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COMPLETELY SQUASHABLE SMOOTH ERGODIC COCYCLES OVER IRRATIONAL ROTATIONS

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Abstract. Let α be an irrational number and the trasformation

$$Tx \mapsto x + \alpha \bmod 1, \quad x \in [0,1),$$

represent an irrational rotation of the unit circle. We construct an ergodic and completely squashable smooth real extension, i.e. we find a real analytic or k time continuously differentiable real function F such that for every $\lambda \neq 0$ there exists a commutor S_λ of T such that $F \circ S_\lambda$ is T-cohomologous to $\lambda\varphi$ and the skew product $T_F(x,y) = (Tx, y + F(x))$ is ergodic.

1. Completely squashable skew products

Let U be an ergodic measure preserving transformation of a σ-finite measure space $(\Omega, \mathcal{A}, \mu)$. A *commutor* Q of U is a nonsingular transformation $Q\colon \Omega \to \Omega$ such that $UQ = QU$. The *centraliser* $C(U)$ is the collection of all invertible commutors. For a commutor Q of an ergodic measure preserving transformation U, the measure $\mu \circ Q^{-1}$ is μ-absolutely continuous and U-invariant, hence has a constant density. The *dilation* of a measure multiplying transformation Q is defined by

$$D(Q) = \frac{d\mu \circ Q}{d\mu} \in (0, \infty].$$

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Evidently $D: C(T) \to \mathbb{R}_+$ is a multiplicative homomorphism. Set $\Delta_0(T) = D(C(T))$. The group $\Delta_0(T)$ was first considered in [8] (see also [3]). Recall from [2] that the transformation U is called *squashable* if it has a commutor with non-unit dilation. If U has the property that $\Delta_0(U) = \mathbb{R}_+$ we call it *completely squashable*.

If μ is an ergodic probability measure, the ergodic means $(1/n)\sum_{i=0}^{n-1}\chi(A)$ converge by the Birkhoff theorem for every measurable set A almost surely to $\mu(A)$. Limit theorems have been studied also for dynamical systems with infinite measure. In such a case, the ergodic theorem gives us a convergence to the constant 0 only, so that using the ergodic theorem we cannot "reconstruct" the measure as before. Hence, for such dynamical systems, it was developed the notion of the "law of large numbers" (see [2]): by the law of large numbers we understand a function $L: \{0,1\}^{\mathbb{N}} \to [0,\infty)$ such that for a measurable set A of finite measure we have $L(\chi(A), \chi(TA), \chi(T^2A), \dots)(\omega) = \mu(A)$ a.s. (μ). The squashability is a condition which excludes the existence of a law of large numbers.

Let T be an ergodic probability preserving transformation of the probability space $(X, \mathcal{B}, m)$ and G an Abelian, locally compact and second countable topological group.

Let $F: X \to G$ be a measurable mapping. The *skew product* or *G-extension* $T_F: X \times G \to X \times G$ is defined by

$$T_F(x, y) = (Tx, F(x) + y).$$

The skew product preserves the measure $\mu = m \times m_G$ where m_G is the Haar measure on G.

By Proposition 1.1 of [4], if T is a Kronecker transformation and T_F is ergodic, then every commutor of T_F is of the form

$$Q(x, y) = (Sx, g(x) + w(y))$$

where $w: G \to G$ is a surjective, continuous group endomorphism, S is a commutor of T, and $g: X \to G$ is measurable.

Suppose that T_F is a real skew product over T; Q is a commutor of T_F if and only if there exists a measurable fuction g, a commutor S of T, and a $\lambda \in \mathbb{R}$ such that

$$g \circ T - g = F \circ S - \lambda F, \qquad Q(x, y) = (Sx, g(x) + \lambda y).$$

The number $|\lambda|$ is then the dilation of Q.

The aim of this paper is to construct smooth completely squashable real cocycles F over an irrational rotation of a circle. In [1] it is proved the existence of completely squashable and ergodic cocycles over odometers. A part of the results on irrational circle rotations (for a reduced set of irrationals) can also be established using [9] and [10].

An irrational rotation of the unit circle can be represented by the transformation $x \mapsto x + \alpha \bmod 1$ on $[0, 1)$. The irrational number α can be represented by the continued fraction expansion $\alpha = [0; a_1, a_2, \dots]$ where the positive integers a_n are called partial quotients. The convergents p_n/q_n of α are given by recurrent formulas

$$\begin{aligned} p_0 &= 0, & p_1 &= 1, & p_n &= a_n p_{n-1} + p_{n-2}, \\ q_0 &= 1, & q_1 &= a_1, & q_n &= a_n q_{n-1} + q_{n-2}. \end{aligned}$$

Classically, a commutor of an irrational rotation of the unit circle is again a rotation. Suppose that S commutes with T. Then for all x, $S(x + \alpha) = S(x) + \alpha \bmod 1$, hence $S(Tx) - Tx = S(x) - x \bmod 1$. Because T is ergodic, the function $x \mapsto S(x) - x$ is constant (cf. [7]).

