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PERIODIC POINTS OF MULTI-VALUED ε-CONTRACTIVE MAPS

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Dedicated to Professor Andrzej Granas

Abstract. Let (X,d) be a nonempty metric space, and let $(2^X, H_d)$ be the hyperspace of all nonempty compact subsets of X with the Hausdorff metric. Let $F: X \to 2^X$ be an ε-contractive map. A general condition is given that guarantees the existence of a periodic point of F (the theorem extends a result of Edelstein to multi-valued maps). The condition holds when X is compact; hence, F has a periodic point when X is compact. It is shown that F has a fixed point (a point $p \in F(p)$) if X is a continuum. Applications to single-valued ε-expansive maps are given.

1. Introduction

Edelstein in [2] proved the following two results (definitions are in Section 2): Let (X,d) be a metric space, and let $f: X \to X$ be a map such that for some point $x \in X$, some subsequence of the sequence $\{f^n(x)\}_{n=1}^{\infty}$ of iterates converges to a point $p \in X$. If f is contractive, then p is a fixed point of f; if f is ε-contractive, then p is a periodic point of f.

Edelstein's fixed point result for contractive maps was extended to multi-valued maps in [4, p. 664]; however, Edelstein's periodic point result for ε-contractive maps was not extended to multi-valued maps in [4]. As a co-author of [4], I can affirm that Edelstein's result for ε-contractive maps was not extended to multi-valued maps for the simple reason that we could not prove the

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extended version. In this paper we prove the generalization to multi-valued ε-contractive maps of Edelstein's result for single-valued ε-contractive maps; we also prove a fixed point theorem for multi-valued ε-contractive maps, and we give applications to single-valued ε-expansive maps. Our main results are Theorem 3.2, Corollary 3.3 and Theorem 4.3; our applications are in Theorem 5.2 and Theorem 5.3.

Theorem 3.2 is for multi-valued maps whose values are nonempty compact sets; we will show at the end of Section 3 that the theorem does not generalize to maps whose values are nonempty, closed and bounded sets.

2. Definitions and preliminary results

We present the basic terminology and notation; we then include a few minor results that we use several times.

Throughout the paper, X denotes a nonempty metric space with a given metric d. For a point $x \in X$ and a nonempty subset A of X,

$$d(x, A) = \inf_{a \in A} d(x, a)$$

A *continuum* is a nonempty compact connected metric space. A *map* is a continuous function.

The hyperspaces $CB(X)$ and 2^X are the spaces

$$CB(X) = \{A \subset X : A \text{ is nonempty, closed and bounded}\}$$

and

$$2^X = \{A \subset X : A \text{ is nonempty and compact}\}$$

with the *Hausdorff metric H_d induced by the metric d*, defined by

$$H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}.$$

This definition of H_d is equivalent to another frequently given definition [5, p. 11] (also, 2.7 of [5, p. 14]).

For any map $F: X \to 2^X$, there is a natural *induced map* $\widehat{F}$ defined on 2^X as follows: For each $A \in 2^X$,

$$\widehat{F}(A) = \bigcup_{a \in A} F(a) = \bigcup F(A)$$

(see Proposition 2.2).

Let $\varepsilon > 0$. A map $f: X \to X$ is said to be *contractive* (*ε-contractive*) provided that for all $x, y \in X$ with $x \neq y$ (and $d(x, y) < \varepsilon$, respectively), $d(f(x), f(y)) < d(x, y)$ (see [2]). A *multi-valued contractive* (*ε-contractive*) *map* is a map $F: X \to CB(X)$ such that for all $x, y \in X$ with $x \neq y$ (and $d(x, y) < \varepsilon$, respectively), $H_d(F(x), F(y)) < d(x, y)$.

Confusion between the single-valued and multi-valued notions we just defined could occur in connection with the induced map $\widehat{F}$ since $\widehat{F}$ maps to a hyperspace. *When we say $\widehat{F}$ is contractive or ε-contractive, we always mean in the single-valued sense.* We remind the reader of this in two ways: we always include the modifier *multi-valued* when referring to a multi-valued map, and we often include the domain and range of a map to emphasize the type of map being considered. The distinction between multi-valued maps to hyperspaces and single-valued maps of hyperspaces to hyperspaces is important when it comes to considering fixed points and periodic points (defined below).

