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DIMENSION AND INFINITESIMAL GROUPS OF CANTOR MINIMAL SYSTEMS

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Abstract. The dimension and infinitesimal groups of a Cantor dynamical system (X,T) are inductive limits of sequences of homomorphisms defined by a proper Bratteli diagram of (X,T). A method of selecting sequences of homomorphisms determining the dimension and the infinitesimal groups of (X,T) based on non-proper Bratteli diagrams is described. The dimension and infinitesimal groups of Rudin–Shapiro, Morse and Chacon flows are computed.

1. Introduction

By a Cantor minimal system (C.m.s.) we mean a pair (X,T), where X is a Cantor set and T is a homeomorphism of X. Let $B_T = \{f - f \circ T^{-1}, f \in C(X,\mathbb{Z})\}$ be the coboundary subgroup of the group $C(X,\mathbb{Z})$ of continuous functions $f\colon X \to \mathbb{Z}$ with integer values. The dimension group $K^0(X,T)$ of a Cantor minimal system (X,T) is the quotient group $C(X,\mathbb{Z})/B_T$. Let $M(X,T)$ be the set of all T-invariant Borel probability measures on X. Let

$$N(X,T) = \left\{ f \in C(X,\mathbb{Z}) : \int_X f\,d\mu = 0 \text{ for every } \mu \in M(X,T) \right\}.$$

Of course, $B_T \subset N(X,T)$. The infinitesimal group $\operatorname{Inf}(X,T)$ of (X,T) is the quotient group $N(X,T)/B_T$. The dimension group $K^0(X,T)$ is an ordered group with positive cone $K^0(X,T)^+$ and a distinguished order unit [1],

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where $K^0(X,T)^+ = C(X,\mathbb{Z}^+)/B_T$ and [1] is the coset of the constant function equal to one. The dimension groups $K^0(X,T)$ and the infinitesimal groups $\mathrm{Inf}(X,T)$ play important roles in the orbital theory of Cantor minimal systems. It was proved in [5] that $K^0(X,T)$ as an ordered group characterizes the strong orbit equivalence class of (X,T). At the same time the ordered group $\widehat{K}^0(X,T) \simeq C(X,\mathbb{Z})/N(X,\mathbb{Z})$ characterizes the orbit equivalence class of (X,T). The dimension group $K^0(X,T)$ and the infinitesimal group $\mathrm{Inf}(X,T)$ are inductive limits arising from a sequence of Kakutani–Rokhlin partitions. A Kakutani–Rokhlin partition is a partition ξ of X into clopen sets of the form

$$\xi = \{T^k(D_{0,v}),\ 0 \le k \le h(v)-1,\ v \in V\},$$

where V is a finite set. In other words, X is partitioned into $|V|$ ($|V|$ is the cardinality of a set V) disjoint clopen T-towers ξ_v, $v \in V$, $\xi_v = \{T^k(D_{0,v}),\ 0 \le k \le h(v)-1\}$.

The set $B(\xi) = \bigcup_{v\in V} D_{0,v}$ is called the base of ξ. Let $C(\xi) \subset C(X,\mathbb{Z})$ be the set of functions which are constant on each set $T^k(D_{0,v})$, $0 \le k \le h(v)-1$, $v \in V$. Of course $C(\xi)$ is a subgroup of $C(X,\mathbb{Z})$. Let $B_T(\xi)$ denote the subgroup of $C(\xi)$ consisting of all coboundary functions $f - f\circ T^{-1}$, $f \in C(\xi)$.

Let (X,T) be a Cantor minimal system and let $\xi^{(n)}, n \in \mathbb{N}$ be a sequence of Kakutani–Rokhlin partitions, $\xi^{(n)} = \{T^k(D^{(n)}_{0,v}),\ 0 \le k \le h(n,v)-1,\ v \in V_n\}$ satisfying the following conditions:

$$\xi^{(n+1)} \succ \xi^{(n)},\ \text{i.e. } \xi^{(n+1)} \text{ refines } \xi^n \text{ and } B(\xi^{(n+1)}) \subset B(\xi^{(n)}), \tag{1.1}$$

$$\text{the partitions } \xi^{(n)} \text{ span the clopen topology of } X. \tag{1.2}$$

We have the inclusions: $C(\xi^{(n)}) \subset C(\xi^{(n+1)})$ and $B_T(\xi^{(n)}) \subset B_T(\xi^{(n+1)})$. These inclusions determine the natural homomorphisms

$$F_n\colon C(\xi^{(n)})/B_T(\xi^{(n)}) \to C(\xi^{(n+1)})/B_T(\xi^{(n+1)}).$$

The dimension group of (X,T) is the inductive limit of the homomorphisms F_n i.e.

$$\begin{aligned} K^0(X,T) &= C(X,\mathbb{Z})/B_T \\ &= \varinjlim\{F_n\colon C(\xi^{(n)})/B_T(\xi^{(n)}) \to C(\xi^{(n+1)})/B_T(\xi^{(n+1)})\}. \end{aligned} \tag{1.3}$$

If a sequence $\xi^{(n)}$ satisfies an additional condition

$$\bigcap_{n=1}^{\infty}\Big(\bigcup_{v\in V_n} D_{0,v}\Big) \quad \text{is a single point of } X \tag{1.4}$$

then

$$K^0(X,T) = \varinjlim\{F_n\colon \mathbb{Z}^{V_n} \to \mathbb{Z}^{V_{n+1}}\}. \tag{1.5}$$

In this case, the T-towers determine a proper Bratteli diagram of (X,T). However, there exist many examples of Cantor minimal systems with natural sequences of Kakutani–Rokhlin partitions not satisfying (1.4) and generating the topologies. In those cases the homomorphisms F_n are defined also by the T-towers $\xi^{(n)}$, however (1.5) is not valid. To get (1.5), we must replace the group $\mathbb{Z}^{V_n}$ by quotient groups $\mathbb{Z}^{V_n}/Z_c^{(n)}$. In this paper we propose a method of describing groups $\mathbb{Z}^{V_n}/Z_c^{(n)}$ and changing a sequence of the homomorphisms $\{F_n\}$ to get (1.5).

The group $C(\xi^{(n)})/B_T(\xi^{(n)})$ can be identified with the group $\mathbb{Z}^{V_n}/Z_c^{(n)}$. To define a subgroup $Z_c^{(n)} \subset \mathbb{Z}^{V_n}$, let us consider a subgroup $H_n \subset C(\xi^{(n)})$ consisting of those functions $f \in C(\xi^{(n)})$ which have the null sum over each tower $\xi_v^{(n)}$, $v \in V_n$. Given $f \in C(\xi^{(n)})$ we can associate a vector $\overline{x}_f = \langle x_v \rangle \in \mathbb{Z}^{V_n}$ as follows:

$$\overline{x}_f = \langle x_v \rangle, \quad v \in V_n, \quad \text{where } x_v = \sum_{k=0}^{h-1} f(T^k(D_{0,v})), h = h(n,v). \tag{1.6}$$

The map $f \to \overline{x}_f$ is a homomorphism from $C(\xi^{(n)})$ onto $\mathbb{Z}^{V_n}$ and its kernel is the group H_n. Therefore,

$$C(\xi^{(n)})/H_n \simeq \mathbb{Z}^{V_n}. \tag{1.7}$$

It is easy to check that every function $f \in H_n$ is a coboundary. Then we have $H_n \subset B_T(\xi^{(n)}) \subset C(\xi^{(n)})$, which implies

$$C(\xi^{(n)})/B_T(\xi^{(n)}) \simeq C(\xi^{(n)})/H_n : B_T(\xi^{(n)})/H_n. \tag{1.8}$$

By (1.7) we can identify $C(\xi^{(n)})/H_n$ with $\mathbb{Z}^{V_n}$ and $B_T(\xi^{(n)})/H_n$ with a subgroup $Z_c^{(n)}$ of $\mathbb{Z}^{V_n}$. Hence, (1.8) and (1.3) can be written in the form

$$C(\xi^{(n)})/B_T(\xi^{(n)}) = \mathbb{Z}^{V_n}/Z_c^{(n)} \tag{1.9}$$

and

$$K^0(X,T) = \varinjlim\{G_n \colon \mathbb{Z}^{V_n}/Z_c^{(n)} \to \mathbb{Z}^{V_{n+1}}/Z_c^{(n+1)}\} \tag{1.10}$$

where G_n are homomorphisms determined by F_n.

In a similar way we can represent the infinitesimal group $\mathrm{Inf}(X,T)$ as an inductive limit. For every $\mu \in M(X,T)$, let us denote by μ_v the measure of the tower ξ_v i.e. $\mu_v = \mu(\bigcup_{i=1}^{h-1} T^i D_{0,v})$, $h = h(n,v)$ and let $\mathcal{N}_n = \{\langle x_v \rangle \in \mathbb{Z}^{V_n} : \sum_{v \in V_n} x_v \mu_v = 0 \text{ for every } \mu \in M(X,T)\}$. Then

$$\mathrm{Inf}(X,T) = \varinjlim\{G_n : \mathcal{N}_n/Z_c^{(n)} \to \mathcal{N}_{n+1}/Z_c^{(n+1)}\}. \tag{1.11}$$

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2. Graphs $(\alpha_n, \widetilde{V}_n)$ determined by the Kakuthani–Rokhlin partitions

To describe the subgroup $Z_c^{(n)}$ of $\mathbb{Z}^{V_n}$, we define a partition α_n of the set V_n (the set of T-towers $\xi_v^{(n)}$).

We say that $v \overset{*}{\sim} v'$ if there exists $x_1 \in D_{0,v}$, $x_2 \in D_{0,v'}$ such that $T^{h(n,v)}(x_1)$ and $T^{h(n,v')}(x_2)$ belong to the same $D_{0,v''}^{(n)}$ for some $v'' \in V_n$. We say that v, v' are α_n-equivalent ($v \overset{\alpha_n}{\sim} v'$) if there exists a finite sequence $v, v_1, \ldots, v_s, v'$ such that $v \overset{*}{\sim} v_1$, $v_1 \overset{*}{\sim} v_2, \ldots, v_{s-1} \overset{*}{\sim} v_s$ and $v_s \overset{*}{\sim} v'$. Notice that α_n is an equivalence relation so it partitions the set V_n into disjoint subsets. Let us denote this partition by α_n as well. Now we define an oriented graph Γ_n whose vertices are the elements $J \in \alpha_n$ and the arrows are elements of V_n. Every $v \in V_n$ determines an arrow $\widetilde{v}$ going from a unique I to a unique J in such a way that $v \in I$ and $T^{h(n,v)}(D_{0,v}^{(n)}) \in \bigcup_{v' \in J} D_{0,v'}^{(n)}$. We will write $\widetilde{v} = (I, J)$. By $\widetilde{V}_n$ we denote the set of all arrows $\widetilde{v}$, $v \in V_n$.

In this manner, we have defined an oriented graph $(\alpha_n, \widetilde{V}_n)$. It is easy to remark that the minimality of T implies that $(\alpha_n, \widetilde{V}_n)$ is a connected graph. Every element $\overline{x} = \langle x_v \rangle_{v \in V_n} \in \mathbb{Z}^{V_n}$ is an integer valued function defined on $\widetilde{V}_n$. By a cycle Γ of $(\alpha_n, \widetilde{V}_n)$ we mean a closed path $\Gamma = \{\widetilde{v}_1, \ldots, \widetilde{v}_s, \widetilde{v}_1\}$ of $(\alpha_n, \widetilde{V}_n)$ without loops. The following characterization of the subgroup $Z_c^{(n)} \subset \mathbb{Z}^{V_n}$ follows from [1] (see Theorems 2.6 and 4.8).

Theorem 2.1. *An element $\overline{x} = \langle x_v \rangle_{v \in V_n} \in \mathbb{Z}^{V_n}$ belongs to $Z_c^{(n)}$ if and only if*

$$\sum_{v \in \Gamma} x_v = 0 \quad \textit{for every cycle } \Gamma \textit{ of } (\alpha_n, \widetilde{V}_n). \tag{2.1}$$

To give a better description of the subgroup $Z_c^{(n)}$ of $\mathbb{Z}^{V_n}$, $n \in \mathbb{N}$ consider linear space $\mathbb{Q}^{V_n}$, $Q_c^{(n)}$ and $Q_{cc}^{(n)}$ over the field $\mathbb{Q}$ ($\mathbb{Q}$ is the set of all rational numbers), where

$$Q_c^{(n)} = \Big\{ \overline{x} = \langle x_v \rangle_{v \in V_n} \in \mathbb{Q}^{V_n} \text{ such that } \sum_{v \in \Gamma} x_v = 0 \text{ for every cycle } \Gamma \text{ of } (\alpha_n, \widetilde{V_n}) \Big\}, \tag{2.2}$$

and

$$Q_{cc}^{(n)} = (Q_c^{(n)})^{\perp}. \tag{2.3}$$

Of course $\mathbb{Z}^{V_n} \subset \mathbb{Q}^{V_n}$, $Z_c^{(n)} \subset Q_c^{(n)}$.

We will define vectors $\overline{x}_J \in \mathbb{Z}^{V_n}$ for each $J \in \alpha_n$. To do this, let us denote by J^+ the set of all arrows arriving to J and by J^- the set of all arrows leaving J.

Let us define $\overline{x}_J = \langle x_v \rangle$ as follows:

$$x_v = \begin{cases} 1 & \text{for } \widetilde{v} \in J^+, \\ -1 & \text{for } \widetilde{v} \in J^-, \\ 0 & \text{for the remaining } v. \end{cases} \tag{2.4}$$

Theorem 2.2.

(a) *The vectors* $\overline{x_J}$*,* $J \in \alpha_n$ *generate the space* $Q_c^{(n)}$*.*

(b) $Q_{cc}^{(n)} = \{\overline{x} = \langle x_v \rangle_{v \in V_n} \in \mathbb{Q}^{V_n}$ *such that* $\sum_{v \in J^+} x_v = \sum_{v \in J^-} x_v$ *for every* $J \in \alpha_n\}$*.*

(c) $\dim Q_c^{(n)} = |\alpha_n| - 1$, $\dim Q_{cc}^{(n)} = |V_n| - |\alpha_n| + 1$.

Proof. It is evident that $\overline{x}_J \in Q_c^{(n)}$. In fact every cycle Γ either does not pass by J (then all coordinates x_v, $v \in \Gamma$ of $\overline{x}_J$ are zero) or Γ pass through J and then $x_v = 1$, $x_{v'} = -1$ for some $v, v' \in V_n$ and $x_v = 0$ for the remaining v. Thus $\overline{x}_J$ satisfies the condition (2.1) i.e. $\overline{x}_J \in Q_c^{(n)}$.

Let L be a subspace of $Q_c^{(n)}$ generated by the vectors $\overline{x}_J$. We have proved that $L \subset Q_c^{(n)}$. This implies

$$L^\perp \supset (Q_c^{(n)})^\perp = Q_{cc}^{(n)}.$$

We will prove the equality

$$L^\perp = Q_{cc}^{(n)}. \tag{2.5}$$

Let us take $\overline{x} = \langle x_v \rangle \in L^\perp$. Then $\overline{x} \perp \overline{x}_J$ for every $J \in \alpha_n$. The last condition is equivalent to the following equalities

$$\sum_{v \in J^+} x_v = \sum_{v \in J^-} x_v \quad \text{for every } J \in \alpha_n. \tag{2.6}$$

Define vectors $\overline{w}_\Gamma = \langle w_v \rangle$, $v \in V_n$ where Γ is a cycle of $(\alpha_n, \widetilde{V_n})$ by

$$w_v = \begin{cases} 1 & \text{for } v \in \Gamma, \\ 0 & \text{for } v \notin \Gamma. \end{cases}$$

It follows from (2.2) that the space $Q_{cc}^{(n)}$ is generated by the vectors $\overline{w}_\Gamma$.

It is easy to see that $\overline{w}_\Gamma$ satisfies condition (2.6) for every $J \in \alpha_n$. To prove (2.5), it is enough to show that every vector $\overline{x}$ satisfying (2.6) is a linear combination of the vectors $\overline{w}_\Gamma$.

First, let us assume that $x_v \geq 0$ for every $v \in V_n$ and $\overline{x} \neq 0$. Choose v with $x_v > 0$. It is not hard to see that there exists a cycle Γ containing v. It follows from (2.6) that $x_{v'} > 0$ for every $v' \in \Gamma$. Let us put $\varepsilon = \min_{v' \in \Gamma} x_{v'}$ and let $\overline{y} = \overline{x} - \varepsilon \cdot \overline{w}_\Gamma$. Then, $\overline{y}$ satisfies (2.6); moreover, $y_v \geq 0$. If $y_v > 0$ for some v

then we repeat the above procedure. It is not hard to see that this procedure is finite which implies that the vector $\overline{x}$ is of the form

$$\overline{x} = \sum a_\Gamma \overline{w}_\Gamma, \quad \Gamma \text{ runs over all cycles of } (\alpha_n, \widetilde{V_n}). \tag{2.7}$$

Now, let us assume that there exists v such that $x_v < 0$. Choose v in such a way that $x_v = \min_{x_{v'}<0} x_{v'}$. Let us again choose a cycle Γ containing v. The vector $\overline{y} = \overline{x} - x_v \cdot \overline{w}_\Gamma$ satisfies the condition (2.6) and $\min_{v' \in V_n} y_{v'} > \min_{v' \in V_n} x_{v'}$.

Repeating the above reasoning we find a vector $\overline{y} = \langle y_v \rangle$ with non-negative coordinates y_v such that $\overline{y} = \overline{x} - \sum_\Gamma b_\Gamma \cdot \overline{w}_\Gamma$. Using (2.7) for the vector $\overline{y}$ we see that every $\overline{x} \in \mathbb{Q}^{V_n}$ satisfying (2.6) is of the form (2.7). In this manner the equality (2.5) is proved which implies (a) and (b).

Now we are in a position to prove (c). Consider the matrix $A = (a(J,v))$, $J \in \alpha_n$, $v \in \widetilde{V_n}$ of the system of equations (2.6). Then $a(J,v)$ are equal to 1, -1, or 0. Moreover, each column of A corresponding to $\widetilde{v} = (I,J)$ either consists of all zeros if $I = J$ or contains a unique 1, a unique -1 and the remaining entries are equal to zero if $I \neq J$. Using the same arguments as in the transportation problem it is easy to check that $\operatorname{rank}(A) = |\alpha_n| - 1$. This implies (c), so the theorem is proved. □

3. The groups $\mathbb{Z}^{V_n}/Z_c^{(n)}$ and the dimension group $K^0(X,T)$

In this part we describe the sequence of groups $\mathbb{Z}^{V_n}$ and of homomorphisms G_n from (1.10). The group $\mathbb{Z}^{V_n}/Z_c^{(n)}$ is isomorphic to the image of $\mathbb{Z}^{V_n}$ in $Q_{cc}^{(n)}$ by the natural projection Π_n of $\mathbb{Q}^{V_n}$ onto $Q_{cc}^{(n)}$. To find this image, we need a basis of the space $Q_{cc}^{(n)}$. To do this we use spanning trees of the non-oriented graph $(\alpha_n, \widetilde{\widetilde{V}}_n)$. (We treat every arrow $\widetilde{v} \in V_n$ as a non-oriented edge $\widetilde{\widetilde{v}}$.) By a path in $(\alpha_n, \widetilde{\widetilde{V}}_n)$ we mean a sequence of edges $P = (\widetilde{\widetilde{v}}_1, \ldots, \widetilde{\widetilde{v}}_s)$ such that $\widetilde{\widetilde{v}}_i$ and $\widetilde{\widetilde{v}}_{i+1}$ have a common vertex, $i = 0, \ldots, s-1$. A path P is closed if $\widetilde{\widetilde{v}}_1 = \widetilde{\widetilde{v}}_s$. Using the similar arguments as in the transportation problem, it is easy to prove the following characterization for a subset $E \subset V_n$ of the columns of A to be linearly independent.

