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MULTIPLE SOLUTIONS OF COMPACT H-SURFACES IN EUCLIDEAN SPACE

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Abstract. We prove here the multiplicity results for the solutions of compact H-surfaces in Euclidean space. Some minimax methods and topological arguments are used for the existence of such solutions in multiply connected domains.

1. Introduction

Let Ω be a smooth and bounded domain in $\mathbb{R}^2$. We denote $V = \{a \in H^1(\Omega),\ a \neq \text{constant}\}$. Given two functions $a, b \in V$, we denote by φ the unique solution in $W^{1,1}(\Omega)$ of the Dirichlet problem

$$\begin{cases} -\Delta\varphi = \{a, b\} & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\{a, b\} = a_x b_y - a_y b_x$ and subscripts denote partial differentiation with respect to coordinates.

Thanks to the works of H. Wente ([16]), H. Brezis and J.-M. Coron ([3]), we have the following estimates:

$$\|\varphi\|_{L^\infty(\Omega)} + \|\nabla\varphi\|_{L^2(\Omega)} \leq C_0(\Omega)\|\nabla a\|_{L^2(\Omega)}\|\nabla b\|_{L^2(\Omega)} \tag{1.2}$$

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for some constant $C_0(\Omega) > 0$. Later on it was proved by F. Bethuel and J. M. Ghidaglia ([1]) that $C_0(\Omega)$ does not depend on Ω. This leads to consider the best constant involving the L^2-norm in the estimations analogous to (1.2). More precisely, Y. Ge has obtained in [6] the following

$$C_2(\Omega) := \sup_{a,b\in V} \frac{\|\nabla\varphi\|_2^2}{\|\nabla a\|_2^2\|\nabla b\|_2^2} = \frac{3}{16\pi}, \tag{1.3}$$

where $\|\cdot\|_2$ denotes the usual norm in $L^2(\Omega)$. Moreover, this best constant is achieved if and only if Ω is simply connected. In fact the study of the best constant involving the L^2-norm can be also done as follows (see [6]): For any $a, b \in V$ and φ defined by (1.1), we define the following energy functional

$$E(a,b,\Omega) = \frac{\|\nabla a\|_2^2 + \|\nabla b\|_2^2}{2\|\nabla\varphi\|_2}$$

or equivalently,

$$E_1(a,b,\Omega) = \frac{1}{2}(\|\nabla a\|_2^2 + \|\nabla b\|_2^2), \quad \text{defined for all } (a,b)\in M,$$

where $M = \{(a,b) \in H^1(\Omega)\times H^1(\Omega) : \|\nabla\varphi\|_2 = 1\}$ is a complete C^2-Finsler manifold. The critical points (a,b,φ) of this functional satisfies the following Euler–Lagrange equation:

$$\begin{cases} -\Delta u = u_x \wedge u_y & \text{in } \Omega, \\ \varphi = \dfrac{\partial a}{\partial n} = \dfrac{\partial b}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $u := (\lambda a, \lambda b, \lambda^2\varphi)$ for $\lambda = \sqrt{(\|\nabla a\|_2^2 + \|\nabla b\|_2^2)/(2\|\nabla\varphi\|_2^2)}$ and $n = (n_1, n_2)$ is the normal vector on $\partial\Omega$. Note that the functional E and its critical points are invariant by conformal transformations of the domain Ω. So this variational problem depends only on the complex structure of Ω. Moreover, the boundary conditions permit to construct a solution of H-system $\widetilde{u}$ from a compact oriented Riemannian surface in $\mathbb{R}^3$ by gluing two copies of Ω. More precisely, we construct $N := \Omega \cup_{\partial\Omega} \widetilde{\Omega}$, where $\widetilde{\Omega}$ is a copy of Ω, provided with opposing orientation and a smooth map $\widetilde{u}$ from N into $\mathbb{R}^3$ which is defined by $\widetilde{u} = u$ on Ω and $\widetilde{u} = (\lambda a, \lambda b, -\lambda^2\varphi)$ on $\widetilde{\Omega}$. Therefore $\widetilde{u}$ satisfies

$$-\Delta\widetilde{u} = \widetilde{u}_x \wedge \widetilde{u}_y \quad \text{in } N.$$

If $\widetilde{u}$ is conformal, that is, the Hopf differential

$$\omega := (|\widetilde{u}_x|^2 - |\widetilde{u}_y|^2 - 2i\langle\widetilde{u}_x, \widetilde{u}_y\rangle)\, dz\otimes dz = 0,$$

it would be a constant mean curvature branched immersion from N into $\mathbb{R}^3$. This motivates the search for critical points of E. Unfortunately, we can not obtain it directly by the standard minimization method since the energy of any minimizing sequence concentrates around some point on the boundary of Ω. In

[6], we proved an existence result for a perforated domain with small holes and this result is generalized for any annular domain in [7]. In this paper, we will deal with this procedure in order to find the multiple solutions with different energy for some multiply connected domains. We will see that the special conformal structure of domains causes this multiply solutions result. More precisely, let $B(z,r) = \{z' \in \mathbb{C} : |z'-z| < r\}$ be the disc in $\mathbb{R}^2 \approx \mathbb{C}$ centered at z of radius r and $\overline{B(z,r)}$ its closure. Let $z_1, \dots, z_k \in B(0,1/2)$ be fixed such that for some $r > 0$, $B(z_i,r) \subset B(0,\frac{1}{2})$ for all $1 \le i \le k$ and $\overline{B(z_i,r)} \cap \overline{B(z_j,r)} = \emptyset$ for any $1 \le i \ne j \le k$. Taking $r/2 \ge r_1 \ge \dots \ge r_k > 0$, we have the following result:

