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CONLEY INDEX CONTINUATION FOR SINGULARLY PERTURBED HYPERBOLIC EQUATIONS

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To my father Reinhold Rybakowski (1929–2003) in gratitude

Abstract. Let $\Omega \subset \mathbb{R}^N$, $N \leq 3$, be a bounded domain with smooth boundary, $\gamma \in L^2(\Omega)$ be arbitrary and $\phi: \mathbb{R} \to \mathbb{R}$ be a C^1-function satisfying a subcritical growth condition. For every $\varepsilon \in]0, \infty[$ consider the semiflow π_ε on $H^1_0(\Omega) \times L^2(\Omega)$ generated by the damped wave equation

$$\begin{aligned} \varepsilon\partial_{tt}u + \partial_t u &= \Delta u + \phi(u) + \gamma(x) && x \in \Omega, \quad t > 0, \\ u(x,t) &= 0 && x \in \partial\Omega, \ t > 0. \end{aligned}$$

Moreover, let π' be the semiflow on $H^1_0(\Omega)$ generated by the parabolic equation

$$\begin{aligned} \partial_t u &= \Delta u + \phi(u) + \gamma(x) && x \in \Omega, \quad t > 0, \\ u(x,t) &= 0 && x \in \partial\Omega, \ t > 0. \end{aligned}$$

Let $\Gamma: H^2(\Omega) \to H^1_0(\Omega) \times L^2(\Omega)$ be the imbedding $u \mapsto (u, \Delta u + \phi(u) + \gamma)$. We prove in this paper that every compact isolated π'-invariant set K' lies in $H^2(\Omega)$ and the imbedded set $K_0 = \Gamma(K')$ continues to a family K_ε, $\varepsilon \geq 0$ small, of isolated π_ε-invariant sets having the same Conley index as K'. This family is upper-semicontinuous at $\varepsilon = 0$. Moreover, any (partially ordered) Morse-decomposition of K', imbedded into $H^1_0(\Omega) \times L^2(\Omega)$ via Γ, continues to a family of Morse decompositions of K_ε, for $\varepsilon \geq 0$ small. This family is again upper-semicontinuous at $\varepsilon = 0$.

These results extend and refine some upper semicontinuity results for attractors obtained previously by Hale and Raugel.

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1. Introduction

Let $N \in \{1,2,3\}$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, $\gamma \in L^2(\Omega)$ be arbitrary and $\phi: \mathbb{R} \to \mathbb{R}$ be a C^1-function such that, for $N \geq 2$, there are constants $\overline{r}$ and $\overline{C} \in [0,\infty[$ with $|\phi'(u)| \leq \overline{C}(1+|u|^{\overline{r}})$ for $u \in \mathbb{R}$. If $N = 3$ we also assume that $\overline{r} < 2$, i.e. that ϕ has subcritical growth.

For every $\varepsilon \in]0,\infty[$ consider the following damped wave equation

$$(\mathrm{Hyp}_\varepsilon) \qquad \begin{aligned} \varepsilon\partial_{tt}u + \partial_t u &= \Delta u + \phi(u) + \gamma(x) && x \in \Omega, \quad t > 0, \\ u(x,t) &= 0 && x \in \partial\Omega,\ t > 0. \end{aligned}$$

It is well-known that equation $(\mathrm{Hyp}_\varepsilon)$ generates a local semiflow (actually, a local flow) π_ε on $H_0^1(\Omega) \times L^2(\Omega)$.

Setting, formally, $\varepsilon = 0$ in equation $(\mathrm{Hyp}_\varepsilon)$ we obtain the parabolic equation

$$(\mathrm{Par}) \qquad \begin{aligned} \partial_t u &= \Delta u + \phi(u) + \gamma(x) && x \in \Omega, \quad t > 0, \\ u(x,t) &= 0 && x \in \partial\Omega,\ t > 0. \end{aligned}$$

Again it is well-known that equation (Par) generates a local semiflow π' on $H_0^1(\Omega)$.

It is a natural question to ask whether, for $\varepsilon \to 0$, solutions of π_ε converge, in some sense, to solutions of π', properly imbedded into $H_0^1(\Omega) \times L^2(\Omega)$. This question was considered in the context of attractors by Hale and Raugel [9], who used some ideas and results by Haraux and by Babin and Vishik. (See [10], [1] and the references cited therein.) Let us briefly describe the main result from [9]. To this end we need some notation. Let $\Gamma: H^2(\Omega) \to H_0^1(\Omega) \times L^2(\Omega)$ be the map defined by $\Gamma(u) := (u,v)$ where $v(x) := \Delta u(x) + \phi(u(x)) + \gamma(x)$, for $u \in H^2(\Omega)$ and $x \in \Omega$.

Under some additional assumptions on ϕ the semiflow π' has a global attractor $\mathcal{A}'$ and, for all $\varepsilon > 0$, the semiflow π_ε has a global attractor $\mathcal{A}_\varepsilon$. It turns out that $\mathcal{A}' \subset H^2(\Omega)$ and the family $(\mathcal{A}_\varepsilon)_{\varepsilon \geq 0}$, where $\mathcal{A}_0 = \Gamma(\mathcal{A}')$, is upper semicontinuous at $\varepsilon = 0$ in $H_0^1(\Omega) \times L^2(\Omega)$, i.e. $\lim_{\varepsilon\to 0}\sup_{y\in\mathcal{A}_\varepsilon}\inf_{z\in\mathcal{A}_0}|y-z|_{H_0^1\times L^2} = 0$.

Now note that $\mathcal{A}'$ and $\mathcal{A}_\varepsilon$, $\varepsilon > 0$, are isolated invariant sets (relative to the corresponding semiflows) with Conley index Σ^0. Thus the above result shows that the compact isolated invariant set $\mathcal{A}'$ continues, after its imbedding into $H_0^1(\Omega) \times L^2(\Omega)$, to a family $\mathcal{A}_\varepsilon$, $\varepsilon > 0$ of isolated invariant sets with the same index.

One of the objectives of this paper is to show that an analogous result holds for arbitrary compact isolated invariant sets of the local semiflow π'.

More precisely, we have the following result:

Theorem A. *Let K' be a compact (in $H_0^1(\Omega)$) isolated invariant set relative to π'. Then $K' \subset H^2(\Omega)$, and there is an $\varepsilon_0 > 0$ and for every $\varepsilon \in]0,\varepsilon_0]$ there is a compact isolated invariant set K_ε relative to π_ε such that the Conley* *index $h(\pi_\varepsilon, K_\varepsilon)$ of K_ε is equal to the Conley index $h(\pi', K')$ of K'. Moreover, the family $(K_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$, where $K_0 = \Gamma(K')$, is upper semicontinuous at $\varepsilon = 0$ in $H^1_0(\Omega) \times L^2(\Omega)$, i.e.* $\lim_{\varepsilon\to 0} \sup_{y\in K_\varepsilon} \inf_{z\in K_0} |y - z|_{H^1_0\times L^2} = 0$.

(See Theorem 6.1 below for a complete statement of a more general result).

A naive approach to the proof of Theorem A would be to make a change of variables

$$\Phi\colon (u, v) \mapsto (u, w) := (u, v - \Delta u - \phi(u) - \gamma)$$

in (Hyp_ε), consider the corresponding conjugate semiflows $\widetilde{\pi}_\varepsilon = \Phi^*\pi_\varepsilon$, $\varepsilon > 0$, and then apply an abstract singular Conley index continuation principle established [4] to the family $\widetilde{\pi}_\varepsilon$, $\varepsilon \geq 0$, where $\widetilde{\pi}_0 = \pi'$. However, there is an inherent difficulty in the present situation due to the fact that the transformation Φ is defined on the space $H^2(\Omega) \times L^2(\Omega)$ which is only a subset of the phase space $H^1_0(\Omega) \times L^2(\Omega)$ of the semiflows π_ε, so $\widetilde{\pi}_\varepsilon$ is not well-defined for $\varepsilon > 0$.

That is why we first study, in Section 3, a model finite-dimensional singular perturbation problem (equations (3.1) and (3.2) below), to which a variable transformation like Φ is applicable. Results from [4] then yield a singular Conley index continuation result for the corresponding family of finite-dimensional semiflows (Theorems 3.1 and 3.9 below).

In Section 4 we establish some compactness and smoothing results for parabolic equations, which, in particular, imply that the Conley index in Theorem A for the semiflow π' is equal to a Conley index for equation (3.2) on an appropriate finite dimensional subspace E of $H^1_0(\Omega)$.

Then, in Section 5, which is based on ideas from [1], [10] and [9], we establish some boundedness and smoothing results for damped wave equations, which, in particular, imply that the Conley index in Theorem A for the semiflow π_ε is equal to a Conley index for equation (3.1) on $E \times E$, uniformly in ε for $\varepsilon > 0$ small.

Combining all these results we then obtain (in Section 7) our first main result, Theorem 6.1, which implies Theorem A above.

In the last section of this paper we use some recent results from [6] and [7] and show that Morse decompositions of the invariant set K', relative to the semiflow π', continue to appropriate Morse decompositions of K_ε relative to the semiflows π_ε, for $\varepsilon > 0$ small.

2. Notation

In this paper we use the letter N to denote various sets (mostly isolating neighbourhoods) as well as the dimension of the open set Ω. This should not lead to confusion.

If a and $b \in \mathbb{R}$ then we write $[\![a, b]\!] := [a, b] \cap \mathbb{Z}$.

If E is a normed space, $I \subset \mathbb{R}$, $t \in I \cap \mathrm{Cl}_{\mathbb{R}}(I \setminus \{t\})$ and $u: I \to E$ is a map which is differentiable at t, then we often use the symbol $\partial^{(E)} u(t)$ to denote the derivative of u at t. This notation is more useful than the traditional $u'(t)$ or $\partial u(t)$ in cases in which it is important to indicate the norm $|\cdot|_E$ with respect to which u is differentiated.

Whenever Z is a set and $h: Z \times \mathbb{R} \to \mathbb{R}$, $u: Z \to \mathbb{R}$ are arbitrary maps, then $\widehat{h}(u): Z \to \mathbb{R}$ is the map defined by $\widehat{h}(u)(x) := h(x, u(x))$, $x \in Z$.

Finally, given a local semiflow π on a metric space X and a strongly admissible isolating neighbourhood N, relative to π, of an isolated π-invariant set K then we write interchangeably $h(\pi, K)$ or $h(\pi, N)$ to denote the Conley index of K (cf. [13] or [14]).