Theorem 3.1. *A subset $E = E_n \subset V_n$ of the columns of A is linearly independent if and only if E does not contain any closed path of $(\alpha_n, \widetilde{\widetilde{V}}_n)$.*

A spanning tree is a sub-graph (α_n, E) of $(\alpha_n, \widetilde{\widetilde{V}}_n)$ such that $|E| = |\alpha_n| - 1$, E does not contain any closed path and for every $v \notin E$ there exists a unique path P_v without loops and such that $\widetilde{\widetilde{v}} \in P_v$ and the remaining edges of P_v belong to E. The paths P_v, $\widetilde{\widetilde{v}} \in \widetilde{\widetilde{V}}_n \setminus E$ allow to define vectors $\overline{u}_v = \langle u_{v'} \rangle \in \mathbb{Q}^{V_n}$. For this reason, we partition P_v into subsets P_v^+ and P_v^-. Every edge $\widetilde{\widetilde{v}} \in P_v$ is simultaneously an arrow of the graph $(\alpha_n, \widetilde{V_n})$. We define P_v^+ as the set of all

$\widetilde{\widetilde{v'}} \in P_v$ such that $\widetilde{v'}$ has the same orientation as $\widetilde{v}$ and P_v^- are those which has the inverse orientation than $\widetilde{v}$. Now define $\overline{x}(v) = \langle x_{v'} \rangle$, $v' \in V_n$ as follows

$$x_{v'} = \begin{cases} 1 & \text{for } \widetilde{\widetilde{v'}} \in P_v^+, \\ -1 & \text{for } \widetilde{\widetilde{v'}} \in P_v^-, \\ 0 & \text{for the remaining } v' \in V_n \setminus E. \end{cases} \tag{3.1}$$

Theorem 3.2. *If (α_n, E) is a spanning tree of $(\alpha_n, \widetilde{\widetilde{V_n}})$, then the vectors $\overline{x}(v)$, $v \in V_n \setminus E$ form a basis of the space $Q_{cc}^{(n)}$.*

Proof. The subspace $Q_{cc}^{(n)}$ is determined by the system of linear equations (2.6). It follows from the Theorems 2.2 and 3.1 that the columns of the matrix A corresponding to $\widetilde{\widetilde{v}} \in E$ form a minor having the order equal to $|\alpha_n| - 1$. For every $v \notin E$ there exists a unique solution $\overline{y}$ of (2.6) such that $y_v = 1$ and $y_{v'} = 0$ if $v' \neq v$ and $v' \notin E$. The family of such solutions forms a basis of $Q_{cc}^{(n)}$ if v runs all over $V_n \setminus E$. It follows from (3.1) that $\overline{y} = \overline{x}(v)$, for $v \in V_n \setminus E$. □

Let $\overline{x} \bullet \overline{y} = \sum_{v \in V_n} x_v \cdot y_v$ be the inner product of vectors $\overline{x}, \overline{y} \in \mathbb{Q}^{V_n}$. The basis $\{\overline{x}(v)\}$, $v \in V_n \setminus E$ allows us to define a homomorphism $I_n \colon \mathbb{Q}^{V_n} \to \mathbb{Q}^{V_n \setminus E}$, $E := E_n$.

For $\overline{x} = \langle x_{v'} \rangle \in \mathbb{Q}^{V_n}$ define $I_n(\overline{x}) \in \mathbb{Q}^{V_n \setminus E}$ as follows

$$I_n(\overline{x}) = \langle \overline{x} \bullet \overline{x}_v \rangle, \quad v \in V_n \setminus E. \tag{3.2}$$

Theorem 3.3.

(a) $\ker(I_n) = \mathbb{Q}_c^{(n)}$,

(b) $\operatorname{Im}(I_n) = \mathbb{Q}^{V_n \setminus E}$,

(c) $Q_{cc}^{(n)} \simeq \mathbb{Q}^{V_n} / Q_c^{(n)} \overset{\widehat{I}_n}{\simeq} \mathbb{Q}^{V_n \setminus E}$ *where $\widehat{I}_n$ is the induced isomorphism.*

Proof. (a) We have $\ker(I_n) = \{\overline{x} \in \mathbb{Q}^{V_n} : \overline{x} \bullet \overline{x}_v = 0\}$ for every $v \in V_n \setminus E\} = Q_c^{(n)}$, because the vectors $\overline{x_v}$, form a basis of $(Q_c^{(n)})^{\perp}$ (see Theorem 3.2).

(b) Let $e_{v'}$, $v' \in V_n$ be the standard basis of $\mathbb{Q}^{V_n}$ (i.e. $e_{v'}$ is the vector with the coordinate 1 on the position v' and 0 for the remaining positions) and let $\widehat{e}_v$, $v \in V_n \setminus E$ be the standard basis of $\mathbb{Q}^{V_n \setminus E}$. It follows from (3.2) that $I_n(e_v) = \widehat{e}_v$ whenever $v \in V_n \setminus E$. This implies (b).

The property (c) is a consequence of (a) and (b). □

Corollary 3.4.

(a') $\ker(I_n | \mathbb{Z}^{V_n}) = \mathbb{Z}_c^{(n)}$,

(b') $\operatorname{Im}(\mathbb{Z}^{V_n}) = \mathbb{Z}^{V_n \setminus E}$,

(c') $\mathbb{Z}^{V_n} / Z_c^{(n)} \simeq \mathbb{Z}^{V_n \setminus E}$.

Proof. If $\overline{x} \in \mathbb{Z}^{V_n}$, then $I_n(\overline{x}) \in \mathbb{Z}^{V_n \setminus E} \subset \mathbb{Q}^{V_n \setminus E}$ because the vectors $\overline{x}_v$, $v \in V_n \setminus E$, have integer coordinates. Further $\ker(I_n|\mathbb{Z}^{V_n}) = \ker(I_n) \cap \mathbb{Z}^{V_n} = Q_c^{(n)} \cap \mathbb{Z}^{V_n} = Z_c^{(n)}$ Thus (a') and (b') are valid. The property (c') is an evident consequence of (a') and (b'). □

The above considerations lead to *a tower algorithm* (TA) of computing the dimension and the infinitesimal groups of C.m.s. (X,T). To formulate it, we reconstruct the sequence (1.10).

Using the isomorphisms $\widehat{I}_n$, $n = 1,2,\ldots$ we can replace $\mathbb{Z}^{V_n}/Z_c^{(n)}$ in (1.10) by $\mathbb{Z}^{V_n \setminus E}$ and then

$$K^0(X,T) = \lim\{\widehat{G}_n\colon \mathbb{Z}^{V_n \setminus E_n} \to \mathbb{Z}^{V_{n+1} \setminus E_{n+1}}\},$$

where $\widehat{G}_n$ are homomorphisms determined by G_n (or F_n).

Now, we describe the homomorphisms G_n. First, we recall the definitions of the F_n's. We have homomorphisms $F_n\colon \mathbb{Z}^{V_n} \to \mathbb{Z}^{V_{n+1}}$ determined by natural homomorphisms $C(\xi^{(n)})/H_n \to C(\xi^{(n+1)})/H_{n+1}$. Let us remind that the elements $v \in V_n$ correspond to the T-towers $\xi_v^{(n)} = \{D_{0,v}^{(n)}, \ldots, T^{h-1}(D_{0,v}^{(n)})\}$ $h = h(n,v)$. Every T-towers $\xi_w^{(n+1)}$, $w \in V_{n+1}$ consists of some sub T-towers $\eta_v^{(n)}$ of $\xi_v^{(n)}$. Thus, we can write $\xi_w^{(n+1)} = \bigcup_{v \in S_w} \eta_v^{(n)}$, where $S_w = (v_1, \ldots, v_t)$ is a unique sequence of v. Define a matrix $\{b_{w,v}\} = B$, $w \in V_{n+1}$, $v \in V_n$, where $b_{wv} = \#\{1 \le i \le t : v = v_i\}$. Using the map $f \to \overline{x}_f$ defined by (1.6) it is easy to conclude that $F_n(\overline{x}) = B \cdot \overline{x}$.

We have the following commuting diagrams

$$\begin{array}{ccc}
\mathbb{Z}^{V_n} & \xrightarrow{F_n} & \mathbb{Z}^{V_{n+1}} \\
\Pi_n \big\downarrow & & \big\downarrow \Pi_{n+1} \\
\mathbb{Z}^{V_n}/Z_c^{(n)} & \xrightarrow{G_n} & \mathbb{Z}^{V_{n+1}}/Z_c^{(n+1)} \\
\widehat{I_n} \big\downarrow & & \big\downarrow \widehat{I_{n+1}} \\
\mathbb{Z}^{V_n \setminus E_n} & \xrightarrow[\widehat{G_n}]{} & \mathbb{Z}^{V_{n+1} \setminus E_{n+1}}
\end{array} \tag{3.3}$$

where Π_n, Π_{n+1} are the natural homomorphisms and $\widehat{G}_n = \widehat{I}_{n+1} \circ G_n \circ (\widehat{I}_n)^{-1}$, $n = 1,2,\ldots$ Now we describe $\widehat{G}_n$. To do this we find spanning trees (α_n, E_n) of $(\alpha_n, \widetilde{\widetilde{V}}_n)$ and (α_{n+1}, E_{n+1}) of $(\alpha_{n+1}, \widetilde{\widetilde{V}}_{n+1})$. Now for every n we define a matrix $\widehat{B} = \widehat{B}_n$ as follows:

(3.4) Take $v \in V_n \setminus E_n$. The v-th column of the matrix B_n is a vector

$$\overline{b}(v) = \langle b_{w,v} \rangle \in \mathbb{Z}^{V_{n+1}}, \quad w \in V_{n+1}.$$

We compute vectors $I_{n+1}(\overline{b}(v)) = \langle \widehat{b}_{w,v} \rangle$, $w \in V_{n+1} \setminus E_{n+1}$, $v \in V_n \setminus E_n$, using (3.2) and then $\widehat{B} = \{\widehat{b}_{w,v}\}$, $w \in V_{n+1} \setminus E_{n+1}$, $v \in V_n \setminus E_n$.

Theorem 3.5. *For every* $\overline{z} = \langle z_v \rangle \in \mathbb{Z}^{V_n \setminus E_n}$, $v \in V_n \setminus E_n$, *it holds*

$$\widehat{G}_n(\overline{z}) = \widehat{B} \cdot \overline{z}. \tag{3.5}$$

Proof. It is enough to check (3.5) for $\overline{z} = \widehat{e}_v$, $v \in V_n \setminus E_n$. From the equalities $I_n(e_v) = \widehat{e}_v$ and $F_n(e_v) = \overline{b}(v)$, $v \in V_n \setminus E_n$ and from the commute of (3.3) it follows that

$$\widehat{G}_n(\widehat{e}_v) = (I_{n+1} \circ F_n)(e_v) = I_{n+1}(\overline{b}(v)) = \langle \widehat{b}_{wv} \rangle,$$

$w \in V_{n+1} \setminus E_{n+1}$. This gives (3.5) for every $\overline{z} \in \mathbb{Z}^{V_n \setminus E_n}$. □

Using the above theorem, (3.3) and (1.10) we get

Corollary 3.6.

$$K^0(X,T) = \varinjlim \{\widehat{G}_n : \mathbb{Z}^{V_n \setminus E_n} \to \mathbb{Z}^{V_{n+1} \setminus E_{n+1}}\}. \tag{3.6}$$

To describe the cone $K^0(X,T)^+ = C(X, \mathbb{Z}^+)/B_T$, we must regard all vectors $I_n(e_v) \in \mathbb{Z}^{V_n \setminus E_n}$ when v runs over all V_n. Let us denote $Z_+^{(n)} = \{\sum_{v \in V_n} a_v \cdot I_n(e_v)$ with $a_v \in \mathbb{Z}^+\}$. Then, we have $\widehat{G}_n(Z_+^{(n)}) \subset Z_+^{(n+1)}$. There are natural homomorphisms $\widehat{\widehat{G}}_n : \mathbb{Z}^{V_n \setminus E_n} \to K^0(X,T)$. Then, we can identify the cone $K^0(X,T)^+$ with the set $\bigcup_{n=1}^{\infty} \widehat{\widehat{G}}_n(Z_+^{(n)})$. In a similar way we can replace the sequence (1.10) by a sequence of $\widehat{G}_n$ on the subspace $\mathrm{Inf}(n) = \widehat{I}_n(\mathcal{N}_n / Z_c^{(n)}) \subset \mathbb{Z}^{V_n \setminus E_n}$. Thus $\mathrm{Inf}(X,T) = \varinjlim \{\widehat{G}_n : \mathrm{Inf}(n) \to \mathrm{Inf}(n+1)\}$.

Before formulating an algorithm (TA) we illustrate previous consideration using the Chacon flow.

3.1. Chacon flow. We remind briefly the definition of Chacon flow. For this, we start with the sequence $\{B_n\}$ of blocks over two symbols $(0, s)$:

$$B_0 = 0, \quad B_{n+1} = B_n B_n s B_n, \quad n > 0.$$

Then

$$|B_n| = \frac{1}{2}(3^{n+1} - 1) = r_n.$$

Let ω be a one-sided sequence defined by the blocks B_n as follows: $\omega[0, r_n - 1] = B_n$, $n > 0$. We take the subset $Y \subset \{0, s\}^{\mathbb{Z}}$ which is the closure of T-orbit of ω with respect to the left shift. The Cantor minimal system (Y, T) is called the Chacon flow. For $n > 0$, we denote

$$\begin{aligned}
D_{00}^{(n)} &= \{x \in Y : x[-r_n, 2r_n - 1] = B_n B_n B_n\}, \\
D_{s0}^{(n)} &= \{x \in Y : x[-r_n - 1, 2r_n - 1] = B_n s B_n B_n\}, \\
D_{0s}^{(n)} &= \{x \in Y : x[-r_n, 2r_n] = B_n B_n s B_n\}, \\
D_{ss}^{(n)} &= \{x \in Y : x[-r_n - 1, 2r_n] = B_n s B_n s B_n\}.
\end{aligned}$$

Let $D_n = \bigcup_{p,q=0,s} D_{pq}^{(n)}$. Let us take Kakutani–Rokhlin partition $\xi^{(n)}$ built by the base D_n and the return time function. Then $\xi^{(n)}$ has four towers $\xi_{pq}^{(n)}$, $p,q = 0,s$ corresponding to the sets $D_{pq}^{(n)}$. We have $h(00,n) = h(s0,n) = r_n$ and $h(0s,n) = h(ss,n) = r_n + 1$ (see Figure 3.1).

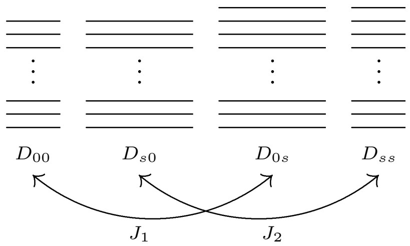

Figure 3.1

It is known that (Y,T) is strictly ergodic and the values of the unique T-invariant measure μ on the sets $D_{pq}^{(n)}$, $n > 0$ are $\mu(D_{00}^{(n)}) = \mu(D_{ss}^{(n)}) = 1/3^{n+2}$, $\mu(D_{0s}^{(n)}) = \mu(D_{s0}^{(n)}) = 2/3^{n+2}$. Taking $(\xi^{(n)})$ as above, one can point out partitions α_n introduced in Section 2 (they do not depend on n). We get $V_n = \{00, s0, 0s, ss\}$ and $J_1 = \{00, 0s\}$ $J_2 = \{s0, ss\}$. Then $\alpha_n = \{J_1, J_2\}$. The graph $(\alpha_n, \widetilde{V}_n)$ determined by the T-towers $D_{pq}^{(n)}$ is presented in Figure 3.2.

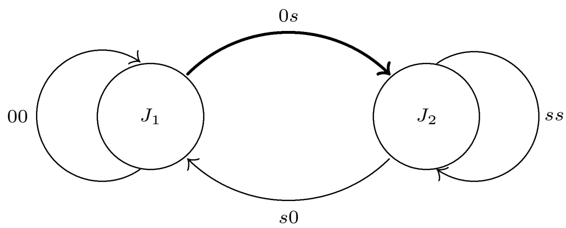

Figure 3.2

Let us notice that there are three cycles of $(\alpha_n \widetilde{V}_n)$ namely: $\Gamma_1 = \{00\}$, $\Gamma_2 = \{ss\}$ and $\Gamma_3 = \{0s, s0\}$. As a spanning tree of the graph $(\alpha_n, \widetilde{V}_n)$ we can take $(\alpha_n, \{0s\})$ (see Figure 3.2), i.e. $E_n = \{0s\}$. Any vector $\overline{x} \in \mathbb{Z}^{V_n}$ we will write as $\overline{x} = \langle x_{00}, x_{s0}, x_{0s}, x_{ss} \rangle$.

Now for each path P_v, $v \in V_n \setminus E_n = \{00, s0, ss\}$ we define vectors $\overline{x}(v)$ as in the Section 3. We have:

$$\overline{x}(00) = \langle 1,0,0,0 \rangle, \quad \overline{x}(s0) = \langle 0,1,1,0 \rangle, \quad \overline{x}(ss) = \langle 0,0,0,1 \rangle.$$

Then the homomorphisms $I_n \colon \mathbb{Q}^4 \to \mathbb{Q}^3$ have the form

$$I_n(\overline{x}) = \langle x_{00}, x_{s0} + x_{0s}, x_{ss} \rangle = \langle y_{00}, y_{s0}, y_{ss} \rangle \in \mathbb{Q}^3$$

for $\overline{x} = \langle x_{00}, x_{s0}, x_{0s}, x_{ss} \rangle \in \mathbb{Q}^4$.