Let Z be a set, let $f: Z \to Z$ be a function, and let $n \geq 1$ be an integer. Then f^n denotes the *n-th iterate of f* (i.e. the composition $f \circ \ldots \circ f$ of f with itself $n-1$ times).

A *periodic point of a (single-valued) map $f: X \to X$* is a point $p \in X$ such that $f^n(p) = p$ for some integer $n \geq 1$.

A *fixed point of a multi-valued map $F: X \to CB(X)$* is a point $p \in X$ such that $p \in F(p)$.

A *periodic point of a multi-valued map $F: X \to 2^X$* is a point $p \in X$ such that $p \in \widehat{F}^n(\{p\})$ for some integer $n \geq 1$. When F maps X to $CB(X)$, the definition of a periodic point must be done more carefully since the analogue of $\widehat{F}$ for $CB(X)$ may not map to $CB(X)$: p is a *periodic point of $F: X \to CB(X)$* provided that there are finitely many points $p_0 = p, p_1, \ldots, p_n$ such that $p_i \in F(p_{i-1})$ for each $i = 1, \ldots, n$ and $p \in F(p_n)$.

We often use one or another of the four propositions below. For a proof of the following result, see 3.5 of [5, p. 18] or 4.13 of [7, p. 59].

PROPOSITION 2.1. *If Z is a compact metric space, then 2^Z is compact.*

PROPOSITION 2.2. *If $F: X \to 2^X$ is a map, then $\widehat{F}$ maps 2^X back into 2^X and is continuous.*

PROOF. Since F is continuous, $\widehat{F}$ maps 2^X back into 2^X because the union of a compact subset of 2^X is compact [5, 11.5(1), p. 91]. Since F and the union map $\cup: 2^{2^X} \to 2^X$ are continuous [5, 11.5(2), p. 91]), we see that $\widehat{F}$ is continuous. $\square$

PROPOSITION 2.3. *Let $Y \in 2^X$. If $F: X \to 2^X$ is a map such that $\widehat{F}^m(Y) \subset Y$ for some integer $m \geq 1$, then $\widehat{F}^m|2^Y$ maps 2^Y to 2^Y.*

PROOF. We prove the result for the case when $m = 1$; the result for any m then follows using Proposition 2.2 (with $X = Y$).

For any $A \in 2^Y$, $\widehat{F}(A) = \bigcup_{a \in A} F(a) \subset \bigcup_{y \in Y} F(y) = \widehat{F}(Y)$; thus, since $\widehat{F}(Y) \subset Y$ by assumption, $\widehat{F}(A) \subset Y$. Therefore, since $\widehat{F}(A) \in 2^X$ by Proposition 2.2, $\widehat{F}(A) \in 2^Y$. $\square$

PROPOSITION 2.4. *If $F: X \to 2^X$ is a multi-valued ε-contractive map, then $\widehat{F}^m: 2^X \to 2^X$ is an ε-contractive map for each integer $m \geq 1$.*

PROOF. First note that $\widehat{F}^m$ does map 2^X to 2^X by Proposition 2.2.

We prove that $\widehat{F}^m$ is ε-contractive when $m = 1$; the proof for any m is then an easy induction which we omit.

Let $A, B \in 2^X$ such that $A \neq B$ and $H_d(A, B) < \varepsilon$. By the symmetry in the definition of H_d, we need only prove that

$$(*) \qquad \sup_{x \in \widehat{F}(A)} d(x, \widehat{F}(B)) < H_d(A, B).$$

Let $x \in \widehat{F}(A)$. Then $x \in F(a_0)$ for some $a_0 \in A$. Hence,

(1) $d(x, F(b_0)) \leq H_d(F(a_0), F(b_0))$.