THEOREM 1.1. *Let α have unbounded partial quotients. Then there exists a real smooth ergodic cocycle F such that for every $\lambda \neq 0$ there exists a rotation S for which*

$$F \circ S - \lambda \cdot F \tag{1.1}$$

is a (T)-coboundary:

(i) *If $1 \le p < \infty$ is an integer and $\limsup_{n\to\infty} q_{n+1}/q_n^p = \infty$, the cocycle F can be found in $\mathcal{C}^p$. (For $p = 1$ the condition means unbounded partial quotients.)*
(ii) *If $\limsup_{n\to\infty} q_{n+1}/q_n^p = \infty$ for all positive integers p, F can be found in $\mathcal{C}^\infty$.*
(iii) *If $\limsup_{n\to\infty} \log q_n/q_{n-1} = \infty$, there exists an analytic, ergodic and squashable cocycle.*

2. Essential value conditions

Let φ be a cocycle over $(X, \mathcal{B}, m)$. We note $S_n(\varphi) = \sum_{i=0}^{n-1} \varphi \circ T^i$ for $n > 0$, $S_n(\varphi) = 0$ for $n = 0$, and $S_n(\varphi) = -S_{-n}(\varphi)$ for $n < 0$.

We say that $y \in G$ is an *essential value* of the cocycle φ if and only if for every $\varepsilon > 0$ and $A \in \mathcal{B}$, $m(A) > 0$, there is an integer n such that

$$m(A \cap T^n A \cap \{S_n(\varphi) \in \mathcal{U}_\varepsilon(y)\}) > 0 \tag{2.1}$$

where $\mathcal{U}_\varepsilon(y)$ denotes the ε-neighbourhood of y. The set of all essential values is a closed subgroup of G (cf. [12]); we'll denote it by $E(\varphi)$. The skew product T_φ is ergodic if and only if every $y \in G$ is an essential value of φ. The ergodicity of T_φ will be proved this way.

If for the transformation T there exists a rigidity sequence $n_k \to \infty$ (i.e. T^{n_k} converge to the indentity transformation), one can prove that $y \in G$ is an essential value of φ if there exists a sequence $n_k \to \infty$ such that $S_{n_k}(\varphi) \to y$ in probability (cf. [5]). This property, however, contradicts squashability. (2.1) will thus be verified using a much finer Aaronson's "essential value criterium" (see [5]). We use it in a special form here:

DEFINITION. We say that the partitions $\{\mathcal{A}_k : k \geq 1\}$ *approximately generate* $\mathcal{B}$ if

$$\forall B \in \mathcal{B}(X),\ \varepsilon > 0\ \exists k_0 \geq 1\ \forall k \geq k_0\ \exists A_k \in \sigma(\mathcal{A}_k),\quad m(B\Delta A_k) < \varepsilon.$$

Here $\sigma(\mathcal{A})$ denotes the algebra generated by $\mathcal{A}$.

It is not hard to see that the partitions $\{\mathcal{A}_k : k \geq 1\}$ approximately generate $\mathcal{B}$, if and only if $E(1_B|\sigma(\mathcal{A}_k)) \to 1_B$ in probability for all $B \in \mathcal{B}$, and in this case,

$$\forall \varepsilon > 0,\ B \in \mathcal{B},\ \exists k_0\ \forall k \geq k_0 \sum_{A\in\mathcal{A}_k,\ 1-m(B|A)\leq\varepsilon} m(A) \geq (1-\varepsilon)m(B).$$

PROPOSITION 2.1. *Let $\varphi: X \to G$ be a cocycle, and let $y \in G$. If for all $\varepsilon > 0$ exists $\delta_k \to 0$, $0 \leq a < d$, $b > 0$, and a sequence of partitions $\mathcal{A}_k$ which approximately generate $\mathcal{B}$, such that for every $k \geq 1$, for δ_k-almost every $A \in \mathcal{A}_k$, exists $n = n(A)$ and $E \in A \cap \mathcal{B}$, $m(E) > b\, m(A)$, such that*

$$m(E\Delta T^{-n}E) < a\, m(E),\ \text{and}\ m(E \cap [S_n(\varphi) \in \mathcal{U}_\varepsilon(y)]) > d\, m(E),$$

then $y \in E(\varphi)$.

The Proposition 2.1 follows from §3 in [5]. For reader's convenience we present an elemementary proof (without using the notion of the full group).

PROOF. We have to verify (2.1). Let $C \in \mathcal{B}$, $m(C) > 0$, $y \in G$, $\varepsilon > 0$. Because $\mathcal{A}_k$ approximately generate $\mathcal{B}$, for every $\delta > 0$ there is a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ there is a set $A_k \in \sigma(\mathcal{A}_k)$ such that $m(A_k) > (1-\delta)m(C)$ and for every $A \in \sigma(\mathcal{A}_k)$ with $A \subset A_k$, $m(C|A) > 1-\delta$.