To find matrices B_n, we must express the T-towers $\xi_{pq}^{(n+1)}$ by some sub-T-towers $\eta^{(n)}$ of $\xi^{(n)}$. We have:

$$\begin{aligned} \xi_{00}^{(n+1)} &= \eta_{00}^{(n)} \cup \eta_{0s}^{(n)} \cup \eta_{s0}^{(n)}, & \xi_{s0}^{(n+1)} &= \eta_{s0}^{(n)} \cup \eta_{0s}^{(n)} \cup \eta_{s0}^{(n)}, \\ \xi_{0s}^{(n+1)} &= \eta_{00}^{(n)} \cup \eta_{0s}^{(n)} \cup \eta_{ss}^{(n)}, & \xi_{ss}^{(n+1)} &= \eta_{s0}^{(n)} \cup \eta_{0s}^{(n)} \cup \eta_{ss}^{(n)}. \end{aligned}$$

This leads to the following matrices B_n

$$B_n = \begin{array}{c} \\ 00 \\ s0 \\ 0s \\ ss \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 00 & s0 & 0s & ss \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

For each column $\overline{b}(v)$ of the matrix B_n, $v \in V_n \setminus E_n = \{00, s0, ss\}$ we calculate $\widehat{I}_{(n+1)}(\overline{b}(v))$. We have:

$$\overline{b}(00) = \langle 1,0,1,0 \rangle, \quad \overline{b}(s0) = \langle 1,2,0,1 \rangle, \quad \overline{b}(ss) = \langle 0,0,1,1 \rangle$$

and

$$\widehat{I}_{n+1}(\overline{b}(00)) = \langle 1,1,0 \rangle, \quad \widehat{I}_{n+1}(\overline{b}(s0)) = \langle 1,2,1 \rangle, \quad \widehat{I}_{n+1}(\overline{b}(ss)) = \langle 0,1,1 \rangle.$$

Thus,

$$\widehat{B}_n = \widehat{B} = \begin{array}{c} \\ 00 \\ s0 \\ ss \end{array} \!\!\begin{array}{c} \begin{array}{ccc} 00 & s0 & 0s \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

defines the map $\widehat{G}_n \colon \mathbb{Z}^3 \to \mathbb{Z}^3$ (see (3.4)). Then Corollary 3.6 results in $K^0(X,T) = \varinjlim\{\widehat{G}_n \colon \mathbb{Z}^3 \to \mathbb{Z}^3\}$. We have $\det \widehat{B} = 0$ so $\dim \widehat{G}_n(\mathbb{Q}^3) < 3$. To indicate $\varinjlim\{\widehat{G}_n \colon \mathbb{Z}^3 \to \mathbb{Z}^3\}$ we must find $\widehat{G}_n(\mathbb{Z}^3)$ and describe the action $\widehat{G}_n$ on it. It is easy to remark that $\widehat{G}_n(\mathbb{Z}^3) \subset \pi \subset \mathbb{Q}^3$, where $\pi = \{\overline{y} = \langle y_{00}, y_{s0}, y_{ss} \rangle \in \mathbb{Q}^3 : y_{00} + y_{ss} = y_{s0}\}$. Moreover, $\widehat{G}_n \colon \pi \to \pi$ is one-to-one. Then,

$$\varinjlim\{\widehat{G}_n \colon \mathbb{Z}^3 \to \mathbb{Z}^3\} = \varinjlim\{\widehat{G}_n \colon \pi \cap \mathbb{Z}^3 \to \pi \cap \mathbb{Z}^3\}.$$

It is convenient to replace the subgroup $\pi \cap \mathbb{Z}^3$ by $\mathbb{Z}^2 = \{\overline{z} = \langle z_{00}, z_{ss} \rangle, z_{00}, z_{ss} \in \mathbb{Z}\}$ by putting $z_{00} = y_{00}, z_{ss} = y_{ss}$, $\langle y_{00}, y_{00} + y_{ss}, y_{ss} \rangle \in \pi$. Let $H = H_n \colon \mathbb{Z}^2 \to \mathbb{Z}^2$ be the isomorphisms determined by $\widehat{G}_n$ on π. Then, H_n is defined by the matrix $C_n = C = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]$. We have the inclusions

$$\ldots \supset H^{-n}(\mathbb{Z}^2) \supset H^{-n+1}(\mathbb{Z}^2) \supset \ldots \supset H^{-1}(\mathbb{Z}^2) \supset \mathbb{Z}^2.$$

Thus we have

$$K^0(X,T) = \varinjlim\{H_n\colon \mathbb{Z}^2 \to \mathbb{Z}^2\} = \bigcup_{n=0}^{\infty} H^{-n}(\mathbb{Z}^2) \subset \mathbb{Q}^2.$$

Now we describe the group $\mathrm{Inf}(X,T)$. Consider the subgroups $\mathcal{N}_n \subset \mathbb{Z}^4$,

$$\begin{aligned}\mathcal{N}_n &= \left\{\overline{x}, \frac{1}{3^{n+2}}(x_{00}+x_{ss}) + \frac{2}{3^{n+2}}(x_{s0}+x_{0s}) = 0\right\} \\ &= \{\overline{x}, x_{00}+x_{ss}+2(x_{s0}+x_{0s}) = 0\}.\end{aligned}$$

We have $I_n(\mathcal{N}_n) = \{\langle y_{00}, y_{s0}, y_{ss}\rangle \subset \mathbb{Z}^3 : y_{00} + 2y_{s0} + y_{ss} = 0\} \in \mathbb{Z}^3$ and next $\widehat{G}_n(I_n(\mathcal{N}_n)) = \{\langle y_{00}, y_{s0}, y_{ss}\rangle \in \pi \cap \mathbb{Z}^3 : y_{00} + y_{ss} = 0\}$. Passing to the coordinates z_{00}, z_{ss} we can identify $\widehat{G}_n(I_n(\mathcal{N}_n))$ with the subgroup $Z_0 = \{\overline{z} = \langle z_{00}, z_{ss}\rangle \in \mathbb{Z}^2,\ z_{00}+z_{ss} = 0\}$. Further $H_n|Z_0 = \mathrm{id}$. So we can identify $\mathrm{Inf}(X,T)$ with $Z_0 \subset \bigcup_{n=0}^{\infty} H^{-n}(\mathbb{Z}^2)$.

To describe the cone $K^0(X,T)^+$, let us remark that

$$\begin{aligned} I_n(e_{00}) &= \widehat{e}_{00} \in \mathbb{Z}^3, & I_n(e_{s0}) &= \widehat{e}_{s0} \in \mathbb{Z}^3, \\ I_n(e_{0s}) &= \widehat{e}_{s0} \in \mathbb{Z}^3, & I_n(e_{ss}) &= \widehat{e}_{ss} \in \mathbb{Z}^3. \end{aligned}$$

Then $Z_+^{(n)} = \mathbb{Z}_+^3$, $\widehat{G}_n(\mathbb{Z}_+^3) = \mathbb{Z}_+^2 = \{\langle z_{00}, z_{ss}\rangle \in \mathbb{Z}^2 : z_{00}, z_{ss} \geq 0\}$. Thus,

$$K^0(X,T)^+ = \bigcup_{n=0}^{\infty} H^{-n}(\mathbb{Z}_+^2) = \{\langle z_{00}, z_{ss}\rangle \in K^0(X,T) : z_{00} + z_{ss} \geq 0\}.$$

Now we are in a position to formulate the algorithm (TA). We distinguish Stages I and II of it. The Stage I contains all calculations to get the sequence (3.6). The Stage II is a simplification of the sequence (3.6) if, for infinitely many n, $\dim(\mathrm{Im}(\widehat{G_n})) < |V_n \setminus E_n|$.

3.2. Algorithm (TA).

Stage I.

Step 1. For a given sequence of towers $\{\xi_n\}$ find homomorphisms

$$\mathbb{Z}^{V_1} \xrightarrow{F_1} \mathbb{Z}^{V_2} \xrightarrow{F_2} \mathbb{Z}^{V_3} \longrightarrow \cdots$$

Step 2. For every T-invariant measure μ calculate the measures μ_v of every T-tower $\xi_v^{(n)}$. Then define subspaces

$$\mathcal{N}_n = \left\{\langle x_v\rangle \in \mathbb{Z}^{V_n} : \sum_{v\in V_n} x_v\mu_v = 0 \text{ for every } \mu \in M(X,T)\right\}. \tag{3.7}$$

Step 3. Find the partitions α_n of the towers $\{\xi_n\}$. For $(\alpha_n, \tilde{\tilde{V}}_n)$ choose spanning trees (α_n, E_n) and calculate homomorphisms I_n using (3.2). Let us calculate sequence of homomorphisms $\widehat{G}_n$ (and matrices $\widehat{B}_n$) by (3.4) and (3.5). Then,

$$K^0(X,T) = \varinjlim\{\widehat{G}_n : \mathbb{Z}^{V_n \setminus E_n} \to \mathbb{Z}^{V_{n+1}\setminus E_{n+1}}\}.$$

If $\det(\widehat{B}_n) \neq 0$ for sufficiently large n then we can identify $(\simeq) K^0(X,T)$ with the subgroup $\bigcup_{n=1}^{\infty} \widehat{G}_1^{-1} \circ \ldots \circ \widehat{G}_n^{-1}(\mathbb{Z}^{V_{n+1}\setminus E_{n+1}})$ of the additive group $\mathbb{Q}^{\max|V_n \setminus E_n|}$.

Step 4. To find the cone $K^0(X,T)^+$ consider subsets

$$Z_+^{(n)} = \left\{ \sum_{v \in V_n} a_v \cdot I_n(e_v) \text{ with } a_v \in \mathbb{Z}^+ \right\} \subset \mathbb{Z}^{V_n \setminus E_n}, \quad n \geq 1.$$

Because $I_n(e_v) = \widehat{e}_v$, for $v \in V_n \setminus E_n$ then

$$Z_+^{(n)} = \left\{ Z_+^{V_n \setminus E_n} + \sum_{v \in E_n} a_v \cdot I_n(e_v),\ a_v \geq 0 \right\}.$$

Then

$$K^0(X,T)^+ \simeq \bigcup_{n=1}^{\infty} \widehat{G}_1^{-1} \circ \ldots \circ \widehat{G}_n^{-1}(Z_+^{(n+1)}). \tag{3.8}$$

Step 5. Set $\mathrm{Inf}(n) \simeq I_n(\mathcal{N}_n)$. Then

$$\mathrm{Inf}(X,T) = \varinjlim\{\widehat{G}_n : \mathrm{Inf}(n) \to \mathrm{Inf}(n+1)\}. \tag{3.9}$$

Stage II. Inspecting the sequence (3.6) and simplifying.

Step 6. If $\det(\widehat{B}_n) = 0$ for infinitely many n, then we can reconstruct the sequence (3.6) to the other one. We replace (3.6) by the sequence

$$\cdots \longrightarrow \mathrm{Im}(\widehat{G}_{n-1}) \xrightarrow{\widehat{G}_n} \mathrm{Im}(\widehat{G}_n) \longrightarrow \cdots$$

which has a form (3.6), too. Namely, find a basis V_n' of $\mathrm{Im}(\widehat{G}_{n-1})$ and let $J_n : \mathbb{Z}^{V_n'} \to \mathrm{Im}(\widehat{G}_{n-1})$ be the natural isomorphism. We obtain the following commuting diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "\widehat{G}_{n-1}|_{\mathrm{Im}(\widehat{G}_{n-2})}"] & \mathrm{Im}(\widehat{G}_{n-1}) \arrow[r, "\widehat{G}_n|_{\mathrm{Im}(\widehat{G}_{n-1})}"] & \mathrm{Im}(\widehat{G}_n) \arrow[r, "\widehat{G}_{n+1}|_{\mathrm{Im}(G_n)}"] & \cdots \\
\cdots \arrow[r, "G_{n-1}'"'] & \mathbb{Z}^{V_n'} \arrow[u, "J_n"'] \arrow[r, "G_n'"'] & \mathbb{Z}^{V_{n+1}'} \arrow[u, "J_{n+1}"'] \arrow[r, "G_{n+1}'"'] & \cdots
\end{tikzcd}$$

where $G_n' = J_{n+1}^{-1} \circ \widehat{G}_n \circ J_n$. Then, $K^0(X,T) = \varinjlim\{\mathbb{Z}^{V_n'} \xrightarrow{G_n'} \mathbb{Z}^{V_{n+1}{}'}\}$ and $\mathrm{Inf}(X,T) = \varinjlim\{\mathrm{Inf}'(n) \xrightarrow{G_n'} \mathrm{Inf}'(n+1)\}$, where $\mathrm{Inf}'(n) = J_n^{-1}(\mathrm{Inf}(n))$.

Step 6a. To find $K^0(X,T)^+$ we choose $Z_+^{(n)'} = J_n^{-1}(Z_+^{(n)})$ and

$$K^0(X,T)^+ \simeq \bigcup_{n=1}^{\infty} {G'_1}^{-1} \circ \ldots \circ {G'_n}^{-1}(Z_+^{(n)'}).$$

Step 7. If $\dim(\mathrm{Im}(G'_{n-1})) < |V'_n|$ for infinitely many n, then we repeat Step 6.

Step 8. If $\dim(\mathrm{Im}(G'_{n-1})) = |V'_n|$ for sufficiently large n, then we compute $K^0(X,T)$, $K^0(X,T)^+$, and $\mathrm{Inf}(X,T)$ by (3.6), (3.8) and (3.9) with $V_n \setminus E_n := V'_n$, $\widehat{G}_n := G'_n$, $Z_+^{(n)} := Z_+^{(n)'}$, $\mathrm{Inf}(n) := \mathrm{Inf}'(n)$.

Remark 3.7. In the sequel, we present examples of the topological flows. We compute their dimension groups applying the algorithm (TA). After applying the Stage I, we get a sequence of the following form

$$\mathbb{Z}^k \xrightarrow{\widehat{G}} \mathbb{Z}^k \xrightarrow{\widehat{G}} \mathbb{Z}^k \xrightarrow{\widehat{G}} \cdots \tag{3.10}$$

where $\widehat{G}_n = \widehat{G}$ for $n = 1,2,\ldots$ If $\det(\widehat{G}) = 0$ then we apply the Stage II. In this case we choose the positive integer $l > 1$, such that $\mathrm{rank}(\widehat{G}^l) = \mathrm{rank}(\widehat{G}^{l+1})$. Then we choose a group isomorphism $J' \colon \widehat{G}^l(\mathbb{Z}^k) \to \mathbb{Z}^s$, where $s = \mathrm{rank}(\widehat{G}^l)$ and we replace the sequence from the Step 6 by the sequence

$$\mathbb{Z}^s \xrightarrow{\widehat{G}''} \mathbb{Z}^s \xrightarrow{\widehat{G}''} \mathbb{Z}^s \xrightarrow{\widehat{G}''} \cdots$$

where $G'' = J' \circ \widehat{G} \circ (J')^{-1}$ and $\det(G'') \neq 0$. Finally, we have $K^0(X,T) = \varinjlim\{\mathbb{Z}^s \xrightarrow{\widehat{G''}} \mathbb{Z}^s\} \subset \mathbb{Q}^s$.

3.3. Relation between our results and results of papers [4] **and** [3]**.** In this section we would like to indicate some relations between some results of our paper and the main result of [4] and also to compare our algorithm (TA) and the algorithm described in [3] for computing the dimension groups for substitutions.

Let $\xi^{(n)} = \{T^k(D_{0,v}^{(n)}),\ 0 \le k \le h(n,v)-1,\ v \in V_n\}$ be a sequence of Kakutani–Rokhlin partitions of a Cantor minimal system (X,T) satisfying the conditions (1.1) and (1.2). The sequence $\{\xi^{(n)}\}$ determines a Bratteli diagram $I = (V_n, E_n)$ (for the definition see [3] and [4]). The diagram I is proper if additionally (1.4) holds. Assuming that (1.4) is not satisfied and the set $N' = \bigcap_{n=1}^{\infty}(\bigcup_{v \in V_n} D_{0,v}^{(n)})$ is finite, a natural question regarding relations between our considerations and Theorem 9 of [4] arises. The theorem 9 states the following isomorphism

$$\frac{K_0(I)}{Q} \oplus Z^{\nu} \equiv K^0(\Pi, S), \tag{3.11}$$

where $K_0(I)$ is the dimension group of the diagram I, Q is a subgroup of $K_0(I)$, ν is a non-negative integer, and (Π, S) is the path-sequence dynamical system (for the needed definitions see [4]) and $K^0(\Pi, S)$ is its dimension group.

It is not difficult to see that in our case we can identify the path-sequence dynamical system (Π, S) with (X, T). At the same time

$$K_0(I) = \varinjlim \{\mathbb{Z}^{V_n} \xrightarrow{F_n} \mathbb{Z}^{V_{n+1}}\},$$

where F_n are homomorphisms from Step 1 of the algorithm (TA). In [4] the following formula for ν is given

$$\nu = e - v + c,$$

where e is the number of $\sim$ equivalent pairs of maximal paths and minimal paths, v is the number of maximal or minimal paths and c is the number of components in the graph whose vertices are maximal or minimal paths connected by the $\sim$ relation. It is not difficult to remark that in our case $\nu = 0$.

In fact, we identify the set N of minimal paths of (Π, S) with the set $\bigcap_{n=1}^{\infty}(\bigcup_{v\in V_n} D_{0,v}^{(n)}) = N'$. While the set M of maximal paths is identified with the set

$$\bigcap_{n=1}^{\infty}\left(\bigcup_{v\in V_n} T^{h(n,v)-1}(D_{0,v}^{(n)})\right) = M' = T^{-1}(N').$$

Moreover, a pair (x, y), $x \in M'$, $y \in N'$, is "$\sim$" if and only if $y = T(x)$. Then $e = \#N'$, $v = 2 \cdot (\#N')$ and $c = \#N'$. Thus, we have $\nu = 0$. Then (3.11) has the following form

$$\varinjlim \{Z^{V_n} \xrightarrow{F_n} Z^{V_{n+1}}\}/Q = K^0(X, T). \tag{3.12}$$

Let us remark that (1.10) can be written in the form

$$\begin{aligned} &C(X,T)/B_T \\ &\quad = \varinjlim \{Z^{V_n} \xrightarrow{F_n} Z^{V_{n+1}}\}/\varinjlim \{Z_c^{(n)} \xrightarrow{F_n} Z_c^{(n+1)}\} = K^0(X, T). \end{aligned} \tag{3.13}$$

The formulas (3.12) and (3.13) lead us to ask what the relations between subgroups Q and $B_T = \varinjlim \{Z_c^{(n)} \xrightarrow{F_n} Z_c^{(n+1)}\}$ are. To answer this question we must adapt the description of Q from [4] to the case considered here. First we distinguish subsets $\widetilde{B}_n \subset \mathbb{Z}^{V_n}$, $n \geq 1$. Take any element $x \in X$ and consider its T-trajectory $O_T(x) = \{T^n(x),\ n \in \mathbb{Z}\}$. The trajectory x passes through the T-towers $\xi_{n,v} = \{T^k(D_{0,v}^{(n)}),\ 0 \leq k \leq h(n,v) - 1\}$, $v \in V_n$. Let us mark the places in $0_T(x)$ whenever it passes through the top $T^{h(n,v)-1}(D_{0,v}^{(n)})$ and denote by $Z_{n,v}$ the set of such places. Now, take any finite fragment $x[i, i+k] = u$ of x and define the vector $\overline{x}_u = \langle x_v \rangle \in \mathbb{Z}^{V_n}$ as follows: $x_v = \#Z_{n,v} \cap u$.