Since $a_0 \in A$, $d(a_0, B) \leq H_d(A, B)$. Let $b_0 \in B$ such that $d(a_0, b_0) = d(a_0, B)$. Then $d(a_0, b_0) \leq H_d(A, B)$. Hence, $d(a_0, b_0) < \varepsilon$. Thus, considering the cases when $a_0 \neq b_0$ and when $a_0 = b_0$ separately, we have that

(2) $H_d(F(a_0), F(b_0)) < H_d(A, B)$.

Since $b_0 \in B$, $F(b_0) \subset \widehat{F}(B)$; hence, clearly, $d(x, \widehat{F}(B)) \leq d(x, F(b_0))$. Thus, by (1) and (2), we have

(3) $d(x, \widehat{F}(B)) < H_d(A, B)$.

Finally, note that $\widehat{F}(A)$ is compact by Proposition 2.2; therefore, having proved (3) for all $x \in \widehat{F}(A)$, $(*)$ follows from the compactness of $\widehat{F}(A)$. □

3. Existence of periodic points

The following lemma is elementary; nevertheless, it is a key observation for the proof of our main theorem (Theorem 3.2).

LEMMA 3.1. *If X is compact and $f: X \to X$ is an ε-contractive map, then f has only finitely many periodic points.*

PROOF. Assume that p and q are periodic points of f with $p \neq q$ such that $d(p, q) < \varepsilon$. Let $k, \ell \geq 1$ be integers such that $f^k(p) = p$ and $f^\ell(q) = q$. Note that $p = f^{k\ell}(p)$ and $q = f^{k\ell}(q)$. Thus, since $f^{k\ell}$ is ε-contractive,

$$d(p, q) = d(f^{k\ell}(p), f^{k\ell}(q)) < d(p, q),$$

which is impossible. Therefore, we have proved that any two periodic points of f must be at least ε apart. The lemma now follows from the compactness of X. □

THEOREM 3.2. *Let $F: X \to 2^X$ be a multi-valued ε-contractive map. Assume that for some $A \in 2^X$, a subsequence $\{\widehat{F}^{n_i}(A)\}_{i=1}^{\infty}$ of $\{\widehat{F}^n(A)\}_{n=1}^{\infty}$ converges to* *a point $B \in 2^X$. Then there is a point $b_0 \in B$ such that b_0 is a periodic point of F.*

Proof. By Proposition 2.4, $\widehat{F}: 2^X \to 2^X$ is an ε-contractive map. Hence, by Theorem 2 of [2], B is a periodic point of $\widehat{F}$, say $\widehat{F}^k(B) = B$ (k a positive integer).

Since $B \in 2^X$ and $\widehat{F}^k(B) = B$, we see by Proposition 2.3 that $\widehat{F}^k|2^B$ maps 2^B to 2^B; furthermore, by Proposition 2.4, $\widehat{F}^k|2^B: 2^B \to 2^B$ is ε-contractive. Thus, since 2^B is compact (Proposition 2.1), we see from Lemma 3.1 that $\widehat{F}^k|2^B$ has only finitely many periodic points, say $B_1 = B, B_2, \ldots, B_n$. At least one of the sets $B_1, \ldots, B_n$ does not contain any of the others. Since we will have no further use for the assumption in our theorem that $\{\widehat{F}^{n_i}(A)\}_{i=1}^{\infty} \to B$, we can assume without loss of generality that B itself is such a minimal set; that is, no compact proper subset of B is a periodic point of $\widehat{F}^k$.

Let $p \in B$. Note that $\widehat{F}^{kn}(B) = B$ for each integer $n \geq 1$; hence, by Proposition 2.3, $\widehat{F}^{kn}(\{p\}) \in 2^B$ for each integer $n \geq 1$. Thus, since 2^B is compact (by Proposition 2.1), the sequence $\{\widehat{F}^{kn}(\{p\})\}_{n=1}^{\infty}$ has a convergent subsequence $\{\widehat{F}^{kn_i}(\{p\})\}_{i=1}^{\infty}$, say

$$\{\widehat{F}^{kn_i}(\{p\})\}_{i=1}^{\infty} \to C \in 2^B.$$

Then, since $\widehat{F}^k|2^B: 2^B \to 2^B$ is ε-contractive, Theorem 2 of [2] gives us that C is a periodic point of $\widehat{F}^k$. Therefore, since C is a compact subset of B, $C = B$ by the minimality of B. Hence,

$$\{\widehat{F}^{kn_i}(\{p\})\}_{i=1}^{\infty} \to B.$$

Therefore, there is an integer $\ell \geq 1$ such that $d(p, \widehat{F}^{kn_\ell}(\{p\})) < \varepsilon$.

