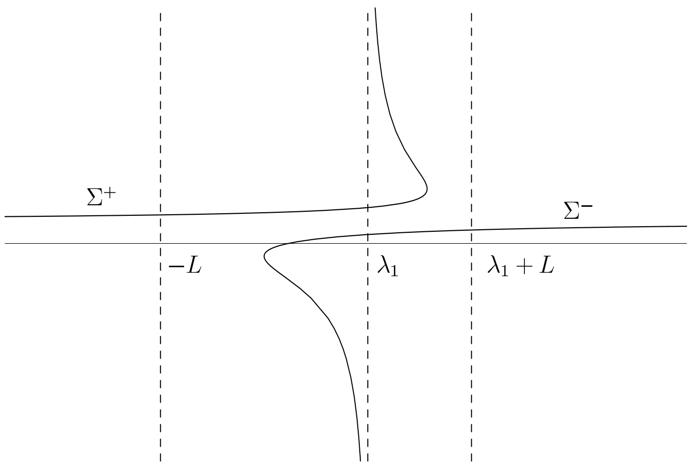

Figure 1. (H1), (H2), (G1), (G2+)

Next theorem refers to the global behavior of Σ^-.

THEOREM 3.3 (Figure 1). *Assume* (H1), (H2), (G1) *and* (G2+). *Then,*

$$]\lambda_1, \lambda_2[\subset \pi(\Sigma^-) \subset [-L, \infty[.$$

PROOF. It is known that if $\lambda < -L$ the function $s \mapsto \lambda s + g(x, s)$ is decreasing and then using the maximum principle one has the solution of problem (3.1) is unique. By previous theorem, there are solutions $(\lambda, u) \in \Sigma^+$ for any $\lambda < -L$. Therefore, $\pi(\Sigma^-) \subset [-L, \infty[$.

Using the Global Bifurcation Theorem of Rabinowitz, either $\pi(\Sigma^-)$ is unbounded (from above), or Σ^- meets another bifurcation point from infinity different from (λ_1, ∞). In both cases $]\lambda_1, \lambda_2[\subset \pi(\Sigma^-)$. $\square$

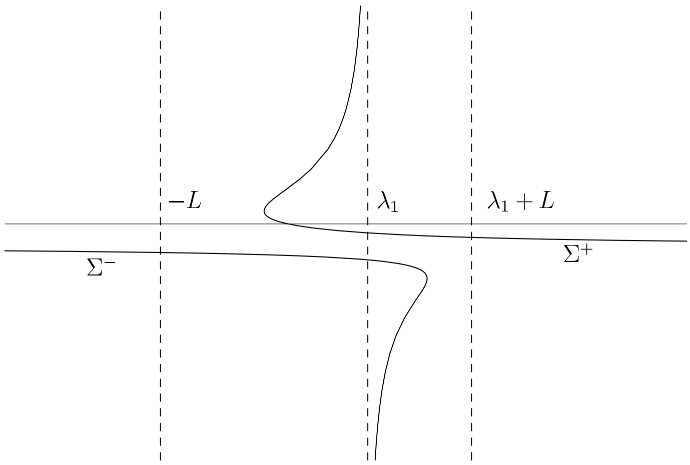

FIGURE 2. (H1), (H2), (G1), (G2−)

REMARKS 3.4. (a) Using the hypothesis (G2−) instead of (G2+) we obtain similar results changing Σ^+ by Σ^- (see Figure 2).

(b) Roughly speaking, under (H1), (H2), (G1) and (G2), globally, one of the two continua goes to the left and the other one goes to the right.

We have seen the global behavior of Σ^+ and Σ^-. Now, we can use those results to study the resonant case. The next result has been proved in [2]. We add this result in this paper because it is a direct consequence of Theorems 3.1–3.3 and Remarks 3.4. We also point out that our solutions of the resonant problem belong to the continua bifurcating from infinity.