We choose δ and k_0 so that $\delta < b(d-a)/2$ and for $k \geq k_0$, $\delta + \delta_k < m(C)$. Then there exists $A \in \mathcal{A}_k$ such that $m(C|A) > 1-\delta$ and there is an $E \in A \cap \mathcal{B}$ such that

$$m(E) > bm(A),\quad m(E\Delta T^{-n}E) < am(E),\quad m(E \cap [S_n(\varphi) \in \mathcal{U}_\varepsilon(y)]) > dm(E).$$

We then have

$$m(E \cap T^{-n}E \cap [S_n(\varphi) \in \mathcal{U}_\varepsilon(y)]) > (d-a)m(E).$$

Hence,

$$m((A\cap C)\cap T^{-n}(A\cap C)\cap[S_n(\varphi)\in\mathcal{U}_\varepsilon(y)])>(b(d-a)-2\delta)m(A).$$

Therefore, $m(C\cap T^{-n}C\cap[S_n(\varphi)\in\mathcal{U}_\varepsilon(y)])>0$. □

Remark. For $\mathcal{A}_k$, instead of partitions, we can have collection of disjoint sets A such that

$$m\Big(\bigcup_{A\in\mathcal{A}_k}A\Big)>1-\delta_k.$$

We shall actually use the following version of Proposition 2.1:

Proposition 2.2. *Let $\mathcal{A}_k\nearrow\mathcal{A}$ be a sequence of partitions. If*

$$\forall a\in\mathbb{R}\ \forall\varepsilon>0\ \exists k_\varepsilon\in\mathbb{N}\ \forall k\ge k_\varepsilon\ \forall A\in\mathcal{A}_k,\ m(A)>0\ \exists n\in\mathbb{N},\ E\subset A,$$
$$m(E\cap T^{-n}E)>0,\ 9m(E),\ m(E)>m(A)/6,\ E\subset\{x:S_n(f)(x)\in\mathcal{U}_\varepsilon(a)\},$$

then f is an ergodic cocycle.

From now on, let us suppose that T is an irrational rotation of the unit circle (equipped with the Lebesgue probability measure) and $G=\mathbb{R}$. G is equipped with the Lebesgue measure $m_G=m$. From Proposition 2.1 it immediately follows

Proposition 2.3. *Let f be a cocycle and $n_k\to\infty$ be a rigid sequence. If the distributions of $S_{n_k}(f)$ weakly converge to a continuous distribution then f is ergodic.*

As it is shown in [11], there exist dense G_δ sets of absolutely continuous, Lipschitz, k times continuously or infinitely differentiable functions f with zero mean on $\mathbb{T}$ for which the distributions of $S_{n_k}(f)$ weakly converge to a continuous distribution along a rigidity sequence $n_k\to\infty$ as soon as the space contains a nontrivial cocycle, i.e. a cocycle which is not cohomologous to a constant.

For irrational rotations with bounded partial quotients, nontrivial cocycles exist in the space of absolutely continuous functions while every zero mean Lipschitz function is a coboundary. In the space $\mathcal{C}_0^p$ of p-times continuously differentiable cocycles, $1\le p<\infty$, there exists a nontrivial cocycle (and hence a dense G_δ set of them) if and only if $\limsup_{n\to\infty}q_{n+1}/q_n^p=\infty$; in $\mathcal{C}_0^\infty$ they exist if and only if the limes superior is infinite for all p (cf. also [6]).

We thus have dense G_δ sets of cocycles which satisfy the assumptions of Proposition 2.3, hence are ergodic. On the other hand, for any measure preserving commutor S of T, the distributions of $S_{n_k}(f\circ S)$ converge to the same limit as the distributions of $S_{n_k}(f)$ while due to the rigidity of (T^{n_k}), $S_{n_k}(g-g\circ T)$ go for every measurable g to zero (in measure), hence (in view of the result from [11] quoted above) we can see that the smooth cocycles are generically not squashable, and in the proof of Theorem 1.1 we cannot use Proposition 2.3.

For a rotation with bounded partial quotients, it is not known whether there exists an ergodic and squashable absolutely continuous cocycle. It is not hard to see that an ergodic (hence nontrivial) real analytic cocycle exists if and only if $\limsup_{n\to\infty} \log q_n/q_{n-1} > 0$. It is not known whether there exists an ergodic and squashable real analytic cocycle if $\limsup_{n\to\infty} \log q_n/q_{n-1}$ is positive and finite.

3. Proof of Theorem 1.1

We shall construct F as a sum $F = \sum_{k\in\mathcal{K}} F_k$ where $\mathcal{K}$ is a countable subset of the set of positive integers and the cocycles F_k are coboundaries. We shall first construct step cocycles (coboundaries) f_k, $k = 1, 2, \ldots$, and then we shall approximate them by $\mathcal{C}^\infty$ coboundaries or real trigonometric polynomials. In doing so we use Theorems 1 and 2 from [13].