Let $\widetilde{B}_n$ be the set of all vectors $\overline{x}_u$ when u is any finite fragment of x. Next we define

$$Q_n = \{\overline{y} \in \mathbb{Z}^{V_n} : |\overline{y} \bullet \overline{x}_u| < \infty,\ \text{the supremum is taken over all } \overline{x}_u \in \widetilde{B}_n\}.$$

Using the minimality of (X,T), it is easy to check that Q_n does not depend on x. Then Q_n is a subgroup of $\mathbb{Z}^{V_n}$ and $F_n(Q_n) \subset Q_{n+1}$. The group Q is defined as

$$Q = \varinjlim \{Q_n \xrightarrow{F_n} Q_{n+1}\}.$$

Now we will show that $Z_c^{(n)} \subset Q_n$ for every $n \geq 1$, whenever $\sup_n \#V_n < \infty$. Take $x \in X$ and assume that its zero position is marked. Then the positive part $O_T^+(x)$ of the trajectory $O_T(x)$ determines an infinite path of the graph $(\alpha_n, \widetilde{V}_n)$. Let u be a finite fragment of x. First, let us assume that u determines a closed cycle Γ of $(\alpha_n, \widetilde{V}_n)$ without loops. Then $\overline{x}_u = \overline{w}_\Gamma$ (see proof of Theorem 2.2). For $\overline{y} \in Z_c^{(n)}$ we have $\overline{y} \bullet \overline{x}_u = 0$. If u determines a finite numbers of closed cycles Γ_s then $\overline{x}_u = \sum_s \overline{w}_{\Gamma_s}$ and again we have $\overline{y} \bullet \overline{x}_u = 0$. In general, $\overline{x}_u$ has a form

$$\overline{x}_u = \sum_s \overline{w}_{\Gamma_s} + \overline{x}',$$

where the coordinates x'_v of $\overline{x}'$ are bounded by the number $\sup_n \#V_n$. Then, for every $\overline{y} \in Z_c^{(n)}$ we have

$$|\overline{y} \bullet \overline{x}_u| \leq \sup_n \#V_n \cdot \sum_{v \in V_n} |y_v|.$$

In this manner we get $Z_c^{(n)} \subset Q_n$. As a consequence we have $B_T \subset Q$. We are unable to answer whether $Z_c^{(n)} = Q_n$ for every n and also whether $B_T = Q$.

Now we want to compare the algorithm (TA) with the algorithm from [3]. For the necessary definitions and notions we refer the reader to [3] including the definitions of substitution minimal systems.

Let $\sigma: A \to A^+$, be a primitive, aperiodic substitution on a finite alphabet A and let (X_σ, T) be the corresponding minimal Cantor system. The substitution σ determines the matrix $M_\sigma = (m_{ab})$, $a, b \in A$, where

$$m_{ab} = \text{the number of occurrences of } a \text{ in } \sigma(b).$$

We can define sets $D_{0,a}^{(n)}$,

$$D_{0,a}^{(n)} = \{x \in X_\sigma : x[0, \lambda_n - 1] = \sigma^n(a)\}, \quad a \in A \text{ and } \lambda_n = |\sigma^n(a)|.$$

The sets $D_{0,a}^{(n)}$ define disjoint T-towers $\xi_a^{(n)'} = \{T^k(D_{0,a}^{(n)}),\ 0 \leq k \leq \lambda_n - 1\}$. A substitution σ is proper if there exists a pair $r, l \in A$ such that for every $a \in A$, r is the last letter $\sigma(a)$ and l is the first letter of $\sigma(a)$. If σ is proper, then the towers $\xi_a^{(n)'}$, $a \in A$ satisfy the conditions (1.1), (1.2) and (1.4). In this case

$$K^0(X_\sigma, T) = \varinjlim \{\mathbb{Z}^A \xrightarrow{M_\sigma} \mathbb{Z}^A\}.$$

The algorithm presented in [3] consists in an associating a primitive, aperiodic and proper substitution τ to a given primitive, aperiodic substitution σ in such a way that (X_τ, T) is topologically isomorphic to (X_σ, T). Then $K^0(X_\sigma, T) \equiv$

$K^0(X_\tau, T)$. We can also apply the algorithm (TA) for substitutions. Then V_n is the set of all triples abc appearing in X_σ and the sets $D^{(n)}_{abc}$ are defined in the following way

$$D^{(n)}_{abc} = \{x \in X_\sigma : x[-|\sigma^n(a)|, |\sigma^n(b)| + |\sigma^n(b)| - 1] = \sigma^n(a)\sigma^n(b)\sigma^n(c)\}.$$

We have $F_n = F = (f_{a'b'c',abc})$, where

$$f_{a'b'c',abc} = \text{ the number of occurrences of the triple } abc \text{ in the sequence } u'_1\sigma(b')u'_2,$$

where u'_1 is the last letter of $\sigma(a')$ and u'_2 is the first letter of $\sigma(c')$. In general, the algorithms presented in [3] and (TA) are independent. We illustrate both algorithms using the example from [3].

3.3.1. Example of a substitution. Let σ be the substitution defined on the alphabet $A = \{a, b\}$ by $\sigma(a) = aba$, $\sigma(b) = baab$. A substitution τ associated with σ is the following: $\tau(1) = 112$, $\tau(2) = 1212$. Then

$$M_\tau = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \quad \text{and} \quad K^0(X_\sigma, T) = K^0(X_\tau, T) = \varinjlim\{\mathbb{Z}^2 \xrightarrow{M_\tau} \mathbb{Z}^2\}.$$

Now we apply the algorithm (TA). We have $V_n = \{aba, bab, baa, abb\}$. The graphs $(\alpha_n, \widetilde{V}_n)$ have the following form

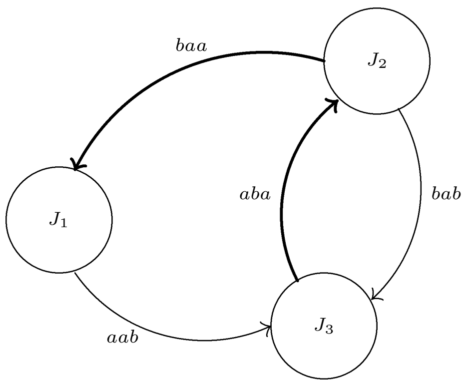

Figure 3.3

where $J_1 = \{aab\}, J_2 = \{baa, bab\}, J_3 = \{aba\}$. Next, we have

$$F_n = F = \begin{array}{c} \\ aab \\ aba \\ baa \\ bab \end{array}\!\!\begin{array}{c} \begin{array}{cccc} aab & aba & baa & bab \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix} \end{array}.$$

As a spanning tree we take (α_n, baa, aba). Applying Step 3 of (TA) we get

$$\widehat{F}_n = \widehat{F} = \begin{array}{c c} & \begin{array}{cc} bab & aab \end{array} \\ \begin{array}{c} bab \\ aab \end{array} & \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} \end{array}.$$

Thus $\widehat{F} = M_\tau$.

4. Examples

4.1. Teoplitz and Teoplitz–Morse flows. Let us remind briefly the definition of Teoplitz flow. Let S be a finite alphabet with at least two symbols. Consider the sequence of positive integers $\lambda_n \geq 2$ and let $p_n = \lambda_1 \cdot \ldots \cdot \lambda_n$, $n \geq 1$. Let us take a sequence of blocks $A^{(n)}$ over $S \cup \{-\}$ ("$-$" is the empty symbol or the hole) such that $|A^{(n)}| = p_n$ and the block $A^{(n+1)}$ is obtained by a concatenation of λ_n copies of $A^{(n)}$, where some "holes" are filled by symbols of S. Let us denote

$$\begin{aligned} k_n &= \max\{k : A^{(n)}[i] \in S, \text{ for all } 0 \leq i \leq k\}, \\ l_n &= \max\{l : A^{(n)}[p_n - i] \in S, \text{ for all } 1 \leq i \leq l\}. \end{aligned}$$

We make an additional assumption that $k_n, l_n \to \infty$.

Now we define a bisequence $\omega \in S^{\mathbb{Z}}$ as follows: $\omega[kp_n, (k+1)p_n - 1] = A^{(n)}$ for each $n \geq 1$.

Let $X = \overline{O(\omega)}$ and let T be the left shift. The dynamical system (X, T) is called a Teoplitz flow if it is not-periodic.

Now we are in a position to define n-symbols. By a n-symbol we mean each block B of the form $B = \omega[kp_n, (k+1)p_n - 1]$ for some $k \in \mathbb{Z}$, so each n-symbol coincides with the block $A^{(n)}$ at every position i such that $A^{(n)}[i] \in S$ and the remaining positions i (the empty positions in $A^{(n)}$) are filled in some way by the alphabet S. The set V_n of all n-symbols $B_v^{(n)}$ is finite. Now we define bases of Kakuthani–Rokhlin partitions ξ_n in the following way:

$$D_{0,v}^{(n)} = \{x \in X : x[0, p_n - 1] = B_v^{(n)},\ v \in V_n\}.$$

The partition ξ_n consist of $|V_n|$ T-towers

$$\xi_v^{(n)} = \{T^i(D_{0,v}^{(n)}),\ i = 0, \ldots, p_n - 1\}, \quad v \in V_n.$$

Let us note that

$$\operatorname{diam}\biggl(\bigcup_{v \in V_n} D_{0,v}^{(n)}\biggr) \leq \max\left(\frac{1}{k_n}, \frac{1}{l_n}\right) \to 0,$$

hence (1.4) is satisfied, so we have

$$K^0(X, T) = \varinjlim\{\mathbb{Z}^{V_n} \xrightarrow{F_n} \mathbb{Z}^{V_{n+1}}\}.$$

The homomorphism F_n is given by the matrix $B = [b_{vw}]$, $w \in V_n$, $v \in V_{n+1}$, where b_{vw} is the number of the appearance of n-symbols $B_w^{(n)}$ in the $(n+1)$-symbol $B_v^{(n+1)}$.

Take $S = \mathbb{Z}_2$, $\lambda_n = 5$ and let $A^{(1)} = 01-10$. For $n \geq 2$ we define blocks $A^{(n)}$ with $|A^{(n)}| = 5^n$ in such a way that $A^{(n)}$ has the only hole in its middle. Let $A_i^{(n)}$, ($i = 0$ or 1) be the block $A^{(n)}$ filled by i in the middle. We define the block $A^{(n+1)} = A_0^{(n)} A_1^{(n)} A^{(n)} A_1^{(n)} A_0^{(n)}$. In this case, we have only two n-symbols: $A_0^{(n)}$ and $A_1^{(n)}$, and $V_n = \{A_0^{(n)}, A_1^{(n)}\}$. Moreover, $A_0^{(n+1)} = A_0^{(n)} A_1^{(n)} A_0^{(n)} A_1^{(n)} A_0^{(n)}$, $A_1^{(n+1)} = A_0^{(n)} A_1^{(n)} A_1^{(n)} A_1^{(n)} A_0^{(n)}$. Thus, the homomorphisms $F_n (= F)$ are given by the matrix

$$\begin{array}{cc} & \begin{array}{cc} A_0^{(n)} & A_1^{(n)} \end{array} \\ \begin{array}{c} A_0^{(n+1)} \\ A_1^{(n+1)} \end{array} & \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \end{array}.$$

The unique T-invariant measure μ is given by $\mu(A_0^{(n)}) = 1/(2 \cdot 5^n) = \mu(A_1^{(n)})$, $p_n = 5^n$, $n = 1, 2, \dots$ Because $\det(F) = 5$, then

$$\begin{aligned} K^0(X,T) &= \bigcup_{n=0}^{\infty} F^{-n}(\mathbb{Z}^2), \\ \operatorname{Inf}(X,T) &= \{\langle x, y\rangle \in \mathbb{Z}^2, x + y = 0\}, \\ K^0(X,T)^+ &= \bigcup_{n=0}^{\infty} F^{-n}(\mathbb{Z}_+^2) = \{\langle x, y\rangle \in K^0(X,T), x + y \geq 0\} \end{aligned}$$

(see algorithm (TA), Stage I).

There are Teoplitz flows defined by a sequence of blocks $A^{(n)}$ not satisfying the condition $\min(k_n, l_n) \to \infty$. An example of such flows are Teoplitz–Morse flows. Consider sequence of blocks a_n such that $a_n[i] \in S$, $i = 0, \dots, \lambda_{n-2}$ and $a_n[\lambda_n - 1] =$"$-$" and each element $s \in S$ appears in every a_n. Define inductively a sequence of blocks $(A^{(n)})_{n \geq 0}$ as follows:

$$\begin{aligned} A^{(0)} &= a_0, \\ A^{(n+1)} &= A^{(n)} \underline{\ {\scriptstyle a_{n+1}[0]}\ } A^{(n)} \underline{\ {\scriptstyle a_{n+1}[1]}\ } \dots A^{(n)} \underline{\ {\scriptstyle a_{n+1}[\lambda-2]}\ } A_n -, \end{aligned}$$

$n \geq 0$, $\lambda = \lambda_{n+1}$. Thus, $A^{(n+1)}$ is obtained as the concatenation of λ_{n+1} copies of $A^{(n)}$ with holes filled by the successive elements of a_{n+1} except of the latest hole.

Define a bisequence ω over $S \cup \{-\}$ as $\omega[kp_n, (k+1)p_n - 1] = A^{(n)}$ for every $k \in \mathbb{Z}$ and $n \geq 0$. The sequence ω has all positions filled by symbols from S except for the position "-1", $\omega[-1] =$"$-$". We will write $\omega = a_0 * a_1 * \dots$

Now let $X \subset S^{\mathbb{Z}}$ be the closure of T-orbit of ω. We get a topological flow (X,T) which is called a Teoplitz–Morse flow. We define the sequence of Kakuthani–Rokhlin partitions $\{\xi_n\}$. As the bases of the towers we take sets

$$D_{0,gh}^{(n)} = \{x \in X : x[-p_n, p_n - 1] = A^{(n)} \underline{\ \ {\scriptstyle g}\ \ } A^{(n)} \underline{\ \ {\scriptstyle h}\ \ }\}.$$

Let $V_n \subset S \times S$ be the set of all pairs $gh \in S \times S$ that appear in the sequence $\omega_{n+1} = a_{n+1} * a_{n+2} * \ldots$ Then V_n is the set of the towers of the partition ξ_n. It is easy to see that $\operatorname{diam}(\bigcup_{(gh)\in V_n} D^{(n)}_{0,gh}) = 1/2$, so (1.4) is not satisfied. Thus we apply the algorithm (TA).

In this case, homomorphisms F_n are given by matrix

$$B_n = [b_{ghuv}]_{gh\in V_{n+1},\, uv\in V_n}$$

where b_{ghuv} equals the number of the appearance of uv in the block $ga_{n+1}h$.

Consider the following example: $S = \{0,1\}$, $a_{2k} = 1-$, $a_{2k+1} = 0-$, for $k \geq 0$. Then,

$$\begin{aligned} \omega_{2k} &= a_0 * a_1 * \ldots = 1011101010111101110\ldots, \\ \omega_{2k+1} &= a_1 * a_2 * \ldots = 010001\ldots \end{aligned}$$

We find that $V_{2k} = \{01, 10, 00\}$ and $V_{2k+1} = \{01, 10, 11\}$. Thus, homomorphisms F_n are given by the matrices:

$$\begin{array}{cc} & V_{2k} \quad 00 \quad 01 \quad 10 \\ V_{2k+1} & \\ \begin{array}{c} 01 \\ 10 \\ 11 \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & V_{2k+1} \quad 01 \quad 10 \quad 11 \\ V_{2k} & \\ \begin{array}{c} 00 \\ 01 \\ 10 \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}.$$

Define $J_g = \{gh : gh \in V_n\}$, $g \in \{0,1\}$, then the partitions $\alpha_n = \{J_0, J_1\}$. The graphs (α_n, V_n) are presented in Figure 4.1.

We choose spanning trees (α_n, E_n), $E_n = \{01\}$ in both cases (Figure 4.1) and calculate the homomorphisms I_n. We have

$$\begin{aligned} I_{2k}\langle x_{00}, x_{01}, x_{10}\rangle &= \langle \underbrace{x_{00}}_{y_{00}}, \underbrace{x_{01} + x_{10}}_{y_{10}} \rangle, \\ I_{2k+1}\langle x_{01}, x_{10}, x_{11}\rangle &= \langle \underbrace{x_{01} + x_{10}}_{y_{10}}, \underbrace{x_{11}}_{y_{11}} \rangle. \end{aligned}$$

Now we calculate the homomorphisms $\widehat{G} = \widehat{G}_n$. They are given by the matrices:

$$\begin{array}{cc} & \begin{array}{cc} 00 & 10 \end{array} \\ \begin{array}{c} 10 \\ 11 \end{array} & \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{array}{cc} 10 & 11 \end{array} \\ \begin{array}{c} 00 \\ 10 \end{array} & \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \end{array}.$$

It is convenient to consider the sequence

$$\mathbb{Z}^2 \xrightarrow{\widehat{G}^2_n} \mathbb{Z}^2 \xrightarrow{\widehat{G}^2_n} \mathbb{Z}^2 \xrightarrow{\widehat{G}^2_n} \ldots$$

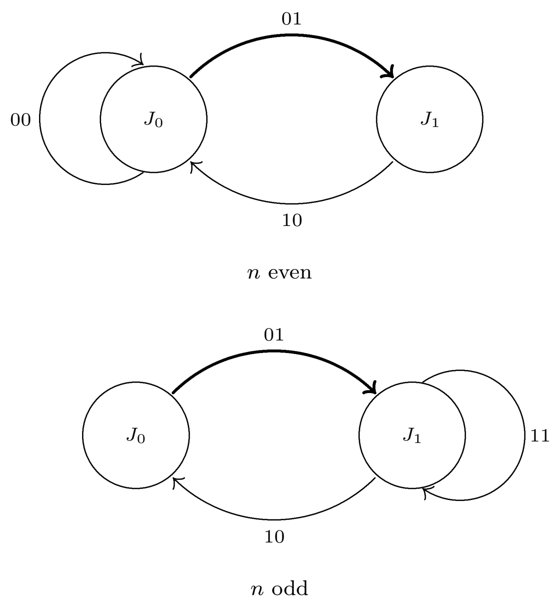

Figure 4.1

instead of the sequence (3.6). The homomorphisms $\widehat{G}_n^2$ are given by the matrix $F = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$. Thus, we have

$$K^0(X,T) = \bigcup_{n=0}^{\infty} F^{-n}(\mathbb{Z}^2).$$

The unique invariant measure is given by $\mu(D_{0,gh}^{(n)}) = 1/(3 \cdot 2^n)$ for $(gh) \in V_n$, $n = 1, 2, \ldots$ Thus,

$$\begin{aligned} \mathcal{N}_{2k} &= \{\langle x_{00}, x_{01}, x_{10}\rangle \in \mathbb{Z}^3 : x_{00} + x_{01} + x_{10} = 0\}, \\ \mathcal{N}_{2k+1} &= \{\langle x_{01}, x_{10}, x_{11}\rangle \in \mathbb{Z}^3 : x_{01} + x_{10} + x_{11} = 0\}. \end{aligned}$$

Then

$$\begin{aligned} I_{2k}(\mathcal{N}_{2k}) &= \{\langle y_{00}, y_{10}\rangle \in Z^2 : y_{00} + y_{10} = 0\}, \\ I_{2k+1}(\mathcal{N}_{2k+1}) &= \{\langle y_{10}, y_{11}\rangle \in \mathbb{Z}^2 : y_{10} + y_{11} = 0\} \end{aligned}$$

and

$$\mathrm{Inf}(X,T) = \{\langle x, y\rangle \in \mathbb{Z}^2 : x + y = 0\} \subset K^0(X,T).$$

We have also

$$K^0(X,T)^+ = \bigcup_{n=0}^{\infty} F^{-n}(\mathbb{Z}^2_+) = \{\langle x, y\rangle \in K^0(X,T) : x + y \geq 0\}.$$

4.2. Morse flow. Let G be a nontrivial finite abelian group. Let B, C be blocks over G, ($|B| = \lambda_B$, $|C| = \lambda_C$), $g \in G$. Then, $B + g$ denotes the block $(B[0] + g, B[1] + g, \ldots, B[\lambda_B - 1] + g)$ and $B \times C$ is the block defined as the concatenation $(B + C[0])(B + C[1]) \ldots (B + C[\lambda_C])$.

