Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 22, 2003, 369–386

ON A MULTIVALUED VERSION OF THE SHARKOVSKII THEOREM AND ITS APPLICATION TO DIFFERENTIAL INCLUSIONS, III

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Abstract. An extension of the celebrated Sharkovskiĭ cycle coexisting theorem (see [14]) is given for (strongly) admissible multivalued self-maps in the sense of [8], on a Cartesian product of linear continua. Vectors of admissible self-maps have a triangular structure as in [10]. Thus, we make a joint generalization of the results in [2], [5], [6] (a multivalued case), in [10] (a multidimensional case), and in [15] (a linear continuum case). The obtained results can be applied, unlike in the single-valued case, to differential equations and inclusions.

1. Introduction

The classical Sharkovskiĭ cycle coexisting theorem ([14]) says that *if a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ has a point of period n with $n \rhd k$, where $n \rhd k$ denotes that n is greater than k in the Sharkovskiĭ ordering of positive integers, namely*

$$\begin{aligned}3 \rhd 5 \rhd 7 \rhd \ldots \rhd 2\cdot 3 \rhd 2\cdot 5 \rhd 2\cdot 7 \rhd \ldots \rhd 2^2\cdot 3 \rhd 2^2\cdot 5 \rhd 2^2\cdot 7 \rhd \ldots\\ \ldots \rhd 2^n\cdot 3 \rhd 2^n\cdot 5 \rhd 2^n\cdot 7 \rhd \ldots \rhd 2^{n+1}\cdot 3 \rhd 2^{n+1}\cdot 5 \rhd 2^{n+1}\cdot 7 \rhd \ldots\\ \ldots \rhd 2^{n+1} \rhd 2^n \rhd \ldots \rhd 2^2 \rhd 2 \rhd 1,\end{aligned}$$

then it has also a point of period k.

2000 *Mathematics Subject Classification.* 34A60, 37E05, 47H04, 47H10, 58C30.

Key words and phrases. Sharkovskiĭ theorem, multivalued version, multidimensional version, linear continuum setting, infinitely many orbits, subharmonics of differential systems.

Supported by the Council of Czech Goverment (J14/98:153100011).

By a *point* $x_0 \in \mathbb{R}$ *of period* $m \in \mathbb{N}$ (shortly, an m-*periodic point*) to f we mean the fixed-point of the m-th iterate f^m of f (i.e. of the m-fold composition of f with itself), but $x_0 \neq f^j(x_0)$, for $1 \leq j < m$.

This deep result was extended in various directions (see e.g. [2], [5], [6], [10], [15], and the references therein).

In [15], it was shown by H. Schirmer that $\mathbb{R}$ can be replaced by a linear continuum $\mathbb{L}$ and that this replacement is in a certain sense the only possible one.

Definition 1.1. We say that a (linearly) ordered set $\mathbb{L}$ with more than one point is a *linear continuum*, whenever

(a) $\mathbb{L}$ has the least upper bound property,
(b) $\mathbb{L}$ is order dense, i.e. if $x < y$, then there exists z so that $x < z < y$,

and we endow $\mathbb{L}$ with the order topology, by which $\mathbb{L}$ becomes a topological (Hausdorff) space.

The linear continuum is not a continuum as usual. As typical examples of linear continua are usually mentioned the real line, any interval or the unit square in the lexicographical order. We also recall that an ordered set in the order topology is a linear continuum if and only if it is connected.

Theorem 1.2 ([15]). *Let* $\mathbb{L}$ *be a linear continuum and let* $f\colon \mathbb{L} \to \mathbb{L}$ *be a continuous function. If* f *has an* n*-periodic point, then* f *has also a* k*-periodic point, for every* $k \lhd n$.

Because of well-known counter-examples (cf. [10]), $\mathbb{R}$ cannot be replaced by $\mathbb{R}^N$, where $N > 1$, in general. Nevertheless, for the maps $f\colon \mathbb{R}^N \to \mathbb{R}^N$ having a special "triangular" structure, the following theorem was obtained by P. E. Kloeden in [10].

Theorem 1.3 ([10, Section 3]). *Let* I *be a compact subset of* $\mathbb{R}^N$ *of the form*

$$I = \prod_{i=1}^{N} I_i,$$

where $I_i \subset \mathbb{R}$ *is a compact interval, for* $i = 1, \ldots, N$, $N \in \mathbb{N}$, *and let* $f\colon I \to I$ *be a continuous mapping of the form*

$$f_i(x_1, \ldots, x_N) = f_i(x_1, \ldots, x_i),$$

for $i = 1, 2, \ldots, N$, *i.e. a mapping for which the* i*-th component* f_i *depends only on the first independent variables* $x_1, \ldots, x_i$. *If* f *has an* n*-periodic point, then* f *has also a* k*-periodic point, for every* $k \lhd n$.

Remark 1.4. By the same reasons as in $\mathbb{R}$ (see [13]), the assertion of Theorem 1.3 holds, without any change, for $f\colon \mathbb{R}^n \to \mathbb{R}^n$ as well.

In [2], [5], [6], a multivalued version of Sharkovskiĭ's theorem was presented in terms of orbits for upper-semicontinuous maps whose sets of values are either single points or closed intervals in $\mathbb{R}$ (i.e. in particular, compact and convex sets) which we called as *M-maps*. Let us recall that a multivalued mapping $\varphi\colon X \rightsquigarrow Y$ (i.e. $\varphi\colon X \to 2^Y \setminus \{\emptyset\}$) is *upper-semicontinuous* if $\varphi^{-1}(U) = \{x \in X\colon \varphi(x) \subset U\}$ is open in X, for every open subset U of Y.

By an *orbit of k-th order* (shortly, a *k-orbit*) to an M-map φ, we mean a sequence $\{x_i\}_{i=0}^{\infty}$ such that

(a) $x_{i+1} \in \varphi(x_i)$, $i = 0, 1, \ldots,$
(b) $x_i = x_{i+k}$, $i = 0, 1, \ldots,$
(c) this orbit is not a product orbit formed by going p-times around a shorter orbit of m-th order, where $mp = k$.

If still

(d) $x_i \neq x_j$, for $i \neq j$, $i, j = 0, \ldots, k-1$,

then we speak about a *primary orbit of k-th order* (shortly, a *primary k-orbit*).

Theorem 1.5 ([6, Corollary 7]). *Let an M-map $\varphi\colon \mathbb{R} \rightsquigarrow \mathbb{R}$ have an n-orbit, where $n = 2^m q$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and q is odd, and let n be maximal in the Sharkovskiĭ ordering.*

(a) *If $q > 3$, then φ has a k-orbit, for every $k \lhd n$, except $k = 2^{m+2}$.*
(b) *If $q = 3$, then φ has a k-orbit, for every $k \lhd n$, except $k = 2^{m+1}3, 2^{m+2}, 2^{m+1}$.*
(c) *If $q = 1$, then φ has a k-orbit, for every $k \lhd n$.*

Corollary 1.6. *Let an M-map $\varphi\colon \mathbb{R} \rightsquigarrow \mathbb{R}$ have an n-orbit, $n \in \mathbb{N}$. Then φ has also a k-orbit, for every $k \lhd n$, with the exception of at most three orbits.*

Let us note that (at most three) exceptional orbits in Theorem 1.5 and Corollary 1.6 are due to counter-examples (see [2], [6]) by which a full multivalued analogy to classical (single-valued) Sharkovskiĭ's theorem fails, in general.