Since we will use the map $\widehat{F}^{kn_\ell}|2^B$ many times throughout the rest of the proof, let us denote $\widehat{F}^{kn_\ell}|2^B$ by G and list three relevant properties of G that we already know:

(1) G maps 2^B to 2^B (by Proposition 2.3 since $\widehat{F}^k(B) = B$),

(2) G is ε-contractive (by Proposition 2.4),

(3) $d(p, G(\{p\})) < \varepsilon$ (by our choice of ℓ).

Now, let $r = \inf_{b \in B} d(b, G(\{b\}))$. Note from (3) the following important fact:

(4) $r < \varepsilon$.

Since B is compact and G is continuous, we see that

(5) $r = d(b_0, G(\{b_0\}))$ for some point $b_0 \in B$.

We show that the point b_0 in (5) satisfies the conclusion of our theorem. Since we already know that $b_0 \in B$ (by (5)), we are left to show that b_0 is a periodic point of F. We show this by proving that

$$(*) \qquad r = 0.$$

Since $G(\{b_0\})$ is nonempty and compact (by (1)), we see from (5) that

(6) $r = d(b_0, y)$ for some point $y \in G(\{b_0\})$.

Proof of $(*)$. Now, suppose by way of contradiction that $(*)$ is false, i.e. $r > 0$. Then, by (6), $b_0 \neq y$; in addition, $d(b_0, y) < \varepsilon$ by (4) and (6). Hence, by (2) and (6), we have that

(7) $H_d(G(\{b_0\}), G(\{y\})) < d(b_0, y) = r$.

Since $y \in G(\{b_0\})$ (by (6)), it follows from the definition of H_d that

$$d(y, G(\{y\})) \leq H_d(G(\{b_0\}), G(\{y\})),$$

hence, by (7), we have that

(8) $d(y, G(\{y\})) < r$.

Now, note that $y \in B$ (since $y \in G(\{b_0\}) \subset B$ by (6) and (1)). Thus, (8) contradicts the fact that $r = \inf_{b \in B} d(b, G(\{b\}))$. Therefore, we have proved $(*)$.

Finally, since $r = 0$, we see from (5) that $b_0 \in G(\{b_0\}) = \widehat{F}^{kn_\ell}(\{b_0\})$, which proves that b_0 is a periodic point of F. □

COROLLARY 3.3. *If X is compact and $F: X \to 2^X$ is a multi-valued ε-contractive map, then F has a periodic point.*

PROOF. There exists $A \in 2^X$ (recall from section 2 that $X \neq \emptyset$). Therefore, since 2^X is compact (Proposition 2.1), the corollary follows from Theorem 3.2.□

We prove in the next section that the map F in Corollary 3.3 has a fixed point when X is a continuum.

The generalization of the Banach Contraction Mapping Theorem to multi-valued maps with values in the general space $CB(X)$ was proved in Theorem 5 of [8, p. 479]. However, our Theorem 3.2 would be false for maps with values in $CB(X)$.

EXAMPLE 3.4. The map $F: X \to CB(X)$ in the example in [4, p. 665] is contractive (hence ε-contractive for any $\varepsilon > 0$). For a particular point, denoted by y in [4], the sequence $\{\widehat{F}^n(\{y\})\}_{n=1}^{\infty}$ of iterates in $CB(X)$ is constant, hence convergent. However, it is easy to see that the map F has no periodic point.