Now take the sequence of blocks a_n over G, such that a_n contains every symbol from G and $a[0] = 0$ for $n = 0, 1, \ldots$ and set $\omega = a_0 \times a_1 \times \ldots$ The one-sided sequence ω is called a generalized Morse sequence if it is not periodic.

Let $X = \overline{O(\omega)}$ and T be the shift. The dynamical system (X, T) is called a Morse flow. We define also sequences $\omega_n = a_n \times a_{n+1} \times \ldots$ and blocks $B^{(n)} = a_0 \times \ldots \times a_n$, $|B^{(n)}| = p_n$, $(\omega = B^{(n)} \times \omega_n)$.

Now we define the bases of Kakuthani–Rokhlin partitions ξ_n in the following way:

$$D^{(n)}_{0,vgh} = \{x \in X : x[-p_n, 2p_n - 1] = (B^{(n)} + v)(B^{(n)} + g)(B^{(n)} + h)\}, \quad v, g, h \in G.$$

Then V_n is the set of all triples vgh that appear in the sequence ω_{n+1}. The partition ξ_n consists of $|V_n|$ T-towers $\xi^{(n)}_{vgh} = \{T^i(D^{(n)}_{0,vgh}) : i = 0, \ldots, p_n - 1\}$, $vgh \in V_n$.

The homomorphism F_n is given by the matrix $B = [b_{uw}]$, $v'g'h' = w \in V_n$, $vgh = u \in V_{n+1}$, where b_{uw} is the number of appearance of the triple $v'g'h'$ in the block

$$(a[\lambda - 1] + v)(a + g)h = (a[\lambda - 1] + v), g, (a[1] + g), \ldots, (a[\lambda - 1] + g), h,$$

where $a = a_{n+1}$, $\lambda = \lambda_{n+1}$.

The partition α_n consists of sets J_{gh}, where gh is a pair of elements of G such, that a triple $(ghv) \in V_n$ for some $v \in G$ and $J_{gh} = \{ghv \in V_n\}$.

Let $G = \mathbb{Z}_2$, $a_n = a = 01$. Then

$$\omega_n = \omega = 0110100110010110100101100110 1001 \ldots$$

and $V_n = \{001, 010, 011, 100, 101, 110\}$.

To find the homomorphisms F_n, we must know triples $v'g'h' = w \in V_n$ appearing in the block $(v + 1)(a + g)h = (v + 1)g(g + 1)h$, $v, g, h \in \mathbb{Z}_2$. We have $\xi^{(n+1)}_{vgh} = \eta^{(n)}_{(v+1)g(g+1)} \cup \eta^{(n)}_{g(g+1)h}$. The homomorphism $F_n = F$ is given by the matrix:

$$\begin{array}{c} \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \end{array} \begin{array}{c} \begin{array}{cccccc} 001 & 010 & 011 & 100 & 101 & 110 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

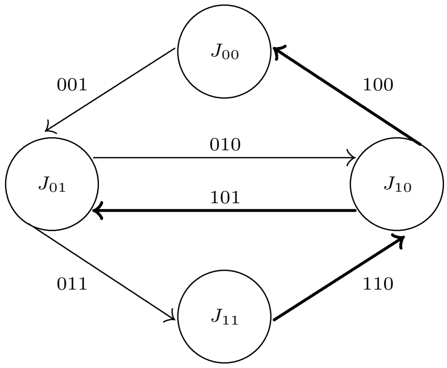

FIGURE 4.2

The Morse system is uniquely ergodic, the only measure μ is given by $\mu(D^{(n)}_{0,vgh}) = 1/(3\cdot 2^n)$, $n = 1,2,\ldots,$ $vgh \in V_n$. The partition α_n has the following form $\alpha_n = \{J_{00}, J_{01}, J_{10}, J_{11}\}$. The graph $(\alpha_n, \widetilde{V_n})$ is presented in Figure 4.2.

We select a spanning tree (α_n, E_n) with $E_n = \{100, 101, 110\}$ (see Figure 4.2). The homomorphisms $I_n = I$ are defined as follows:

$$I_n\langle x_{001}, x_{010}, x_{011}, x_{100}, x_{101}, x_{110}\rangle$$
$$= \langle x_{001} + x_{100} - x_{101}, x_{010} + x_{101}, x_{011} + x_{110} + x_{101}\rangle = \langle y_{001}, y_{010}, y_{011}\rangle.$$

Hence the sequence of homomorphisms $\widehat{G}_n = \widehat{G}$ is defined by the matrix:

$$\begin{array}{c} \\ 001 \\ 010 \\ 011 \end{array} \begin{array}{c} \begin{array}{ccc} 001 & 010 & 011 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}.$$

Because $\det(\widehat{G}) = 0$ we apply the Stage II of the algorithm (TA). We have

$$\widehat{G}(\mathbb{Z}^3) = \{\langle y_{001}, y_{010}, y_{011}\rangle \in \mathbb{Z}^3 : y_{011} = y_{001} + y_{010}\} \overset{J_n}{\simeq} \mathbb{Z}^2 = \{\langle y_{001}, y_{010}\rangle\}.$$

The homomorphism $G'_n = G'$ is given by the matrix:

$$G' = \begin{array}{c} \\ 001 \\ 010 \end{array} \begin{array}{c} \begin{array}{cc} 001 & 010 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \end{array}.$$

Because $\det(G'_n) = -2$ thus $K^0(X,T) \simeq \bigcup_{n=0}^{\infty}(G'^{-n})(\mathbb{Z}^2)$.

Next we have

$$\mathcal{N}_n = \{\overline{x} \in \mathbb{Z}^6 : x_{001} + x_{010} + x_{011} + x_{100} + x_{101} + x_{110} = 0\}$$

and

$$\mathrm{Inf}(n) = I_n(\mathcal{N}_n) = \{\langle y_{001}, y_{010}, y_{011}\rangle \in \mathbb{Z}^3 : y_{001} + y_{010} + y_{011} = 0\}.$$

Moreover,

$$J_n^{-1}(\mathrm{Inf}(n)) = \{\overline{y} \in \mathbb{Z}^2 : y_{001} + y_{010} = 0\}.$$

Finally

$$\mathrm{Inf}(X,T) \simeq \{\langle y_{001}, y_{010}\rangle \in \mathbb{Z}^2 : y_{001} + y_{010} = 0\} \simeq \mathbb{Z}.$$

To describe the cone $K^0(X,T)^+$ we find

$$\begin{aligned} I_n(e_{100}) &= \widehat{e}_{001}, & I_n(e_{101}) &= \langle -1,1,1\rangle, \\ I_n(e_{110}) &= \widehat{e}_{011}, & I_n(e_{vgh}) &= \widehat{e}_{vgh}, \end{aligned}$$

$(vgh) \in V_n \setminus E_n$. Then $Z_+^{(n)} = \{\mathbb{Z}_+^3 + a\langle -1,1,1\rangle,\ a \in \mathbb{Z}_+\}$ for every $n \geq 1$. Next we have

$$\begin{aligned} Z_+^{(n)'} &= J_n^{-1}(Z_+^{(n)}) \\ &= \{J_n^{-1}(\mathbb{Z}_+^3) + aJ_n^{-1}(\langle -1,1,1\rangle),\ a \in \mathbb{Z}_+\} = \{\mathbb{Z}_+^2 + a\langle -1,1\rangle,\ a \in \mathbb{Z}_+\}. \end{aligned}$$

Then

$$K^0(X,T)^+ \simeq \bigcup_{n=0}^{\infty} G'^{-n}(Z_+^{(n)'}) = \{\langle x,y\rangle \in K^0(X,T) : x + y \geq 0\}.$$

4.3. Rudin–Shapiro flow. The Rudin–Shapiro flow (X,T) is a symbolic topological flow, where $X = \overline{O(\omega)} \subset \{0,1\}^{\mathbb{Z}}$, T is the shift and ω is the Rudin–Shapiro sequence. To define the sequence ω, we consider the binary expansion $n = \sum_{i=0}^{k_n} 2^i \cdot \varepsilon_i$, $\varepsilon_i = 0,1$, $\varepsilon_{k_n} = 1$ of every positive integer n. Then, we compute a sequence $\{a[n]\}_{n=0}^{\infty}$, where $a[n] =$ the number of the appearance of the pair 11 in the block $\varepsilon_0 \ldots \varepsilon_{k_n}$ for $n \geq 1$ and $a[0] = 0$. The sequence $\omega[n]$ is defined as

$$\omega[n] = \begin{cases} 1 & \text{if } a_n \text{ is odd,} \\ 0 & \text{if } a_n \text{ is even.} \end{cases}$$

Then $\omega = 0001001000011101\ldots$ The sequence ω is determined also by a sequence of blocks $A_0^{(n)}, A_1^{(n)}$ over $\{0,1\} = \mathbb{Z}_2$, $n = 0,1,\ldots$, such that $A_0^{(0)} = A_1^{(0)} = 0$ and $A_0^{(n+1)} = A_0^{(n)}A_1^{(n)}$, $A_1^{(n+1)} = A_0^{(n)}(A_1^{(n)})^1$, where $B^0 = B = B + 0$ and $B^1 = B + 1$ in $\mathbb{Z}_2$ (B^1 is called also the mirror of the block B). Then,

$$|A_0^{(n)}| = |A_1^{(n)}| = 2^n \quad \text{and} \quad \omega[0, 2^{n+1} - 1] = A_0^{(n)}A_1^{(n)}$$

for every $n \geq 0$. The blocks $A_0^{(n)}, A_1^{(n)}$ define partitions $\xi^{(n)}$, $n \geq 0$, on T-towers $\xi_{ghuv}^{(n)} = \{T^i(D_{ghuv}^{(n)}),\ i = 0,\ldots, 2^{n+1} - 1,\ g,h,u,v \in \mathbb{Z}_2\}$, where

$$D_{ghuv}^{(n)} = \{x \in X : x[-2^n, 3 \cdot 2^n - 1] = (A_1^{(n)})^g(A_0^{(n)})^h(A_1^{(n)})^u(A_0^{(n)})^v\}.$$

Then $V_n = \mathbb{Z}_2^4$ and $\mathbb{Z}^{V_n} = \mathbb{Z}^{16}$.

To find a matrix $F_n \colon \mathbb{Z}^{16} \to \mathbb{Z}^{16}$ we remark that

$$\xi_{ghuv}^{(n+1)} = \eta_{g+1,hhu}^{(n)} \cup \eta_{hu,u+1,v}^{(n)} \quad \text{for every } ghuv \in \mathbb{Z}^4.$$

We have $F_n = F$, where F has the following form:

$$
\begin{array}{c}
\begin{array}{cccccccccccccccc}
\scriptstyle 0000 & \scriptstyle 0001 & \scriptstyle 0010 & \scriptstyle 0011 & \scriptstyle 0100 & \scriptstyle 0101 & \scriptstyle 0110 & \scriptstyle 0111 & \scriptstyle 1000 & \scriptstyle 1001 & \scriptstyle 1010 & \scriptstyle 1011 & \scriptstyle 1100 & \scriptstyle 1101 & \scriptstyle 1110 & \scriptstyle 1111
\end{array} \\
\begin{array}{c}
\scriptstyle 0000 \\ \scriptstyle 0001 \\ \scriptstyle 0010 \\ \scriptstyle 0011 \\ \scriptstyle 0100 \\ \scriptstyle 0101 \\ \scriptstyle 0110 \\ \scriptstyle 0111 \\ \scriptstyle 1000 \\ \scriptstyle 1001 \\ \scriptstyle 1010 \\ \scriptstyle 1011 \\ \scriptstyle 1100 \\ \scriptstyle 1101 \\ \scriptstyle 1110 \\ \scriptstyle 1111
\end{array}
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\end{array}.
$$

The partition α_n of V_n has a form $\alpha_n = \{J_{gh}, g, h \in \mathbb{Z}_2\}$, where $J_{gh} = \{ghuv \in V_n\}$. The graph $(\alpha_n, \widetilde{V}_n)$ is presented in Figure 4.3.

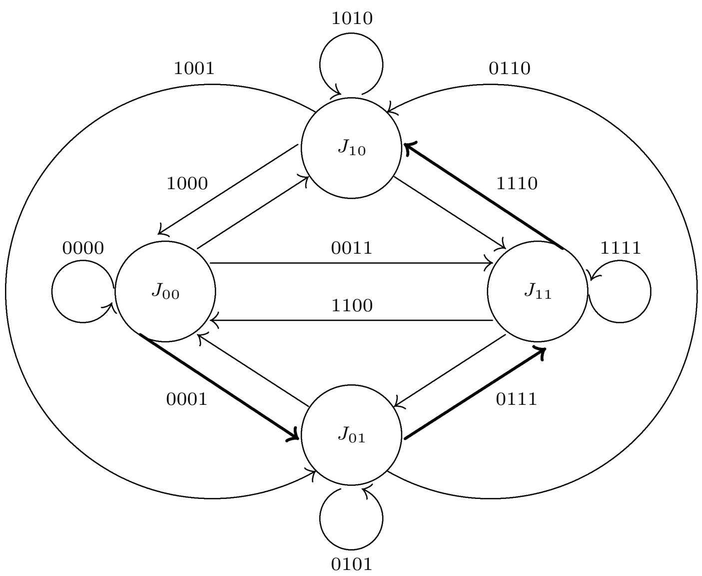

FIGURE 4.3

To simplify notations, we will write sometimes 0 instead of 0000, 1 instead of 0001, ..., 15 instead of 1111. In particular $x_{0000} := x_0$, $y_{0000} := y_0$, ..., $x_{1111} := x_{15}$, $y_{1111} := y_{15}$.

As a spanning tree we take $E_n = E = \{1, 7, 14\}$ (see Figure 4.3). Then by (3.2) we have $I\langle x_0, \ldots, x_{15}\rangle = \langle y_0, \ldots, y_{15}\rangle$, $I_n = I$, where

$$
\begin{aligned}
y_0 &= x_0, & y_9 &= x_9 + x_7 + x_{14},\\
y_2 &= x_2 - x_1 - x_7 - x_{14}, & y_{10} &= x_{10},\\
y_3 &= x_3 - x_1 - x_7, & y_{11} &= x_{11} + x_{14},\\
y_4 &= x_4 + x_1, & y_{12} &= x_{12} + x_1 + x_7,\\
y_5 &= x_5, & y_{13} &= x_{13} + x_7,\\
y_6 &= x_6 - x_7 - x_{14}, & y_{15} &= x_{15}.\\
y_8 &= x_8 + x_1 + x_7 + x_{14},
\end{aligned}
$$

Next, we compute the homomorphisms $\widehat{G}_n$ by (3.4) and (3.5). We have $\widehat{G}_n = \widehat{G}$ for every $n \geq 1$ and $\widehat{G}\colon \mathbb{Z}^{13} \to \mathbb{Z}^{13}$ is given by the following matrix $\widehat{G}$ (denoted by the same symbol).

$$
\begin{array}{c}
\\ 0\\ 2\\ 3\\ 4\\ 5\\ 6\\ 8\\ 9\\ 10\\ 11\\ 12\\ 13\\ 15
\end{array}
\begin{array}{c}
\begin{array}{ccccccccccccc}
0 & 2 & 3 & 4 & 5 & 6 & 8 & 9 & 10 & 11 & 12 & 13 & 15
\end{array}\\
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & -1 & -1\\
0 & 0 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & -1\\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1\\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\end{array}.
$$

Here $\operatorname{rank}(\widehat{G})) = 9$, so we apply Stage II of the algorithm (TA). Moreover,

$$
\begin{aligned}
\text{(4.1)}\quad \operatorname{Im}(\widehat{G}) = \{\overline{y} \in \mathbb{Z}^{13} : y_9 = -y_0 + y_8,\ y_{11} = -y_2 + y_3 + y_{10},\\
y_{13} = -y_4 + y_5 + y_{12},\ y_{15} = -y_6\}.
\end{aligned}
$$

Thus $\operatorname{Im}(\widehat{G}) \simeq \mathbb{Z}^9 = \{\langle y_0, y_2, y_3, y_4, y_5, y_6, y_8, y_{10}, y_{12}\rangle\}$ (we omit the coordinates $y_9, y_{11}, y_{13}, y_{15}$), i.e. $V'_n = \{0, 2, 3, 4, 5, 6, 8, 10, 12\}$ and $J_n\colon \mathbb{Z}^9 \to \operatorname{Im}(\widehat{G})$ (see Step 6 of the algorithm (TA)) is given by $J_n(e_i) = e_i$, $i \in V'_n$.