For simplicity suppose that n is odd (the case with n even is similar). From the continued fraction expansion we get two Rohlin towers:

$$\begin{aligned} &[\{j\alpha\}, \{(j+q_{n-1})\alpha\}), && \text{for } j = 0, \ldots, q_n - 1,\\ &[\{q_n\alpha\}, 1),\ [\{(j+q_n)\alpha\}, \{j\alpha\}), && \text{for } j = 1, \ldots, q_{n-1} - 1. \end{aligned}$$

By $\|x\|$ we denote $\min\{x, 1-x\}$ $(x \in [0,1))$. For $0 \le x < 1 - \|q_{n-1}\alpha\|$ we have $T^{q_{n-1}}x = x + \|q_{n-1}\alpha\|$.

Let us denote $I_0 = [0, \|q_{n-1}\alpha\|)$, $I_i = T^i I_0$, $i = 1, \ldots, q_n - 1$. Notice that for $j = 0, \ldots, q_{n-1} - 1$, the intervals $I_{j+q_{n-1}}, I_{j+2q_{n-1}}, \ldots I_{j+a_n q_{n-1}}$ are adjacent.

3.1. Construction of the step cocycles. For $k = 1, 2, \ldots$ we define

$$c_k = k^3 e^k, \quad r_k = [k^4 e^k],$$

where $[x]$ denotes the integer part of x.

Let a_n be the partial quotients, q_n the convergents. We define

$$\ell_{k,n} = 2[a_n/2r_k], \quad \ell'_{k,n} = \ell_{k,n}/2, \quad d_{k,n} = e^k k^6/q_n.$$

From the definition of r_k it follows that for every k, $1 - r_k\ell_{k,n}/a_n \to 0$ as $a_n \to \infty$.

For $j = 0, \ldots, r_k - 1$ and $u = 0, \ldots, q_{n-1} - 1$ we define

$$\begin{aligned} J'_{0,0} &= \bigcup_{i=0}^{\ell'_{k,n}-1} I_{i\cdot q_{n-1}}, \\ J'_{0,j} &= T^{j\cdot\ell_{k,n}q_{n-1}} J'_{0,0}, \qquad J''_{0,j} = T^{\ell'_{k,n}q_{n-1}} J'_{0,j}, \\ J'_{u,j} &= T^u J'_{0,j}, \qquad\qquad\ \ J''_{u,j} = T^u J''_{0,j}, \\ J_{u,j} &= J'_{u,j} \cup J''_{u,j}, \\ J_u &= \bigcup_{j=0}^{r_k-1} J_{u,j}. \end{aligned} \tag{3.1}$$

Notice that for every u the sets $J_{u,j}$, $0 \le j \le r_k - 1$, are adjacent intervals and their union equals

$$J_u = \bigcup_{j=0}^{r_k \ell_{k,n} - 1} I_{u + j q_{n-1}} = [\{u\alpha\}, r_k \ell_{k,n} \| q_{n-1}\alpha \|),$$

each $J_{u,j}$ is cut in the middle into $J'_{u,j}$ and $J''_{u,j}$.

We define

$$f_{k,n} = \begin{cases} (-1)^j \left(1 + \dfrac{1}{c_k}\right)^j & \text{on the sets } J'_{u,j}, \\ & 0 \le u \le q_{n-1} - 1,\ 0 \le j \le r_k - 1, \\ -(-1)^j \left(1 + \dfrac{1}{c_k}\right)^j & \text{on the sets } J''_{u,j}, \\ & 0 \le u \le q_{n-1} - 1,\ 0 \le j \le r_k - 1, \\ 0 & \text{on the rest of the circle.} \end{cases}$$

From the definition of $f_{k,n}$ we get that for $x \in I_{j \cdot \ell_{k,n} q_{n-1}}$, $0 \le j \le r_k - 1$,

$$\sum_{i=0}^{\ell'_{k,n} q_{n-1} - 1} f_{k,n}(T^i x) = - \sum_{i=\ell'_{k,n} q_{n-1}}^{\ell_{k,n} q_{n-1} - 1} f_{k,n}(T^i x), \tag{3.2}$$

hence for every $0 \le j \le r_k - 1$ and $x \in I_0$,

$$\sum_{i=j\ell_{k,n} q_{n-1}}^{(j+1)\ell_{k,n} q_{n-1} - 1} f_{k,n}(T^i x) = 0. \tag{3.2'}$$

Therefore, $f_{k,n} = g_{k,n} - g_{k,n} \circ T$ where

$$\begin{aligned} g_{k,n}(T^u x) &= - \sum_{i=0}^{u-1} f_{k,n}(T^i x) \quad \text{for } x \in I_0,\ u = 0, \dots, q_n - 1, \\ g_{k,n}(x) &= 0 \qquad\qquad\qquad\qquad \text{for } x \in [0,1) \setminus \bigcup_{i=0}^{q_n - 1} I_i. \end{aligned} \tag{3.3}$$

Let us compute $\sup |g_{k,n}|$. We have

$$\sup |g_{k,n}| = \sup_{x \in I_0} \max \left\{ \left| \sum_{i=0}^{u-1} f_{k,n}(T^i x) \right| : 0 \le u \le q_n - 1 \right\}.$$