THEOREM. *Let $\Omega = B(z_1,\dots,z_k; r_1,\dots,r_k) = B(0,1) \setminus (\bigcup_{i=1}^k \overline{B(z_i,r_i)})$. Then there exist $\overline{r}_1 > 2\overline{r}_2 > \dots > 2^{k-1}\overline{r}_k$ such that if $r_i \in (\overline{r}_i/2, \overline{r}_i)$ for all $1 \le i \le k$, there exist k distinct critical points of E_1 with different energy in Ω.*

This is a generalization of the previous result of [6] (Theorem 11) which is similar to an earlier work of J.-M. Coron ([4]) concerning the critical Sobolev exponent problem. Here we use the same strategy. For $t \in \mathbb{R}$, we denote $E_M^t = \{(a,b) \in M : E_1(a,b) \le t\}$ the level set of E_1. We see that the topology of E_M^γ is equivalent to $\partial\Omega$ when γ is near the value $G(\Omega) := \inf_{(a,b)\in M} E_1(a,b,\Omega) = \sqrt{16\pi/3}$ and the topology of the level set changes k times for $t \in (G(\Omega), \sqrt{2}G(\Omega))$. To establish the result we argue by contradiction. We construct a topological disc Δ in $E_M^{\sqrt{2}G(\Omega)}$ whose boundary is a non contractible circle $\partial\Delta$ in $E_M^{G(\Omega)+\mu}$ for some small $\mu > 0$. And if the system (1.4) does not admit a solution in $E_M^{\sqrt{2}G(\Omega)}$, then it implies that there exists a contraction h of Δ onto $\partial\Delta$, which is a contradiction. Iterating this procedure we can find the second minimax critical value between the first one and $G(\Omega)$ and so on. This method has been exploited to search several critical points by D. Passaseo in [12] and P. Padilla in [13] for semi-linear problems involving critical Sobolev exponent and by F. Takahashi in [15] for H-systems with homogenous boundary conditions.

In the following section, we will prove some technical lemmas which are needed in the proof of the main theorem. In all this paper, C denotes generic positive constant independent of the solutions, even its value could be changed from one line to another one.

2. The proof of Theorem

The proof is divided into several steps.

Step 1. We introduct a map Q from $H^1(\Omega) \times H^1(\Omega)$ into $\mathbb{R}^2$,

$$Q\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}^2,$$
$$(a,b) \mapsto \frac{\sqrt{3}}{8\sqrt{\pi}} \int_\Omega (x,y)\cdot(|\nabla a|^2 + |\nabla b|^2)\,dx\,dy \in \mathbb{R}^2.$$

It is easy to prove that Q is continuous. Our first result is the following

Lemma 2.1. *Let $1 \le m < k$. For any $\overline{r} \in (0, r/2)$, there exist positive numbers $\varepsilon > 0$ and $\delta > 0$ such that for the domain $\Omega = B(z_1, \dots, z_k; r_1, \dots, r_k)$ with $r/2 \ge r_1 \ge \dots \ge r_m > \overline{r}$ and $\delta > r_{m+1} \ge r_{m+2} \ge \dots \ge r_k > 0$, if $(a, b) \in M$ with $E(a, b, \Omega) < \sqrt{16\pi/3} + \varepsilon$, then $Q(a, b) \in B(z_1, \dots, z_m; \overline{r}/2, \dots, \overline{r}/2)$.*

Proof. We argue by contradiction. Suppose that the statement fails. Then there exist some positive number $\overline{r}$ with $r/2 > \overline{r} > 0$ and a sequence of domains $\Omega_n = B(z_1, \dots, z_k; r_{1,n} \dots, r_{k,n})$ and $(a_n, b_n) \in M(\Omega_n)$ with

$$\begin{gathered} r_{i,n} \ge \overline{r} \quad \text{for all } n \in \mathbb{N},\ 1 \le i \le m, \\ r_{i,n} \to 0 \quad \text{as } n \to \infty \text{ for any } m+1 \le i \le k, \\ E(a_n, b_n, \Omega_n) \to \sqrt{16\pi/3} \quad \text{as } n \to \infty, \\ Q(a_n, b_n) \notin B(z_1, \dots, z_m; \overline{r}/2, \dots, \overline{r}/2). \end{gathered}$$

Without loss of generality, we can assume that for any $1 \le i \le m$,

$$r_{i,n} \to \overline{r}_i \quad \text{as } n \to \infty.$$

Thus $\Omega^* = B(0,1) \setminus (\bigcup_{i=1}^m \overline{B(z_i, \overline{r}_i)} \cup (\bigcup_{i=m+1}^k \{z_i\}))$ is the limit domain of Ω_n. Setting $\Omega_\theta = B(0,1) \setminus (\bigcup_{i=1}^k \overline{B(z_i, r/2)})$, we can suppose that

$$\int_{\Omega_\theta} a_n = \int_{\Omega_\theta} b_n = 0,$$

otherwise, we take

$$\widetilde{a}_n = a_n - \frac{1}{|\Omega_\theta|} \int_{\Omega_\theta} a_n \quad \text{and} \quad \widetilde{b}_n = b_n - \frac{1}{|\Omega_\theta|} \int_{\Omega_\theta} b_n$$

instead of a_n and b_n if necessary. By virtue of Poincaré's inequality, we get

$$\|a_n\|_{L^2(\Omega_\theta)} \le C \|\nabla a_n\|_{L^2(\Omega_\theta)} \le C$$

and

$$\|b_n\|_{L^2(\Omega_\theta)} \le C \|\nabla b_n\|_{L^2(\Omega_\theta)} \le C.$$