3. A finite dimensional singular perturbation problem

In this section let $(E, |\cdot|)$ be a finite dimensional Banach space. Given a C^1-map $g: E \to E$ and $\varepsilon > 0$ let $\pi_{\varepsilon,g}$ be the local (semi)flow generated by the following ordinary differential equation on $E \times E$:

$$\dot{u} = v, \quad \dot{v} = (1/\varepsilon)(-v + g(u)), \quad (u, v) \in E. \tag{3.1}$$

Furthermore, let π'_g be the local (semi)flow on E generated by the following ordinary differential equation on E:

$$\dot{u} = g(u), \quad u \in E. \tag{3.2}$$

One of the goals of this section is the proof of the following result.

Theorem 3.1. *Let $N' \subset E$ be a compact isolating neighbourhood relative to π'_g. Then for every $\beta > 0$ there is an $\varepsilon_0 > 0$ such that for every $\varepsilon \in]0, \varepsilon_0]$ the set*

$$N'_\beta = N'_{\beta,g} := \{(u, v) \mid u \in N',\ |v - g(u)| \le \beta\}$$

is an isolating neighbourhood relative to $\pi_{\varepsilon,g}$ and

$$h(\pi_{\varepsilon,g}, N'_\beta) = h(\pi'_g, N').$$

The proof of Theorem 3.1 will be based on a singular Conley index continuation result established in [4] (cf. also [3]). In order to state this result, we shall need a few definitions and notations.

Let (X_0, d_0) be a metric space. Let $\varepsilon_0 > 0$ and for each $\varepsilon \in]0, \varepsilon_0]$ let $(Y_\varepsilon, d_\varepsilon)$ be a metric space and $\theta_\varepsilon \in Y_\varepsilon$ be a distinguished point of Y_ε. The open ball in Y_ε of center in y and radius $\beta > 0$ is denoted by $B_\varepsilon(y, \beta)$.

For each $\varepsilon \in]0, \varepsilon_0]$ define the set $Z_\varepsilon := X_0 \times Y_\varepsilon$. Endow Z_ε with the metric

$$\Gamma_\varepsilon((x, y), (x', y')) := \max\{d_0(x, x'), d_\varepsilon(y, y')\} \quad \text{for } (x, y), (x', y') \in Z_\varepsilon.$$

Given a subset V of X_0, $\beta > 0$ and $\varepsilon \in \,]0,\varepsilon_0]$ define

$$[V]_{\varepsilon,\beta} = \{(x,y) \in Z_\varepsilon \mid x \in V \text{ and } y \in \mathrm{Cl}_\varepsilon B_\varepsilon(\theta_\varepsilon,\beta)\}.$$

Let π_0 be a local semiflow on X_0 and for every $\varepsilon \in \,]0,\varepsilon_0]$ let π_ε be a local semiflow on Z_ε.

Definition 3.2. We say that the family $(\pi_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ of local semiflows *converges singularly to* the local semiflow π_0 if whenever (ε_n) and (t_n) are sequences in $]0,\varepsilon_0]$ and $[0,\infty[$, respectively such that $\varepsilon_n \to 0$ and $t_n \to t_0$ for some $t_0 \in [0,\infty[$ and whenever $x_0 \in X_0$ and $z_n \in Z_{\varepsilon_n}$, $n \in \mathbb{N}$ are such that $\Gamma_{\varepsilon_n}(z_n,(x_0,\theta_{\varepsilon_n})) \to 0$ and $x_0\pi_0 t_0$ is defined, then there exists an $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $z_n\pi_{\varepsilon_n}t_n$ is defined and $\Gamma_{\varepsilon_n}(z_n\pi_{\varepsilon_n}t_n,(x_0\pi_0t_0,\theta_{\varepsilon_n})) \to 0$.

Definition 3.3. Let β be a positive number and N be a closed subset of X_0. We say that N is a *singularly strongly admissible set with respect to β and the family $(\pi_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$* if the following conditions are satisfied:

(a) N is a strongly π_0-admissible set;
(b) for each $\varepsilon \in \,]0,\varepsilon_0]$ the set $[N]_{\varepsilon,\beta}$ is strongly π_ε-admissible;
(c) whenever (ε_n) and (t_n) are sequences in $]0,\varepsilon_0]$ and $[0,\infty[$ such that $\varepsilon_n \to 0$ and $t_n \to \infty$ and whenever $z_n \in Z_{\varepsilon_n}$, $n \in \mathbb{N}$, are such that $z_n\pi_{\varepsilon_n}[0,t_n] \subset [N]_{\varepsilon_n,\beta}$, $n \in \mathbb{N}$, then there exist a $x_0 \in N$ and a subsequence of the sequence $(z_n\pi_{\varepsilon_n}t_n)$ of endpoints, denoted again by $(z_n\pi_{\varepsilon_n}t_n)$, such that $\Gamma_{\varepsilon_n}(z_n\pi_{\varepsilon_n}t_n,(x_0,\theta_{\varepsilon_n})) \to 0$.

Theorem 3.4 ([4]). *Suppose that there exists an $\eta_0 > 0$ such that for all $\varepsilon \in \,]0,\varepsilon_0]$ and all $\eta \in \,]0,\eta_0]$ the set $\mathrm{Cl}_\varepsilon B_\varepsilon(\theta_\varepsilon,\eta)$ is contractible to the point θ_ε.*

Let $\beta \in \,]0,\infty[$ be arbitrary. Suppose $(\pi_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ is a family of local semiflows that converges singularly to the local semiflow π_0 and N is a singularly strongly admissible set with respect to β and $(\pi_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$. Assume that N is an isolating neighbourhood for π_0.

Then for every $\eta \in \,]0,\widetilde{\eta}_0]$, where $\widetilde{\eta}_0 < \min\{\eta_0,\beta\}$, there exists an $\varepsilon^{\mathrm{c}} = \varepsilon^{\mathrm{c}}(\eta) > 0$ such that for every $\varepsilon \in \,]0,\varepsilon^{\mathrm{c}}]$ the set $[N]_{\varepsilon,\eta}$ is a strongly admissible isolating neighbourhood relative to π_ε and

$$h(\pi_\varepsilon,[N]_{\varepsilon,\eta}) = h(\pi_0,N).$$

Remark 3.5. Recall that a topological space Y is called *contractible to the point $p \in Y$* if there is a continuous map $H\colon Y \times [0,1] \to Y$ such that $H(y,0) = y$ and $H(y,1) = p$ for all $y \in Y$.

Proof of Theorem 3.1. Let U be a bounded open neighbourhood of N' and $\widetilde{g}\colon E \to E$ be a C^1-map such that $g|U = \widetilde{g}|U$ and $\sup_{u\in E}(|\widetilde{g}(u)| + |D\widetilde{g}(u)|) < \infty$. The existence of $\widetilde{g}$ follows since E is finite-dimensional.

Since the differential equations defining $\pi_{\varepsilon,g}$ and $\pi_{\varepsilon,\widetilde{g}}$ coincide on the open neighbourhood $U \times E$ of N'_β in $E \times E$ it follows that N'_β is an isolating neighbourhood relative to $\pi_{\varepsilon,g}$ if and only if N'_β is an isolating neighbourhood relative to $\pi_{\varepsilon,\widetilde{g}}$ and then

$$h(\pi_{\varepsilon,g}, N'_\beta) = h(\pi_{\varepsilon,\widetilde{g}}, N'_\beta).$$

Similarly, N' is an isolating neighbourhood relative to π'_g if and only if N' is an isolating neighbourhood relative to $\pi'_{\widetilde{g}}$ and then

$$h(\pi'_g, N') = h(\pi'_{\widetilde{g}}, N').$$

It follows that we may assume, without loss of generality, that

$$\sup_{u\in E}(|g(u)| + |Dg(u)|) < \infty. \tag{3.3}$$

In particular, g is globally Lipschitzian and so both $\pi_{\varepsilon,g}$, $\varepsilon > 0$, and π'_g are global semiflows. We write $\pi_\varepsilon := \pi_{\varepsilon,g}$, $\varepsilon > 0$, and $\pi' := \pi'_g$ for short.

Notice that the map $\Phi: E\times E \to E\times E$, $\Phi(u,v) = (u,w) := (u, v - g(u))$ is a C^1-diffeomorphism with inverse Φ^{-1} given by $\Phi^{-1}(u,w) = (u,v) := (u, w+g(u))$. Let $\widetilde{\pi}_\varepsilon$ be the conjugate of π_ε via Φ i.e. $(u,w)\widetilde{\pi}_\varepsilon t := \Phi((\Phi^{-1}(u,w))\pi_\varepsilon t)$, $(u,w) \in E\times E$, $t \in [0,\infty[$. Note that $\widetilde{\pi}_\varepsilon$ is the semiflow generated by the equation

$$\begin{aligned} \dot{u} &= w + g(u), \\ \dot{w} &= -(1/\varepsilon)w - Dg(u)(w + g(u)). \end{aligned} \tag{3.4}$$

Let B_β be the closed ball in E with radius β centered at zero. Since $\Phi(N'_\beta) = N' \times B_\beta$, and since the Conley index is invariant under semiflow conjugation, it follows that, for $\varepsilon > 0$, the set N'_β is an isolating neighbourhood relative to π_ε if and only if $N' \times B_\beta$ is an isolating neighbourhood relative to $\widetilde{\pi}_\varepsilon$ and then

$$h(\pi_\varepsilon, N'_\beta) = h(\widetilde{\pi}_\varepsilon, N' \times B_\beta).$$

Thus, in order to prove Theorem 3.1, we only have to establish the validity of the following lemma.