The aim of the present paper is to make a joint generalization of Theorems 1.2, 1.3 and 1.5. The obtained results (Theorem 4.1 and Corollary 4.3) are then applied to differential systems.

In the first stage, when replacing $\mathbb{R}$ by $\mathbb{L}$ in Theorem 1.5 (see Theorem 3.8 and Corollary 3.9), the notion of M-maps will be understood in a more general setting.

Definition 1.7. An upper-semicontinuous mapping $\varphi\colon \mathbb{L} \rightsquigarrow \mathbb{L}$ is called an *M-map*, whenever the sets of values of φ are either single points or (nonempty) closed intervals of $\mathbb{L}$.

The generalized notion of convexity on $\mathbb{L}$ will be understood in a similar way, namely it is nothing else than connectedness.

On the other hand, on (finite) Cartesian products $\mathbb{L}\times\ldots\times\mathbb{L}$ of $\mathbb{L}$, the most natural extension of M-maps is, in view of applications to differential equations in Chapter 5, the class of (strongly) admissible maps in the sense of L. Górniewicz ([3], [8]).

Definition 1.8. Assume that X, Y are topological Hausdorff spaces and that $\varphi\colon X \rightsquigarrow Y$. We say that φ is *admissible* if there exists a topological Hausdorff space Γ and two continuous (single-valued) mappings $p\colon \Gamma \Rightarrow X$, $q\colon \Gamma \to Y$ such that:

(a) p is onto,
(b) p is perfect, i.e. p is closed and $p^{-1}(x)$ is compact for all $x \in X$,
(c) $p^{-1}(x)$ is acyclic, i.e. homologically the same as a one point space (for more details, see [3], [8]),
(d) $\varphi(x) = q(p^{-1}(x))$ for all $x \in X$.

Let us note that in [8], where the spaces X, Y were only metric, these maps are called strongly admissible, while by admissible maps those having a (multivalued) strongly admissible selection (i.e. $q(p^{-1}(x)) \subset \varphi(x)$, for all $x \in X$, in (d)) were understood. Moreover, in metric spaces, p can be only proper in (b) (i.e. $p^{-1}(x)$ compact, for all $x \in X$), because the closedness of p follows there automatically.

The closedness of p is essential, because it implies, jointly with (a), the upper-semicontinuity of p^{-1}, and subsequently (see (d)) of φ. Since the continuous q-image of the compact and acyclic (i.e. in particular, connected) set $p^{-1}(x)$ (see (b) and (c)) is compact and connected, the same must be true (see (d)) for φ. Thus, *admissible maps are always upper-semicontinuous with nonempty, compact and connected values.*

There is still another remarkable property of admissible maps, namely that *their class is closed w.r.t. finite compositions.* This follows from the following commutative diagram:

$$\begin{tikzcd}
X \arrow[rr, rightsquigarrow, "\varphi_1"] & & Y \arrow[rr, rightsquigarrow, "\varphi_2"] & & Z \\
& \Gamma_1 \arrow[ul, Rightarrow, "p_1"'] \arrow[ur, "q_1"'] & & \Gamma_2 \arrow[ul, Rightarrow, "p_2"] \arrow[ur, "q_2"'] & \\
& & \Gamma := \{(u,v) \in \Gamma_1 \times \Gamma_2 : q_1(u) = p_2(v)\}, \arrow[ul, Rightarrow, "\widetilde{p}"] \arrow[ur, "\widetilde{q}"'] & &
\end{tikzcd}$$

where $\widetilde{p}, \widetilde{q}$ are natural projections, because

$$q(p^{-1}(x)) = \varphi_2(\varphi_1(x)) \quad \text{for all } x \in X,$$

where $p(u,v) = p_1 \circ \widetilde{p}\,(u,v) = p_1(u)$ and $q(u,v) = q_2 \circ \widetilde{q}\,(u,v) = q_2(v)$. In other words, *a finite composition of admissible maps is also admissible.*

Since admissible maps $\varphi\colon \mathbb{L} \rightsquigarrow \mathbb{L}$ reduce themselves on $\mathbb{L}$ obviously to M-maps in the sense of Definition 1.7, and, vice versa, M-maps on $\mathbb{L}$ can be easily checked to be admissible in the sense of Definition 1.8, we have an equivalence on $\mathbb{L}$, and so their class is again closed w.r.t. finite compositions and, in particular, w.r.t. iterations.

Let us note that we could even consider the class of upper-semicontinuous maps with compact connected values for the same aim. However, then we do not have to our disposal any sufficient conditions guaranteing the existence of the given n-orbit.

Let us finally point out that, in contrast to this, the existence of at least one or several n-orbits to special (e.g. compact) admissible self-maps on ANR-spaces can be guaranteed by the recent results in [4], obtained by means of the Lefschetz and the Nielsen periodic-point theorems.

2. Alternative proof of an important statement

In this section, the alternative proof of Theorem 4 in [2] is mainly presented which is also suitable for a linear continuum extension. Besides Theorem 4 (in [2]), three simple lemmas play a fundamental role in the proof of the above Theorem 1.5. The first one (cf. [2, Lemma 1]) can be easily extended from a real line to a linear continuum in the following way.

Lemma 2.1. *Let $\varphi\colon I \rightsquigarrow \mathbb{L}$ be an M-map, where $I = [a,b] \subset \mathbb{L}$ is a closed interval. If there are points $A \in \varphi(a)$ and $B \in \varphi(b)$ such that $a < A$, $B < b$ or $a > A$, $B > b$, then there exists a fixed-point of φ.*

Proof. At first, we consider the case $a < A$, $B < b$. We define $s := \sup\{t \in [a,b] : \text{there exists } T \in \varphi(t) \text{ such that } T \geq t\}$. Thanks to $a < A$, s is well-defined. Assuming the existence of $S_1 \in \varphi(s)$, $S_1 > s$, and the absence of $S_2 \in \varphi(s)$, $S_2 \leq s$ (what immediately implies $s \neq b$), we obtain a contradiction with the upper-semicontinuity of φ, because $\varphi(s)$ is compact and $\varphi(t) < t$, for every $t > s$.

Assuming the existence of $S_2 \in \varphi(s)$, $S_2 < s$, and the absence of $S_1 \in \varphi(s)$, $S_1 \geq s$ (what immediately implies $s \neq a$), we obtain again, in view of the definition of s, a contradiction to the upper-semicontinuity of φ. Thus, s is a fixed-point of φ.

Now, we consider the case $a > A$, $B > b$. We define $r := \inf\{t \in [a,b] : \text{there exists } T \in \varphi(t) \text{ such that } T \geq t\}$. Assuming the existence of $R_1 \in \varphi(r)$, $R_1 > r$, and the absence of $R_2 \in \varphi(r)$, $R_2 \leq r$ (what immediately implies $r \neq a$), we obtain a contradiction to the upper-semicontinuity of φ, because $\varphi(r)$ is compact and $\varphi(t) < t$, for every $t < r$.