By (3.2'), the partial sums are zero for every $u = j \ell_{k,n} q_{n-1}$, $1 \le j \le r_k$. From this and from (3.2) we get

$$\begin{aligned} |g_{n,k}| &\le \sup \left\{ \left| \sum_{i=0}^{\ell'_{k,n} q_{n-1} - 1} f_{k,n}(T^i x) \right| : x \in T^{j \ell_{k,n} q_{n-1}} I_0,\ 0 \le j \le r_k - 1 \right\} \\ &\le \ell'_{k,n} q_{n-1} \left(1 + \frac{1}{c_k}\right)^{r_k} \le \left[\frac{a_n}{2 r_k}\right] q_{n-1} e^k \le \frac{q_n}{k^4}. \end{aligned} \tag{3.4}$$

3.2. Construction of the smooth and real analytic cocycles. If the assumption (i) or (ii) holds, $\delta_k > 0$, then by [13, Theorem 1], there exist an integer $\eta(k)$ and a number $K(k)$ such that for $q_n/q_{n-1}^p > \eta(k)$ there exists a coboundary $F_{k,n} = G_{k,n} - G_{k,n} \circ T$, $F_{k,n} \in \mathcal{C}^\infty$,

(a) $|F_{k,n}| \le 2 \max |f_{k,n}|$,

(b) $\|F_{k,n}\|_{\mathcal{C}^p} < K(k) q_{n-1}^p$,

(c) for the transfer functions $g_{k,n}$, $G_{k,n}$, of $f_{k,n}$, $F_{k,n}$, we have

$$|g_{k,n} - G_{k,n}| < \delta_k q_n.$$

We choose

$$\delta_k = \frac{1}{e^k k^8}.$$

If (iii) holds, we can by [13, Theorem 2], find an integer $\eta(k)$ and a number $K(k)$ such that for $\log q_n/q_{n-1} > \eta(k)$ there exists a real trigonometric polynomial

$$F_{k,n}(x) = \sum_{\ell=-s_k}^{s_k} b_\ell e^{2\pi i q_{n-1} x},$$

(a) $|F_{k,n}| \le 2 \max |f_{k,n}|$,

(b') for $M_k = \sum_{\ell=-s_k}^{s_k} |b_\ell|$ we have $M_k e^{s_k q_{n-1}}/q_n < \delta_k$ $(= 1/(e^k k^8))$,

(c) for the transfer functions $g_{k,n}$, $G_{k,n}$, of $f_{k,n}$, $F_{k,n}$, we have $|g_{k,n} - G_{k,n}| < \delta_k q_n = q_n/(e^k k^4)$.

Let the assumption (i) or (ii) or (iii) hold. We choose a sequence $n_k \to \infty$ so that $q_{n_k}/q_{n_k-1}^p > \eta(k)$ and

$$\sum_{k=1}^{\infty} d_{k,n_k} K(k) q_{n_k-1}^p < \infty, \quad d_{k,n_k} e^k q_{n_k-1}^p \to 0. \tag{3.5}$$

We denote $\ell_k = \ell_{k,n_k}$, $\ell'_k = \ell'_{k,n_k}$, $d_k = d_{k,n_k}$, $f_k = d_k f_{k,n_k}$, $g_k = d_k g_{k,n_k}$, $F_k = d_k F_{k,n_k}$, $G_k = d_k G_{k,n_k}$. From (b) and (3.5) we have

$$\sum_{k=1}^{\infty} \|F_k\|_{\mathcal{C}^p} < \sum_{k=1}^{\infty} d_k K(k) q_{n_k-1}^p < \infty. \tag{3.6}$$

In the case when (ii) holds we can find the sequence $n_k \to \infty$ such that (3.6) holds for every $p = 1, 2, \ldots$.

If (iii) holds, by (b') we can choose the sequence $n_k \to \infty$ so that

$$\sum_{k=1}^{\infty} d_k M_k e^{s_k q_{n_k-1}} = \sum_{k=1}^{\infty} \frac{e^k k^6}{q_{n_k}} M_k e^{s_k q_{n_k-1}} < \infty, \tag{3.7}$$

hence the sum $\sum_{k=1}^{\infty} F_k$ can be expressed as a trigonometric series

$$F(x) = \sum_{\ell=-\infty}^{\infty} \beta_\ell e^{2\pi i \ell}$$

with $\beta_0 = 0$, $\beta_{-\ell} = \overline{\beta}_\ell$, and $\sum_{\ell=-\infty}^{\infty} |\beta_\ell| e^{|\ell|} < \infty$, i.e. F is a real analytic function on $\mathbb{T}$ (represented as the interval $[0,1)$).