Lemma 3.6. *There is an $\varepsilon_0 > 0$ such that for all $\varepsilon \in \,]0, \varepsilon_0[$ the set $N' \times B_\beta$ is an isolating neighbourhood relative to $\widetilde{\pi}_\varepsilon$ and*

$$h(\widetilde{\pi}_\varepsilon, N' \times B_\beta) = h(\pi', N').$$

Proof. To prove Lemma 3.6, define $X_0 = Z_\varepsilon = E$, $\theta_\varepsilon = 0$ and $d_0(u,u') = d_\varepsilon(u,u') = |u - u'|$ for all $\varepsilon > 0$ and u, $u' \in E$. It follows that $N' \times B_\beta = [N']_{\varepsilon,\beta}$ for all $\varepsilon > 0$. Moreover, let $\langle\,\cdot\,,\,\cdot\,\rangle$ be an arbitrary scalar product on E and $\|\cdot\|$ be the corresponding Euclidean norm. Let $\varepsilon > 0$ be arbitrary and (u,w) be an arbitrary solution of $\widetilde{\pi}_\varepsilon$ on $\mathbb{R}$ (i.e. a *full* solution). Then, for all s and $t \in \mathbb{R}$ with $s \le t$, we have

$$w(t) = e^{-(1/\varepsilon)(t-s)} w(s) - \int_s^t e^{-(1/\varepsilon)(t-r)} Dg(u(r))(w(r) + g(u(r)))\,dr. \tag{3.5}$$

Moreover, for $t \in \mathbb{R}$,

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2 = \langle w(t), \dot{w}(t)\rangle = -\frac{1}{\varepsilon}\|w(t)\|^2 - \langle w(t), Dg(u(t))(w(t) + g(w(t)))\rangle.$$

Since the norms $|\cdot|$ and $\|\cdot\|$ are equivalent we thus obtain from (3.3) that there is a constant $C \in\,]0,\infty[$, independent of the solution (u,w), such that

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2 \le -\frac{1}{\varepsilon}\|w(t)\|^2 + C\|w(t)\|^2 + C^2\|w(t)\|, \quad t \in \mathbb{R}. \tag{3.6}$$

Let $\overline{\varepsilon} > 0$ be such that $-(1/\overline{\varepsilon}) + C + C^2 < 0$ and suppose that $\varepsilon \in\,]0,\overline{\varepsilon}]$. We claim that

$$\|w(t)\| < \|w(0)\| + 1, \quad t \in [0,\infty[\,. \tag{3.7}$$

In fact if this is not true, then there is a smallest $\overline{t} \in [0,\infty[$ such that $\|w(\overline{t})\| = \|w(0)\| + 1$. It follows that $\overline{t} > 0$ and that $\|w(t)\|^2 < \|w(\overline{t})\|^2$ for $t \in \left[0,\overline{t}\right[$. Therefore

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{|t=\overline{t}} \ge 0. \tag{3.8}$$

On the other hand, (3.6) implies that

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{|t=\overline{t}} \le \|w(\overline{t})\|^2\left(-\frac{1}{\varepsilon} + C + C^2\right) < 0,$$

a contradiction which proves (3.7). Thus again there is a constant $C' \in\,]0,\infty[$, independent of the solution (u,w), such that

$$|w(t)| \le C'(|w(0)| + 1), \quad t \in [0,\infty[\,. \tag{3.9}$$

It follows from (3.5) and (3.9) that

$$|w(t)| \le e^{-(1/\varepsilon)t}|w(0)| + \int_0^t e^{-(1/\varepsilon)(t-r)} C(C'(|w(0)|+1) + C)\,dr, \quad t \in [0,\infty[$$

so

$$|w(t)| \le e^{-(1/\varepsilon)t}|w(0)| + \varepsilon C(C'(|w(0)|+1) + C), \quad t \in [0,\infty[\,. \tag{3.10}$$

Suppose now that $\varepsilon_k \to 0$ and, for every $k \in \mathbb{N}$, let (u_k, w_k) be a solution of $\widetilde{\pi}_{\varepsilon_k}$ on $\mathbb{R}$. Assume first that $t_k \to \infty$ and $(u_k(r), w_k(r)) \in N' \times B_\beta$ for $k \in \mathbb{N}$ and $r \in [0,t_k]$. Then, using (3.10) we see that

$$|w_k(t_k)| \le e^{-(1/\varepsilon_k)t_k}\beta + \varepsilon_k C(C'(\beta+1) + C)$$

so

$$|w_k(t_k)| \to 0. \tag{3.11}$$

(3.11) and the compactness of N' imply that there is a sequence $k_m \to \infty$ in $\mathbb{N}$ and a $u_0 \in N'$ such that $(u_{k_m}(t_{k_m}), w_{k_m}(t_{k_m})) \to (u_0, 0)$. This shows that item (c) in Definition 3.3 is satisfied. Items (a) and (b) of that definition are obvious as both N' and $N' \times B_\beta$ are compact. It follows that the set N' is singularly strongly admissible with respect to β and the family $(\widetilde{\pi}_\varepsilon)_{\varepsilon>0}$.

Now suppose that $t_k \to t_0$ in $[0,\infty[$ and $(u_k(0), w_k(0)) \to (u_0, 0)$ for some $u_0 \in E$. Let u be the (uniquely determined) full solution of π' with $u(0) = u_0$. There is a $k_0 \in \mathbb{N}$ such that $\varepsilon_k \le \overline{\varepsilon}$ for all $k \ge k_0$. Let $k \ge k_0$ be arbitrary. By (3.3), (3.10) and by the mean-value theorem we obtain, for all $t \in [0,\infty[$,

$$\begin{aligned}
|u_k(t) - u(t)| &\le |u_k(0) - u_0| + \int_0^t (C|u_k(r) - u(r)| + |w_k(r)|)\,dr \\
&\le |u_k(0) - u_0| + \int_0^t (C|u_k(r) - u(r)| + |w_k(0)| \\
&\quad + \varepsilon_k C(C'(|w_k(0)| + 1) + C)))\,dr \\
&\le |u_k(0) - u_0| + C\int_0^t |u_k(r) - u(r)|\,dr + t(|w_k(0)| \\
&\quad + \varepsilon_k C(C'(|w_k(0)| + 1) + C))).
\end{aligned} \tag{3.12}$$

There is a $k_1 \ge k_0$ such that for all $k \ge k_0$ we have $t_k \le \overline{t} := t_0 + 1$. Thus for all such k we obtain, using (3.12) and Gronwall's inequality, that

$$\begin{aligned}
|u_k(t_k) - u(t)| &\le |u_k(t_k) - u(t_k)| + |u(t_k) - u(t_0)| \\
&\le e^{C\overline{t}}(|u_k(0) - u_0| + \overline{t}(|w_k(0)| \\
&\quad + \varepsilon_k C(C'(|w_k(0)| + 1) + C))) + |u(t_k) - u(t_0)|
\end{aligned}$$

so $u_k(t_k) \to u(t_0)$. Again, using (3.10) we also obtain

$$|w_k(t_k)| \le |w_k(0)| + \varepsilon_k C(C'(|w_k(0)| + 1) + C) \to 0.$$

Altogether we have shown that $(\widetilde{\pi}_\varepsilon)_{\varepsilon>0}$ singularly converges to π'.

The assertion of Lemma 3.6 now follows from Theorem 3.4.

The theorem is proved. □

We shall now generalize Theorem 3.1 to comprise isolating neighbourhoods which are more general than N'_β. To this end, we need the following definition.

Definition 3.7. Let (Y,d) be a metric space and $\mathcal{C} = C(\mathbb{R} \to Y)$ be the set of all continuous maps from $\mathbb{R}$ to Y. Let $\mathcal{T}$ be an arbitrary subset of $\mathcal{C}$ and $N \subset Y$ be arbitrary. Define

$$\operatorname{Inv}_{\mathcal{T}}(N) = \{y \in Y \mid \exists \sigma \in \mathcal{T} \text{ with } \sigma(\mathbb{R}) \subset N \text{ and } y = \sigma(0)\}.$$

We say that N is a *$\mathcal{T}$-isolating neighbourhood* (*of a subset K of Y*) if N is closed in Y and $\mathrm{Inv}_{\mathcal{T}}(N) \subset \mathrm{Int}_Y(N)$ (with $K = \mathrm{Inv}_{\mathcal{T}}(N)$). If $K \subset Y$ and there exists a set $N \subset Y$ such that N is a $\mathcal{T}$-isolating neighbourhood of K then we call K a *$\mathcal{T}$-isolated invariant set*.

Define $\mathcal{T}_g$ to be the set of all functions $z\colon \mathbb{R} \to E \times E$ such that there is a full bounded solution $u\colon \mathbb{R} \to E$ of π' so that $z(t) = (u(t), g(u(t))$ for all $t \in \mathbb{R}$. Thus, defining the map $\Gamma_g\colon E \to E \times E$ by $\Gamma_g(\xi) = (\xi, g(\xi))$, $\xi \in E$, we see that $\mathcal{T}_g$ is the set of all functions $z\colon \mathbb{R} \to E \times E$ such that there is a full bounded solution $u\colon \mathbb{R} \to E$ of π'_g with $z = \Gamma_g \circ u$. Now we have the following lemma.

Lemma 3.8. *The set N'_β defined in Theorem 3.1 is a $\mathcal{T}_g$-isolating neighbourhood of the set $K := \Gamma_g(K')$, where $K' := \mathrm{Inv}_{\pi'_g}(N')$.*

Proof. We have to show that, first, $K \subset \mathrm{Int}_{E\times E}(N'_\beta)$ and, second, that $K = \mathrm{Inv}_{\mathcal{T}_g}(N'_\beta)$. The first assertion follows since $K \subset U$ where U is the set of all (u, v) with $u \in \mathrm{Int}_E(N')$ and $|v - g(u)| < \beta$, U is open in $E \times E$ and $U \subset N'_\beta$. To prove the second assertion, let $(\overline{u}, \overline{v}) \in K$ be arbitrary. Then $\overline{u} \in K'$ and $\overline{v} = g(\overline{u})$. By the π'_g-invariance of K' there is a full solution u of π'_g with $u(0) = \overline{u}$ and lying in $K' \subset N'$. It follows that $z := \Gamma_g \circ u \in \mathcal{T}_g$, $z(0) = (\overline{u}, \overline{v})$ and z lies in N'_β. Thus $(\overline{u}, \overline{v}) \in \mathrm{Inv}_{\mathcal{T}_g}(N'_\beta)$. Conversely, let $(\overline{u}, \overline{v}) \in \mathrm{Inv}_{\mathcal{T}_g}(N'_\beta)$. Then there is a $z \in \mathcal{T}_g$ with $z(0) = (\overline{u}, \overline{v})$ and z lies in N'_β. It follows that there is a full solution u of π' with $z = \Gamma_g \circ u$. Consequently, u lies in N' and so $\overline{u} = u(0) \in \mathrm{Inv}_{\pi'_g}(N') = K'$ and $\overline{v} = g(\overline{u})$. Thus $(\overline{u}, \overline{v}) \in K$. The lemma is proved. $\square$

We can now state the following result.

Theorem 3.9. *Let N' be an arbitrary compact isolating neighbourhood relative to π', $K' := \mathrm{Inv}_{\pi'_g}(N')$ and $K := \Gamma_g(K')$. Then K is a $\mathcal{T}_g$-isolated invariant set and for every compact $\mathcal{T}_g$-isolating neighbourhood N of K there is an $\varepsilon_0 > 0$ such that for all $\varepsilon \in \,]0, \varepsilon_0]$ the set N is an isolating neighbourhood relative to $\pi_{\varepsilon,g}$ and*

$$h(\pi_{\varepsilon,g}, N) = h(\pi'_g, N').$$

Proof. Again write $\pi_\varepsilon := \pi_{\varepsilon,g}$, $\varepsilon > 0$, and $\pi' := \pi'_g$ for short.