Assuming the existence of $R_2 \in \varphi(r)$, $R_2 < r$, and the absence of $R_1 \in \varphi(r)$, $R_1 > r$ (what immediately implies $r \neq b$), we obtain again, in view of the definition of r, a contradiction to the upper-semicontinuity of φ. Thus, $r \in \varphi(r)$. $\square$

In the proof of two further lemmas, only standard properties of the reals, whose analogies hold without any difference on a linear continuum as well, are used (i.e. linear ordering, the density of ordering structure and the least upper bound property). Therefore, we can immediately reformulate Lemma 2 in [2] and Lemma 3 in [6] as follows.

Lemma 2.2. *Let $\varphi\colon \mathbb{L} \rightsquigarrow \mathbb{L}$ be an M-map. Assume that $I_k \subset \mathbb{L}$, $k = 0, \dots, n-1$, are closed intervals such that $I_{k+1} \subset \varphi(I_k)$, for $k = 0, \dots, n-1$, which we write as $I_0 \to I_1 \to \dots \to I_n = I_0$. Then the n-th iterate φ^n of φ has a fixed-point x_0 (i.e. $x_0 \in \varphi^n(x_0)$) with $x_{k+1} \in \varphi(x_k)$, $x_n = x_0$, where $x_k \in I_k$, for $k = 0, \dots, n-1$.*

Lemma 2.3. *Let $\varphi\colon \mathbb{L} \rightsquigarrow \mathbb{L}$ be an M-map and let there exist $a, A, b, B \in \mathbb{L}$ such that $A \in \varphi(a)$, $B \in \varphi(b)$. If $C \in [A, B]$, then there exists a point $c \in [a, b]$ such that $C \in \varphi(c)$.*

The proof of Theorem 4 in [2] (for its linear continuum version see Theorem 2.7 below) is there based on the following approximation lemma, which is a particular case of [7, Lemma 4.5], see also [7, Remark 4.6].

Lemma 2.4 ([2, Lemma 3]). *Let $\varphi\colon I \rightsquigarrow I$ be a composition of M-maps φ_i: $I_{i-1} \rightsquigarrow I_i$, $i = 1, \dots, n$, i.e. $\varphi = \varphi_n \circ \dots \circ \varphi_1$, where $I_0 = I_n = I \subset \mathbb{R}$ and $I, I_1, \dots, I_n$ are closed intervals. Assume that a_k are fixed-points of φ, $a_k \in \varphi(a_k)$, $k = 1, \dots, m$. Then, for every $\varepsilon > 0$, there exists a continuous ε-approximation $f = f_n \circ \dots \circ f_1$ of φ (on the graph of φ), namely $\Gamma_f \subset N_\varepsilon(\Gamma_\varphi)$, where $N_\varepsilon(\Gamma)$ denotes an open neighbourhood of Γ in $\mathbb{R}^2$, such that f_i are continuous $\delta(\varepsilon)$-approximations of φ_i, for every $i = 1, \dots, n$, with $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$ and $a_k = f(a_k)$, $k = 1, \dots, m$. Moreover, there exists a_{ik} with $a_{ik} \in \varphi(a_{i-1,k})$, $a_{0k} = a_k \in \varphi_n(a_{n-1,k})$ such that $a_{ik} = f_i(a_{i-1,k})$, $a_{0k} = a_k = f_n(a_{n-1,k})$, for every $i = 1, \dots, n-1$, and $k = 1, \dots, m$.*

It is not quite clear to us, whether or not we can replace $\mathbb{R}$ by a linear continuum in Lemma 2.4.

Nevertheless, below we give the alternative proof of Theorem 4 in [2] which can be used for a linear continuum version, too.

The following proposition can be proved quite analogously (i.e. by means of Lemmas 2.1–2.3) as in [6], where only M-maps on $\mathbb{R}$ were considered.

Proposition 2.5. *Assume that an M-map $\varphi\colon \mathbb{L} \rightsquigarrow \mathbb{L}$ has a primary p-orbit $\{x_1, \dots, x_p\}$, where p is even and $x_1 \in \varphi(x_1)$, but φ has no l-orbit with $l \triangleright p+1$. Then φ admits a k-orbit, where k is even, $k \le p$, $k \ne 4$.*

Although the proof of the following proposition is similar to the one of Proposition 2.5, we give it, for the sake of completness.

Proposition 2.6. *Assume that an M-map $\varphi\colon \mathbb{L} \rightsquigarrow \mathbb{L}$ has a primary n-orbit $\{x_1, \dots, x_n\}$, where $n > 3$ is odd and $x_1 \in \varphi(x_1)$, but φ has no l-orbit with $l \rhd n$. Then φ admits a k-orbit, for every $k \lhd n$, $k \neq 4$.*

Proof. Obviously, thanks to a fixed-point $\{x_1\}$ (see the hypothesis), φ admits a k-orbit, for every $k \geq n$. So, it suffices to show the existence of a k-orbit, where k is even, $k \leq n$, $k \neq 4$.

We can assume that $x_1 = 0$ and $x_1 < x_2$ (the other cases can be obtained, when translating and changing the orientation of axes, respectively). Furthermore, $x_i \notin [x_1, x_2]$, for every $i \in \{4, \dots, n\}$, because otherwise we have either an odd orbit $\{x_1, x_i, x_{i+1}, \dots, x_n\}$ or an odd orbit $\{x_1, x_1, x_i, x_{i+1}, \dots, x_n\}$, accordingly i is even or odd, but this is a contradiction with the assumption of the maximality of n.

Moreover, we can also suppose that $x_3 \notin [x_1, x_2]$, because otherwise, we have an even orbit $\{x_1, x_3, \dots, x_n\}$ with $x_1 \in \varphi(x_1)$, and it is possible to use Proposition 2.5.

Now, it holds that $\operatorname{sgn}(x_i) = \operatorname{sgn}(x_{i+1})$ implies $\operatorname{sgn}(x_{i+2}) = \operatorname{sgn}(x_i)$, for every $i \in \{2, \dots, n-2\}$. Indeed, suppose conversely that there exists an $i \in \{2, \dots, n-2\}$ with the property $\operatorname{sgn}(x_i) = \operatorname{sgn}(x_{i+1})$, and $\operatorname{sgn}(x_{i+2}) \neq \operatorname{sgn}(x_i)$. Then we consider

$$[x_1, x_2] \to [x_2, x_3] \to \dots \to [x_i, x_{i+1}] \to [x_1, x_2],$$

if i is odd or, if i is even,

$$[x_1, x_2] \to [x_1, x_2] \to [x_2, x_3] \to \dots \to [x_i, x_{i+1}] \to [x_1, x_2].$$

In the both cases, applying Lemma 2.2 and taking into account that

$$[x_i, x_{i+1}] \cap [x_1, x_2] = \emptyset, \quad \text{for } i \in \{3, \dots, n-2\}$$

(we note that the case $i = 2$ must be treated separately), we obtain a shorter odd orbit, which is a contradiction.