From (c) it follows that

$$|g_k - G_k| = d_k |g_{k,n_k} - G_{k,n_k}| \le \delta_k d_k q_{n_k} = \frac{1}{e^k k^8} \frac{e^k k^6}{q_{n_k}} q_{n_k} = \frac{1}{k^2}. \tag{3.8}$$

3.3. Ergodicity. Let $\mathcal{A}_k$ be the partition of $[0,1)$ into the sets $J_{u,j}$ and the complement of their union; as $a_{n_k} \to \infty$, $\mathcal{A}_k \nearrow \mathcal{A}$ (for a subsequence of the numbers k). Let $0 \le u \le q_{n_k-1} - 1$, $0 \le j \le r_k - 1$ be fixed, E be the middle third of the interval $J'_{u,j}$. We shall consider the sums

$$S_{i \cdot q_{n_k-1}}(F_k), \quad 1 \le i \le [\ell'_k/k].$$

Let a be a fixed number of the same sign as $(-1)^j$ and let $\varepsilon > 0$. Without loss of generality we can suppose that j is even, $a \ge 0$. From the definition of f_k and (3.5) it follows

$$|S_{q_{n_k-1}}(f_k)| \le q_{n_k-1} d_k \left(1 + \frac{1}{c_k}\right)^{r_k} \le q_{n_k-1} d_k e^k \to 0.$$

If $k \ge 12$ and $x \in E$, then for $0 \le i \le [\ell'_k/k] q_{n_k-1}$, $f_k(T^i x) = d_k(1 + 1/c_k)^j$. If $k \ge 12$, $0 \le m \le [\ell'_k/k]$, we thus have

$$\chi_E S_{m q_{n_k-1}}(f_k) = m q_{n_k-1} d_k \left(1 + \frac{1}{c_k}\right)^j \chi_E.$$

From (3.5) it follows that $d_k e^k q_{n_k-1} = k^6 e^{2k} q_{n_k-1}/q_{n_k} \to 0$ hence $a_{n_k}/(k^6 e^{2k}) \to \infty$. For $a_{n_k} \ge 4k^5 e^k$,

$$\left[\frac{\ell'_k}{k}\right] d_k q_{n_k-1} \ge \left(\frac{a_{n_k}}{2k^5 e^k} - 1\right) \frac{e^k k^6}{q_{n_k}} q_{n_k-1} \ge \frac{k}{8}.$$

Therefore, if k is bigger than some constant $k(a,\varepsilon)$, then there exists $1 \le m \le \ell'_k/k$ for which $S_{m q_{n_k-1}}(f_k) \in \mathcal{U}_{\varepsilon/2}(a)$ on E. From (3.8) it follows

$$\begin{aligned} |S_{m q_{n_k-1}}(F_k) - S_{m q_{n_k-1}}(f_k)| &= |(g_k - g_k \circ T^{m q_{n_k-1}}) - (G_k - G_k \circ T^{m q_{n_k-1}})| \\ &\le 2 \sup |g_k - G_k| \le \frac{2}{k^2}. \end{aligned}$$

If $k \ge k(a,\varepsilon)$, $k \ge 12$, and $2/k^2 < \varepsilon/2$, we thus have $S_{m q_{n_k-1}}(F_k) \in \mathcal{U}_\varepsilon(a)$ on E.

The rotation $T^{q_{n_k-1}}$ is the shift by $\|q_{n_k-1}\alpha\| \pmod 1$, hence, if $k \ge 30$, we have

$$m(E \cap T^{-i q_{n_k-1}} E) > 0{,}9 m(E) \tag{3.9}$$

for every $1 \le i \le \ell'_k/k$.

In the "p-times differentiable" ($1 \le p \le \infty$) as well as in the "analytic" case we have: As $T^{q_{n_k}-1}$ is the shift (mod 1) by $\|q_{n_k-1}\alpha\|$, $T^{j\cdot q_{n_k-1}}$, $j = 1, \dots, \ell'_k$, $k = 1, 2, \dots$, is a rigid sequence. For any fixed positive integer u we thus get

$$\lim_{k\to\infty} \max_{j=1,\dots,\ell'_k} \left| S_{j\cdot q_{n_k-1}} \left(\sum_{i=0}^{u} F_i \right) \right| = 0 \quad \text{in the measure } m.$$

By (a) and (3.5), $|F_k| = d_k |F_{k,n_k}| \le 2d_k |f_{k,n_k}| \le 2e^{2k}k^6/q_{n_k} \to 0$, hence there exists an infinite subset $\mathcal{K} \subset \mathbb{N}$ such that

$$\lim_{k\in\mathcal{K}, k\to\infty} \max_{j=1,\dots,\ell'_k} |S_{j\cdot q_{n_k-1}}(F_{(\mathcal{K})} - F_k)| = 0 \quad \text{in the measure } m \tag{3.10}$$

where $F_{(\mathcal{K})} = \sum_{k\in\mathcal{K}} F_k$. From Proposition 2.2, (3.10) and (3.9) it follows that $F_{(\mathcal{K})}$ is ergodic. The set $\mathcal{K}$ can be chosen such that

$$\sum_{k\in\mathcal{K}} \frac{1}{k} < \infty.$$

3.4. Squashability. It remains to prove that the set $\mathcal{K}$ can be chosen so that for $F = F_{(\mathcal{K})}$, every $\lambda \ne 0$, there exists a rotation S for which

$$F \circ S - \lambda \cdot F \quad \text{is a } T\text{-coboundary.} \tag{1.1}$$

If $F \circ S' - \lambda' F$ and $F \circ S'' - \lambda'' F$ are coboundaries, then $F \circ S' \circ S'' - \lambda'\lambda'' F = (F \circ S') \circ S'' - \lambda''(F \circ S') + \lambda''(F \circ S' - \lambda' F)$ is a coboundary, too. Similarly we can show that if $F \circ S - \lambda F$ is a coboundary then $F \circ S^{-1} - (1/\lambda)F$ is also a coboundary. Therefore the set of numbers $\lambda \ne 0$ for which (1.1) holds true is a multiplicative semigroup. It thus suffices to find S for an interval of $\lambda > 0$ of positive length and a $\lambda < 0$. We'll consider the interval $[3/2, 2]$.