The first assertion follows from Lemma 3.8. Let us prove the second assertion. For $\alpha > 0$ let B'_α be the closed α-neighbourhood of K' in E. Then we claim that there are α and $\beta \in \,]0, \infty[$ such that

$$B'_{\alpha,\beta} := \{(u, v) \mid u \in B'_\alpha,\ |v - g(u)| \le \beta\} \subset N$$

and B'_α is an isolating neighbourhood of K', relative to π'. In fact, if this claim is not true then, by the definition of $B'_{\alpha,\beta}$ there are sequences (u_k), (u'_k) and (v_k) in E such that $|u_k - u'_k| \to 0$, $|v_k - g(u_k)| \to 0$, $u'_k \in K'$ and $(u_k, v_k) \notin N$

for every $k \in \mathbb{N}$. We may assume that $u'_k \to u'$ for some $u' \in K'$. It follows that $u_k \to u'$ so $g(u_k) \to g(u')$. Hence $v_k \to v' := g(u')$. Thus $(u_k, v_k) \to (u', v') \in K \subset \mathrm{Int}_{E\times E}(N)$ so $(u_k, v_k) \in \mathrm{Int}_{E\times E}(N) \subset N$ for all $k \in \mathbb{N}$ large enough, a contradiction which proves the claim. Let α and β be as in the claim. We also claim that there is an $\varepsilon_1 > 0$ such that

$$\mathrm{Inv}_{\pi_\varepsilon}(B'_{\alpha,\beta}) = \mathrm{Inv}_{\pi_\varepsilon}(N), \quad \varepsilon \in \,]0, \varepsilon_1]\,. \tag{3.13}$$

In fact, by the choice of α and β we have $\mathrm{Inv}_{\pi_\varepsilon}(B'_{\alpha,\beta}) \subset \mathrm{Inv}_{\pi_\varepsilon}(N)$ for all $\varepsilon > 0$. Thus, if there is no $\varepsilon_1 > 0$ for which (3.13) is true, then there is a sequence (ε_k) with $\varepsilon_k \to 0^+$ and for every $k \in \mathbb{N}$ there is a full solution (u_k, v_k) of π_{ε_k} lying in N and such that $(u_k(0), v_k(0)) \notin B'_{\alpha,\beta}$.

For $k \in \mathbb{N}$ and $t \in \mathbb{R}$ set $w_k(t) = v_k(t) - g(u_k(t))$. Then (u_k, w_k) solves the equation

$$\begin{aligned} \dot u_k &= w_k + g(u_k), \\ \dot w_k &= -(1/\varepsilon_k) w_k - Dg(u_k)(w_k + g(u_k)). \end{aligned} \tag{3.14}$$

It follows from (3.14) that, for all s, $t \in \mathbb{R}$ with $s \le t$

$$w_k(t) = e^{-(1/\varepsilon_k)(t-s)} w_k(s) - \int_s^t e^{-(1/\varepsilon_k)(t-r)} (Dg(u_k(r))(w_k(r) + g(u_k(r))\, dr$$

so, setting

$$M = \sup_{(u,v)\in N} (|g(u)| + |Dg(u)| + |v|) < \infty$$

we obtain

$$|w_k(t)| \le e^{-(1/\varepsilon_k)(t-s)} M + \int_s^t e^{-(1/\varepsilon_k)(t-r)} M^2\, dr \le e^{-(1/\varepsilon_k)(t-s)} M + \varepsilon_k M^2.$$

Letting $s \to -\infty$ we thus obtain

$$|w_k(t)| \le \varepsilon_k M^2, \quad k \in \mathbb{N},\ t \in \mathbb{R}. \tag{3.15}$$

Now (3.14) and (3.15) imply that

$$|\dot u_k(t)| \le \varepsilon_k M^2 + M, \quad k \in \mathbb{N},\ t \in \mathbb{R}$$

so the boundedness of N and Arzelà–Ascoli theorem imply that there is a continuous map $u\colon \mathbb{R} \to E$ and a subsequence of $((u_k, w_k))$, denoted by $((u_k, w_k))$ again, so that $u_k(t) \to u(t)$, uniformly on compact subsets of $\mathbb{R}$. It follows from (3.15) that $\dot u_k(t) \to g(u(t))$, uniformly on compact subsets of $\mathbb{R}$ and so u is differentiable and

$$\dot u(t) = g(u(t)), \quad t \in \mathbb{R}.$$

It follows that $(u_k(t), v_k(t)) \to (u(t), v(t))$ in $E \times E$, uniformly on compact subsets of $\mathbb{R}$, where u is a full solution of π' and $v(t) = g(u(t))$ for all $t \in \mathbb{R}$. Since N is closed in $E\times E$ it follows that (u, v) lies in N. Consequently $(u, v) \in \mathcal{T}_g$ and

$(u(0), v(0)) \in \mathrm{Inv}_{\mathcal{T}_g}(N) = K \subset \mathrm{Int}_{E\times E}(B'_{\alpha,\beta})$. It follows that $(u_k(0), v_k(0)) \in \mathrm{Int}_{E\times E}(B'_{\alpha,\beta})$ for all $k \in \mathbb{N}$ large enough, a contradiction which proves (3.13). Now using Theorem 3.1 with N' replaced by B'_α we see that there is an $\varepsilon_2 > 0$ such that

$$h(\pi_\varepsilon, B'_{\alpha,\beta}) = h(\pi', B'_\alpha). \tag{3.16}$$

Since B'_α and N' are both isolating neighbourhoods of K', relative to π', we have that

$$h(\pi', B'_\alpha) = h(\pi', N'). \tag{3.17}$$

Now (3.13), (3.16) and (3.17) imply the second assertion of the theorem. The proof is complete. □

4. Compactness and smoothing for parabolic equations

In this section we study local semiflows π'_f generated by abstract parabolic equations of the form $\dot{u} = Au + f(u)$ where A is a positive self-adjoint operator on a Hilbert space X (generating fractional power spaces X^β, $\beta \in [0,1]$) and $f: X^\alpha \to X$ is a suitable nonlinearity defined on X^α with some $\alpha \in [0,1[$. We establish a compactness result (in X^α) for full bounded solutions of the semiflows π'_{f_κ} for a given sequence of nonlinearities (f_κ) (Theorem 4.3). This result enables us to compute the Conley index of isolating neighbourhoods of π'_f by using finite-dimensional Galerkin approximations of π'_f (Propositions 4.2 and 4.4). We then strengthen Theorem 4.3 to a compactness result in X^1. (Theorem 4.6). This will allow us to imbed compact invariant sets relative to π'_f into the phase space $X^{1/2} \times X$ of damped hyperbolic equations and study some perturbation properties of such imbeddings (Theorems 4.10 and 4.11).

For the rest of this paper, let $(X, \langle\cdot\,,\,\cdot\rangle)$ be a real Hilbert space and $A: D(A) \subset X \to X$ be a positive selfadjoint operator with compact resolvent. Let $(\phi_\nu)_{\nu\in\mathbb{N}}$ be a complete X-orthonormal basis of X consisting of eigenfunctions of A. Let $P_n: X \to X$ be the orthogonal projection of X onto the subspace spanned by the first n eigenfunctions. Moreover, set $Q_n := I - P_n$ where I is the identity map on X. Note that A is sectorial on X and so it generates a family $(X^\alpha)_{\alpha\in[0,\infty[}$ of fractional power spaces. Moreover, for $\alpha \in [0,\infty[$ let $X^{-\alpha} := X^{\alpha *}$ be the dual of X^α. (Here we depart from the usual notation of, say, [11].) For $\alpha \in [0,\infty[$ the formula

$$\langle u, v\rangle_\alpha := \langle A^\alpha u, A^\alpha v\rangle_X, \quad u, v \in X^\alpha,$$

defines a Hilbert product in X^α and A^α is an isometry between the Hilbert spaces X^α and X. Endow $X^{-\alpha} := X^{\alpha *}$ with the dual product. We write $|\cdot|_\alpha$ for the induced norm of X^α, $\alpha \in \mathbb{R}$. For $\alpha \in [0,\infty[$ we also write

$$A^{-\alpha} := (A^\alpha)^{-1}: X \to X^\alpha.$$

It is well-known that for every $\beta \in \mathbb{R}$ the operator A^β can be uniquely extended to a map

$$A^\beta : \bigcup_{\alpha\in\mathbb{R}} X^\alpha \to \bigcup_{\alpha\in\mathbb{R}} X^\alpha$$

such that whenever $\alpha \in \mathbb{R}$ then $A^\beta(X^\alpha) = X^{\alpha-\beta}$ and $A^\beta_{|X^\alpha} : X^\alpha \to X^{\alpha-\beta}$ is an isometry. Moreover, A^0 is the identity on $\bigcup_{\alpha\in\mathbb{R}} X^\alpha$ and $A^\beta \circ A^\gamma = A^{\beta+\gamma}$ for all β and $\gamma \in \mathbb{R}$.

Example 4.1. Set $X := L^2(\Omega)$, $A: D(A) := H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega)$,

$$Au := -\Delta u, \quad u \in D(A).$$

It is well-known that A is positive selfadjoint in $L^2(\Omega)$ and has compact resolvent. In this case, $X^0 = L^2(\Omega)$, $X^{1/2} = H^1_0(\Omega)$ and $X^1 = H^2(\Omega) \cap H^1_0(\Omega)$ (the latter space being regarded as a subspace of $H^2(\Omega)$).

Given $\alpha \in [0, 1[$ and a locally Lipschitzian map $f: X^\alpha \to X$ let π'_f be the local semiflow on X^α generated by the abstract parabolic equation (see [11])

$$\dot{u} = -Au + f(u), \quad u \in X^\alpha.$$

The following result holds.

Proposition 4.2. *Suppose $f(X^\alpha) \subset P_n(X)$ for some $n \in \mathbb{N}$. Then the set $Y_n := P_n(X^\alpha)$ is positively invariant relative to the local semiflow $\pi' := \pi'_f$. Let π'_n be the restriction of π' to Y_n. π'_n is the local semiflow on Y_n generated by the ordinary differential equation*

$$\dot{u} = -Au + f(u), \quad u \in Y_n \tag{4.1}$$

on Y_n. Let $N' \subset X^\alpha$ be closed and bounded. If N' is an isolating neighbourhood relative to π', then $N'_n := N' \cap Y_n$ is an isolating neighbourhood relative to π'_n and

$$h(\pi', N') = h(\pi'_n, N'_n).$$

Proof. Note that $P_n(X^\alpha) = P(X)$. Since A and f map Y_n into itself, it follows that the finite dimensional ODE (4.1) is well defined. Let π''_n be the local semiflow on $Z_n := Q_n(X^\alpha)$ generated by the abstract parabolic equation

$$\dot{u} = -Au, \quad u \in Z_n. \tag{4.2}$$

Note that for every interval $J \subset \mathbb{R}$ and every map $u: J \to X^\alpha$ we have that u is a solution of π'_f if and only if there are maps $u_1: J \to Y_n$ and $u_2: J \to Z_n$ such that $u = u_1 + u_2$ and u_1 is a solution of π'_n while u_2 is a solution of π''_n; u_1 is given by $u_1 = P_n \circ u$ while u_2 is given by $u_2 = Q_n \circ u$. It follows that Y_n is positively invariant with respect to π' and π'_n is generated by equation (4.1). Furthermore, a set $K \subset X^\alpha$ is invariant relative to π' if and only if $P_n(K)$ is invariant relative to π'_n and $Q_n(K)$ is invariant relative to π''_n. Note that there are constants α, $C \in]0,\infty[$ such that

$$|e^{-At}u|_\alpha \le Ce^{-\alpha t}|u|_\alpha, \quad u \in Z_n,\ t \in [0,\infty[\,. \tag{4.3}$$

This implies that every full bounded solution of (4.2) is trivial. Hence every bounded invariant set K relative to π' satisfies the inclusion $Q_n(K) \subset \{0\}$. Thus $K \subset Y_n$ so K is invariant relative to π'_n. Therefore, if N' is an isolating neighbourhood of K relative to π', then N'_n is an isolating neighbourhood of K relative to π'_n.