Now, the proof splits into the following cases:

Case 1. $x_i > x_2$, for every $i \in \{3, \dots, n\}$.

There are two possibilities:

(a) $x_n > x_{n-1}$. Since

$$[x_1, x_{n-1}] \to [x_1, x_{n-1}] \to [x_{n-1}, x_n] \to [x_1, x_{n-1}],$$

we have by Lemma 2.2 either a 3-orbit of φ or $x_{n-1} \in \varphi(x_{n-1})$. In the second case, $[x_{n-1}, x_n] \subset \varphi(x_{n-1})$, and Lemma 2.3 yields the existence of a

$b \in [x_{n-1}, x_n]$ such that $x_2 \in \varphi(b)$. Subsequently, there is an even $(n-1)$-orbit $\{x_{n-1}, b, x_2, x_3, \dots, x_{n-2}\}$ with $x_{n-1} \in \varphi(x_{n-1})$, and we can use Proposition 2.5.

(b) $x_n < x_{n-1}$. There exists $j \in \{2, \dots, n-2\}$ with the property $x_j < x_n < x_{j+1}$. Considering

$$[x_1, x_j] \to [x_1, x_j] \to [x_j, x_n] \to [x_1, x_j],$$

we obtain by Lemma 2.3 either a 3-orbit of φ or $x_j \in \varphi(x_j)$. In the second case, the convexity of $\varphi(x_j)$ and $x_{j+1} \in \varphi(x_j)$ imply $x_n \in \varphi(x_j)$, and so there are the following possibilities. If $j \neq n-2$, then there is either an odd $(j+1)$-orbit $\{x_1, \dots, x_j, x_n\}$ or an odd $(j+2)$-orbit $\{x_1, x_1, x_2, \dots, x_j, x_n\}$, accordingly j is even or odd. Finally, if $j = n-2$, then there is an even $(n-1)$-orbit $\{x_1, \dots, x_{n-2}, x_n\}$ with $x_1 \in \varphi(x_1)$ and Proposition 2.5 completes the proof of this case.

Case 2. $x_i < 0$, for every $i \in \{3, \dots, n\}$.

If $x_3 < x_n$, then there exists $\varepsilon \in (x_n, x_1)$ satisfying $[\varepsilon, x_1] \subset \varphi[x_3, x_n]$. Considering

$$[x_3, x_n] \to [\varepsilon, x_1] \to [x_1, x_2] \to [x_3, x_n]$$

and applying Lemma 2.2, we obtain a 3-orbit. Otherwise, the following possibilities can occur.

(a) $x_n < x_{n-1}$. We can proceed like in Case 1(a). So, since

$$[x_{n-1}, x_1] \to [x_{n-1}, x_1] \to [x_n, x_{n-1}] \to [x_{n-1}, x_1],$$

we have, thanks to Lemma 2.2, either a 3-orbit of φ or $x_{n-1} \in \varphi(x_{n-1})$. In the second case, $[x_n, x_{n-1}] \subset \varphi(x_{n-1})$ and Lemma 2.3 yields the existence of a $b \in [x_n, x_{n-1}]$ such that $x_3 \in \varphi(b)$. Subsequently, there is an odd $(n-2)$-orbit $\{x_{n-1}, b, x_3, \dots, x_{n-2}\}$.

(b) $x_n > x_{n-1}$. We can proceed like in Case 1(b). So, it is obvious that there exists a $j \in \{3, \dots, n-2\}$ with the property $x_j > x_n > x_{j+1}$. Considering

$$[x_j, x_1] \to [x_j, x_1] \to [x_n, x_j] \to [x_j, x_1],$$

we obtain by Lemma 2.2 either a 3-orbit of φ or $x_j \in \varphi(x_j)$. In the second case, the convexity of $\varphi(x_j)$ and $x_{j+1} \in \varphi(x_j)$ imply $x_n \in \varphi(x_j)$, and there are the following possibilities. If $j \neq n-2$, then there is either an odd $(j+1)$-orbit $\{x_1, \dots, x_j, x_n\}$ or an odd $(j+2)$-orbit $\{x_1, x_1, x_2, \dots, x_j, x_n\}$ depending whether j is even or odd. Finally, if $j = n-2$, then there is an even $(n-1)$-orbit $\{x_1, \dots, x_{n-2}, x_n\}$ with $x_1 \in \varphi(x_1)$ and Proposition 2.5 completes the proof of this case.

Case 3. $\operatorname{sgn}(x_i) = -\operatorname{sgn}(x_{i+1})$, for every $i \in \{2, \dots, n-1\}$.

We redenote the set $\{x_1, \dots, x_n\}$ into $\{a_1, \dots, a_n\}$, in order the set $\{a_1, \dots, a_n\}$ to be ordered as $a_n < \dots < a_3 < a_1 < a_2 < \dots < a_{n-1}$, and we consider the map $a\colon \{x_1, \ldots, x_n\} \to \{a_1, \ldots, a_n\}$ such that $a(x_i) = a_j$ if and only if $a_j = x_i$, where $i, j \in \{1, \ldots, n\}$. By the hypothesis, we have $a_1 = x_1$, $a_2 = x_2$ and $a_l := a(x_n) < x_1$. Now, there are two possibilities.

(a) $l = 3$. It can be readily checked that

$$[a_5, a_3] \to [a_1, a_2] \to [a_1, a_2] \to [a_5, a_3],$$

so Lemma 2.2 completes the proof of this case.

(b) $l > 3$. Consider an $a_m \in (a_l, x_1)$ with the property

$$s(a^{-1}(a_m)) = \max\{s(a^{-1}(a_i)) : i = 3, 5, 7, \ldots, l-2\},$$

where $s(x_j) = x_{j+1}$, for every $j = 1, \ldots, n-1$. It can be readily checked that $[a_1, a_{i+1}] \subset \varphi[a_1, a_i]$, for $i \in \{2, \ldots, n-1\}$. Subsequently, it holds $[a_l, a_m] \subset \varphi[a_1, a_{l-1}]$, $[a_1, a_{l-1}] \subset \varphi[a_l, a_m]$, and $[a_1, a_{l-1}] \subset \varphi[a_l, a_m]$.

Now, if $k = 2$, then we consider

$$[a_l, a_m] \to [a_1, a_{l-1}] \to [a_l, a_m].$$

If $k \geq 6$ is even, $k < n$, $k < m + 3$, then we consider

$$\begin{aligned}[a_l, a_m] \to [a_1, a_{l-1}] \to [a_s, a_1] \to [a_1, a_{s+1}]& \\ \to \ldots \to [a_m, a_1] \to [a_1, a_{l-1}] \to [a_l, a_m]&,\end{aligned}$$

where $s = m + 4 - k$.

If $k \geq m + 3$ is even, $k < n$, then we consider

$$\begin{aligned}[a_l, a_m] \to \underbrace{[a_1, a_2] \to \ldots \to [a_1, a_2]}_{(k-m)\text{-times}} \to [a_3, a_1] \to [a_1, a_4]& \\ \to \ldots \to [a_m, a_1] \to [a_1, a_{l-1}] \to [a_l, a_m]&.\end{aligned}$$

In the previous cases, by Lemma 2.2, we obtain the existence of a k-orbit.