4. A Fixed Point Theorem

In Corollary 3.3 we proved that if X is compact and $F: X \to 2^X$ is a multi-valued ε-contractive map, then F has a periodic point. Simple examples show that even single-valued ε-contractive selfmaps of compact metric spaces may not have fixed points (e.g. the fixed point free map of $\{0,1\}$ onto $\{0,1\}$ is 1-contractive). Nevertheless, we prove in Theorem 4.3 that when X is a continuum, a multi-valued ε-contractive map $F: X \to 2^X$ must have a fixed point. Theorem 4.3 for single-valued ε-contractive maps follows from 6.2 of [2, p. 78].

We have tried, but failed, to obtain Theorem 4.3 directly from Theorem 3.2 and Corollary 3.3. The proof we give is based on combining a few facts and techniques in the literature. In essence, the proof is a matter of adjusting part of the proof of Theorem 6 of [8]; the adjustment is made possible by a fact from a proof in [9, p. 216].

We often consider another metric on X along with the original metric d. For clarity (in this section only), we write (X,d) to remind the reader that d denotes the original metric.

We need a definition and some notation.

Let $x, y \in X$. An *δ-chain in X from x to y* is a finite indexed set of points $x_0 = x, x_1, \dots, x_n = y$ of X such that $d(x_i, x_{i+1}) \le \delta$ for all $i = 0, \dots, n-1$ (the usual condition is $d(x_i, x_{i+1}) < \delta$, but the last part of Lemma 4.1 is easier to state if we allow $d(x_i, x_{i+1})$ to be δ). We denote the collection of all δ-chains in X from x to y by $\mathcal{C}_\delta(x,y)$.

Let (X,d) be a continuum, and let $\delta > 0$. Define $d_\delta: X \times X \to \mathbb{R}^1$ as follows:

$$d_\delta(x,y) = \inf\left\{ \sum_{i=0}^{n-1} d(x_i, x_{i+1}) : \{x_0, \dots, x_n\} \in \mathcal{C}_\delta(x,y) \right\}.$$

The idea of using d_δ in connection with changing local Lipschitz maps to global Lipschitz maps seems to have originated in 2.34 of [3, p. 691] (although the germ of the idea is apparent in the proof of the Proposition in [1, p. 8]). The idea was used in [8, p. 481] and then in [9, p. 216].

The lemma below summarizes the general properties of d_δ, its relation to d, and the relation of H_{d_δ} to H_d. For a proof of the parts of the lemma not involving the Hausdorff metrics, see [9, pp. 216–217]; the part involving the Hausdorff metrics is easy (as was noted in the proof of Theorem 6 of [4]).

LEMMA 4.1. *Let (X,d) be a continuum, and let $\delta > 0$. Then d_δ is a metric giving the topology on X, $d \le d_\delta$, $d(x,y) = d_\delta(x,y)$ if $d(x,y) < \delta$, and $H_{d_\delta}(A,B) = H_d(A,B)$ for all $A, B \in 2^X$ such that $H_d(A,B) < \delta$. Furthermore, for any points $x, y \in X$, there exists $\{x_0, \dots, x_n\} \in \mathcal{C}_\delta(x,y)$ such that $d_\delta(x,y) = \sum_{i=0}^{n-1} d(x_i, x_{i+1})$.*

The following theorem is the multi-valued analogue of Theorem 2.1 of [9] (in the presence of compactness, locally contractive as defined in [9] is the same as ε-contractive for some $\varepsilon > 0$, as is readily seen using Lebesgue numbers of covers [6, p. 24]).

THEOREM 4.2. *Let (X,d) be a continuum, and let $F\colon X \to 2^X$ be a multi-valued ε-contractive map with respect to H_d and d. Then, for any δ such that $0 < \delta < \varepsilon$, F is a multi-valued contractive map with respect to the metrics H_{d_δ} and d_δ.*

PROOF. Fix δ such that $0 < \delta < \varepsilon$. Let $x, y \in X$ such that $x \neq y$. By Lemma 4.1, there exists $\{x_0, \dots, x_n\} \in \mathcal{C}_\delta(x,y)$ with $x_i \neq x_{i+1}$ for each $i \leq n-1$ such that

(1) $d_\delta(x,y) = \Sigma_{i=0}^{n-1} d(x_i, x_{i+1})$.