Therefore $\{a_n\}_{n \in \mathbb{N}}$ and $\{b_n\}_{n \in \mathbb{N}}$ are bounded in $H^1(\Omega_\theta)$. Fixing a function $\xi \in C_0^\infty(\mathbb{R}^2)$ such that $\operatorname{supp}(\xi) \subset B(0, r)$, and $\xi|_{B(0,r/2)} = 1$. We define for all $n \in \mathbb{N}$, for all $1 \le i \le k$,

$$a_{n,i}(z) = \begin{cases} \xi(z - z_i) a_n(z) & \text{if } z \in B(z_i, r) \setminus \overline{B(z_i, r_{i,n})}, \\ 0 & \text{if } z \in \mathbb{R}^2 \setminus B(z_i, r), \end{cases}$$

and

$$\overline{a}_{n,i}(z) = \begin{cases} a_{n,i}(z) & \text{if } z \in B(z_i, r) \setminus \overline{B(z_i, r_{i,n})}, \\ a_{n,i}\bigg(\dfrac{(r_{i,n})^2 (z - z_i)}{|z - z_i|^2} + z_i\bigg) & \text{if } z \in B(z_i, r_{i,n}). \end{cases}$$

Hence, for any $1 \le i \le k$, we obtain a sequence $\{\overline{a}_{n,i}\}_{n\in\mathbb{N}}$ in $H_0^1(B(z_i,r))$. Clearly,

$$\|\nabla \overline{a}_{n,i}\|_{L^2(B(z_i,r))} = \sqrt{2}\|\nabla a_{n,i}\|_{L^2(B(z_i,r)\setminus\overline{B(z_i,r_{i,n})})} \le C.$$

It follows again from Poincaré's inequality that $\{\overline{a}_{n,i}\}_{n\in\mathbb{N}}$ is bounded in $H_0^1(B(z_i,r))$. Now set

$$\overline{a}_n(z) = \begin{cases} a_n(z) & \text{if } z \in \Omega_n, \\ \overline{a}_{n,i}(z) & \text{if } z \in B(z_i, r_{i,n}) \text{ with } 1 \le i \le k. \end{cases}$$

We see that $\{\overline{a}_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(B(0,1))$. Similary we can define another bounded sequence $\{\overline{b}_n\}_{n\in\mathbb{N}}$ in $H^1(B(0,1))$. Set

$$\overline{\varphi}_n(z) = \begin{cases} \varphi_n(z) & \text{if } z \in \Omega_n, \\ 0 & \text{if } z \in B(0,1)\setminus\Omega_n, \end{cases}$$

where φ_n is the solution of (1.1) for $a = a_n$ and $b = b_n$ in Ω_n. It follows from (1.2) that $\{\overline{\varphi}_n\}_{n\in\mathbb{N}}$ is bounded in $H_0^1(B(0,1))$. Without loss of generality, we may assume that

- $\overline{a}_n \to \alpha$ weakly in $H^1(B(0,1))$ and strongly in $L^2(B(0,1))$,
- $\overline{b}_n \to \beta$ weakly in $H^1(B(0,1))$ and strongly in $L^2(B(0,1))$,
- $\overline{\varphi}_n \to \psi$ weakly in $H^1(B(0,1))$ and strongly in $L^2(B(0,1))$ and a.e. for $z \in B(0,1)$.

On the other hand, $\overline{\varphi}_n \to 0$ a.e. for $z \in \bigcup_{i=1}^m B(z_i,\overline{r}_i)$. Therefore $\psi = 0$ in $\bigcup_{i=1}^m B(z_i,\overline{r}_i)$ which implies $\psi \in H_0^1(B(0,1)\setminus\bigcup_{i=1}^m \overline{B(z_i,\overline{r}_i)})$. In view of Lemma 7.2 in [6], for any domain $\Omega' \subset\subset \Omega^*$, we have $\{\overline{a}_n,\overline{b}_n\} \to \{\alpha,\beta\}$ in $\mathcal{D}'(\Omega')$. Hence we deduce $-\Delta\psi = \{\alpha,\beta\}$ in $\mathcal{D}'(\Omega^*)$.

For any $m < i \le k$, let $\psi_{i,1}$ and $\psi_{i,2}$ be solutions of the following problems

$$\begin{cases} -\Delta\psi_{i,1} = 0 & \text{in } B(z_i, r/2), \\ \psi_{i,1} = \psi & \text{on } \partial B(z_i, r/2), \end{cases}$$

and

$$\begin{cases} -\Delta\psi_{i,2} = \{\alpha,\beta\} & \text{in } B(z_i, r/2), \\ \psi_{i,2} = 0 & \text{on } \partial B(z_i, r/2). \end{cases}$$

Clearly, $\psi_{i,1} \in H^1(B(z_i,r/2))$ and, by (1.2), $\psi_{i,2} \in H_0^1(B(z_i,r/2))$ so that

$$\psi - \psi_{i,1} - \psi_{i,2} \in H_0^1(B(z_i,r/2))$$

and

$$-\Delta(\psi - \psi_{i,1} - \psi_{i,2}) = 0 \quad \text{in } \mathcal{D}'(B(z_i,r/2)\setminus\{z_i\}).$$