The estimate (4.3) shows that

$$h(\pi''_n, \{0\}) = \Sigma^0. \tag{4.4}$$

Now the homeomorphism $\Phi\colon X^\alpha \to Y_n \times Z_n$, $u \mapsto (P_n u, Q_n u)$, conjugates the local semiflow π' with the product $\pi'_n \times \pi''_n$. Let $K := \mathrm{Inv}_{\pi'}(N')$. It follows that Φ maps the set K onto $K \times \{0\}$ if $K \ne \emptyset$ and onto $\emptyset$ if $K = \emptyset$. Thus, in the first case, using (4.4), we have

$$h(\pi', N') = h(\pi', K) = h(\pi'_n, K) \wedge h(\pi''_n, \{0\}) = h(\pi'_n, K) \wedge \Sigma^0 = h(\pi'_n, N'_n)$$

and, in the second case, $h(\pi', N') = \overline{0} = h(\pi'_n, N'_n)$. The proposition is proved.□

Theorem 4.3. *Let N be a closed subset of X^α which is bounded in X. Suppose f and f_κ, $\kappa \in \mathbb{N}$ are locally Lipschitzian maps from X^α to X such that $f_\kappa(u) \to f(u)$ in X, uniformly for u lying in compact subsets of N. Moreover, suppose*

$$\sup_{\kappa\in\mathbb{N}} \sup_{u \in N} |f_\kappa(u)|_0 < \infty.$$

For every $\kappa \in \mathbb{N}$ let u_κ be a full solution of π'_{f_κ} lying in N. Then there is a sequence (κ_n) with $\kappa_n \to \infty$ and there is a full solution u of π'_f lying in N such that $u_{\kappa_n} \to u$ in X^α, uniformly on compact subsets of $\mathbb{R}$.

Proof. Choose β arbitrary with $\alpha < \beta < 1$. Since

$$u_\kappa(t) = e^{-A(t-r)}u_\kappa(r) + \int_r^t e^{-A(t-s)} f_\kappa(u_\kappa(s))\,ds, \tag{4.5}$$

$\kappa \in \mathbb{N}$, $r, t \in \mathbb{R}$, $r < t$, it follows that

$$|u_\kappa(t)|_\beta \le C_\beta (t-r)^{-\beta}|u_\kappa(r)|_0 + \int_r^t C_\beta (t-s)^{-\beta}|f_\kappa(u_\kappa(s)|_0\,ds,$$

for $\kappa \in \mathbb{N}$, r, $t \in \mathbb{R}$, with $r < t$ so, choosing $r = t-1$ and using our hypotheses, we see that there is a constant $C \in]0,\infty[$ such that

$$|u_\kappa(t)|_\beta \le C, \quad \kappa \in \mathbb{N},\ t \in \mathbb{R}. \tag{4.6}$$

Moreover, (4.5) also implies

$$|u_\kappa(t)-u_\kappa(r)|_\alpha \le |e^{-A(t-r)}u_\kappa(r)-u_\kappa(r)|_\alpha + \int_r^t C_\alpha (t-s)^{-\alpha}|f_\kappa(u_\kappa(s))|_0$$

for $\kappa\in\mathbb{N}$ and r, $t\in\mathbb{R}$ with $r<t$, so noting that $|e^{-A(t-r)}u_\kappa(r)-u_\kappa(r)|_\alpha = |e^{-A(t-r)}A^\alpha u_\kappa(r)-A^\alpha u_\kappa(r)|_0 \le (1/(\beta-\alpha))C_{1-(\beta-\alpha)}(t-r)^{\beta-\alpha}|u_\kappa(r)|_\beta$ we obtain from (4.6) that there is a constant $C'\in\,]0,\infty[$ such that

$$|u_\kappa(t)-u_\kappa(r)|_\alpha \le C'(t-r)^{\beta-\alpha}, \quad \kappa\in\mathbb{N},\ r,t\in\mathbb{R},\ r<t. \tag{4.7}$$

Since A has compact resolvent, (4.6) implies that, for every $t\in\mathbb{R}$, the set $\{u_\kappa(t)\mid\kappa\in\mathbb{N}\}$ lies in a compact subset of X^α so that, by (4.7) and the Arzelà–Ascoli theorem, there is a sequence (κ_n) with $\kappa_n\to\infty$ and there is a continuous mapping $u:\mathbb{R}\to X^\alpha$ such that $u_{\kappa_n}(t)\to u(t)$ in X^α, uniformly for t lying in compact subsets of $\mathbb{R}$. Since N is closed in X^α, we see that u lies in N. It also follows from our hypotheses and (4.5) that

$$u(t)=e^{-A(t-r)}u(r)+\int_r^t e^{-A(t-s)}f(u(s))\,ds, \quad r,t\in\mathbb{R},\ r<t.$$

Hence u is a full solution of π'_f, as claimed. □

Proposition 4.4. *Let $f:X^\alpha\to X$ be Lipschitzian on bounded subsets of X^α and $N\subset X^\alpha$ be bounded and closed. Let (n_κ) be a sequence in $\mathbb{N}$ with $n_\kappa\to\infty$ and (θ_κ) be an arbitrary sequence in $[0,1]$. For $\kappa\in\mathbb{N}$ let $f_\kappa:X^\alpha\to X$ be defined by*

$$f_\kappa(u)=(1-\theta_\kappa)f(u)+\theta_\kappa P_{n_\kappa}f(P_{n_\kappa}u), \quad u\in X^\alpha.$$

Then f and f_κ, $\kappa\in\mathbb{N}$, satisfy the assumptions (*and hence the conclusions*) *of Theorem* 4.3.

Proof. There is a $C\in\,]0,\infty[$ such that $|u|_\alpha\le C$ for $u\in N$. Moreover, there is a constant $L\in\,]0,\infty[$ such that $|f(u)-f(v)|_0\le L|u-v|_\alpha$ for all u and $v\in X^\alpha$ with $|u|_\alpha$, $|v|_\alpha\le C$. It follows that $|f(u)|_0\le|f(0)|_0+LC=:C'$ for all $u\in X^\alpha$ with $|u|_\alpha\le C$. Thus, for $n\in\mathbb{N}$ and $u\in N$ we see that $|P_nu|_\alpha\le|u|_\alpha\le C$ and so, for $\kappa\in\mathbb{N}$, $|f_\kappa(u)|_0\le|f(u)|_0+|f(P_{n_\kappa}u)|_0\le 2C'$. Moreover, if $v_\kappa\to v$ in X^α, then, clearly, $f_\kappa(v_\kappa)\to f(v)$ in X. This completes the proof. □

The following result follows by a careful inspection of the proof of Theorem 3.5.2 in [11].

Lemma 4.5. *For all nonnegative real constants C_1, C_2, L and for all constants α and $\beta\in[0,1[$ there is a $C=C(C_1,C_2,L,\alpha,\beta)\in\,]0,\infty[$ such that whenever $f:X^\alpha\to X$ is Lipschitzian on bounded subsets of X^α, $|f(u)|_0\le C_2$ and $|f(u)-f(v)|_X\le L|u-v|_\alpha$ for all u, v with $|u|_\alpha$, $|v|_\alpha\le C_1$ and whenever*

$u\colon\mathbb{R} \to X^\alpha$ is a full solution of π'_f with $|u(t)|_\alpha \le C_1$ for all $t \in \mathbb{R}$, then u is differentiable into X^β and $|\partial^{(X^\beta)}u(t)|_\beta \le C$ for all $t \in \mathbb{R}$.

We can now strengthen Theorem 4.3 to a compactness result in X^1.

Theorem 4.6. *Let N be a closed subset of X^α which is bounded in X. Suppose f and f_κ, $\kappa \in \mathbb{N}$ are maps from X^α to X such that $f_\kappa(u) \to f(u)$ in X, uniformly for u lying in compact subsets of N. Moreover, suppose $C_2 := \sup_{\kappa\in\mathbb{N}}\sup_{u\in N}|f_\kappa(u)|_0 < \infty$. Furthermore, suppose that the family $(f_\kappa)_{\kappa\in\mathbb{N}}$ is equi-Lipschitzian on bounded subsets of X^α, i.e. for every $C \in]0,\infty[$ there is an $L = L(C)$ such that $|f_\kappa(u) - f_\kappa(v)|_0 \le L|u-v|_\alpha$ for all $\kappa \in \mathbb{N}$ and all u, $v \in X^\alpha$ with $|u|_\alpha$, $|v|_\alpha \le C$. For every $\kappa \in \mathbb{N}$ let u_κ be a full solution of π'_{f_κ} lying in N. Then there is a subsequence (u_{κ_n}) of (π_κ) and there is a full solution u of π'_f lying in N such that $u_{\kappa_n} \to u$ in X^1, uniformly on compact subsets of $\mathbb{R}$.*

Proof. Note that, by our hypotheses, the map f is Lipschitzian on bounded subsets of X^α so the local semiflow π'_f is defined.