Since $d(x_i, x_{i+1}) < \varepsilon$ and $x_i \neq x_{i+1}$ for each i, and since $F\colon X \to 2^X$ is a multi-valued ε-contractive map with respect to H_d and d, we have

(2) $H_d(F(x_i), F(x_{i+1})) < d(x_i, x_{i+1})$ for each $i \leq n-1$.

Since $d(x_i, x_{i+1}) \leq \delta$ for each i, (2) gives us that $H_d(F(x_i), F(x_{i+1})) < \delta$ for each i; hence, by Lemma 4.1, we have

(3) $H_{d_\delta}(F(x_i), F(x_{i+1})) = H_d(F(x_i), F(x_{i+1}))$ for each $i \leq n-1$.

Now, using the triangle inequality, then using (3), (2) and (1) in turn,

$$\begin{aligned} H_{d_\delta}(F(x), F(y)) &\leq \sum_{i=0}^{n-1} H_{d_\delta}(F(x_i), F(x_{i+1})) \\ &= \sum_{i=0}^{n-1} H_d(F(x_i), F(x_{i+1})) < \sum_{i=0}^{n-1} d(x_i, x_{i+1}) = d_\delta(x,y). \qquad \square \end{aligned}$$

THEOREM 4.3. *If X is a continuum and $F\colon X \to 2^X$ is a multi-valued ε-contractive map, then F has a fixed point.*

PROOF. Fix δ such that $0 < \delta < \varepsilon$. Since X with its original metric d is compact, (X, d_δ) is compact by Lemma 4.1. Hence, $(2^X, H_{d_\delta})$ is compact by Proposition 2.1. Therefore, the theorem follows from Theorem 4.2 and Theorem 4 of [4]. $\square$

5. Applications to single-valued ε-expansive maps

Let $Y \subset X$, and let $\varepsilon > 0$. A map $f\colon Y \to X$ is said to be *ε-expansive* provided that for all $y_1, y_2 \in Y$ such that $y_1 \neq y_2$ and $d(y_1, y_2) < \varepsilon$, $d(f(y_1), f(y_2)) > d(y_1, y_2)$.

An *open map* of a space Y onto a space X is a continuous function that takes open sets in Y onto open sets in X.

Using Corollary 3.3 and Theorem 4.3, we prove theorems about the existence of periodic points and fixed points of ε-expansive open maps. The theorems are related to Theorems 7 and 8 of [8] and to Theorem 3.0 of Rosenholtz (see [9]). We state Rosenholtz's theorem and make specific comments about its relation to our theorems after the proof of Theorem 5.2.

Lemma 5.1. *Let X be compact, let Y be a nonempty compact subset of X, and let $p \in X$. If $f: Y \to X$ is a map of Y onto X, then $f^n[\widehat{f^{-1}}^n(\{p\})] = p$ for each integer $n \geq 1$.*

Proof. The proof is by induction. For $n = 1$, $\widehat{f^{-1}}(\{p\}) = \bigcup_{a \in \{p\}} f^{-1}(a) = f^{-1}(p)$; therefore, $f[\widehat{f^{-1}}(\{p\})] = p$.

Now, assume inductively that $f^n[\widehat{f^{-1}}^n(\{p\})] = p$ for some integer $n \geq 1$. Note that

$$\widehat{f^{-1}}^{n+1}(\{p\}) = \widehat{f^{-1}}[\widehat{f^{-1}}^n(\{p\})] = \bigcup_{a \in \widehat{f^{-1}}^n(\{p\})} f^{-1}(a).$$

Hence, if $x \in \widehat{f^{-1}}^{n+1}(\{p\})$, then $x \in f^{-1}(a_0)$ for some $a_0 \in \widehat{f^{-1}}^n(\{p\})$. Thus, $f(x) = a_0$ and, by our inductive assumption, $f^n(a_0) = p$; therefore, $f^{n+1}(x) = p$. This proves that $f^{n+1}[\widehat{f^{-1}}^{n+1}(\{p\})] = p$. □