Therefore, there exists $l_0 \in \mathbb{N}^*$ such that

$$-\Delta(\psi - \psi_{i,1} - \psi_{i,2}) = \sum_{\gamma=(\gamma_1,\gamma_2)\in\mathbb{N}^2, |\gamma|=\gamma_1+\gamma_2\le l_0} C_{\gamma_1\gamma_2}\frac{\partial^{|\gamma|}}{\partial x^{\gamma_1}\partial y^{\gamma_2}}\delta_{z_i}$$

in $\mathcal{D}'(B(z_i, r/2))$, where $C_{\gamma_1\gamma_2} \in \mathbb{R}$ and δ_{z_i} denotes the Dirac measure centered at z_i with unit mass. Remark that for all $\gamma = (\gamma_1, \gamma_2) \in \mathbb{N}^2$, $(\partial^{|\gamma|}/\partial x^{\gamma_1}\partial y^{\gamma_2})\delta_{z_i} \notin H^{-1}(B(z_i, r/2))$. Thus the fact $-\Delta(\psi - \psi_{i,1} - \psi_{i,2}) \in H^{-1}(B(z_i, r/2))$ implies that $C_{\gamma_1\gamma_2} = 0$ for all $\gamma = (\gamma_1, \gamma_2)$ with $|\gamma| \le l_0$. Finally, we have

$$\begin{cases} -\Delta\psi = \{\alpha, \beta\} & \text{in } \widetilde{\Omega}, \\ \psi = 0 & \text{on } \partial\widetilde{\Omega}, \end{cases}$$

where $\widetilde{\Omega} = B(0,1) \setminus \bigcup_{i=1}^m \overline{B(z_i, \overline{r}_i)}$. We claim that

$$\|\nabla a_n\|^2_{L^2(\Omega_n)} = \|\nabla(a_n - \alpha)\|^2_{L^2(\Omega_n)} + \|\nabla\alpha\|^2_{L^2(\widetilde{\Omega})} + o(1), \tag{2.1}$$

$$\|\nabla b_n\|^2_{L^2(\Omega_n)} = \|\nabla(b_n - \beta)\|^2_{L^2(\Omega_n)} + \|\nabla\beta\|^2_{L^2(\widetilde{\Omega})} + o(1), \tag{2.2}$$

$$\|\nabla \varphi_n\|^2_{L^2(\Omega_n)} = \|\nabla(\varphi_n - \psi)\|^2_{L^2(\Omega_n)} + \|\nabla\psi\|^2_{L^2(\widetilde{\Omega})} + o(1). \tag{2.3}$$

For this purpose, we write

$$\begin{aligned} \int_{\Omega_n} |\nabla a_n|^2 &= \int_{\Omega_n} |\nabla(a_n - \alpha)|^2 + \int_{\Omega_n} |\nabla\alpha|^2 + 2\int_{\Omega_n} \nabla(a_n - \alpha)\nabla\alpha \\ &= \int_{\Omega_n} |\nabla(a_n - \alpha)|^2 + \int_{\widetilde{\Omega}} |\nabla\alpha|^2 + 2\int_{\widetilde{\Omega}} \nabla(\overline{a}_n - \alpha)\nabla\alpha \\ &\quad + \int_{\Omega_n\setminus\widetilde{\Omega}} (|\nabla\alpha|^2 + 2\nabla(\overline{a}_n - \alpha)\nabla\alpha) \\ &\quad - \int_{\widetilde{\Omega}\setminus\Omega_n} (|\nabla\alpha|^2 + 2\nabla(\overline{a}_n - \alpha)\nabla\alpha). \end{aligned} \tag{2.4}$$

It is clear that

$$\int_{\widetilde{\Omega}} 2\nabla(\overline{a}_n - \alpha)\nabla\alpha = o(1),$$

since $\overline{a}_n \to \alpha$ weakly in $H^1(B(0,1))$. Moreover, denoting $\Omega_n\Delta\widetilde{\Omega} = (\Omega_n \setminus \widetilde{\Omega}) \cup (\widetilde{\Omega} \setminus \Omega_n)$, we have

$$\begin{aligned} &\int_{\Omega_n\Delta\widetilde{\Omega}} (|\nabla\alpha|^2 + 2|\nabla(\overline{a}_n - \alpha)\nabla\alpha|) \\ &\qquad \le ||\nabla\alpha||_{L^2(\Omega_n\Delta\widetilde{\Omega})}(||\nabla\alpha||_{L^2(\Omega_n\Delta\widetilde{\Omega})} + 2||\nabla(\overline{a}_n - \alpha)||_{L^2(\Omega_n\Delta\widetilde{\Omega})}) \\ &\qquad \le C||\nabla\alpha||_{L^2(\Omega_n\Delta\widetilde{\Omega})}. \end{aligned}$$

As $\operatorname{meas}(\Omega_n\Delta\widetilde{\Omega}) \to 0$, we deduce that

$$\int_{\Omega_n\Delta\widetilde{\Omega}} (|\nabla\alpha|^2 + 2\nabla(\overline{a}_n - \alpha)\nabla\alpha)\, dx = o(1). \tag{2.6}$$

Combining (2.4) to (2.6), we obtain (2.1). Similarly, we establish (2.2) and (2.3). Now denote by $\varphi_{n,1}$ (resp. $\varphi_{n,2}$) the unique solution of equation (1.1) for $a = a_n - \alpha$ and $b = \beta$ (resp. $a = \alpha$ and $b = b_n - \beta$) in Ω_n. So $\gamma_n = \varphi_n - \psi - \varphi_{n,1} - \varphi_{n,2}$ is the unique solution of equation (1.1) for $a = a_n - \alpha$ and $b = b_n - \beta$ in Ω_n.