The homomorphism $\widehat{G}\colon \operatorname{Im}(\widehat{G}) \to \operatorname{Im}(\widehat{G})$ defines a homomorphism $G'\colon \mathbb{Z}^9 \to \mathbb{Z}^9$. The matrix G' is obtained from the matrix G by eliminating y_9, y_{11}, y_{13}, y_{15}

using (4.1). Then, we get the following matrix G'

$$\begin{array}{c} \\ 0 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 8 \\ 10 \\ 12 \end{array}\begin{array}{c} \begin{array}{ccccccccc} 0 & 2 & 3 & 4 & 5 & 6 & 8 & 10 & 12 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 2 & -1 & 1 & 0 & 0 & -2 \\ -1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

We have $\operatorname{rank}(G') = 5$ and

$$\begin{aligned} \operatorname{Im}(G') = \{\overline{y} \in \mathbb{Z}^9 : y_5 = -y_0 + y_4,\ y_6 = y_2 - y_3, \\ y_{10} = -y_0 + y_2 + y_8,\ y_{12} = -y_2 + y_3 + y_4\}. \end{aligned}$$

Repeating the same procedure as above we get $G'' \colon \mathbb{Z}^5 \to \mathbb{Z}^5$ given by the matrix

$$\begin{array}{c} \\ 0 \\ 2 \\ 3 \\ 4 \\ 8 \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 2 & 3 & 4 & 8 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 3 & -4 & -1 & 0 \\ -1 & 2 & -3 & 0 & 0 \\ -1 & 1 & 1 & 0 & 2 \\ 0 & -2 & 4 & 2 & 1 \end{bmatrix} \end{array}.$$

We have $\operatorname{rank}(G'') = 4$ and $\operatorname{Im}(G'') = \{\overline{y} \in \mathbb{Z}^5 : y_8 = 3y_0 - 2y_2 + y_3 - y_4\}$. Then, $\operatorname{Im}(G'') \simeq \mathbb{Z}^4$.

Consequently, we replace the homomorphism $G'' \colon \operatorname{Im}(G'') \to \operatorname{Im}(G'')$ with the homomorphism $G''' \colon \mathbb{Z}^4 \to \mathbb{Z}^4$. It is given by the matrix

$$\begin{array}{c} \\ 0 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 3 & -1 & 1 & -1 \\ 0 & 3 & -4 & -1 \\ -1 & 2 & -3 & 0 \\ 5 & -3 & 3 & -2 \end{bmatrix} \end{array}.$$

We have $\det(G''') = 4$ and then $K^0(X,T) \simeq \bigcup_{n=0}^{\infty} G'''^{-n}(\mathbb{Z}^4)$.

The unique T-invariant measure of (X,T) μ is given by $\mu(D_v^{(n)}) = 1/2^{n+3}$ for every $n \geq 1$ and $v \in V_n$. Then $\mathcal{N}_n = \{\langle x_v \rangle \in \mathbb{Z}^{16} : \sum_{v \in V_n} x_v = 0\}$.

It is not hard to remark that

$$\operatorname{Inf}(n) = I_n(\mathcal{N}_n) = \left\{ \langle y_v \rangle_{v \in V_n \setminus E_n} \in \mathbb{Z}^{13} : \sum_{v \in V_n \setminus E_n} y_v = 0 \right\}.$$

To find $\mathrm{Inf}'(n)$, we remark that $G(\mathrm{Inf}(n) \subset \mathrm{Inf}(n)$. Then, we have $G(\mathrm{Inf}(n)) = G(\mathbb{Z}^{13}) \cap \mathrm{Inf}(n)$. Therefore,

$$\begin{aligned}
\mathrm{Inf}(n)' &= (J_n^{-1} \circ G)(\mathrm{Inf}(n)) = J_n^{-1}(G(\mathbb{Z}^{13})) \cap J_n^{-1}(\mathrm{Inf}(n)) \\
&= \{\overline{y} \in \mathbb{Z}^9 : \overline{y} \text{ satisfies the equation obtained from} \\
&\qquad \sum_{v \in V_n \setminus E_n} y_v = 0 \text{ by using equations from (4.1) } \} \\
&= \{\overline{y} \in \mathbb{Z}^9 : y_3 + y_5 + y_8 + y_{10} + y_{12} = 0\}.
\end{aligned}$$

By the same arguments, we get

$$\mathrm{Inf}''(n) = \{\overline{y} \in \mathbb{Z}^5 : J(\overline{y}) \in \mathrm{Inf}'(n)\} = \{\overline{y} \in \mathbb{Z}^5 : -y_0 + y_3 + y_4 + y_8 = 0\}.$$

Finally, $\mathrm{Inf}'''(n) = \{\overline{y} \in \mathbb{Z}^4 : y_0 - y_2 + y_3 = 0\}$. We have $G'''(\mathrm{Inf}(n)''') \subset \mathrm{Inf}(n)'''$. Thus,

$$\begin{aligned}
\mathrm{Inf}(X, T) &= \varinjlim\{G \colon \mathrm{Inf}'''(n) \to \mathrm{Inf}'''(n+1)\} \\
&= \{\langle y_0, y_2, y_3, y_4\rangle \in K^0(X, T) : y_0 - y_2 + y_3 = 0\} \subset K^0(X, T).
\end{aligned}$$

To describe the cone $K^0(X, T)^+$ we find that $I_n(e_v) = \widehat{e}_v$, $v \in V_n \setminus E_n$, and

$$\begin{aligned}
I_n(e_{0001}) &= \langle 0, -1, -1, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0\rangle, \\
I_n(e_{0111}) &= \langle 0, -1, -1, 0, 0, -1, 1, 1, 0, 0, 1, 1, 0\rangle, \\
I_n(e_{1110}) &= \langle 0, -1, 0, 0, 0, -1, 1, 1, 0, 1, 0, 0, 0\rangle,
\end{aligned}$$

in $\mathbb{Z}^{13}$. Thus,

$$Z_+^{(n)} = \{\mathbb{Z}_+^{13} + a_1 I_n(e_{0001}) + a_2 I_n(e_{0111}) + a_3 I_n(e_{1110}),\ a_1, a_2, a_3 \in \mathbb{Z}_+\} = Z'_+.$$

Let $Z_+^{(n)''} = Z''_+ = (G'' \circ G' \circ G)(Z'_+)$. Then

$$K^0(X, T)^+ = \bigcup_{n=0}^{\infty} (G''')^{-n}(Z''_+).$$

Of course

$$K^0(X, T)^+ \subset \left\{\langle y_0, y_2, y_3, y_4\rangle \in \bigcup_{n=0}^{\infty} G'''^{-n}(\mathbb{Z}^4) : y_0 + y_3 \geq y_2\right\}.$$

We show that

$$\bigcup_{n=0}^{\infty} (G''')^{-n}(Z''_+) = \{\langle y_0, y_2, y_3, y_4\rangle \in K^0(X, T) : y_0 + y_3 \geq y_2\}. \tag{4.2}$$

For $\overline{x} \in \mathbb{Z}^4$ let $L_{\overline{x}}^+$ be a positive line defined by $\overline{x}$ i.e. $L_{\overline{x}}^+ = \{\lambda\overline{x},\ \lambda \in \mathbb{Z}_+\}$. Let us denote $\overline{w}_i = (G'' \circ G' \circ G \circ I_n)(\widehat{e}_i) \in \mathbb{Z}^4$, $\widehat{e}_i \in \mathbb{Z}^{16}$, $i = 0, \ldots, 15$. Of course,

$L^+_{\overline{w}_i}$ are elements of the cone $K^0(X,T)^+$. We need the vectors $\overline{w}_0$, $\overline{w}_2$, $\overline{w}_5$ and $\overline{w}_6$. By direct computations we find:

$$\overline{w}_0 = \langle 1,-1,-1,1\rangle, \quad \overline{w}_2 = \langle 1,0,0,1\rangle, \quad \overline{w}_5 = \langle 0,0,1,0\rangle, \quad \overline{w}_6 = \langle 0,-2,-1,1\rangle.$$

The matrix G''' has four eigenvalues $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = \sqrt{2}$, $\lambda_4 = -\sqrt{2}$, with the eigenvectors:

$$\begin{aligned} \overline{x}_{(-1)} &= \langle 0,1,1,0\rangle, & \overline{x}_{(2)} &= \langle 1,-2,-1,2\rangle,\\ \overline{x}_{(\sqrt{2})} &= \langle \sqrt{2}+1, \sqrt{2}+2, 1, \sqrt{2}\rangle, & \overline{x}_{(-\sqrt{2})} &= \langle 1-\sqrt{2}, 2-\sqrt{2}, 1, -\sqrt{2}\rangle. \end{aligned}$$

The vectors $\overline{x}_{(-1)}$, $\overline{x}_{(\sqrt{2})}$ and $\overline{x}_{(-\sqrt{2})}$ form a basis of the subspace

$$\Pi = \{\langle z_0, z_2, z_3, z_4\rangle : z_0 + z_3 = z_2\} \subset \mathbb{R}^4.$$

Let $\Pi_{\mathrm{int}} = \Pi \cap \mathbb{Z}^4$. For any $\overline{y} = \langle y_0, y_2, y_3, y_4\rangle \in \mathbb{Z}^4$, we have $\overline{y} = a \cdot \overline{x}_{(2)} + \overline{u}$, $\overline{u} \in \Pi$, where $a = (y_0 + y_3 - y_2)/2$. In particular, we have $\overline{w}_i = \overline{x}_{(2)}/2 + \overline{u}_i$, $i = 0,2,5,6$, where

$$\begin{aligned} \overline{u}_0 &= \left\langle \frac{1}{2}, 0, -\frac{1}{2}, 0\right\rangle, & \overline{u}_2 &= \left\langle \frac{1}{2}, 1, \frac{1}{2}, 0\right\rangle,\\ \overline{u}_5 &= \left\langle -\frac{1}{2}, 1, \frac{3}{2}, -1\right\rangle, & \overline{u}_6 &= \left\langle -\frac{1}{2}, -1, -\frac{1}{2}, 0\right\rangle. \end{aligned}$$

The subspace Π is G'''-invariant and the map G''' (in the coordinates z_0, z_3, z_4 of Π; we omit $z_2 = z_0 + z_3$) is given by the matrix:

$$F = \begin{array}{c} \\ 0 \\ 3 \\ 4 \end{array} \!\! \begin{array}{c} \begin{array}{ccc} 0 & \;\;3 & \;\;4 \end{array} \\ \begin{bmatrix} 2 & 0 & -1 \\ 1 & -1 & 0 \\ 2 & 0 & -2 \end{bmatrix} \end{array}.$$

We find that

$$F^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 1 & -1 & -\frac{1}{2} \\ 1 & 0 & -1 \end{bmatrix}.$$

By the induction we check that

$$F^{-2l} = \begin{bmatrix} \frac{1}{2^l} & 0 & 0 \\ \frac{1}{2^l} - 1 & 1 & 1 - \frac{1}{2^l} \\ 0 & 0 & \frac{1}{2^l} \end{bmatrix}, \qquad F^{-(2l+1)} = \begin{bmatrix} \frac{1}{2^l} & 0 & -\frac{1}{2^{l+1}} \\ 1 & -1 & -\frac{1}{2^{l+1}} - 1 \\ \frac{1}{2^l} & 0 & -\frac{1}{2^l} \end{bmatrix},$$

for $l = 0, 1, 2, \ldots$ We find:

$$
\begin{aligned}
F^{-2l}(\overline{u}_2) &= \left\langle \frac{1}{2^{l+1}}, \frac{1}{2^{l+1}}, 0 \right\rangle, & F^{-(2l+1)}(\overline{u}_2) &= \left\langle \frac{1}{2^{l+1}}, 0, \frac{1}{2^{l+1}} \right\rangle, \\
F^{-(2l+1)}(\overline{u}_0) &= \left\langle \frac{1}{2^{l+1}}, 1, \frac{1}{2^{l+1}} \right\rangle, & F^{-2l}(\overline{u}_6) &= \left\langle -\frac{1}{2^{l+1}}, -\frac{1}{2^{l+1}}, 0 \right\rangle, \\
F^{-(2l+1)}(\overline{u}_6) &= \left\langle -\frac{1}{2^{l+1}}, 0, \frac{-1}{2^{l+1}} \right\rangle, & F^{-(2l+1)}(\overline{u}_5) &= \left\langle 0, -1, -\frac{1}{2^{l+1}} \frac{1}{2^{l+1}} \right\rangle,
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
(G''')^{-2l}(\overline{w}_2) &= \frac{1}{2^{2l+1}}\overline{x}_2 + \left\langle \frac{1}{2^{l+1}}, \frac{1}{2^{l}}, \frac{1}{2^{l+1}}, 0 \right\rangle, \\
(G''')^{-2l}(\overline{w}_6) &= \frac{1}{2^{2l+1}}\overline{x}_2 + \left\langle -\frac{1}{2^{l+1}}, -\frac{1}{2^{l}}, -\frac{1}{2^{l}}, 0 \right\rangle, \\
(G''')^{-(2l+1)}(\overline{w}_2) &= \frac{1}{2^{2l+2}}\overline{x}_2 + \left\langle \frac{1}{2^{l+1}}, \frac{1}{2^{l+1}}, 0, \frac{1}{2^{l+1}} \right\rangle, \\
(G''')^{-(2l+1)}(\overline{w}_6) &= \frac{1}{2^{2l+2}}\overline{x}_2 + \left\langle -\frac{1}{2^{l+1}}, -\frac{1}{2^{l+1}}, 0, -\frac{1}{2^{l+1}} \right\rangle, \\
(G''')^{-(2l+1)}(\overline{w}_0) &= \frac{1}{2^{2l+2}}\overline{x}_2 + \left\langle \frac{1}{2^{l+1}}, 1 + \frac{1}{2^{l+1}}, 1, \frac{1}{2^{l+1}} \right\rangle, \\
(G''')^{-(2l+1)}(\overline{w}_5) &= \frac{1}{2^{2l+2}}\overline{x}_2 + \left\langle 0, -1 - \frac{1}{2^{l+1}}, -1 - \frac{1}{2^{l+1}}, \frac{1}{2^{l+1}} \right\rangle.
\end{aligned}
$$

Further, we have

$$
\begin{aligned}
2^{l+1}(G''')^{-2l}(\overline{w}_2) &= \frac{1}{2^l}\overline{x}_{(2)} + \langle 1, 2, 1, 0 \rangle = (\overline{A}_l), \\
2^{l+1}(G''')^{-2l}(\overline{w}_6) &= \frac{1}{2^l}\overline{x}_{(2)} + \langle -1, -2, -1, 0 \rangle = (\overline{B}_l), \\
2^{l+1}(G''')^{-(2l+1)}(\overline{w}_2) &= \frac{1}{2^l}\overline{x}_{(2)} + \langle 1, 1, 0, 1 \rangle = (\overline{C}_l), \\
2^{l+1}(G''')^{-(2l+1)}(\overline{w}_6) &= \frac{1}{2^l}\overline{x}_{(2)} + \langle -1, -1, 0, -1 \rangle = (\overline{D}_l).
\end{aligned}
$$

Of course $\overline{A}_l, \overline{B}_l, \overline{C}_l, \overline{D}_l \in \bigcup_{n=0}^{\infty}(G''')^{-n}(Z''_+)$. Then,

$$
\begin{aligned}
\lim_l \overline{A}_l &= \langle 1, 2, 1, 0 \rangle, & \lim_l \overline{B}_l &= \langle -1, -2, -1, 0 \rangle, \\
\lim_l \overline{C}_l &= \langle 1, 1, 0, 1 \rangle, & \lim_l \overline{D}_l &= \langle -1, -1, 0, -1 \rangle
\end{aligned}
$$

are some elements of the cone $K^0(X,T)^+$. At the same time $\langle 0, 1, 1, 0 \rangle = \lim_l (G''')^{-(2l+1)}(\overline{w}_0)$ and $\langle 0, -1, -1, 0 \rangle = \lim_l (G''')^{-(2l+1)}(\overline{w}_5)$ are also elements of $K^0(X,T)^+$. It follows from above that for any vector $\overline{y} \in \Pi_{\mathrm{int}}$ the positive line $L^+_{\overline{y}}$ is an element of $K^0(X,T)^+$. Then for any $\overline{y} \in \bigcup_{n=0}^{\infty}(G''')^{-n}(\Pi_{\mathrm{int}}) = \{\overline{y} \in K^0(X,T),\ y_0 + y_3 - y_2 = 0\}$ the positive line $L_{\overline{y}}$ is an element of $K^0(X,T)^+$. This implies (4.2).

4.4. Skew product extensions. Let (X,T) be a Cantor minimal system, G a finite abelian group and $c\colon X \to G$ a continuous cocycle. Consider a minimal group extension $(X \times G, T_c)$, where $T_c(x,g) = (T(x), g + c(x))$. Assume that $\{\xi'_n\}$, $\xi'_n = \{T^i(D_{0,v}^{(n)}),\ i = 0,\dots,h(n,v)-1,\ v \in V_n\}$ is a sequence of Kakuthani–Rokhlin partitions of (X,T) generating the topology. The clopen sets $T^i(D_{0,v}^{(n)}) \times g$, $g \in G$, form partitions ξ_n of $(X \times G)$ generating the topology of $X \times G$. Because c is continuous we can choose n_o such that $c = \mathit{const}$ on every $T^i(D_{0,v}^{(n)})$, $i = 0,\dots,h(n,v)-1$, $v \in V_n$. Without loss of generality we assume that this property holds for every $n = 1,2,\dots$ Then the partitions ξ_n, $n \geq 1$, are Kakuthani–Rokhlin partitions of $(X \times G, T_c)$ consisting of T_c-towers

$$\xi_n(v,g) = \{T_c^i(D_{0,v}^{(n)} \times g),\ i = 0,\dots,h(n,v)-1\}, \quad v \in V_n,\ g \in G.$$

Let $(\alpha_n, \widetilde{V}_n)$ be the oriented graphs determined by ξ_n (see Section 2). In addition, we assume that

$$\max_{J \in \alpha_n} \left(\operatorname{diam} \bigcup_{v \in J} D_{0,v}^{(n)} \right) \to 0. \tag{4.3}$$

This implies that

$$\max_{J \in \alpha_n} \left(\operatorname{diam} \bigcup_{v \in J} T^{-1}(D_{0,v}^{(n)}) \right) \to 0. \tag{4.4}$$

The conditions (4.3) and (4.4) imply that the sets $T_c^{-1}(\bigcup_{v \in J} D_{0,v}^{(n)} \times g)$ are ξ_n-sets for every $J \in \alpha_n$ and $g \in G$. Thus, the sets

$$\xi_n(J,g) = \bigcup_{v \in J} \xi_n(v,g), \quad J \in \alpha_n,$$

form a partition of the set $V_n \times G$ and it is the smallest partition having this property. Thus, the sets $\xi_n(J,g)$, $J \in \alpha_n$, $g \in G$ form a partition of $V_n \times G$ defined in the Section 2. Denote it by $\alpha_n \times G$. We have an oriented graph $(\alpha_n \times G, \widetilde{V_n \times G})$. For $v \in V_n$ let

$$d(v) = \sum_{i=0}^{h(v)-1} c(T^i x), \quad h(v) = h(n,v),$$

where $x \in D_{v,0}^{(n)}$. The arrows of $(\alpha_n \times G, V_n \times G)$ are characterized by the following property:

(4.5) Let $(J,g) \in \alpha_n \times G$ and $(J',h) \in \alpha_n \times G$. Then, there exists an arrow $v \times g$ joining the vertices (J,g) and (J',h) if and only if $v = (J,J')$ and $h = g + d(v)$.