Summarizing the above conclusions, φ has a k-orbit, for every $k \lhd n$, except $k = 4$. $\square$

Now, we are ready to formulate the linear continuum version of Theorem 4 in [2].

Teorem 2.7. *Let L be a linear continuum. Assume that an M-map $\varphi\colon \mathbb{L} \rightsquigarrow \mathbb{L}$ has a primary n-orbit $\{x_1, \ldots, x_n\}$, where $n > 3$ is odd, but φ has no l-orbit with $l \rhd n$. Then φ admits a k-orbit, for each $k \lhd n$, $k \neq 4$.*

Proof. Assuming that $x_i \notin \varphi(x_i)$, for every $i = 1, \ldots, n$, we can proceed exactly as in [13, Proposition 1.7], which is based on [13, Lemma 1.6] (only, instead of $\mathbb{R}$, we take $\mathbb{L}$, and, instead of a continuous function, we take an M-map). If $x_i \in \varphi(x_i)$, for some $i = 1, \ldots, n$, then we could have a problem to define a in the proof of [13, Lemma 1.6], but this situation is covered by Proposition 2.6. $\square$

3. Extension for M-maps on linear continua

The purpose of this section is to prove the linear continuum extension of Theorem 1.5.

Analysing the assertions leading to the proof of Theorem 1.5, one can see that they are only based on three fundamental lemmas above (see Lemmas 2.1–2.3 for their linear continuum versions) and on some properties of $\mathbb{R}$ which hold for linear continua as well. Hence, we can immediately state the series of results for M-maps on linear continua which can be proved exactly as in their $\mathbb{R}$-versions (see [5] and [6]). In this series, φ and g denote M-maps from a linear continuum into itself.

LEMMA 3.1. *Let φ has no primary 3-orbits. If φ has an n-orbit, where $n \neq 1$, then φ admits a 2-orbit.*

LEMMA 3.2. *Assume the existence of a 3-orbit $\{a, b, c\}$ of φ such that $a < \min_{x\in\varphi(a)} x \leq \max_{y\in\varphi(b)} y < \min_{y\in\varphi(b)} y$ or $a > \max_{x\in\varphi(a)} x \geq \min_{x\in\varphi(a)} x > \max_{y\in\varphi(b)} y$. Then φ has also a k-orbit, for every $k \in \mathbb{N} \setminus \{4, 6\}$.*

LEMMA 3.3. *If φ has a 3-orbit, then φ has also a k-orbit, for every $k \triangleleft n$, except $k = 2, 4, 6$.*

LEMMA 3.4. *If φ has an n-orbit, where n is odd, then φ has also a k-orbit, for every $k \triangleleft n$, except $k = 2, 4, 6$.*

LEMMA 3.5. *Let $g = \varphi^l$, where $l = 2^s$, for some $s \in \mathbb{N}$.*

(a) *If g has a q-orbit, where q is odd, then φ has also a q-orbit or an lq-orbit.*
(b) *If g has a q-orbit, where $q = 2^r$, for some $r \in \mathbb{N}$, then φ has a 2^{r+s}-orbit.*

LEMMA 3.6. *Let $g = \varphi^l$, where $l \in \mathbb{N}$, and let $q \in \mathbb{N}$. If φ has an lq-orbit, then g has an m-orbit, where q is devided by m and $m \neq 1$.*

LEMMA 3.7. *Let $g = \varphi^l$, where $l \in \mathbb{N}$, and let q be odd. If φ has an lq-orbit, then g has a q-orbit.*

Thanks to Theorem 2.7 and Lemmas 3.1–3.7, the following linear continuum extension of Theorem 1.5 can now be completed as in [6] (see the proof of Theorem 4 and Corollary 7 there).

THEOREM 3.8. *Let $\mathbb{L}$ be a linear continuum, let an M-map $\varphi\colon \mathbb{L} \rightsquigarrow \mathbb{L}$ have an n-orbit, where $n = 2^m q$, $m \in \mathbb{N}_0$, q is odd, and n be maximal in the Sharkovskiĭ ordering.*

(a) *If $q > 3$, then φ has a k-orbit, for every $k \triangleleft n$, except $k = 2^{m+2}$.*

(b) *If $q = 3$, then φ has a k-orbit, for every $k \lhd n$, except $k = 2^{m+1}3$, 2^{m+2}, 2^{m+1}.*

(c) *If $q = 1$, then φ has a k-orbit, for every $k \lhd n$.*

Corollary 3.9. *Let $\mathbb{L}$ be a linear continuum and let an M-map $\varphi\colon \mathbb{L} \rightsquigarrow \mathbb{L}$ have an n-orbit, $n \in \mathbb{N}$. Then φ has also a k-orbit, for every $k \lhd n$, with the exception of at most three orbits.*

We conclude this section by an almost evident, but useful fact.

Lemma 3.10. *Let $\mathbb{L}$ be a linear continuum, $I = [a,b] \subset \mathbb{L}$ be a compact interval, and $\varphi\colon I \rightsquigarrow I$ be an M-map satisfying $\varphi(I) \subset I$. Then φ has a fixed-point.*

Proof. We define $s := \sup\{t \in [a,b] :$ there exists $T \in \varphi(t)$ such that $T \geq t\}$. Observe that because of $\varphi(I) \subset I$, s is well-defined.

Assuming the existence of $S_1 \in \varphi(s)$, $S_1 > s$ (what immediately implies $s \neq b$), and the absence of $S_2 \in \varphi(s)$, $S_2 \leq s$, we obtain a contradiction to the upper-semicontinuity of φ, because $\varphi(s)$ is compact and $\varphi(t) < t$, for every $t > s$.

Assuming the existence of $S_2 \in \varphi(s)$, $S_2 < s$ (what immediately implies $s \neq a$), and the absence of $S_1 \in \varphi(s)$, $S_1 \geq s$, we obtain again, in view of the definition of s, a contradiction to the upper-semicontinuity of φ. Thus, s is a fixed-point of φ. $\square$

4. Main result

Throughout this section, let $\mathbb{L}_i$, $i = 1, \ldots, N$, will be linear continua, and $\widetilde{\mathbb{L}} = \mathbb{L}_1 \times \ldots \times \mathbb{L}_N$, $N \in \mathbb{N}$, denotes their Cartesian product. Furthermore, $\varphi\colon \widetilde{\mathbb{L}} \rightsquigarrow \widetilde{\mathbb{L}}$ will be an admissible mapping (in the sense of Definition 1.8) of the form

$$\varphi = (\varphi_1, \ldots, \varphi_N), \quad \varphi_i(x_1, \ldots, x_N) = \varphi_i(x_1, \ldots, x_i), \tag{4.1}$$

for every $i = 1, \ldots, N$. We say that the difference inclusion

$$x^{n+1} \in \varphi(x^n)$$

has an *n-orbit* $x^0, \ldots, x^{n-1}$, whenever $\{x^0, \ldots, x^{n-1}\}$ is an n-orbit of φ.

Now, we state a joint generalization of the results in [2], [5], [6], [10], [15].