Denote by $\widetilde{\varphi}_{n,1}$ the unique solution of equation (1.1) for $a = \overline{a}_n - \alpha$ and $b = \beta$ in $B(0,1)$ and set

$$\overline{\varphi}_{n,1}(z) = \begin{cases} \varphi_{n,1}(z) & \text{if } z \in \Omega_n, \\ 0 & \text{if } z \in B(0,1) \setminus \Omega_n. \end{cases}$$

As $\widetilde{\varphi}_{n,1}$ minimizes the energy functional

$$E_2(\varphi) = \frac{1}{2}\int_{B(0,1)} (|\nabla\varphi|^2\,dx - 2\{\overline{a}_n - \alpha, \beta\}\varphi)\,dx,$$

for all $\varphi \in H_0^1(B(0,1))$. We have

$$\begin{aligned} \frac{1}{2}\int_{\Omega_n} |\nabla\varphi_{n,1}|^2 &= -\frac{1}{2}\int_{B(0,1)} |\nabla\overline{\varphi}_{n,1}|^2 + \int_{B(0,1)} \{\overline{a}_n - \alpha, \beta\}\overline{\varphi}_{n,1} \\ &\le -\frac{1}{2}\int_{B(0,1)} |\nabla\widetilde{\varphi}_{n,1}|^2 + \int_{B(0,1)} \{\overline{a}_n - \alpha, \beta\}\widetilde{\varphi}_{n,1} \\ &= \frac{1}{2}\int_{B(0,1)} |\nabla\widetilde{\varphi}_{n,1}|^2 = \frac{1}{2}\int_{B(0,1)} \{\overline{a}_n - \alpha, \beta\}\widetilde{\varphi}_{n,1}. \end{aligned}$$

Using Lemma 7.2 of [6], we obtain

$$\int_{\Omega_n} |\nabla\varphi_{n,1}|^2 = o(1).$$

With the same argument, we get

$$\int_{\Omega_n} |\nabla\varphi_{n,2}|^2 = o(1),$$

so that

$$1 = \|\nabla\varphi_n\|^2_{L^2(\Omega_n)} = \|\nabla\gamma_n\|^2_{L^2(\Omega_n)} + \|\nabla\psi\|^2_{L^2(\widetilde{\Omega})} + o(1).$$

Thanks to Theorem 1.3 in [6], we have

$$\begin{aligned} \sqrt{\frac{16\pi}{3}} &= \frac{1}{2}(\|\nabla a_n\|^2_{L^2(\Omega_n)} + \|\nabla b_n\|^2_{L^2(\Omega_n)}) + o(1) \\ &= \frac{1}{2}(\|\nabla(a_n - \alpha)\|^2_{L^2(\Omega_n)} + \|\nabla(b_n - \beta)\|^2_{L^2(\Omega_n)}) \\ &\quad + \frac{1}{2}(\|\nabla\alpha\|^2_{L^2(\widetilde{\Omega})} + \|\nabla\beta\|^2_{L^2(\widetilde{\Omega})}) + o(1) \\ &\ge \sqrt{\frac{16\pi}{3}}(\|\nabla\gamma_n\|_{L^2(\Omega_n)} + \|\nabla\psi\|_{L^2(\widetilde{\Omega})}) + o(1) \\ &= \sqrt{\frac{16\pi}{3}}\left(\sqrt{1 - \|\nabla\psi\|^2_{L^2(\widetilde{\Omega})}} + \|\nabla\psi\|_{L^2(\widetilde{\Omega})}\right) + o(1). \end{aligned}$$

Passing to the limit as $n \to \infty$, there holds

$$\sqrt{\frac{16\pi}{3}} \ge \sqrt{\frac{16\pi}{3}}\left(\sqrt{1 - \|\nabla\psi\|^2_{L^2(\widetilde{\Omega})}} + \|\nabla\psi\|_{L^2(\widetilde{\Omega})}\right).$$

That is, $\|\nabla\psi\|_{L^2(\widetilde{\Omega})} = 0$, or $\|\nabla\psi\|_{L^2(\widetilde{\Omega})} = 1$. In the later case, we infer that

$$\frac{1}{2}(\|\nabla\alpha\|^2_{L^2(\widetilde{\Omega})} + \|\nabla\beta\|^2_{L^2(\widetilde{\Omega})}) = \sqrt{\frac{16\pi}{3}}$$

which contradicts Theorem 1.3 in [6]. Therefore $\alpha = \beta = \gamma = 0$.

Now denote by $\mathcal{M}(\mathbb{R}^2)$ the space of non-negative measures on $\mathbb{R}^2$ with finite mass. Set

$$\mu_n(z) = \begin{cases} \dfrac{1}{2}(|\nabla a_n|^2 + |\nabla b_n|^2)(z)\,dx\,dy & \text{if } z \in \Omega_n, \\ 0 & \text{if } z \notin \Omega_n, \end{cases}$$

and

$$\nu_n(z) = \begin{cases} |\nabla\varphi_n|^2\,dx\,dy & \text{if } z \in \Omega_n, \\ 0 & \text{if } z \notin \Omega_n. \end{cases}$$

Clearly, $\{\mu_n\}_{n\in\mathbb{N}}$ and $\{\nu_n\}_{n\in\mathbb{N}}$ are bounded in $\mathcal{M}(\mathbb{R}^2)$. Without loss of generality, we suppose that $\mu_n \rightharpoonup \mu$, $\nu_n \rightharpoonup \nu$ weakly in the sense of measure for some bounded non-negative measures μ and ν on $\mathbb{R}^2$. Fixing some $\eta \in C_0^\infty(\mathbb{R}^2)$. Denote by ψ_n the unique solution of equation (1.1) for $a = \eta a_n$ and $b = \eta b_n$ in Ω_n. Set

$$\overline{\psi}_n(z) = \begin{cases} \psi_n(z) & \text{if } z \in \Omega_n, \\ 0 & \text{if } z \notin \Omega_n. \end{cases}$$