Let $\beta \in]0,1[$ be arbitrary. Proceeding as in the proof of Theorem 4.3 we see that

$$C_1 := \sup_{\kappa\in\mathbb{N}}\sup_{t\in\mathbb{R}}|u_\kappa(t)|_\alpha < \infty.$$

Let $L := L(C_1)$. Finally, let $C = C(C_1, C_2, L, \alpha, \beta)$ be as in Lemma 4.5. By Theorem 4.3 there is a sequence (κ_n) with $\kappa_n \to \infty$ and there is a full solution u of π'_f lying in N such that $u_{\kappa_n} \to u$ in X^α, uniformly on compact subsets of $\mathbb{R}$. We claim that $u_{\kappa_n} \to u$ in X^1, uniformly on compact subsets of $\mathbb{R}$. Suppose this claim is not true. Then, choosing a subsequence of (κ_n) if necessary, we may assume that there is a sequence (t_n) with $t_n \to t$ in $[0,\infty[$ and a $\delta \in]0,\infty[$ such that, setting $v_n := u_{\kappa_n}(t_n)$, $n \in \mathbb{N}$ and $v := u(t)$, we have

$$|v_n - v|_1 \ge \delta. \tag{4.8}$$

By Lemma 4.5 we obtain that, for every $\kappa \in \mathbb{N}$, the solution u_κ is differentiable into X^β and $|\partial^{(X^\beta)}u_\kappa(t)|_\beta \le C$ for all $\kappa \in \mathbb{N}$ and all $t \in \mathbb{R}$. Set $w_n := \partial^{(X^\beta)}u_{\kappa_n}(t_n)$, $n \in \mathbb{N}$. It follows that the set $\{w_n \mid n \in \mathbb{N}\}$ is included in a compact subset of $X^0 = X$ so we may assume that $w_n \to w$ in X for some $w \in X$. Since $v_n \to v$ in X^α it follows that $Av_n \to Av$ in $X^{\alpha-1}$ and $f_{\kappa_n}(v_n) \to f(v)$ in X^0. Hence $w_n = Av_n + f_{\kappa_n}(v_n) \to Av + f(v)$ in $X^{\alpha-1}$. Thus $w = Av + f(v)$ and so $Av_n = w_n - f_{\kappa_n}(v_n) \to w - f(v) = Av$ in X^0. This implies that $v_n \to v$ in X^1, contradicting (4.8). The theorem is proved. □

Corollary 4.7. *Suppose $f\colon X^\alpha \to X$ is Lipschitzian on bounded subsets of X^α. Then every full solution of π'_f which is bounded in X^α, is bounded in X^1. Moreover, every compact subset of X^α which is invariant relative to π'_f, is compact in X^1.*

PROOF. Actually, the proof of the first assertion is contained in the proof of Theorem 4.6. However, it also follows from the following argument. Let u be a full solution of π'_f which is bounded in X^α. If u is not bounded in X^1, then there is a sequence (t_n) such that

$$|\overline{u}_n|_1 \to \infty, \tag{4.9}$$

where $\overline{u}_n := u(t_n)$, $n \in \mathbb{N}$. Set $f_n \equiv f$ and $u_n(t) := u(t+t_n)$, $n \in \mathbb{N}$, $t \in \mathbb{R}$. An application of Theorem 4.6 shows that a subsequence (u_{n_k}) converges in X^1, uniformly on compact subsets of $\mathbb{R}$, to a full solution v of π'_f. In particular, $\overline{u}_{n_k} = u_{n_k}(0) \to v(0)$ in X^1, a contradiction proving the first assertion. Now let K be compact in X^α and invariant relative to π'_f. Let (a_n) be an arbitrary sequence in K. For every $n \in \mathbb{N}$ there is a full solution u_n of π'_f lying in K with $u_n(0) = a_n$. Again an application of Theorem 4.6 with $f_n \equiv f$ shows that a subsequence (u_{n_k}) converges in X^1, uniformly on compact subsets of $\mathbb{R}$, to a full solution u of π'_f. In particular, $a_{n_k} \to u(0)$ in X^1. Since K is closed in X^α, we see that $v(0) \in K$ so K is compact in X^1, as claimed. □

PROPOSITION 4.8. *Let f satisfy the assumptions of Proposition* 4.4 *and f_κ, $\kappa \in \mathbb{N}$, be as in that proposition. Then $(f_\kappa)_{\kappa\in\mathbb{N}}$ is equi-Lipschitzian on bounded subsets of X^α.*

PROOF. This follows from the estimate

$$|f_\kappa(u) - f_\kappa(v)|_0 \le |f(u) - f(v)|_0 + |f(P_{n_\kappa}u) - f(P_{n_\kappa}v)|_0$$

and the fact that f is Lipschitzian on bounded subsets of X^α. □

COROLLARY 4.9. *Let $f\colon X^\alpha \to X$ be Lipschitzian on bounded subsets of X^α. For $n \in \mathbb{N}$ and $\theta \in [0,1]$ let $f_{n,\theta}\colon X^\alpha \to X$ be defined by*

$$f_{n,\theta}(u) = (1-\theta)f(u) + \theta P_n f(P_n u), \quad u \in X^\alpha.$$

Set $\pi'_{n,\theta} := \pi'_{f_n,\theta}$, $n \in \mathbb{N}$, $\theta \in [0,1]$. Let (n_κ) and (θ_κ) be sequences such that $n_\kappa \to \infty$ and $\theta_\kappa \in [0,1]$ for every $\kappa \in \mathbb{N}$. For every $\kappa \in \mathbb{N}$ let u_κ be a full solution of $\pi'_{n_\kappa,\theta_\kappa}$ such that

$$\sup_{\kappa\in\mathbb{N}} \sup_{t\in\mathbb{R}} |u_\kappa(t)|_\alpha < \infty.$$

Then there is a subsequence of (u_κ), denoted again by (u_κ), and there is a full solution u_0 of π' lying in X^1 such that $u_\kappa \to u_0$ in X^1, uniformly on compact subsets of $\mathbb{R}$.

PROOF. This follows immediately from Proposition 4.4, Proposition 4.8 and Theorem 4.3. □

In the sequel, unless otherwise specified, we use the following notation. If $f\colon X^\alpha \to X$ is locally Lipschitzian, then $\Gamma_f\colon X^1 \to X^\alpha \times X$ is the map defined by

$$\Gamma_f(u) := (u, -Au + f(u)), \quad u \in X^1.$$

Moreover, by $\mathcal{T}_f$ we denote the set of all maps $z\colon \mathbb{R} \to X^\alpha \times X$ for which there is a full bounded solution u of π'_f such that $z(t) = \Gamma_f(u(t))$, $t \in \mathbb{R}$. In view of Corollary 4.7, the definition of $\mathcal{T}_f$ makes sense.

The following result describes the behavior of isolated invariant sets of π'_f under the imbedding Γ_f.

Theorem 4.10. *Let $f\colon X^\alpha \to X$ be Lipschitzian on bounded subsets of X^α. Let $K' \subset X^\alpha$ be compact in X^α and isolated invariant relative to π'_f. Then K' is compact in X^1. Set $K := \Gamma_f(K')$. Then K is compact in $X^\alpha \times X$ and K is a $\mathcal{T}_f$-isolated invariant set.*

Proof. K' is compact in X^1 by Corollary 4.7 and so the continuity of the map Γ_f implies that K is compact in $X^\alpha \times X$. Set $\pi' := \pi'_f$, $\mathcal{T} := \mathcal{T}_f$ and $\Gamma := \Gamma_f$. Let $\beta \in \,]0, \infty[$ be arbitrary. Let $N' \subset X^\alpha$ be closed and bounded in X^α and such that N' is an isolating neighbourhood of K' relative to π'. Set $U' := \mathrm{Int}_{X^\alpha}(N')$. Let U (resp. N) be the set of all $(u, v) \in X^\alpha \times X$ such that $u \in U'$ (resp. $u \in N'$) and there is a $u' \in K'$ such that $|(u, v) - \Gamma(u')|_{X^\alpha \times X} < \beta$ (resp. $|(u, v) - \Gamma(u')|_{X^\alpha \times X} \le \beta$). It is clear that U is open in $X^\alpha \times X$ while the compactness of K' in X^1 and the continuity of Γ imply that N is closed in $X^\alpha \times X$. If $(u, v) \in K$ then $u \in K' \subset U'$ and $(u, v) = \Gamma(u)$. Thus, choosing $u' := u$ we see that $(u, v) \in U \subset \mathrm{Int}_{X^\alpha \times X}(N)$. It follows that $K \subset U \subset \mathrm{Int}_{X^\alpha \times X}(N)$. Therefore, in order to complete the proof, we must show that

$$K = \mathrm{Inv}_{\mathcal{T}}(N). \tag{4.10}$$

Now, if $(\overline{u}, \overline{v}) \in K$ then $\overline{u} \in K'$ and so there is a full solution u of π' lying in K' and such that $u(0) = \overline{u}$. Thus $z := \Gamma \circ u \in \mathcal{T}$ and z lies in $K \subset N$. Since $(\overline{u}, \overline{v}) = z(0)$, it follows that $(\overline{u}, \overline{v}) \in \mathrm{Inv}_{\mathcal{T}}(N)$. Conversely, if $(\overline{u}, \overline{v}) \in \mathrm{Inv}_{\mathcal{T}}(N)$, then there is a $z \in \mathcal{T}$ lying in N, such that $z(0) = (\overline{u}, \overline{v})$. Therefore there is a full bounded solution u of π' such that $z = \Gamma \circ u$. Since z lies in N we have that u lies in N', and so u lies in K'. Thus z lies in K and so, in particular, $(\overline{u}, \overline{v}) \in K$. Formula (4.10) is proved. □

We now establish a stability property of the imbeddings Γ_f under perturbations of the nonlinearity f.

Theorem 4.11. *Suppose f and f_n, $n \in \mathbb{N}$, are maps from X^α to X such that $f_n(u) \to f(u)$ in X, uniformly for u lying in compact subsets of N. Furthermore, suppose that the family $(f_n)_{n\in\mathbb{N}}$ is equi-Lipschitzian on bounded subsets of X^α. Let K' be compact in X^α and isolated invariant relative to π'_f. Moreover, let N'*

be a bounded subset of X^α which is an isolating neighbourhood of K' relative to π'_f. For every $n \in \mathbb{N}$ let $K'_n := \operatorname{Inv}_{\pi'_{f_n}}(N')$. Then K' and K'_n, $n \in \mathbb{N}$ are included in X^1. Set $K := \Gamma_f(K')$ and $K_n := \Gamma_{f_n}(K'_n)$, $n \in \mathbb{N}$. Then K is a $\mathcal{T}_f$-isolated invariant set. Let $N \subset X^\alpha \times X$ be any bounded $\mathcal{T}_f$-isolating neighbourhood of K. Then there is an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the set N is $\mathcal{T}_{f_n}$-isolating neighbourhood of K_n.