We will construct a spanning tree $(\alpha_n \times G, E'_n)$ of the non-oriented graph $(\alpha_n \times G, \widetilde{\widetilde{V_n \times G}})$ using a modification of the Kruskal algorithm (see [2]). To do this, take a spanning tree (α_n, E), $E = E_n$ of $(\alpha_n, \widetilde{\widetilde{V}}_n)$. Let P_v, $v \in V_n \setminus E$, be the cycle of $(\alpha_n, \widetilde{\widetilde{V}}_n)$ defined in Section 3. Set

$$d(P_v) = \sum_{v' \in P_v^+} d(v') - \sum_{v' \in P_v^-} d(v'). \tag{4.6}$$

Let $P_{JJ'}$, $J, J' \in \alpha_n$, be the unique path joining the vertices J and J' inside E. Let

$$d(P_{JJ'}) = \sum_{v' \in P_{JJ'}^+} d(v') - \sum_{v' \in P_{JJ'}^-} d(v'), \tag{4.7}$$

where $P_{JJ'}^+$ is the set of all $v' \in P_{JJ'}$ having the same orientation as the direction from J to J' and $P_{JJ'}^- = P_{JJ'} \setminus P_{JJ'}^+$. In the sequel we need two facts:

(F1) The minimality of $(X \times G, T_c)$ implies that the elements $d(P_v)$, $v \in V_n \setminus E$ generate the group G for every $n \geq 1$ (this fact is easy to prove).

(F2) A connected acyclic subgraph (α, E) of a connected not oriented graph (α, V) is a spanning tree if and only if $|E| = |\alpha| - 1$ (see [2]).

4.4.1. Construction of a spanning tree of the graph $(\alpha_n \times G, \widetilde{\widetilde{V_n \times G}})$. Using (F1) we choose $v_1, \ldots, v_s \in V_n \setminus E$ such that

$$\begin{aligned} G = H_1 = G(a_1, \ldots, a_s) \supsetneq H_2 = G(a_2, \ldots, a_s) \supsetneq \ldots \\ \supsetneq H_s = G(a_s) \supsetneq H_{s+1} = \{0\}, \end{aligned}$$

where $a_i = d(P_{v_i})$.

Let $r_i = \operatorname{rank} a_i$ in H_i / H_{i+1}, $i = 1, \ldots, s$. Of course $r_i > 1$. Then, every $h \in G$ has the unique decomposition $h = \tau_1 \cdot a_1 + \ldots + \tau_s \cdot a_s$, $0 \leq \tau_i \leq r_i - 1$. Inductively, we construct a family of connected, acyclic subgraphs $S_g^{(m)}(J) = (\alpha_g^{(m)}(J), E_g^{(m)}(J))$, $g \in H_m$, $J \in \alpha_n$, $m = 1, \ldots, s$, satisfying

$$\begin{cases} \text{(a) } \alpha_g^{(m)}(J) \cap \alpha_h^{(m)}(J) = \emptyset, \ g \neq h, \ g, h \in H_m, \\ \text{(b) } \bigcup_{g \in H_m} \alpha_g^{(m)} = \alpha_n \times G, \text{ for every } J \in \alpha_n, \\ \text{(c) } \xi_n(J, h) \in \alpha_g^{(m)}(J) \Leftrightarrow h = \tau_1 \cdot a_1 + \ldots + \tau_{m-1} \cdot a_{m-1} + g, \\ \qquad h \in G, \ g \in H_m, \\ \text{(d) } \alpha_g^{(m)}(J) = \alpha_{g+d_m}^{(m)}(J'), \ E_g^{(m)}(J) = E_{g+d_m}^{(m)}(J'), g \in H_m, \end{cases} \tag{4.8}$$

where $d_m \in H_m$ is chosen from the conditions $d_1 = d(P_{JJ'})$ and $d(P_{JJ'}) = \tau_1 \cdot a_1 + \ldots + \tau_{m-1} \cdot a_{m-1} + d_m$, $m > 1$.

Step 1. For $J \in \alpha_n$ and $g \in G$ set $\alpha_g^{(1)}(J) = \{(J', g + d(P_{JJ'})), J' \in \alpha_n\} \subset \alpha_n \times G$, and $E_g^{(1)}(J) = \{\xi_n(v, g + d(P_{JJ'}) + d(v)) : (J', J'') = v \in E\}$.

It is easy to see that $\alpha_g^{(1)}$ and $E_g^{(1)}$ satisfy (4.8). We define a family $S_g^{(1)}(J)$ of subgraphs of $(\alpha_n \times G, V_n \times G)$ as follows:

$$S_g^{(1)}(J) = (\alpha_g^{(1)}(J), E_g^{(1)}(J)).$$

The graphs $S_g^{(1)}(J)$ are connected and acyclic.

Step $m+1$. Assume that we have constructed a family $S_g^{(m)}(J)$, $m < s+1$, $g \in H_m$, of subgraphs satisfying (4.8). For $g \in H_{m+1}$ and $J \in \alpha_n$ define

$$\begin{aligned} \alpha_g^{(m+1)}(J) &= \bigcup_{k=0}^{r_m-1} \alpha_{g+k\cdot a_m}^{(m)}(J), \\ E_g^{(m+1)}(J) &= \bigcup_{k=0}^{r_m-1} E_{g+k\cdot a_m}^{(m)}(J) \cup \bigcup_{k=0}^{r_m-2} \xi_n(v_m, g+k\cdot a_m), \\ S_g^{(m+1)}(J) &= (\alpha_g^{(m+1)}(J), E_g^{(m+1)}(J)). \end{aligned}$$

Now we check that $\alpha^{(m+1)}$ and $E^{(m+1)}$ satisfy (4.8). Let $v_m = (J, J')$, $J, J' \in \alpha_n$. It follows from (4.5) that the arrow $\xi_n(v_m, g+k\cdot a_m)$, $g \in H_{m+1}$, $0 \le k \le r_m - 2$, joins the vertices $\xi_n(J, g+k\cdot a_m)$ and $\xi_n(J', g+k\cdot a_m + d(v_m))$. Of course $\xi_n(J, g+k\cdot a_m) \in \alpha_{g+k\cdot a_m}(J)$. We will check that $\xi_n(J', g+k\cdot a_m + d(v_m)) \in \alpha_{g+(k+1)\cdot a_m}(J)$. We use the equality $d(v_m) - d(P_{JJ'}) = d(P_{v_m})$. We have

$$\begin{aligned} \xi_n(J', g+k\cdot a_m + d(v_m)) &= \xi_n(J', g+(k+1)a_m + d(P_{JJ'})) \\ &\overset{(4.8)(c)}{\in} \alpha_{g+(k+1)a_m+d_m}^{(m)}(J') \overset{(4.8)(d)}{=} \alpha_{g+(k+1)a_m}^{(m)}(J). \end{aligned}$$

This means that the arrow $\xi_n(J, g+k\cdot a_m)$ joins the subgraphs $S_{g+k\cdot a_m}^{(m)}(J)$ and $S_{g+(k+1)a_m}^{(m)}(J)$, $k = 0, \ldots, r_m - 2$.

Each $S_g^{(m+1)}(J)$, $g \in H_{m+1}$ is an acyclic, connected subgraph of $(\alpha_n \times G, \widetilde{\widetilde{V_n \times G}})$. For any $J'' \in \alpha_n$ we define $S_g^{(m+1)}(J'')$ by (4.8)(d). It is not hard to see that the family $\{S_g^{(m+1)}(J'')\}$, $g \in H_{m+1}$ satisfies the conditions (4.8) for every $J'' \in \alpha_n$.

4.4.2. A spanning tree of $(\alpha_n \times G, \widetilde{\widetilde{V_n \times G}})$. We finish the construction of the families $\{S_g^{(m)}(J)\}$, $g \in H_m$, $J \in \alpha_n$, when $m = s+1$. Then we have an acyclic, connected subgraph $S_0^{(s+1)}(J)$ of $(\alpha_n \times G, \widetilde{\widetilde{V_n \times G}})$.

To prove that $S_0^{(s+1)}(J)$ is a spanning tree, we use (F2). Of course, $|E_g^{(1)}(J)| = |\alpha_n| - 1$. Assume that

$$|E_{g+k\cdot a_m}^{(m)}(J)| = |\alpha_{g+k\cdot a_m}^{(m)}(J)| - 1 = |\alpha_g^{(m)}(J)| - 1,$$

for $g \in H_{m+1}, k = 0, \ldots, r_m - 1$. We have

$$|E_g^{(m+1)}(J)| = |E_{g+ka_m}^{(m)}(J)| \cdot r_m + r_m - 1 = r_m|\alpha_g^{(m)}(J)| - 1 = |\alpha_g^{(m+1)}(J)| - 1.$$

In this way it holds $|E_g^{(m)}(J)| = |\alpha_g^{(m)}(J)| - 1$ for every $m = 1, \dots, s+1$ and $g \in H_m$. In particular

$$|E_g^{(s+1)}(J)| = |\alpha_g^{(s+1)}(J)| - 1 = |\alpha_n \times G| - 1.$$

Therefore, because of (F2), $S_0^{(s+1)}(J) = (\alpha_n \times G, E'_n)$ is a spanning tree of $(\alpha_n \times G, V_n \times G)$.

4.5. A group extension of the Chacon flow. Consider a Chacon sequence ω over two symbols $0, 1$ (we replace the symbol "s" by "1") treated as the elements of the group $\mathbb{Z}_3 = \{0,1,2\}$. Define a cocycle $c\colon X = \overline{O(\omega)} \to \mathbb{Z}_3$ as follows: $c(x) := x[0]$, $x \in X$. We have $V_n = \{00, 01, 10, 11\}$ (see Subsection 3.1). Then,

$$d(00) = d(10) = \sum_{i=0}^{r_n - 1} B_n[i] = 1, \qquad d(01) = d(11) = 1 + \sum_{i=0}^{r_n-1} B_n[i] = 2,$$

for every $n \geq 1$. To find the homomorphism F'_n from $\mathbb{Z}^{V_n \times \mathbb{Z}_3} \to \mathbb{Z}^{V_n \times \mathbb{Z}_3}$, let us note that

$$\begin{aligned}
\xi^{(n+1)}(00, g) &= \eta^{(n)}(00, g) \cup \eta^{(n)}(01, g+1) \cup \eta^{(n)}(10, g),\\
\xi^{(n+1)}(10, g) &= \eta^{(n)}(10, g) \cup \eta^{(n)}(01, g+1) \cup \eta^{(n)}(10, g),\\
\xi^{(n+1)}(01, g) &= \eta^{(n)}c(00, g) \cup \eta^{(n)}(01, g+1) \cup \eta^{(n)}(11, g),\\
\xi^{(n+1)}(11, g) &= \eta^{(n)}(10, g) \cup \eta^{(n)}(01, g+1) \cup \eta^{(n)}(11, g),
\end{aligned}$$

for $g \in \mathbb{Z}_3$. So F'_n is given by the matrix

$$\begin{array}{c|cccccccccccc|}
 & 00{\times}0 & 00{\times}1 & 00{\times}2 & 01{\times}0 & 01{\times}1 & 01{\times}2 & 10{\times}0 & 10{\times}1 & 10{\times}2 & 11{\times}0 & 11{\times}1 & 11{\times}2 \\
00{\times}0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
00{\times}1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
00{\times}2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
01{\times}0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
01{\times}1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
01{\times}2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
10{\times}0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
10{\times}1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\
10{\times}2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
11{\times}0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
11{\times}1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
11{\times}2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1
\end{array}.$$

According to (4.5) the graph $(\alpha_n \times \mathbb{Z}_3, \widetilde{V_n \times \mathbb{Z}_3})$ (see Figure 4.4) has the following arrows

$$\begin{aligned}
(01) \times g &= ((J_1, g) \to (J_2, g+2)), &\qquad (10) \times g &= ((J_2, g) \to (J_1, g+1)),\\
(00) \times g &= ((J_1, g) \to (J_1, g+1)), &\qquad (11) \times g &= ((J_2, g) \to (J_2, g+2)),
\end{aligned}$$

$g \in \mathbb{Z}_3$, $J_1 = \{00, 01\}$, $J_2 = \{10, 11\}$.

To calculate I_n, we find a spanning tree using (4.4.1) and (4.4.2). We select the same spanning tree (α_n, E) $E = E_n = \{01\}$ as in Subsection 3.1. According to (4.6) we have $a_{00} = d(P_{00}) = 1$, $a_{11} = d(P_{11}) = 2$, $a_{10} = d(P_{10}) = 0$.

Next, we have

$$P_{J_1 J_1} = P_{J_2 J_2} = P^-_{J_1 J_2} = P^+_{J_2 J_1} = \{\emptyset\} \quad \text{and} \quad P^+_{J_1 J_2} = P^-_{J_2 J_1} = \{01\}.$$

Thus, (4.7) gives $d(P_{J_1 J_1}) = d(P_{J_2 J_2}) = 0$, $d(P_{J_1 J_2}) = 2$, $d(P_{J_2 J_1}) = 1$.

Now, we apply the procedures 4.4.1 and 4.4.2. Because $a_{00} = 1$ generates $\mathbb{Z}_3$, we have $s = 1$, $v_1 = (00)$, $\mathbb{Z}_3 = H_1 \supsetneq H_0 = \{0\}$, $r_1 = 3$, $\alpha_g^{(1)}(J_1) = \{(J_1, g), (J_2, g+2)\}$, $E_g^{(1)}(J_1) = 01 \times g = \{((J_1, g) \to (J_2, g+2))\}$. The graphs $S_g^{(1)}(J_1) = (\alpha_g^{(1)}(J_1), E_g^{(1)}(J_1))$ are marked in Figure 4.4 with thick lines.

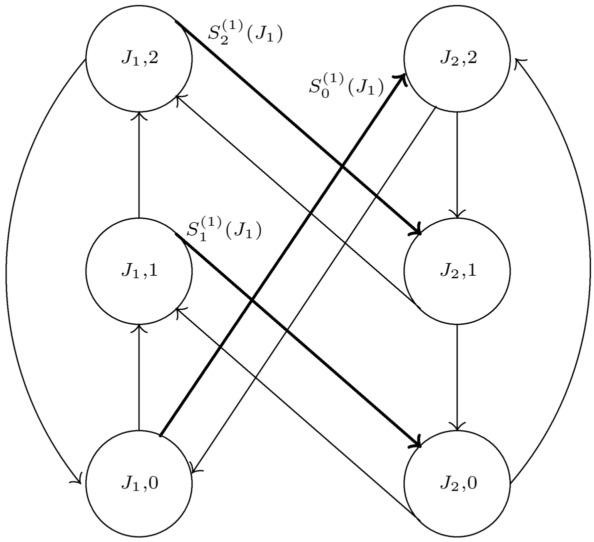

FIGURE 4.4

Next, we have

$$\begin{aligned} \alpha_g^{(2)}(J_1) &= \alpha_g^{(1)}(J_1) \cup \alpha_{g+1}^{(1)}(J_1) \cup \alpha_{g+2}^{(1)}(J_1) = \alpha \times V, \quad V = V_n, \\ E'_n &= E' = E_g^{(2)}(J_1) \\ &= E_g^{(1)}(J_1) \cup E_{g+1}^{(1)}(J_1) \cup E_{g+2}^{(1)}(J_1) \cup \{(00 \times 0)\} \cup \{(00 \times 1)\} \end{aligned}$$

for every $g \in \mathbb{Z}_3$, $E'_n = E' = \{01 \times \mathbb{Z}_3, 00 \times 0, 00 \times 1\}$. The tree $(\alpha \times G, E')$ is presented on Figure 4.5.

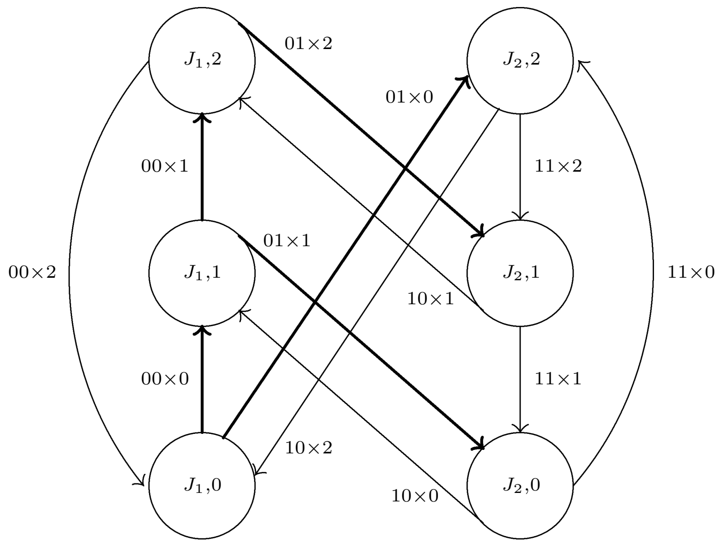

Figure 4.5

Next, we have $I_n = I\langle x_{00\times 0}, \ldots, x_{11\times 2}\rangle = \langle y_{00\times 2}, \ldots, y_{11\times 2}\rangle$, where

$$\begin{aligned}
y_{00\times 2} &= x_{00\times 2} + x_{00\times 0} + x_{00\times 1},\\
y_{10\times 0} &= x_{10\times 0} + x_{01\times 1},\\
y_{10\times 1} &= x_{10\times 1} + x_{01\times 2},\\
y_{10\times 2} &= x_{10\times 2} + x_{01\times 0},\\
y_{11\times 0} &= x_{11\times 0} + x_{00\times 0} - x_{01\times 0} + x_{01\times 1},\\
y_{11\times 1} &= x_{11\times 1} + x_{00\times 1} - x_{01\times 1} + x_{01\times 2},\\
y_{11\times 2} &= x_{11\times 2} - x_{00\times 0} - x_{00\times 1} + x_{01\times 0} - x_{01\times 2}.
\end{aligned}$$

Thus, the homomorphisms $\widehat{G}_n$ are given by the matrices $B = B_n$

$$\begin{array}{c c}
 & \begin{array}{ccccccc} 00{\times}2 & 10{\times}0 & 10{\times}1 & 10{\times}2 & 11{\times}0 & 11{\times}1 & 11{\times}2 \end{array} \\
\begin{array}{c} 00{\times}2 \\ 10{\times}0 \\ 10{\times}1 \\ 10{\times}2 \\ 11{\times}0 \\ 11{\times}1 \\ 11{\times}2 \end{array} &
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0\\
0 & 2 & 0 & 0 & 0 & 1 & 0\\
1 & 0 & 2 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 2 & 1 & 0 & 0\\
0 & 2 & 0 & 0 & 0 & 1 & 0\\
1 & 0 & 2 & 0 & 0 & 0 & 1\\
-1 & -1 & -1 & 1 & 1 & 0 & 0
\end{bmatrix}
\end{array}.$$

The determinant of the matrix is 0. To find $K^0(X\times\mathbb{Z}_3, T_c)$, we apply the Stage II of the algorithm. In this case $\operatorname{rank}(\widehat{G}^2) = \operatorname{rank}(\widehat{G}^3) = 4$. We have $y_{11\times 0} = y_{10\times 0}$, $y_{11\times 2} = y_{10\times 2}$, $y_{11\times 2} = y_{10\times 2} - y_{00\times 2}$.