Theorem 4.1. *Let φ have an n-orbit, where $n = 2^m q$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and q is odd, and let n be maximal in the Sharkovskiĭ ordering.*

(a) *If $q > 3$, then φ has a k-orbit, for every $k \lhd n$, except $k = 2^{m+2}$.*

(b) *If $q = 3$, then φ has a k-orbit, for every $k \lhd n$, except $k = 2^{m+1}3, 2^{m+2}$, 2^{m+1}.*

(c) *If $q = 1$, then φ has a k-orbit, for every $k \lhd n$.*

In the proof of Theorem 4.1 below, we proceed analogously as in the particular case of a continuous function $f\colon \mathbb{R}^N \to \mathbb{R}^N$, treated in [10]. Our approach is, however, far from to be obvious.

We need the following important

LEMMA 4.2. *Let us consider the difference inclusion*

$$x^{n+1} \in \varphi(x^n) \tag{4.2}$$

and the truncated one

$$\widehat{x}^{n+1} \in \widehat{\varphi}(\widehat{x}^n), \tag{4.3}$$

where $\widehat{x} = (x_1, \dots, x_{N-1})$ *and* $\widehat{\varphi} = (\varphi_1, \dots, \varphi_{N-1})$. *If inclusion* (4.3) *has, for any* $p = 1, \dots,$ *a* p*-orbit, then the same is true for inclusion* (4.2).

PROOF. For the admissible mapping φ of the form (4.1) and $x = (\widehat{x}, x_N)$, inclusion (4.2) can be written as

$$\begin{cases} \widehat{x}^{n+1} \in \widehat{\varphi}(\widehat{x}^n), \\ x_N^{n+1} \in \varphi_N(\widehat{x}^n, x_N^n). \end{cases} \tag{4.4}$$

Let $\widehat{\eta}^0, \dots, \widehat{\eta}^{p-1}$ be a p-orbit of inclusion (4.3), and define the set-valued mapping $h\colon \mathbb{L}_N \rightsquigarrow \mathbb{L}_N$ by

$$h(x_N) = \varphi_N(\widehat{\eta}^{p-1}, \varphi_N(\widehat{\eta}^{p-2}, \dots, \varphi_N(\widehat{\eta}^0, x_N) \dots)), \tag{4.5}$$

for all $x_N \in \mathbb{L}_N$. Then h is an M-map from $\mathbb{L}_N$ into itself. Indeed, since for an arbitrary $\widehat{x} \in \mathbb{L}_{N-1}$, a multivalued mapping $x_N \rightsquigarrow \varphi_N(\widehat{x}, x_N)$ is a composition of a continuous mapping $x_N \to (\widehat{x}, x_N)$, an admissible mapping φ and a continuous projection $\pi_N\colon \widetilde{\mathbb{L}} \to \mathbb{L}_N$, i.e. a composition of three (strongly) admissible maps, it follows from the properties of (strongly) admissible maps that $x_N \rightsquigarrow \varphi_N(\widehat{x}, x_N)$ is admissible in $\widetilde{\mathbb{L}}$, i.e. an M-map. In particular, M-maps preserve their character under iterates. For more details of (strongly) admissible maps, see [3], [8].

By Lemma 3.10, we obtain that h has a fixed-point $\eta^* \in h(\eta^*)$. Putting $\eta_N^0 = \eta^*$, it holds

$$\eta_N^0 \in h(\eta_N^0) = \varphi_N(\widehat{\eta}^{p-1}, \varphi_N(\widehat{\eta}^{p-2}, \dots, \varphi_N(\widehat{\eta}^0, \eta_N^0) \dots)),$$

and there exist $\eta_N^1, \dots, \eta_N^{p-1} \in I_N$ satisfying

$$\eta_N^1 \in \varphi_N(\widehat{\eta}^0, \eta_N^0), \eta_N^2 \in \varphi_N(\widehat{\eta}^1, \eta_N^1), \dots, \eta_N^{p-1} \in \varphi_N(\widehat{\eta}^{p-2}, \eta_N^{p-2}),$$

and

$$\eta_N^0 \in \varphi_N(\widehat{\eta}^{p-1}, \eta_N^{p-1}).$$

Thus,

$$\eta^0 = (\widehat{\eta}^0, \eta_N^0), \eta^1 = (\widehat{\eta}^1, \eta_N^1), \dots, \eta^{p-1} = (\widehat{\eta}^{p-1}, \eta_N^{p-1})$$

is a p-orbit of inclusion (4.4), i.e. of inclusion (4.2). $\square$

Proof of Theorem 4.1. We prove the theorem by induction on $\mathbb{N}$. For $N = 1$, Theorem 4.1 is just Theorem 3.8. We suppose that $N \geq 2$ and that the existence of an $n = 2^m q$-orbit of inclusion (4.3), where $m \in \mathbb{N}_0$, q is odd and n is maximal in the Sharkovskiĭ ordering for inclusion (4.3), implies

(a) the existence of a k-orbit, for every $k \lhd n$, $k \neq 2^{m+2}$, if $q > 3$,
(b) the existence of a k-orbit, for every $k \lhd n$, $k \neq 2^{m+1}3, 2^{m+2}, 2^{m+1}$, if $q = 3$,
(c) the existence of a k-orbit, for every $k \lhd n$, if $q = 1$.

Using the above induction assumption, we show the same for inclusion (4.2).

Hence, let $\eta^0, \dots, \eta^{n-1}$ be an n-orbit of inclusion (4.2), where $n = (2k+1) \cdot 2^l$, $k \in \mathbb{N}_0$, $l \in \mathbb{N}_0$, and n is maximal in the Sharkovskiĭ ordering for inclusion (4.2). Then inclusion (4.3) has a p-orbit $\widehat{\eta}^0, \dots, \widehat{\eta}^{p-1}$, where $\eta^0 = (\widehat{\eta}^0, \eta_N^0), \eta^1 = (\widehat{\eta}^1, \eta_N^1), \dots, \eta^{p-1} = (\widehat{\eta}^{p-1}, \eta_N^{p-1}), \eta^p = (\widehat{\eta}^0, \eta_N^p), \dots, \eta^{n-1} = (\widehat{\eta}^{p-1}, \eta_N^{n-1})$, $p = (2j+1) \cdot 2^i$ divides n and $0 \leq j \leq k$, $0 \leq i \leq l$. Moreover, Lemma 4.2, the induction assumption and the maximality argument specify that either $p = n$ or $j = 0$. The first case can be verified immediately by Lemma 4.2. Thus, it suffices to consider $p = 2^i$.

Using Lemma 4.2 again, we can suppose that the number of the maximal orbits of inclusion (4.3) is less than n in the Sharkovskiĭ ordering. Hence, by the induction assumption, inclusion (4.3) has the orbits related to the numbers

$$2^{i-1} \rhd 2^{i-2} \rhd \ldots \rhd 2 \rhd 1. \tag{4.6}$$

Subsequently, according to Lemma 4.2, inclusion (4.2) has also the orbits related to the same numbers.