Proof. Set $\pi' := \pi'_f$, $\mathcal{T} := \mathcal{T}_f$, $\Gamma := \Gamma_f$, $\pi'_n := \pi'_{f_n}$, $\mathcal{T}_n := \mathcal{T}_{f_n}$ and $\Gamma_n := \Gamma_{f_n}$, $n \in \mathbb{N}$. From our preceding results we know that K is a $\mathcal{T}$-isolated invariant set. Let N be as in the assumptions of this theorem. We first claim that there is an $n_1 \in \mathbb{N}$ such that

$$K_n \subset \operatorname{Int}(N) := \operatorname{Int}_{X^\alpha \times X}(N), \quad n \geq n_1. \tag{4.11}$$

Indeed, otherwise there is a sequence (n_k) with $n_k \to \infty$ such that $K_{n_k} \not\subset \operatorname{Int}(N)$ for all $k \in \mathbb{N}$. Hence, for every $k \in \mathbb{N}$, there is a full solution u_k of π'_{n_k} lying in N' and such that $(\overline{u}_k, \overline{v}_k) := \Gamma_{n_k}(u_k(0)) \notin N$. Using Theorem 4.3 we may assume that there is a full solution u_0 of π' such that $u_k \to u_0$ in X^1, uniformly on compact subsets of $\mathbb{R}$. Thus u_0 lies in N' and so u_0 is bounded in X^α which, by Corollary 4.7, implies that u_0 is bounded in X^1. Since u lies in K', it follows that $z := \Gamma \circ u_0 \in \mathcal{T}$ and z lies in N. Hence, by our hypothesis, z lies in $\operatorname{Int}(N)$. In particular, $z(0) \in \operatorname{Int}(N)$. Since $(\overline{u}_k, \overline{v}_k) = \Gamma_{n_k}(u_k(0)) \to \Gamma(u_0(0)) = z(0)$ in $X^\alpha \times X$ it follows that $(\overline{u}_k, \overline{v}_k) \in \operatorname{Int}(N) \subset N$ for all $k \in \mathbb{N}$ large enough, a contradiction which proves our first claim.

We next claim that there is an $n_2 \in \mathbb{N}$ such that

$$K_n = \operatorname{Inv}_{\mathcal{T}_n}(N), \quad n \geq n_2. \tag{4.12}$$

In fact, let n_1 be as in (4.11) and $n \geq n_1$ be arbitrary. Moreover, let $(\overline{u}, \overline{v}) \in K_n$ be arbitrary. Then there is a full solution u of π'_n lying in N' with $u(0) = \overline{u}$. Thus, by Corollary 4.7, $z := \Gamma_n \circ u \in \mathcal{T}_n$ and, by our choice of n, we see that $z(t) \in K_n \subset N$ for all $t \in \mathbb{R}$. Hence $(\overline{u}, \overline{v}) = z(0) \in \operatorname{Inv}_{\mathcal{T}_n}(N)$. It follows that

$$K_n \subset \operatorname{Inv}_{\mathcal{T}_n}(N), \quad n \geq n_1.$$

Therefore, if there is no $n_2 \in \mathbb{N}$ so that (4.12) holds, then there is a sequence (n_k) with $n_k \to \infty$ such that $\operatorname{Inv}_{\mathcal{T}_{n_k}}(N) \not\subset K_{n_k}$ for all $k \in \mathbb{N}$. Therefore there is a sequence (u_k) such that, for every $k \in \mathbb{N}$, u_k is a full bounded solution of π'_{n_k}, $z_k := \Gamma_{n_k} \circ u_k$ lies in N and $u_k(0) \notin N'$. Since N is bounded in $X^\alpha \times X$, it follows that

$$\sup_{k \in \mathbb{N}} \sup_{t \in \mathbb{R}} |u_k(t)|_\alpha < \infty$$

so, using Theorem 4.6, we may assume that there is a full solution u_0 of π', bounded in X^α, such that $u_k \to u_0$ in X^1, uniformly on compact subsets of $\mathbb{R}$. This implies that $z_k = \Gamma_{n_k} \circ u_k \to z_0 := \Gamma \circ u_0$ in $X^\alpha \times X$, uniformly on compact subsets of $\mathbb{R}$. Since u_0 is bounded in X^α, we have that $z_0 \in \mathcal{T}$. Since N is closed (in $X^\alpha \times X$) it follows that z_0 lies in N. Hence, in particular, $z_0(0) \in \mathrm{Inv}_{\mathcal{T}}(N) = K = \Gamma(K')$. Thus $u_0(0) \in K' \subset \mathrm{Int}_{X^\alpha}(N')$ so $u_k(0) \in \mathrm{Int}_{X^\alpha}(N')$ for all $k \in \mathbb{N}$ large enough, a contradiction which proves the claim. Taking $n_0 := \sup(n_1, n_2)$ we now see that, for all $n \geq n_0$, the set N is a $\mathcal{T}_n$-isolating neighbourhood of K_n. The theorem is proved. □

5. Compactness and smoothing for damped hyperbolic equations

In this section we study local semiflows $\pi_{\varepsilon,f}$ on $X^{1/2} \times X$ generated by second-order equations of the type

$$\dot{u} = v, \quad \dot{v} = (1/\varepsilon)(-v - Au + f(u)), \quad (u, v) \in X^{1/2} \times X^0$$

where $\varepsilon \in \,]0, \infty[$ and $f\colon X^{1/2} \to X$ is an appropriate nonlinearity. After recalling some basic properties of $\pi_{\varepsilon,f}$ we establish a preliminary abstract smoothing property for full bounded solutions of $\pi_{\varepsilon,f}$ (Proposition 5.4). Then, using Proposition 5.4 and ideas of Haraux ([10]) and Babin and Vishik ([1]) we prove a smoothing property of full bounded solutions of $\pi_{\varepsilon,f_{n,\theta}}$, the map $f_{n,\theta}$ having the special form

$$f_{n,\theta}(u) = (1-\theta)(\widehat{\phi}(u) + \gamma) + \theta P_n(\widehat{\phi}(P_n u) + \gamma), u \in X^{1/2} \tag{5.1}$$

where $\varepsilon \in \,]0, \infty[$, $n \in \mathbb{N}$ and $\theta \in [0,1]$ are arbitrary (but fixed) and ϕ and γ satisfy the properties listed in the Introduction (Theorem 5.9).

Using this latter result and following the arguments from the paper [9] by Hale and Raugel we then prove a *uniform* boundedness and smoothing property of full bounded solutions of $\pi_{\varepsilon,f_{n,\theta}}$ for ε small (Theorem 5.11).

Theorem 5.11 implies a singular compactness result for bounded sequences (u_κ, v_κ) of full bounded solutions of $\pi_{\varepsilon_\kappa, f_{n_\kappa,\theta_\kappa}}$, where $\varepsilon_k \to 0$ and $n_k \to \infty$ (Theorem 5.13). This latter result is an important step in the proof of the main results of this paper (Theorems 6.1 and 7.4).

For every $\beta \in \mathbb{R}$ set

$$Z_\beta := X^{\beta+(1/2)} \times X^\beta.$$

Endow Z_β with the (complete) scalar product

$$\langle (u_1, u_2), (v_1, v_2) \rangle_{Z_\beta} := \langle u_1, v_1 \rangle_{\beta+(1/2)} + \langle u_2, v_2 \rangle_\beta.$$

For every $\varepsilon \in \,]0, \infty[$ and $\beta \in \mathbb{R}$ define the operator $B_{\varepsilon,\beta}\colon Z_{\beta+(1/2)} \to Z_\beta$ by

$$B_{\varepsilon,\beta}(u, v) = (-v, (1/\varepsilon)(v + Au)), \quad (u, v) \in Z_{\beta+(1/2)}.$$

It is well-known ([12]) that $-B_{\varepsilon,\beta}$ is the infinitesimal generator of a C^0-group $e^{-tB_{\varepsilon,\beta}}$, $t \in \mathbb{R}$, of operators on Z_β. There are constants $M_{\varepsilon,\beta}$, $\alpha_{\varepsilon,\beta} \in \,]0, \infty[$ such that

$$|e^{-tB_{\varepsilon,\beta}}(u,v)|_{Z_\beta} \leq M_{\varepsilon,\beta}e^{-\alpha_{\varepsilon,\beta}t}|u|_{Z_\beta}, \quad t \geq 0,\ (u,v) \in Z_\beta. \tag{5.2}$$

Moreover, if β_1, $\beta_2 \in \mathbb{R}$ and $\beta_1 \leq \beta_2$ then

$$B_{\varepsilon,\beta_1}(u,v) = B_{\varepsilon,\beta_2}(u,v), \quad (u,v) \in Z_{\beta_2+(1/2)} \tag{5.3}$$

and

$$e^{-tB_{\varepsilon,\beta_1}}(u,v) = e^{-tB_{\varepsilon,\beta_2}}(u,v), \quad t \in [0,\infty[\,,\ (u,v) \in Z_{\beta_2}. \tag{5.4}$$

The following two propositions follow from results in [12].

Proposition 5.1. *Let $\varepsilon \in\,]0,\infty[$, $\beta \in \mathbb{R}$ be arbitrary, $J \subset \mathbb{R}$ be an interval and $g: J \to X^\beta$ be continuous. Moreover, $z: J \to Z_\beta$ be arbitrary. The following properties are equivalent:*

(a) *$z(t) = e^{-B_{\varepsilon,\beta}(t-t_0)}z(t_0) + \int_{t_0}^t e^{-B_{\varepsilon,\beta}(t-s)}(0,(1/\varepsilon)g(s))\,ds$ for all t_0 and $t \in J$ with $t_0 \leq t$,*

(b) *z is differentiable into $Z := Z_{\beta-(1/2)}$ and*

$$\partial^{(Z)}z(t) = -B_{\varepsilon,\beta-(1/2)}z(t) + (0,(1/\varepsilon)g(t)), \quad t \in J.$$

If, in addition, $J = \mathbb{R}$, $\sup_{t\in\mathbb{R}}|g(t)|_\beta < \infty$ and $\sup_{t\in\mathbb{R}}|z(t)|_{Z_\beta} < \infty$ then the following properties are equivalent:

(c) *$z(t) = \int_{-\infty}^t e^{-B_{\varepsilon,\beta}(t-s)}(0,(1/\varepsilon)g(s))\,ds$ for all $t \in \mathbb{R}$,*

(d) *z is differentiable into $Z := Z_{\beta-(1/2)}$ and*

$$\partial^{(Z)}z(t) = -B_{\varepsilon,\beta-(1/2)}z(t) + (0,(1/\varepsilon)g(t)), \quad t \in \mathbb{R}.$$

Proposition 5.2. *Let J be an interval in $\mathbb{R}$, $\varepsilon \in\,]0,\infty[$ be arbitrary and $g: J \to X$ be continuous. Suppose $z: J \to Z_0$, $z(t) = (u(t),v(t))$, $t \in J$, is differentiable into $Z := Z_{-(1/2)}$ with*

$$\partial^{(Z)}z(t) = -B_{\varepsilon,-(1/2)}z(t) + (0,(1/\varepsilon)g(t)), \quad t \in J.$$