We come to the homomorphisms $G'_n = G'$ given by the matrix

$$\begin{array}{c} \\ 00\times 2 \\ 10\times 0 \\ 10\times 1 \\ 10\times 2 \end{array} \begin{array}{c} \begin{array}{cccc} 00\times 2 & 10\times 0 & 10\times 1 & 10\times 2 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix} \end{array}.$$

Since $\det(G'_n) = 9$, $K^0(X,T) = \bigcup_{n=0}^{\infty}(G')^{-n}(\mathbb{Z}^4)$. In a sequel, we replace 00×2 by 1, 10×0 by 2, 10×1 by 3 and 10×2 by 4. By a similar reasoning as in Subsection 4.3 we find $\mathrm{Inf}(X,T) = \{\overline{y} = \langle y_1, y_2, y_3, y_4\rangle \in K^0(X,T) : y_2 + y_3 + y_4 = 0\}$.

To describe the cone $K^0(X,T)^+$, we must find the images $\overline{w}_v$ of the vectors $\widehat{e}_v \in \mathbb{Z}^{12}$, $v \in V_n \times \mathbb{Z}_3$ via I_n and via $\widehat{G}_n^2$ to $\widehat{G}_n^2(\mathbb{Z}^{12}) \simeq \mathbb{Z}^4$. We find

$$\begin{aligned} \overline{w}_{00\times 0} &= \langle 1,0,0,1\rangle, & \overline{w}_{00\times 1} &= \langle 1,1,0,0\rangle, \\ \overline{w}_{00\times 2} &= \langle 1,0,1,0\rangle, & \overline{w}_{11\times 0} &= \langle 0,0,0,1\rangle, \\ \overline{w}_{11\times 1} &= \langle 0,1,0,0\rangle, & \overline{w}_{11\times 2} &= \langle 0,0,1,0\rangle. \end{aligned}$$

The matrix G' has the eigenvalue $\lambda = 3$ with the eigenvector $\overline{x}_3 = \langle 3/2, 1, 1, 1\rangle$.

Let $\Pi \subset \mathbb{R}^4$ be the subspace defined by $\Pi = \{\overline{y} = \langle y_1, y_2, y_3, y_4\rangle : y_2 + y_3 + y_4 = 0\}$. Then every $\overline{y} \in \mathbb{Z}^4$ has the unique decomposition $\overline{y} = a\cdot \overline{x}_3 + \overline{u}$, $\overline{u} \in \Pi$, $a = (y_2+y_3+y_4)/3$. In particular $\overline{w}_v = \overline{x}_3/3 + \overline{u}_v$, $v = 00\times 0$, 00×1, 00×2, 11×0, 1×1, 11×2. We have

$$\begin{aligned} \overline{u}_{00\times 0} &= \left\langle \frac{1}{2}, -\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\right\rangle, & \overline{u}_{00\times 1} &= \left\langle \frac{1}{2}, \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right\rangle, \\ \overline{u}_{00\times 2} &= \left\langle \frac{1}{2}, -\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right\rangle, & \overline{u}_{11\times 0} &= \left\langle -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\right\rangle, \\ \overline{u}_{11\times 1} &= \left\langle -\frac{1}{2}, \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right\rangle, & \overline{u}_{11\times 2} &= \left\langle -\frac{1}{2}, -\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right\rangle. \end{aligned}$$

The subspace Π is isomorphic to $\mathbb{R}^3 = \{\overline{z} = \langle z_1, z_2, z_3\rangle, z_i \in \mathbb{R}\}$ by the mapping $y_1 = z_1$, $y_2 = z_2$, $y_3 = z_3$, $y_4 = -(z_1 + z_3)$. Then the homomorphism $G'|\Pi = F\colon \mathbb{Z}^3 \to \mathbb{Z}^3$ is defined by the matrix

$$F = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 1 \end{bmatrix} \end{array}.$$

The matrix F has the eigenvalues $\lambda_1 = 1$, $\lambda_2 = (3 + i\sqrt{3})/2$, $\lambda_3 = (3 - i\sqrt{3})/2$, with the eigenvectors $\langle 1,0,0\rangle$, $\langle 0,1,(i\sqrt{3}-1)/2\rangle$, $\langle 0,1,(-1-i\sqrt{3})/2\rangle$. Then,

$$F^{-k} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(\lambda_1^{-k} + \lambda_2^{-k}) + \frac{i\sqrt{3}}{6}(\lambda_2^{-k} - \lambda_1^{-k}), & \frac{i\sqrt{3}}{2}(\lambda_2^{-k} - \lambda_1^{-k}) \\ 0 & \frac{i\sqrt{3}}{2}(\lambda_1^{-k} - \lambda_2^{-k}) & \lambda_1^{-k}\frac{3+i\sqrt{3}}{6} + \lambda_2^{-k}\frac{i\sqrt{3}-3}{6} \end{bmatrix}$$

for $k = 0,1,\ldots$ Taking $k = 12l$ and $k = 12l+6$ we get

$$F^{-12l} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3^{6l}} & 0 \\ 0 & 0 & \frac{1}{3^{6l}} \end{bmatrix}, \qquad F^{-12l-6} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3^{6l+3}} & 0 \\ 0 & 0 & -\frac{1}{3^{6l+3}} \end{bmatrix}.$$

Next, we have

$$\begin{aligned}
(G')^{-12l}(\overline{w}_{00\times 0}) &= \frac{1}{3^{12l+1}}\overline{x}_3 + \left\langle \frac{1}{2}, -\frac{1}{3^{6l+1}}, -\frac{1}{3^{6l+1}}, \frac{2}{3^{6l+1}} \right\rangle, \\
(G')^{-12l}(\overline{w}_{00\times 1}) &= \frac{1}{3^{12l+1}}\overline{x}_3 + \left\langle \frac{1}{2}, \frac{2}{3^{6l+1}}, -\frac{1}{3^{6l+1}}, -\frac{1}{3^{6l+1}} \right\rangle, \\
(G')^{-12l}(\overline{w}_{00\times 2}) &= \frac{1}{3^{12l+1}}\overline{x}_3 + \left\langle \frac{1}{2}, -\frac{1}{3^{6l+1}}, \frac{2}{3^{6l+1}}, -\frac{1}{3^{6l+1}} \right\rangle, \\
(G')^{-12l}(\overline{w}_{11\times 0}) &= \frac{1}{3^{12l+1}}\overline{x}_3 + \left\langle -\frac{1}{2}, -\frac{1}{3^{6l+1}}, -\frac{1}{3^{6l+1}}, \frac{2}{3^{6l+1}} \right\rangle, \\
(G')^{-12l}(\overline{w}_{11\times 1}) &= \frac{1}{3^{12l+1}}\overline{x}_3 + \left\langle -\frac{1}{2}, \frac{2}{3^{6l+1}}, -\frac{1}{3^{6l+1}}, -\frac{1}{3^{6l+1}} \right\rangle, \\
(G')^{-12l}(\overline{w}_{11\times 2}) &= \frac{1}{3^{12l+1}}\overline{x}_3 + \left\langle -\frac{1}{2}, -\frac{1}{3^{6l+1}}, \frac{2}{3^{6l+1}}, -\frac{1}{3^{6l+1}} \right\rangle.
\end{aligned}$$

Then, the vectors

$$\overline{A} = \langle 1,0,0,0\rangle = \lim_{l\to\infty} 2(G')^{-12l}(\overline{w}_{00\times 0})$$

and

$$-\overline{A} = \langle -1,0,0,0\rangle = \lim_{l\to\infty} 2(G')^{-12l}(\overline{w}_{11\times 0})$$

are some elements of the cone $K^0(X,T)^+$. At the same time the vectors

$$\begin{aligned}
\overline{B}_1 = \langle 0,0,-3,3\rangle = \lim_{l\to\infty} 3^{6l+1}[&(G')^{-12l}(\overline{w}_{00\times 0}) + (G')^{-12l}(\overline{w}_{00\times 1}) \\
&+ (G')^{-12l}(\overline{w}_{00\times 2}) + 2(G')^{-12l}(\overline{w}_{11\times 0}) + (G')^{-12l}(\overline{w}_{11\times 1})], \\
\overline{C}_1 = \langle 0,-3,0,3\rangle = \lim_{l\to\infty} 3^{6l+1}[&(G')^{-12l}(\overline{w}_{00\times 0}) + (G')^{-12l}(\overline{w}_{00\times 1}) \\
&+ (G')^{-12l}(\overline{w}_{00\times 2}) + 2(G')^{-12l}(\overline{w}_{11\times 0}) + (G')^{-12l}(\overline{w}_{11\times 2})]
\end{aligned}$$

are elements of $K^0(X,T)^+$. Then, the vectors

$$3^5(G')^{-12}(\overline{B}_1) = \langle 0,0,-1,1\rangle = \overline{B}, \quad 3^5(G')^{-12}(\overline{C}_1) = \langle 0,-1,0,1\rangle = \overline{C}$$

are elements of $K^0(X,T)^+$. Next, we take the images of $\overline{B}, \overline{C}$ via $(G')^{-18}$. We get

$$3^3(G')^{-6}(\overline{B}) = \langle 0,0,1,-1\rangle = -\overline{B}, \quad 3^3(G')^{-6}(\overline{B}) = \langle 0,1,0,-1\rangle = -\overline{C}.$$

Because the vectors $\overline{A}$, $\overline{B}$, $\overline{C}$ form a base of the group $\Pi_{\text{int}} = \Pi \cap \mathbb{Z}^4$ and $\pm\overline{A}$, $\pm\overline{B}$, $\pm\overline{C}$ are elements of the cone $K^0(X,T)^+$, then the set of all elements of $K^0(X,T)^+$ is

$$\bigcup_{n=0}^{\infty} (G')^{-n}(\Pi_{\text{int}}) = \{\overline{y} = \langle y_1,y_2,y_3,y_4\rangle \in K(X,T) : y_2+y_3+y_4 = 0\}.$$

In this way, we proved that

$$K^0(X,T)^+ = \{\overline{y} = \langle y_1,y_2,y_3,y_4\rangle \in K(X,T) : y_2+y_3+y_4 \geq 0\}.$$

4.6. Orbit equivalence and strong orbit equivalence. Now, we can examine the topological orbit equivalence and the topological strong orbit equivalence of the topological flows from the Section 4. For the Cantor minimal systems, there are known complete invariants of the above orbit equivalences ([5]). We will use the following theorems for Cantor minimal systems (X,T) and (Y,S).

THEOREM 4.1. *The following statements are equivalent:*

(a) (X,T) *and* (Y,S) *are strong orbit equivalent.*
(b) $K^0(X,T)$ *is order isomorphic to* $K^0(Y,S)$ *by a map preserving the distinguished order units.*

Now let (X,T) and (Y,S) be strictly ergodic.

THEOREM 4.2. *The following are equivalent:*

(a) (X,T) *and* (Y,S) *are orbit equivalent.*
(b) $\widehat{K}^0(X,T)$ *is order isomorphic to* $\widehat{K}^0(Y,S)$ *by a map preserving the distinguished order units.*
(c) *The set of the values* $\{\mu(U) : U$ *is a clopen set of* $X\}$ *is equal to the set of values* $\{\nu(U) : V$ *is a clopen set of* $Y\}$*, where* μ *and* ν *are the unique* T*-invariant and* S*-invariant measures.*

The dimension group $C(X,\mathbb{Z})/B_T$ of each example of the Section 4 is of the form $\bigcup_{n=0}^{\infty} F^{-n}(\mathbb{Z}^k) \subset \mathbb{Q}^k$, where $F\colon \mathbb{Z}^k \to \mathbb{Z}^k$ is a homomorphism given by a matrix F with non-negative integer entries and $|\det(F)| > 1$. The natural question arises when two groups of such kind are isomorphic. The group $\bigcup_{n=0}^{\infty} F^{-n}(\mathbb{Z}^k)$

is an union of increasing chain of groups $F^{-n}(\mathbb{Z}^k)$, $n \in \mathbb{N}$, each of them is a free abelian group of rank k. Thus, two groups of such kind $\bigcup_{n=0}^{\infty} F^{-n}(\mathbb{Z}^k)$ and $\bigcup_{n=0}^{\infty} G^{-n}(\mathbb{Z}^l)$ are not isomorphic if $k \neq l$. It is easy to give a necessary condition for two groups $\bigcup_{n=0}^{\infty} F^{-n}(\mathbb{Z}^k)$ and $\bigcup_{n=0}^{\infty} G^{-n}(\mathbb{Z}^k)$ to be isomorphic. Let $\mathcal{P}_F$ be the set of prime numbers appearing in the decomposition of $\det(F)$ into a product of primes.

Fact 4.3. *If the group $\bigcup_{n=0}^{\infty} F^{-n}(\mathbb{Z}^k)$ and $\bigcup_{n=0}^{\infty} G^{-n}(\mathbb{Z}^k)$ are isomorphic, where F, G are matrices with non-negative integers such that $|\det(F)| > 1$ and $|\det(G)| > 1$, then $\mathcal{P}_F = \mathcal{P}_G$.*

Proof. It is enough to analyze the groups $F^{-n}(\mathbb{Z}^k)/\mathbb{Z}^k$ and $G^{-n}(\mathbb{Z}^k)/\mathbb{Z}^k$, $n = 1, 2, \ldots$ The quotient group $F^{-n}(\mathbb{Z}^k)/\mathbb{Z}^k$ is a finite abelian group of the order $|\det(F)|^n$ and it is a direct product of some p-groups $G_p(n)$, $p \in \mathcal{P}_F$. It is evident that if $V : \bigcup_{n=0}^{\infty} F^{-n}(\mathbb{Z}^k) \to \bigcup_{n=0}^{\infty} G^{-n}(\mathbb{Z}^k)$ is a group isomorphism then the quotient groups $F^{-n}(\mathbb{Z}^k)/\mathbb{Z}^k$ and $G^{-n}(\mathbb{Z}^k)/\mathbb{Z}^k$ contains the same quantity of p-groups. Thus $\mathcal{P}_F = \mathcal{P}_G$. $\square$

Remark 4.4. Observe that by the general theory of free abelian groups, a structure of all $F^n(\mathbb{Z}^k)$ groups is known. Of course $F^n(\mathbb{Z}^k)$ is a free subgroup of a rank k of $\mathbb{Z}^k$.

(4.9) Then, there is a base $u_1, \ldots, u_k$ of $\mathbb{Z}^k$ and positive integers $d_1, \ldots, d_k$ such that $v_1 = d_1 \cdot u_1, \ldots, v_k = d_k \cdot u_k$ is a base of $F(\mathbb{Z}^k)$ and d_i is a divisor of d_{i+1} for each $i = 1, \ldots, k-1$. Moreover, the numbers $d_1, \ldots, d_k$ are unique (see [6]).

The quotient group $\mathbb{Z}^k/F(\mathbb{Z}^k)$ is isomorphic to the group $\mathbb{Z}_{d_1} \times \ldots \times \mathbb{Z}_{d_k}$. Of course $d_1 \cdot \ldots \cdot d_k = |\det(F)|$.

A procedure of effective finding the numbers $d_1, \ldots, d_k$ consists of using the elementary transformations of two types:

- adding linear combinations of some rows (columns) with integer coefficients to the other ones,
- changing rows (columns).

We demonstrate this procedure on the matrix $F = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ from the Subsection 3.1. The successive steps of the procedure we mark by the sign "$\to$". We have

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} \text{subtracting the second} \\ \text{column from the first one} \end{pmatrix} \to \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{pmatrix} \text{adding the first row} \\ \text{to the second one} \end{pmatrix}$$
$$\to \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \begin{pmatrix} \text{subtracting the first} \\ \text{column from the second one} \end{pmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

Then $d_1 = 1$, $d_2 = 3$ and $\mathcal{P}_F = \{3\}$.

Proceeding in the same way we find:

Example 4.5. Teoplitz flow:

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}, \quad d_1 = 1,\ d_2 = 5,\ \mathcal{P}_F = \{5\}.$$

Teoplitz-Morse flow:

$$\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad d_1 = 1,\ d_2 = 4,\ \mathcal{P}_F = \{2\}.$$

Example 4.6.

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad d_1 = 1,\ d_2 = 2,\ \mathcal{P}_F = \{2\}.$$

Example 4.7.

$$\begin{bmatrix} 3 & -1 & 1 & -1 \\ 0 & 3 & -4 & -1 \\ -1 & 2 & -3 & 0 \\ 5 & -3 & 3 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix},$$

$d_1 = 1$, $d_2 = 1$, $d_3 = 1$, $d_4 = 4$, $\mathcal{P}_F = \{2\}$.

Example 4.8.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix},$$

$d_1 = 1$, $d_2 = 1$, $d_3 = 1$, $d_4 = 9$, $\mathcal{P}_F = \{3\}$.

Using the Theorem 4.2 and the Fact 4.3 we can answer which of the topological flows described in Subsection 3.1 and Examples 4.1, 4.2 and 4.5 are strong orbit equivalent.

Corollary 4.9. *The Teoplitz–Morse flow from Subsection* 4.1 *and the Morse flow from Subsection* 4.2 *are strong orbitally equivalent. The topological flows from the remaining examples are not strong orbitally equivalent.*

To describe which of the topological flows are orbitally equivalent, we use the Theorem 4.1. It follows from our previous computations:

Example (from Subsection 3.1). $\widehat{K}^0(X,T) \simeq \{a/3^n,\ a \in \mathbb{Z},\ n = 0,1,\dots\}$.

Example 4.5.

$$\widehat{K}^0(X,T) \simeq \{a/5^n,\ a \in \mathbb{Z},\ n = 0,1,\dots\} \quad \text{(Teoplitz flow)},$$
$$\widehat{K}^0(X,T) \simeq \{a/2^n,\ a \in \mathbb{Z},\ n = 0,1,\dots\} \quad \text{(Teoplitz–Morse flow)}.$$

Example 4.6. $\widehat{K}^0(X,T) \simeq \{a/2^n,\ a \in \mathbb{Z},\ n = 0,1,\dots\}$.

Example 4.7. $\widehat{K}^0(X,T) \simeq \{a/2^n,\ a \in \mathbb{Z},\ n = 0,1,\dots\}$.

Example 4.8. $\widehat{K}^0(X,T) \simeq \{a/3^n,\ a \in \mathbb{Z},\ n = 0,1,\dots\}$.

Corollary 4.10. *The Chacon flow is orbitally equivalent to the topological flow in Subsection* 4.5. *The Teoplitz–Morse flow from the Example* 4.6 *and the topological flows from Subsection* 4.2 *are orbitally equivalent.*

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