Now, we define the M-mapping $h\colon \mathbb{L}_N \rightsquigarrow \mathbb{L}_N$ in the same way as in (4.5). Since

$$\begin{gathered}\eta_N^1 \in \varphi_N(\widehat{\eta}^0, \eta_N^0), \eta_N^2 \in \varphi_N(\widehat{\eta}^1, \eta_N^1), \dots, \eta_N^p \in \varphi_N(\widehat{\eta}^{p-1}, \eta_N^{p-1}), \\ \eta_N^{p+1} \in \varphi_N(\widehat{\eta}^0, \eta_N^p), \dots, \eta_N^n = \eta_N^0 \in \varphi_N(\widehat{\eta}^{p-1}, \eta_N^{n-1}),\end{gathered}$$

we obtain that the difference inclusion

$$x_N^{n+1} \in h(x_N^n) \tag{4.7}$$

admits, in a view of the form of h, an $n/p = (2k+1) \cdot 2^{l-i}$-orbit. We note that n/p is maximal in the Sharkovskiĭ ordering for inclusion (4.7). Indeed, the existence of an u-orbit, $u \rhd n/p$, of inclusion (4.7) yields the existence of an up-orbit of inclusion (4.2). One gets that $up \rhd n$, but it is a contradiction to the maximality of n.

Applying Theorem 3.8, we obtain that inclusion (4.7) has also the orbits related to the numbers

$$(2k+3)\cdot 2^{l-i} \triangleright (2k+5)\cdot 2^{l-i} \triangleright \ldots \triangleright 2 \triangleright 1,$$

except 2^{l-i+2}, if $k > 1$;

$$(2k+3)\cdot 2^{l-i} \triangleright (2k+5)\cdot 2^{l-i} \triangleright \ldots \triangleright 2 \triangleright 1,$$

except $2^{l-i+1}3$, 2^{l-i+2} and 2^{l-i+1}, if $k = 1$; and, if $k = 0$,

$$2^{l-i-1} \triangleright 2^{l-i-2} \triangleright \ldots \triangleright 2 \triangleright 1.$$

Let $x_N^0, x_N^1, \ldots, x_N^{r-1}$ be an r-orbit of inclusion (4.7) and define

$$\xi_N^{sp} = x_N^s,$$

for $s = 0, \ldots, r-1$. Then we can find $\xi_N^{sp+t} \in \mathbb{L}_N$, for $s = 0, \ldots, r-1$, and $t = 0, \ldots, p-1$ satisfying

$$\xi_N^{sp+t+1} \in \varphi_N(\widehat{\eta}^t, \xi_N^{sp+t}),$$

for $s = 0, \ldots, r-1$, and $t = 0, \ldots, p-1$, where $p = 2^i$. Then

$$(\widehat{\eta}^0, \xi_N^0), (\widehat{\eta}^1, \xi_N^1), \ldots, (\widehat{\eta}^{p-1}, \xi_N^{p-1}), (\widehat{\eta}^0, \xi_N^p), \ldots, (\widehat{\eta}^{p-1}, \xi_N^{rp-1})$$

is an rp-orbit of inclusion (4.2).

Doing this for each r for which inclusion (4.7) has an r-orbit demonstrates that inclusion (4.2) has the orbits related to the numbers

$$(2k+3)\cdot 2^{l} \triangleright (2k+5)\cdot 2^{l} \triangleright \ldots \triangleright 2^{i+1} \triangleright 2^i,$$

except 2^{l+2}, if $k > 1$;

$$(2k+3)\cdot 2^{l} \triangleright (2k+5)\cdot 2^{l} \triangleright \ldots \triangleright 2^{i+1} \triangleright 2^i,$$

except $2^{l+1}3, 2^{l+2}, 2^{l+1}$, if $k = 1$; and, if $k = 0$,

$$2^{l-1} \triangleright 2^{l-2} \triangleright \ldots \triangleright 2^{i+1} \triangleright 2^i.$$

Summarizing with (4.6), we obtain that inclusion (4.2) has the orbits related to the numbers

$$(2k+3)\cdot 2^{l} \triangleright (2k+5)\cdot 2^{l} \triangleright \ldots \triangleright 2^{i} \triangleright 2^{i-1} \triangleright \ldots \triangleright 2 \triangleright 1,$$

except 2^{l+2}, if $k > 1$;

$$(2k+3)\cdot 2^{l} \triangleright (2k+5)\cdot 2^{l} \triangleright \ldots \triangleright 2^{i} \triangleright 2^{i-1} \triangleright \ldots \triangleright 2 \triangleright 1,$$

except $2^{l+1}3, 2^{l+2}, 2^{l+1}$, if $k = 1$; and

$$2^{l-1} \triangleright 2^{l-2} \triangleright \ldots \triangleright 2^{i} \triangleright 2^{i-1} \triangleright \ldots \triangleright 2 \triangleright 1,$$

if $k = 0$, which was to prove. $\square$

COROLLARY 4.3. *If φ has an n-orbit, then φ has also a k-orbit, for every $k \lhd n$, with the exception of at most three orbits.*

REMARK 4.4. If φ in Theorem 4.1 and Corollary 4.3 is a self-mapping of $\mathbb{I} = \prod_{i=1}^{N} \mathbb{I}_i$, i.e. $\varphi\colon \mathbb{I} \leadsto \mathbb{I}$, where $\mathbb{I}_i \subset \mathbb{L}_i$ are bounded closed subintervals of $\mathbb{L}_i$, $i = 1, \dots, N$; then the assertions of Theorem 4.1 and Corollary 4.3 hold obviously as well, because $\mathbb{I}$ is again a Cartesian product of linear continua $\mathbb{I}_i$. In any case, Theorem 4.1 extends Theorem 1.3 (see also Remark 1.4).

5. Application to differential inclusions

Now, we would like to apply Theorem 4.1 and Corollary 4.3 to differential systems. Unfortunately, as explained in [2], the classical Sharkovskiĭ theorem, and subsequently also Theorem 1.3, cannot be applied to differential equations, because only empty assertions are available. Moreover, we do not have any appropriate result concerning differential equations or inclusions on a linear continuum, in general, in order to apply Theorem 3.8. For a survey of the results in this field, see e.g. [1]. The application of Theorem 1.2 has very probably not much meaning as well.

Therefore, we restrict ourselves to the system

$$X' \in F(t, X), \quad F(t + \omega, X) \equiv F(t, X), \quad \omega > 0, \tag{5.1}$$

where $F\colon [0, \omega] \times \mathbb{R}^N \leadsto \mathbb{R}^N$ is an (upper) Carathéodory map with nonempty, convex and compact values:

(a) $F(\,\cdot\,, X)\colon [0, \omega] \leadsto \mathbb{R}^N$ is measurable, for every $X \in \mathbb{R}^N$, i.e. $\{t \in [0, \omega] : F(t, X) \subset V\}$ is open in $[0, \omega]$, for every $X \in \mathbb{R}^N$, whenever V is open in $\mathbb{R}^N$,

(b) $F(t, \cdot\,)\colon \mathbb{R}^N \leadsto \mathbb{R}^N$ is upper-semicontinuous, for a.a. $t \in [0, \omega]$,

(c) $|F(t, X)| \le \alpha|X| + \beta$, for a.a. $t \in [0, \omega]$ and every $X \in \mathbb{R}^N$, where α, β are suitable nonnegative constants.