Theorem 5.2. *Let X be compact, and let Y be a nonempty compact subset of X. If $f: Y \to X$ is an ε-expansive open map of Y onto X, then f has a periodic point.*

Proof. Since $f: Y \to X$ is an open map of Y onto X, $f^{-1}: X \to 2^Y$ is continuous [7, p. 280]. Thus, since X is compact, f^{-1} is uniformly continuous. Hence, there exists $\delta > 0$ such that

(1) $H_d(f^{-1}(x_1), f^{-1}(x_2)) < \varepsilon$ for all $x_1, x_2 \in X$ such that $d(x_1, x_2) < \delta$.

We show that $f^{-1}: X \to 2^Y$ is a multi-valued δ-contractive map. Fix points $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $d(x_1, x_2) < \delta$. It follows from the definition of the Hausdorff metric that there are points $y_1 \in f^{-1}(x_1)$ and $y_2 \in f^{-1}(x_2)$ such that

(2) $H_d(f^{-1}(x_1), f^{-1}(x_2)) = d(y_1, y_2)$.

Since $d(x_1, x_2) < \delta$, we see from (1) and (2) that $d(y_1, y_2) < \varepsilon$; also, $y_1 \neq y_2$ since $x_1 \neq x_2$. Thus, since f is ε-expansive, $d(f(y_1), f(y_2)) > d(y_1, y_2)$. Therefore, since $f(y_i) = x_i$, we see from (2) that

$$H_d(f^{-1}(x_1), f^{-1}(x_2)) < d(f(y_1), f(y_2)) = d(x_1, x_2).$$

This proves that f^{-1} is a multi-valued δ-contractive map.

We can now apply Corollary 3.3 to see that f^{-1} has a periodic point p. This means that for some integer $n \geq 1$,

$$p \in \widehat{f^{-1}}^n(\{p\}).$$

Therefore, $f^n(p) = p$ by Lemma 5.1. □

Rosenholtz [9, p. 217] proved the following result: *An ε-expansive open map of a continuum onto itself has a fixed point.* (Rosenholtz's theorem is stated for locally expansive maps; however, for compact spaces, locally expansive as defined in [9] is equivalent to ε-expansive for some $\varepsilon > 0$, as is seen using Lebesgue numbers of covers [6, p. 24].)

In comparing Theorem 5.2 with Rosenholtz's theorem, we find it particularly interesting that when connectedness is dropped from Rosenholtz's theorem, the first cousins of fixed points – periodic points – still exist, and this happens even when the map f is not defined on all of X.

Our next theorem shows that Rosenholtz's theorem can be extended to the situation when the map is not defined on the entire continuum; in fact, we do not even require the domain of the map to *be* a continuum (see the last comment below).

Theorem 5.3. *Let X be continuum, and let Y be a nonempty compact subset of X. If $f\colon Y \to X$ is an ε-expansive open map of Y onto X, then f has a fixed point.*

Proof. As in the proof of Theorem 5.2, there exists $\delta > 0$ such that $f^{-1}\colon X \to 2^Y$ is a multi-valued δ-contractive map. Therefore, by Theorem 4.3, f^{-1} has a fixed point p. Obviously, p is a fixed point of f. □

Theorems 5.2 and 5.3 have applications to n-manifolds that are similar to but more general than the results in [10, p. 3]. The statements of the applications we have in mind are straightforward adjustments of the results in [10, p. 3], so we do not state them here.

It is necessary for f to be open in Theorem 5.3, as is seen from the example in [10, p. 4]. However, we do not know if it is necessary for f to be open in Theorem 5.2.

Note that even though we do not require Y in Theorem 5.3 to be a continuum, there must be a component C of Y that maps onto X (by 13.14 of [7, p. 284]); however, $f|C$ may not be an open map. Thus, recalling that Theorem 5.3 would be false without requiring f to be open, Theorem 5.3 is of interest in the generality stated.

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