Thus $\eta\overline{a}_n \rightharpoonup 0$ and $\eta\overline{b}_n \rightharpoonup 0$ in $H^1(B(0,1))$. Reasoning as before we have

$$\overline{\psi}_n \rightharpoonup 0 \text{ weakly in } H^1(B(0,1)) \text{ and strongly in } L^2(B(0,1)).$$

A direct computation shows that

$$\begin{aligned}
\int_{\Omega_n} &|\nabla(\psi_n - \eta^2\varphi_n)|^2 \\
&= \int_{\Omega_n} (-\Delta(\psi_n - \eta^2\varphi_n))(\psi_n - \eta^2\varphi_n) \\
&= \int_{\Omega_n} (\eta b_n\{a_n,\eta\} + \eta a_n\{\eta, b_n\} + 2\nabla(\eta^2)\nabla\varphi_n + (\Delta\eta^2)\varphi_n)(\psi_n - \eta^2\varphi_n) \\
&\le (\|b_n\|_{L^2(\Omega_n)} + \|a_n\|_{L^2(\Omega_n)})(\|\nabla b_n\|_{L^2(\Omega_n)} + \|\nabla a_n\|_{L^2(\Omega_n)}) \\
&\quad \times \|\eta\|^2_{C^1(B(0,1))}\|\psi_n - \eta^2\varphi_n\|_{L^\infty(B(0,1))} \\
&\quad + \|\eta\|^2_{C^2(B(0,1))}\|\varphi_n\|_{L^2(\Omega_n)}\|\psi_n - \eta^2\varphi_n\|_{L^2(\Omega_n)} \\
&\quad + \|\eta\|^2_{C^1(B(0,1))}\|\nabla\varphi_n\|_{L^2(\Omega_n)}\|\psi_n - \eta^2\varphi_n\|_{L^2(\Omega_n)},
\end{aligned}$$

which implies

$$\lim_{n\to\infty} \|\nabla(\psi_n - \eta^2\varphi_n)\|_{L^2(\Omega_n)} = 0.$$

Therefore

$$\sqrt{\frac{16\pi}{3}}\|\nabla(\eta^2\varphi_n)\|_{L^2(\Omega_n)} + o(1) \le \frac{1}{2}(\|\nabla(\eta a_n)\|^2_{L^2(\Omega_n)} + \|\nabla(\eta b_n)\|^2_{L^2(\Omega_n)}).$$

Letting $n \to \infty$, we deduce that

$$\sqrt{\frac{16\pi}{3}}\sqrt{\int_{\mathbb{R}^2} \eta^4 \, d\nu} \leq \int_{\mathbb{R}^2} \eta^2 \, d\mu, \quad \text{for all } \eta \in C_0^\infty(\mathbb{R}^2). \tag{2.7}$$

Reasoning as in [6], there exists $z_0 \in \overline{\overline{\Omega}}$ such that $\nu = \delta_{z_0}$ and $\mu = \sqrt{16\pi/3}\delta_{z_0}$. Thus for n large enough, we have

$$Q(a_n, b_n) \in B(z_1, \dots, z_m; \overline{r}/2, \dots, \overline{r}/2).$$

This contradiction yields the desired result. □

Step 2. We need the following lemma.

Lemma 2.2 ([6, Theorem 6.3]). *E_1 satisfies the Palais–Smale condition for all $c \in (\sqrt{16\pi/3}, \sqrt{32\pi/3})$.*

Step 3. Let

$$(a, b, \varphi) = \left(\frac{2 \times 3^{1/4} x}{\pi^{1/4}(1 + x^2 + y^2)}, \frac{2 \times 3^{1/4} y}{\pi^{1/4}(1 + x^2 + y^2)}, \frac{\sqrt{3}(1 - x^2 - y^2)}{2\sqrt{\pi}(1 + x^2 + y^2)} \right)$$

be a minimizer of E_1 for the unit disc. Denote $\sigma_{s,t}(z) = (z + ts)/(1 + t\overline{s}z)$ where $s = s^1 + is^2 \in S^1$, the unit circle and $t \in [0, 1)$. We set

$$P_{z_i, r_i}(z) = \frac{r_i(z - z_i)}{|z - z_i|^2}, \quad \text{for all } 1 \leq i \leq k.$$

Now, we define another continuous maps T_i from $B(0, 1)$ to M such that

$$T_i(s, t; \Omega) = e(a \circ \sigma_{s,t} \circ P_{z_i, r_i}, \ b \circ \sigma_{s,t} \circ P_{z_i, r_i})|_\Omega,$$

for all $s \in S^1$, for all $t \in [0, 1)$ where $e \in \mathbb{R}$ is well choosen such that $T_i(s, t; \Omega) \in M$. Then we have the following lemma.

Lemma 2.3. *For all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $s \in S^1$, for all $t \in [0, 1)$, if $r_i < \delta$, we have*

$$E_1(T_i(s, t; \Omega)) \leq \sqrt{\frac{16\pi}{3}} + \varepsilon.$$

Proof. Set $a_{s,t} = a \circ \sigma_{s,t} \circ P_{z_i, r_i}$, $b_{s,t} = b \circ \sigma_{s,t} \circ P_{z_i, r_i}$ and $\varphi_{s,t} = \varphi \circ \sigma_{s,t} \circ P_{z_i, r_i}$. Hence $\varphi_{s,t}$ satisfies $-\Delta \varphi_{s,t} = \{a_{s,t}, b_{s,t}\}$ in $\mathbb{R}^2$. In particular, there holds $-\Delta \varphi_{s,t} = \{a_{s,t}, b_{s,t}\}$ in Ω. We decompose now $\varphi_{s,t}$ into its harmonic $\theta_{s,t}$ and non-harmonic $\psi_{s,t}$ components as $\varphi_{s,t} = \theta_{s,t} + \psi_{s,t}$, where

$$\begin{cases} -\Delta \theta_{s,t} = 0 & \text{in } \Omega, \\ \theta_{s,t} = \varphi_{s,t} & \text{on } \partial\Omega, \end{cases} \tag{2.8}$$

and

$$
(2.9)\qquad \begin{cases} -\Delta\psi_{s,t} = \{a_{s,t}, b_{s,t}\} & \text{in } \Omega, \\ \psi_{s,t} = 0 & \text{on } \partial\Omega. \end{cases}
$$