Furthermore, we assume that $F = (F_1, \dots, F_N)$ has a special triangular structure, namely

(d) $F_i(X) = F_i(x_1, \dots, x_N) = F_i(x_1, \dots, x_i), \quad i = 1, \dots, N.$

Thus, the (well-defined) associated Poincaré translation operator $T_{k\omega}\colon \mathbb{R}^N \leadsto \mathbb{R}^N$, $k \in \mathbb{N}$, takes the form

$$\begin{aligned} T_{k\omega}(X^\circ) := \{X(k\omega; X^\circ) = (&x_1(k\omega; x_1^\circ), \dots, x_i(k\omega; x_1^\circ, \dots, x_i^\circ), \dots, \\ &x_N(k\omega; x_1^\circ, \dots, x_N^\circ)) : X(\,\cdot\,; X^\circ) \in AC([0; k\omega], \mathbb{R}^N) \\ &\text{is a solution of (5.1) with } X(0; X^\circ) = X^\circ\}. \end{aligned} \tag{5.2}$$

Observe that $T_{k\omega}(X^\circ) \equiv T_\omega^k(X^\circ)$, $k \in \mathbb{N}$. Moreover, the operators in (5.2) are well-known (see e.g. [1], [3], [8]) to be admissible in the sense of Definition 1.8, satisfying evidently condition (4.1).

Because of an obvious correspondence between (Carathéodory subharmonic) $k\omega$-periodic solutions $X(t) \in AC([0;k\omega],\mathbb{R}^N)$ of (5.1), i.e. $X(t) \equiv X(t+k\omega)$ and $X(t) \not\equiv X(t+l\omega)$, for $1 \leq l < k$, and k-orbits of the Poincaré operator T_ω in (5.2), we can express immediately Theorem 4.1 in terms of differential inclusions as follows.

Theorem 5.1. *Assume that $F\colon [0,\omega]\times\mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ has nonempty, convex and compact values and that it satisfies conditions* (a)–(d) *above. Let system* (5.1) *admit an $n\omega$-periodic solution, where $n = 2^m \cdot q$, $m \in \mathbb{N}_0$ and q is odd, and let n be maximal in the Sharkovskiĭ ordering.*

(a) *If $q > 3$, then* (5.1) *admits a $k\omega$-periodic solution, for every $k \lhd n$, except $k = 2^{m+2}$.*
(b) *If $q = 3$, then* (5.1) *admits a $k\omega$-periodic solution, for every $k \lhd n$, except $k = 2^{m+1}3, 2^{m+2}, 2^{m+1}$.*
(c) *If $q = 1$, then* (5.1) *admits a $k\omega$-periodic solution, for every $k \lhd n$.*

By the same reasons, Corollary 4.3 can be rewritten as follows.

Corollary 5.2. *If the above Carathéodory system* (5.1) *has an $n\omega$-periodic solution, then it also possesses a $k\omega$-periodic solution, for every $k \lhd n$, with the exception of at most three subharmonics. In particular, for $n \neq 2^m$, $m \in \mathbb{N}_0$, system* (5.1) *admits infinitely many subharmonics.*

The notion of linear continua and their Cartesian products can be also related to smooth manifolds. We have to our disposal appropriate results for differential inclusions on so called proximate retracts (see e.g. [8], [9], [12]).

Definition 5.3. A compact subset $A \subset \mathbb{R}^N$ is called *a proximate* (*neighbourhood*) *retract* if there exists an open neighbourhood U of A in $\mathbb{R}^N$ and a continuous mapping (called a proximative retraction) $r\colon U \to A$ such that

$$|r(X) - X| = \operatorname{dist}(X, A) \quad \text{for all } X \in U.$$

Let us note that proximate retracts are always ANR-spaces and as their examples, we can give C^2-manifolds (with or without boundaries) and convex subsets of $\mathbb{R}^N$; for more details, see e.g. [8].

Hence, consider again (4.1), but let F be this time defined only on a proximate retract $A \subset \mathbb{R}^N$ which is at the same time the Cartesian product of linear continua, i.e. $F\colon [0,\omega] \times A \rightsquigarrow \mathbb{R}^N$, satisfying (a)–(d). Furthermore, assume still the Nagumo-type condition

$$F(t,X) \cap T_A(X) \neq \emptyset \quad \text{for all } (t,X) \in [0,\omega] \times A, \tag{5.3}$$

where

$$T_A(X) := \{Y \in \mathbb{R}^N : \liminf_{h \to 0^+}(1/h)[\operatorname{dist}(X + hY, A)] = 0\}$$

is the Bouligand cone to A.

It can be checked (see [8], [12]) that the associated Poincaré translation operator

$$\begin{aligned} T_{k\omega}(X^\circ) := \{X(k\omega; X^\circ) = (&x_1(k\omega; x_1^\circ), \dots, x_i(k\omega; x_1^\circ, \dots, x_i^\circ), \dots, \\ &x_N(k\omega; x_1^\circ, \dots, x_N^\circ)) : X(\,\cdot\,; X^\circ) \in AC([0;\omega], \mathbb{R}^N) \\ &\text{is a solution of (5.1) with } X(0; X^\circ) = X^\circ \\ &\text{and } X(t, X^\circ) \in A, \text{for all } t \in [0, \omega]\} \end{aligned}$$

is again admissible in the sense of Definition 1.8.

By the same arguments as above, Theorem 5.1 and Corollary 5.2 can be easily modified in the following way.

Theorem 5.4. *Let $A \subset \mathbb{R}^N$ be a proximate retract which is at the same time the Cartesian product of linear continua. Assume that $F\colon [0, \omega] \times A \rightsquigarrow \mathbb{R}^N$ has nonempty, convex and compact values and that it satisfies the above conditions* (a)–(d)*, jointly with the Nagumo-type condition* (5.3)*. Let system* (5.1) *admit an $n\omega$-periodic solution whose values are in A, where $n = 2^m \cdot q$, $m \in \mathbb{N}_0$ and q is odd, and let n be maximal in the Sharkovskiĭ ordering. Then the same conclusions as in Theorem* 5.1 *hold and, moreover, the values of all implied subharmonics are in A.*

Corollary 5.5. *Let the above Carathéodory system* (5.1)*, considered on a proximate retract $A \subset \mathbb{R}^N$, which is at the same time the Cartesian product of linear continua, have an $n\omega$-periodic solution. Then the same conclusions as in Corollary* 5.2 *hold, provided F satisfies the Nagumo-type condition* (5.3)*. Moreover, the values of subharmonics are in A.*

Remark 5.6. The further application of Theorem 4.1 and Corollary 4.3 can be given quite analogously to differential inclusions on Hilbert proximate retracts on the basis of the results in [9].

Remark 5.7. We have to our disposal even more abstract appropriate results for the Poincaré operators of differential inclusions (see [1], [3], and the references therein), but they do not seem to be quite adequate to the notion of a linear continuum.

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Manuscript received March 6, 2003

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