For each $\lambda > 0$, let $j(\lambda, k)$ be the greatest positive even integer for which

$$\left(1 + \frac{1}{c_k}\right)^{j(\lambda,k)} < \lambda.$$

For $\lambda > 0$ and nonnegative integers k, v, let us define a number

$$\sigma(\lambda, k, v) = \{(v + j(\lambda, k)\ell_k q_{n_k-1})\alpha\}$$

(where $\{x\}$ denotes the fractional part $x - [x]$ of x) and a rotation

$$x \mapsto x + \sigma(\lambda, k, v) \bmod 1 = T^{v + j(\lambda,k)\ell_k q_{n_k-1}} x$$

(we denote both the rotation and the number by the same symbol).

We'll recursively define a set $\mathcal{K}' = \{k_0 < k_1 < \dots\} \subset \mathcal{K}$, nonnegative integers $v(\lambda, k)$, $k \in \mathcal{K}'$, $3/2 \le \lambda \le 2$, numbers and rotations $\sigma(\lambda, k) = \sigma(\lambda, k, v(\lambda, k))$, $k \in \mathcal{K}'$ (denoted by the same symbol):

- For k_0 we choose the smallest element of $\mathcal{K}$ and define $v(\lambda, k_0) = 0$.

- If k_i, $v(\lambda, k_i)$ have been defined for $i = 0, \dots, n$, $3/2 \le \lambda \le 2$, we define k_{n+1} as the smallest $k \in \mathcal{K}$ such that:
  (1) $k > k_n$.
  (2) For each $3/2 \le \lambda \le 2$ there exists an integer $0 \le v = v(\lambda, k) < q_{n_k - 1}/k$ such that

$$|\sigma(\lambda, k_n) - \sigma(\lambda, k, v)| < \frac{1}{2^n},$$

$$m\biggl(\sup |G_j \circ \sigma(\lambda, k_n) - G_j \circ \sigma(\lambda, k, v)| < \frac{1}{2^n}\biggr) > 1 - \frac{1}{2^n} \quad \text{for } j = k_0, \dots, k_n.$$

Set $\sigma(\lambda, k_{n+1}) = \sigma(k_{n+1}, v(\lambda, k_{n+1}))$. The numbers $\sigma(\lambda, k_n)$ then for $n \to \infty$ converge to a limit $\sigma(\lambda)$. Let us suppose that the number $3/2 \le \lambda \le 2$ is fixed, $\sigma = \sigma(\lambda)$, $\sigma(k) = \sigma(\lambda, k)$, $v(k) = v(\lambda, k)$, and $j(k) = j(\lambda, k)$ for all k. By S we denote the rotation $x \mapsto x + \sigma \bmod 1$. For $k = k_n \in \mathcal{K}'$ we have with probability greater than $1 - 1/2^{n-1}$

$$\begin{aligned}
\sup |G_k \circ S - G_k \circ \sigma(k)| &\le \sum_{i=0}^{\infty} \sup |G_k \circ \sigma(k_{n+i}) - G_k \circ \sigma(k_{n+i+1})| \\
&< \sum_{i=0}^{\infty} \frac{1}{2^{n+i}} = \frac{1}{2^{n-1}},
\end{aligned}$$

hence $\sum_{k \in \mathcal{K}'} \sup |G_k \circ S - G_k \circ \sigma(k)|$ converges almost surely. We have

$$\begin{aligned}
F \circ S - \lambda \cdot F &= \sum_{k \in \mathcal{K}'} (F_k \circ S - \lambda \cdot F_k) \\
&= \sum_{k \in \mathcal{K}'} \biggl((F_k \circ S - F_k \circ \sigma(k)) + \biggl(F_k \circ \sigma(k) - \Bigl(1 + \frac{1}{c_k}\Bigr)^{j(k)} F_k \biggr) \\
&\qquad + \biggl(\Bigl(1 + \frac{1}{c_k}\Bigr)^{j(k)} - \lambda \biggr) F_k \biggr).
\end{aligned}$$

Each of the functions F_k is a coboundary, hence all summands in the last sum are coboundaries, too. For proving (1.1) it suffices to show that the sum of the corresponding transfer functions converges:

(1) $\sum_{k \in \mathcal{K}'} (F_k \circ S - F_k \circ \sigma(k))$.

Every F_k is a coboundary with a transfer function G_k. We have shown that $\sum_{k \in \mathcal{K}'} (G_k \circ S - G_k \circ \sigma(k))$ converges; it is a transfer function of $\sum_{k \in \mathcal{K}'} (F_k \circ S - F_k \circ \sigma(k))$.