First we remark that for all $s \in S^1$, for all $t \in [0,1)$, for all $r_j < r/2$, with $j \neq i$,

$$
\begin{aligned}
\frac{1}{2}\bigg(\int_{\mathbb{R}^2\setminus B(z_i,r)} |\nabla a_{s,t}|^2 &+ \int_{\mathbb{R}^2\setminus B(z_i,r)} |\nabla b_{s,t}|^2\bigg) \\
&= \frac{1}{2}\bigg(\int_{B(0,r_i/r)} |\nabla(a\circ\sigma_{s,t})|^2 + \int_{B(0,r_i/r)} |\nabla(b\circ\sigma_{s,t})|^2\bigg) \\
&= \frac{1}{2}\bigg(\int_{\sigma_{s,t}^{-1}(B(0,r_i/r))} |\nabla a|^2 + \int_{\sigma_{s,t}^{-1}(B(0,r_i/r))} |\nabla b|^2\bigg).
\end{aligned}
$$

Obviously, $\sigma_{s,t}^{-1} = \sigma_{-s,t}$ and $\operatorname{meas}(\sigma_{s,t}(B(0,r_i/r))) \to 0$ uniformly in $s \in S^1$, and $t \in [0,1)$ as $r_i \to 0$, which in turn implies that for all $\varepsilon > 0$, there exists $\eta > 0$ such that for all $s \in S^1$, for all $t \in [0,1)$, for all $r_j < r/2$, with $j \neq i$ if $r_i < \eta$, then

$$
\frac{1}{2}\bigg(\int_{\mathbb{R}^2\setminus B(z_i,r)} |\nabla a_{s,t}|^2\,dx + \int_{\mathbb{R}^2\setminus B(z_i,r)} |\nabla b_{s,t}|^2\,dx\bigg) \leq \varepsilon.
$$

Similarly, for such domain Ω,

$$
(2.10)\qquad \int_{\mathbb{R}^2\setminus B(z_i,r)} |\nabla\varphi_{s,t}|^2\,dx \leq \varepsilon.
$$

On the other hand, for all $z \notin B(z_i,r)$, we have $|P_{z_i,r_i}(z)| \leq r_i/r$, so that for all $s \in S^1$, for all $t \in [0,1)$,

$$
|\varphi_{s,t}(z) - \varphi\circ\sigma_{s,t}(0)| \leq \|\nabla\varphi\|_{L^\infty(B(0,1))} |\sigma_{s,t}(P_{z_i,r_i}(z)) - \sigma_{s,t}(0)| < \varepsilon
$$

provided $r_i < \eta'$ for some sufficiently small $\eta' > 0$. For any $1 \leq j \leq k$, with $j \neq i$, we choose $\chi_j \in C_0^\infty(\mathbb{R}^2)$ such that $\operatorname{supp}(\chi_j) \subset B(z_j,r)$, $\chi_j|_{B(z_j,r/2)} \equiv 1$ and $\|\nabla\chi_j\|_{L^\infty(\mathbb{R}^2)} \leq 3/r$. We also choose $\chi_0 \in C_0^\infty(\mathbb{R}^2)$ such that $\operatorname{supp}(\chi_0) \subset B(0,1)$ with $\chi_0|_{B(0,1/2)} \equiv 1$ and $\|\nabla\chi_0\|_{L^\infty(\mathbb{R}^2)} \leq 3$, then we define $\widetilde{\theta}_{s,t} : \Omega \to \mathbb{R}$ by

$$
\widetilde{\theta}_{s,t}(z) = \begin{cases} (1-\chi_0(z))(\varphi_{s,t}(z) - \varphi(\sigma_{s,t}(0))) & \text{for all } z \in B(0,1)\setminus B(0,1/2), \\ \chi_j(z)(\varphi_{s,t}(z) - \varphi(\sigma_{s,t}(0))) & \text{for all } z \in B(z_j,r)\setminus \overline{B(z_j,r_j)} \\ & \text{with } j \neq i, \\ -\varphi(\sigma_{s,t}(0))\bigg(\ln\dfrac{|z-z_i|}{r}\bigg)\bigg(\ln\dfrac{r_i}{r}\bigg)^{-1} & \text{for all } z \in B(z_i,r)\setminus\overline{B(z_i,r_i)}, \\ 0 & \text{for all} \\ & z \in B(0,1/2)\setminus\bigg(\displaystyle\bigcup_{l=1}^{k} B(z_l,r)\bigg). \end{cases}
$$

A direct calculation leads to, for all $s \in S^1$, $t \in [0,1)$, $\|\nabla\widetilde{\theta}_{s,t}\|^2_{L^2(\Omega)} < \varepsilon$ provided that $r_i < \eta''$ for some small η''. Hence

$$\|\nabla\theta_{s,t}\|^2_{L^2(\Omega)} \le \|\nabla(\widetilde{\theta}_{s,t} + \varphi(\sigma_{s,t}(0)))\|^2_{L^2(\Omega)} = \|\nabla\widetilde{\theta}_{s,t}\|^2_{L^2(\Omega)} < \varepsilon. \tag{2.11}$$