Consider the problem

$$\begin{cases} -\Delta u = \lambda_1 u + g(x, u) & \text{if } x \in \Omega, \\ \text{B.C.} \end{cases} \tag{3.3}$$

COROLLARY 3.5. *Assume* (H1), (H2), (G1) *and* (G2+).

(a) *If the bifurcations from* $(\lambda_1, \pm\infty)$ *in* (3.1) *are both locally to the right or both locally to the left, then* (3.3) *admits at least one solution* u. *Furthermore,* $u \in P$ *if bifurcations are both locally to the right and* $u \notin P$ *if bifurcations are both locally to the left* (*see Figures* 3 *and* 4).

(b) *If the bifurcation from* (λ_1, ∞) *is locally to the right and the bifurcation from* $(\lambda_1, -\infty)$ *is locally to the left in* (3.1), *then* (3.3) *admits at least two solutions* u_1 *and* u_2 *with* $u_1 \in P$ *and* $u_2 \notin P$ (*see Figure* 5).

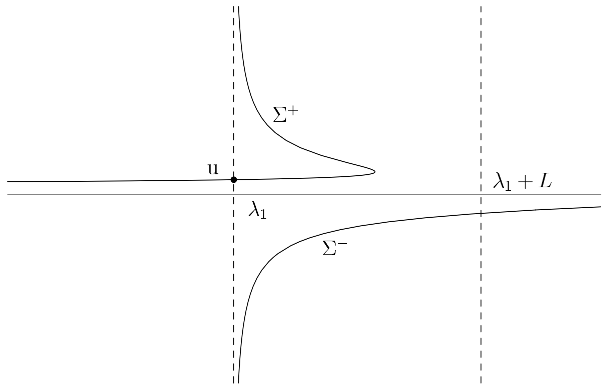

FIGURE 3. Solution $u \in P$

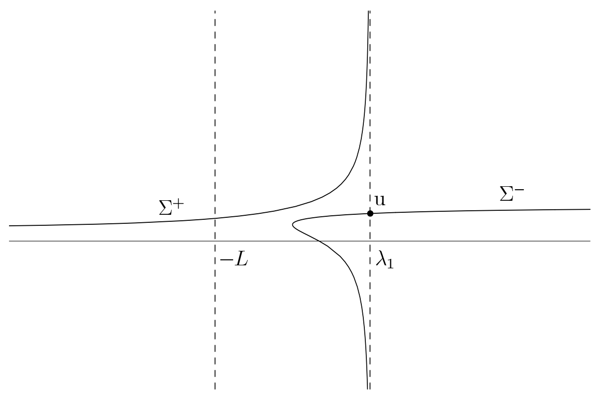

FIGURE 4. Solution $u \notin P$

REMARK 3.6. Note that the corollary hypotheses refer to the local behavior of the bifurcation from $(\lambda_1, \pm\infty)$ which either was studied in [2] with Dirichlet boundary conditions or in Section 2 of this paper with Neumann boundary conditions.

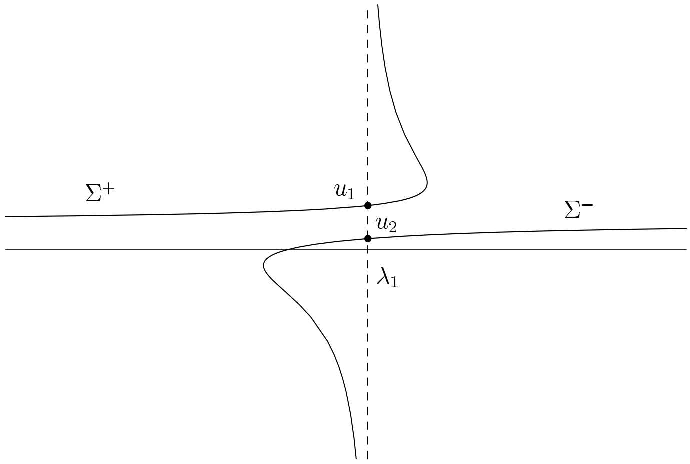

Figure 5. Solutions $u_1 \in P$, $u_2 \notin P$

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