(2) $\sum_{k \in \mathcal{K}'} (F_k \circ \sigma(k) - (1 + 1/c_k)^{j(k)} F_k)$.

The function $F_k \circ \sigma(k) - (1 + 1/c_k)^{j(k)} F_k$ is a coboundary with a transfer function

$$G_k \circ \sigma(k) - \Bigl(1 + \frac{1}{c_k}\Bigr)^{j(k)} G_k.$$

By (3.3) we for $g_k = g_{k,n_k}$ have

$$
\begin{aligned}
g_k(T^u x) &= -\sum_{i=0}^{u-1} f_k(T^i x) \quad && \text{for } x \in I_0,\ u = 0, \dots, q_{n_k} - 1, \\
g_k(x) &= 0 && \text{for } x \in [0,1) \setminus \bigcup_{i=0}^{q_{n_k}-1} I_i.
\end{aligned} \tag{3.3'}
$$

Every integer $0 \le i$ can be uniquely expressed as

$$i = u + w q_{n_k - 1} + t \ell_k q_{n_k - 1}$$

where $0 \le u \le q_{n_k-1} - 1$, $0 \le w \le \ell_k - 1$, $0 \le t$. From the definition of $f_k = d_k f_{k,n_k}$, (3.2'), and (3.3'), it follows that if $u + v(k) \le q_{n_k-1} - 1$ and $t + j(k) \le r_k - 1$, we have

$$g_k(\sigma(k)(x)) = g_k(T^{v(k)+j(k)\ell_k q_{n_k-1}} x) = \left(1 + \frac{1}{c_k}\right)^{j(k)} g_k(x)$$

for $x \in I_i$. For $k \to \infty$ we have

$$\left(1 + \frac{1}{c_k}\right)^{c_k \log \lambda} \to e^{\log \lambda} = \lambda,$$

hence $(j(k) - c_k \log \lambda)/c_k \to 0$. From this and the definition of r_k it follows that there exists a constant C such that

$$j(k) \le C \frac{r_k}{k}.$$

Recall that, by the definition, $v(k) < q_{n_k-1}/k$. Hence, there exists a constant B such that

$$m\left(g_k(\sigma(k)(x)) = \left(1 + \frac{1}{c_k}\right)^{j(k)} g_k\right) \ge 1 - \frac{B}{k}.$$

From this and (3.8) we get

$$m\left(\left|G_k(\sigma(k)(x)) - \left(1 + \frac{1}{c_k}\right)^{j(k)} G_k\right| \le \frac{1+\lambda}{k^2}\right) \ge 1 - \frac{B}{k}.$$

Therefore, by (3.10) and the Borel–Cantelli lemma (recall that $\sum_{k \in \mathcal{K}} 1/k < \infty$)

$$\sum_{k \in \mathcal{K}'} \left(G_k \circ \sigma(k) - \left(1 + \frac{1}{c_k}\right)^{j(k)} G_k\right)$$

converges a.s. hence

$$\sum_{k \in \mathcal{K}'} \left(F_k \circ \sigma(k) - \left(1 + \frac{1}{c_k}\right)^{j(k)} F_k\right)$$

is a coboundary.

(3) $\sum_{k \in \mathcal{K}'} ((1 + 1/c_k)^{j(k)} - \lambda) F_k$.

F_k is a coboundary with a transfer function G_k. By (3.4), $|g_k| = d_k|g_{k,n_k}| \leq d_k q_{n_k}/k^4$, and by (3.8),

$$|G_k| \leq \max|g_k| + \sup|g_k - G_k| \leq e^k k^2 + \frac{1}{k^2}.$$

We can easily see that $\lambda - (1+1/c_k)^{j(k)} \leq 2\lambda/c_k$, hence

$$\sum_{k\in\mathcal{K}'}\left|\left(\left(1+\frac{1}{c_k}\right)^{j(k)} - \lambda\right)G_k\right| \leq \sum_{k\in\mathcal{K}'}\frac{2\lambda}{c_k}\left(e^k k^2 + \frac{1}{k^2}\right) \leq \sum_{k\in\mathcal{K}'}\frac{4\lambda}{k}.$$

From (3.10) and the definition of c_k it follows that the last sum is finite. This finishes the proof that for $3/2 \leq \lambda \leq 2$, hence for all $\lambda > 0$, there exists a rotation S such that (1.1) holds.

To finish the proof it suffices to find a rotation for a single $\lambda < 0$. Let $\lambda < 0$. We define $j(\lambda, k)$ as the biggest odd number for which

$$\left(1+\frac{1}{c_k}\right)^{j(\lambda,k)} < |\lambda|$$

and continue in the definition of the rotation S as in the previous case. We can prove (1.1) similarly as before. Notice that

$$g_k(\sigma(k)(x)) = -\left(1+\frac{1}{c_k}\right)^{j(k)} g_k(x)$$

on a set of measure bigger than $1-(B/k)$. □

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