Finally, we have

$$\begin{aligned} \|\nabla\psi_{s,t}\|_{L^2(\Omega)} &\ge \|\nabla\varphi_{s,t}\|_{L^2(\Omega)} - \|\nabla\theta_{s,t}\|_{L^2(\Omega)} \\ &\ge \|\nabla\varphi_{s,t}\|_{L^2(B(z_i,r)\setminus B(z_i,r_i))} - \|\nabla\theta_{s,t}\|_{L^2(\Omega)} \\ &\ge \|\nabla\varphi_{s,t}\|_{L^2(\mathbb{R}^2\setminus B(z_i,r_i))} \\ &\quad - \|\nabla\varphi_{s,t}\|_{L^2(\mathbb{R}^2\setminus B(z_i,r))} - \|\nabla\theta_{s,t}\|_{L^2(\Omega)} \\ &= 1 - \|\nabla\varphi_{s,t}\|_{L^2(\mathbb{R}^2\setminus B(z_i,r))} - \|\nabla\theta_{s,t}\|_{L^2(\Omega)} \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} &\frac{1}{2}(\|\nabla a_{s,t}\|^2_{L^2(\Omega)} + \|\nabla b_{s,t}\|^2_{L^2(\Omega)}) \\ &\qquad \le \frac{1}{2}(\|\nabla a\|^2_{L^2(B(0,1))} + \|\nabla b\|^2_{L^2(B(0,1))}) = \sqrt{\frac{16\pi}{3}}. \end{aligned} \tag{2.13}$$

Combining (2.10) to (2.13), we prove the result. □

With the same strategy, we can establish the following lemma.

Lemma 2.4. *We have*

$$\lim_{t\to 1} E_1(T_i(s,t;\Omega)) = \sqrt{\frac{16\pi}{3}}$$

uniformly in $(r_1,\dots,r_k) \in (0,r/2)^k$ *and* $s \in S^1$.

Proof of the main theorem completed. We choose $\overline{r}_1 > 0$ such that if $\overline{r}_1 > r_1 \ge \dots \ge r_k$

$$\max_{s\in S^1, t\in[0,1)} E_1(T_1(s,t;\Omega)) < \sqrt{\frac{32\pi}{3}}.$$

In view of Lemma 2.1, there exist $\beta_1 > 0$, $\widetilde{r}_2 \in (0,\overline{r}_1/2)$ such that for the domain $\Omega = B(z_1,\dots,z_k;r_1,\dots,r_k)$ with $r_1 \in (\overline{r}_1/2,\overline{r}_1)$ and $\widetilde{r}_2 > r_2 \ge \dots \ge r_k$, if $(a,b) \in M$ with $E_1(a,b,\Omega) < \sqrt{16\pi/3} + \beta_1$, then

$$Q(a,b) \in B(0,1) \setminus \overline{B(z_1,\overline{r}_1/4)}.$$

By virtue of Lemma 2.4, we can choose $t_1 \in (0,1)$ such that

$$\max_{s\in S^1} E_1(T_1(s,t_1;\Omega)) < \sqrt{\frac{16\pi}{3}} + \frac{\beta_1}{2}.$$

Therefore, we have

$$\min_{h\in H} \max_{s\in S^1, t\in[0,t_1]} E_1(h(s,t)) > \sqrt{\frac{16\pi}{3}} + \beta_1,$$

where H is the set of all continuous maps $h\colon \overline{B(0,t_1)} \to M$, homotopic to $T_1|_{S^1\times[0,t_1]}$ with fixed boundary condition $h|_{S^1\times\{t_1\}} = T_1|_{S^1\times\{t_1\}}$. Indeed, for any such h, $Q(h|_{S^1\times\{t_1\}})$ will be a non trivial lacet in $B(0,1)\setminus\overline{B(z_1,\overline{r}_1/4)}$, however $Q(h|_{S^1\times[0,t_1]})$ will be contraction of such lacet in $\mathbb{R}^2$. Thus we obtain a minimax value (critical value) $> \sqrt{16\pi/3}+\beta_1$. Then we choose $\overline{r}_2 < \widetilde{r}_2$ such that if $\overline{r}_2 > r_2$, $\max_{s\in S^1,t\in[0,1)} E_1(T_2(s,t;\Omega)) < \sqrt{16\pi/3}+\beta_1$. Repeating the above arguments, there exist $\beta_2 \in (0,\beta_1)$, $\widetilde{r}_3 < \overline{r}_2/2$ and $t_2 \in (0,1)$ such that for Ω with $r_2 \in (\overline{r}_2/2, \overline{r}_2)$ and $\widetilde{r}_3 > r_3 \geq \ldots \geq r_k$ we can find the second minimax value

$$\min_{h\in\widetilde{H}} \max_{s\in S^1,t\in[0,t_2]} E_1(h(s,t)) \in \left(\sqrt{\frac{16\pi}{3}}+\beta_2, \sqrt{\frac{16\pi}{3}}+\beta_1\right)$$

where $\widetilde{H}$ is the set of all continuous maps $h\colon \overline{B(0,t_2)} \to M$, homotopic to $T_2|_{S^1\times[0,t_2]}$ with fixed boundary condition $h|_{S^1\times\{t_2\}} = T_2|_{S^1\times\{t_2\}}$ and t_2 is choosen such that

$$\max_{s\in S^1} E_1(T_2(s,t_2;\Omega)) < \sqrt{16\pi/3}+\beta_2/2.$$

Iterating this procedure, we prove the desired result. □

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