For $c \in [0,\infty[$ define the function $V = V_{\varepsilon,c}: Z_0 \to \mathbb{R}$ by

$$V(u,v) = \frac{1}{2}|u|_{1/2}^2 + \frac{1}{2}\varepsilon|v|_0^2 + \varepsilon c\langle u,v\rangle_0, \quad (u,v) \in Z_0.$$

Then $V \circ z: J \to \mathbb{R}$ is continuously differentiable and, for $t \in J$,

$$\begin{aligned}(V\circ z)'(t) = &-(1-\varepsilon c)|v(t)|_0^2 - c|u(t)|_{1/2}^2 - c\langle u(t),v(t)\rangle_0 \\ &+ \langle g(t),v(t)\rangle_0 + c\langle g(t),u(t)\rangle_0.\end{aligned}$$

Let $\varepsilon \in]0,\infty[$ be arbitrary and $f\colon X^{1/2} \to X$ be a locally Lipschitzian map. Given an interval $J \subset \mathbb{R}$ and a continuous map $(u,v)\colon J \to Z_0$ we say that (u,v) is a *solution of the second-order equation*

$$\dot{u} = v, \quad \dot{v} = (1/\varepsilon)(-v - Au + f(u)), \quad (u,v) \in Z_0 = X^{1/2} \times X^0 \tag{5.5}$$

if, setting $B = B_{\varepsilon,0}$ and $z(t) := (u(t),v(t))$, $t \in J$, we have that

$$z(t) = e^{-B(t-t_0)}z(t_0) + \int_{t_0}^{t} e^{-B(t-s)}(0,(1/\varepsilon)f(u(s)))\,ds$$

for all t_0, $t \in J$ with $t_0 < t$. It is well-known (see e.g. [15]) that for every $(u_0,v_0) \in Z_\beta$ there is a unique maximally defined solution

$$(u,v) = (u,v)_{u_0,v_0}\colon [0,\omega_{u_0,v_0}[\to Z_0$$

of (5.5) satisfying $(u,v)_{u_0,v_0}(0) = (u_0,v_0)$. We have that $\omega_{u_0,v_0} \in]0,\infty]$. Setting

$$(u_0,v_0)\pi_{\varepsilon,f}t := (u,v)_{u_0,v_0}(t), \quad t \in [0,\omega_{u_0,v_0}[$$

we obtain a local semiflow $\pi_{\varepsilon,f}$ on Z_0.

Proposition 5.3. *Suppose $f\colon X^{1/2} \to X$ is Lipschitzian on bounded subsets of $X^{1/2}$. Let $\varepsilon \in]0,\infty[$ be arbitrary. Suppose $f(X^{1/2}) \subset P_n(X)$ for some $n \in \mathbb{N}$. Then the set $X_n := P_n(X^{1/2}) \times P_n(X)$ is positively invariant relative to the local semiflow $\pi := \pi_{\varepsilon,f}$. Let π_n be the restriction of π to X_n. π_n is the local semiflow on X_n generated by the ordinary differential equation*

$$\dot{u} = v, \quad \dot{v} = \frac{1}{\varepsilon}(-v - Au + f(u)), \quad (u,v) \in X_n. \tag{5.6}$$

Let $N \subset Z_0$ be closed and bounded. If N is an isolating neighbourhood relative to π, then $N_n := N \cap X_n$ is an isolating neighbourhood relative to π_n and

$$h(\pi,N) = h(\pi_n,N_n).$$

Proof. The proof is completely analogous to the proof of Proposition 4.2. Details are omitted. □

We can now state a first, abstract, smoothing result for full bounded solutions of $\pi_{\varepsilon,f}$.

Proposition 5.4. *Let $\varepsilon \in]0,\infty[$ be arbitrary and $f\colon X^{1/2} \to X$ be Lipschitzian on bounded subsets of $X^{1/2}$. Assume that the following hypothesis holds:*

(5.7) *Whenever (u,v) is a full bounded solution of $\pi_{\varepsilon,f}$, then the map $f \circ u$ is differentiable into X^{-1} and $\sup_{t\in\mathbb{R}} |\partial^{(X^{-1})}(f \circ u)(t)|_{-1} < \infty$.*

Then the following properties are satisfied:

(a) *For every $r \in]0,\infty[$ and $\beta \in [0,\infty[$ there is a $C(r,\beta) \in]0,\infty[$ such that whenever (u,v) is a full bounded solution of $\pi_{\varepsilon,f}$ such that $\partial^{(X^{-1})}(f \circ u)$*

is defined and continuous as a map from $\mathbb{R}$ *to* $X^{-\beta}$, $\sup_{t\in\mathbb{R}} |\partial^{(X^{-1})}(f \circ u)(t)|_{-\beta} \leq r$ *and* $\sup_{t\in\mathbb{R}} |u(t)|_{1/2} \leq r$ *then* (u, v, w) *is defined and continuous as a map from* $\mathbb{R}$ *into* $X^{-\beta+1} \times X^{-\beta+(1/2)} \times X^{-\beta}$ *and*

$$\sup_{t\in\mathbb{R}}(|u(t)|^2_{-\beta+1} + |v(t)|^2_{-\beta+(1/2)} + |w(t)|^2_{-\beta}) \leq C(r, \beta)^{1/2}.$$

Here, $w(t) := \partial^{(X^{-1/2})}v(t)$ *for all* $t \in \mathbb{R}$.

(b) *Whenever* $\beta \in [0, \infty[$, (u, v) *and* (u_k, v_k), $k \in \mathbb{N}$, *are full bounded solutions of* $\pi_{\varepsilon,f}$ *such that* $\partial^{(X^{-1})}(f \circ u)$ *and* $\partial^{(X^{-1})}(f \circ u_k)$, $k \in \mathbb{N}$, *are defined and continuous as maps from* $\mathbb{R}$ *to* $X^{-\beta}$,

$$\sup_{k\in\mathbb{N}} \sup_{t\in\mathbb{R}} |\partial^{(X^{-1})}(f \circ u_k)(t)|_{-\beta} < \infty$$

and if $(f \circ u_k)(t) \to (f \circ u)(t)$ *and* $\partial^{(X^{-1})}(f \circ u_k)(t) \to \partial^{(X^{-1})}(f \circ u)(t)$ *in* $X^{-\beta}$ *for every* $t \in \mathbb{R}$, *then* $(u_k(t), v_k(t), w_k(t)) \to (u(t), v(t), w(t))$ *in* $X^{-\beta+1} \times X^{-\beta+(1/2)} \times X^{-\beta}$ *for all* $t \in \mathbb{R}$. *Here,* $w(t) := \partial^{(X^{-1/2})}v(t)$ *and* $w_k(t) := \partial^{(X^{-1/2})}v_k(t)$ *for all* $k \in \mathbb{N}$ *and* $t \in \mathbb{R}$.

REMARK. The constant $C(r, \beta)$ also depends on ε, θ and n but since these latter numbers are fixed, we do not need to indicate this dependence explicitly.

PROOF. Let (u, v) be a full bounded solution of $\pi_{\varepsilon,f}$. Using Proposition 5.1 we see that (u, v) is continuously differentiable into $Z_{-(1/2)}$. Let $w := \partial^{(X^{-1/2})}v$. Since, again by Proposition 5.1,

$$\varepsilon w(t) = -v(t) - Au(t) + f(u(t)), \quad t \in \mathbb{R} \tag{5.8}$$

it follows from hypothesis (5.7) that, first, $\sup_{t\in\mathbb{R}} |(v(t), w(t))|_{Z_{-(1/2)}} < \infty$ and that w is differentiable into X^{-1} and

$$\varepsilon\partial^{(X^{-1})}w(t) = -w(t) - Av(t) + g(t), \quad t \in \mathbb{R}$$

where $g := \partial^{(X^{-1})}(f \circ u)$. Proposition 5.1 now implies that

$$(v(t), w(t)) = \int_{-\infty}^{t} e^{-(t-s)B_{\varepsilon,-1}}(0, (1/\varepsilon)g(s))\, ds, \quad t \in \mathbb{R}. \tag{5.9}$$

Now suppose that $r \in]0, \infty[$, $\beta \in [0, \infty[$, $\partial^{(X^{-1})}(f \circ u)$ is defined and continuous as a map from $\mathbb{R}$ to $X^{-\beta}$, $\sup_{t\in\mathbb{R}} |\partial^{(X^{-1})}(f\circ u)(t)|_{-\beta} \leq r$ and $\sup_{t\in\mathbb{R}} |u(t)|_{1/2} \leq r$. Then by (5.4) and (5.9)

$$(v(t), w(t)) = \int_{-\infty}^{t} e^{-(t-s)B_{\varepsilon,-\beta}}(0, (1/\varepsilon)g(s))\, ds, \quad t \in \mathbb{R}$$

so, first, (v,w) is defined and continuous as a map from $\mathbb{R}$ into $Z_{-\beta}$ and second, by (5.2),

$$\begin{aligned}\sup_{t\in\mathbb{R}}|(u(t),v(t))|_{Z_{-\beta}} &\le M_{\varepsilon,-\beta}(1/\alpha_{\varepsilon,-\beta})\sup_{s\in\mathbb{R}}(1/\varepsilon)|g(s)|_{-\beta}\\ &\le rM_{\varepsilon,-\beta}/(\varepsilon\alpha_{\varepsilon,-\beta}).\end{aligned}\tag{5.10}$$

Now (5.8) and (5.10) imply that for some constant $C'_\beta\in[0,\infty[$, independent of r or (v,w), and for all $t\in\mathbb{R}$

$$\begin{aligned}|Au(t)|_{-\beta} &\le |\varepsilon w(t)|_{-\beta}+|v(t)|_{-\beta}+|f(u(t))|_0\\ &\le C'_\beta(|(v(t),w(t))|_{Z_{-\beta}}+|f(u(t))|_0)\\ &\le C'_\beta(rM_{\varepsilon,-\beta}(1/\alpha_{\varepsilon,-\beta})+\sup_{|a|_{1/2}\le r}|f(a)|_0).\end{aligned}\tag{5.11}$$

Moreover, (5.8) also implies that u is continuous as a map from $\mathbb{R}$ into $X^{-\beta+1}$ and so (5.10) and (5.11) prove part (a) of the proposition. Let β, (u,v), and (u_k,v_k), $k\in\mathbb{N}$, satisfy the hypothesis of part (b) of the proposition. Set $g:=\partial^{(X^{-1})}(f\circ u)$, $w:=\partial^{(X^{-1/2})}v$ and $g_k:=\partial^{(X^{-1})}(f\circ u_k)$, $w_k:=\partial^{(X^{-1/2})}v_k$, $k\in\mathbb{N}$. By our assumption, part (a) of this proposition, and the dominated convergence theorem we have that, for all $t\in\mathbb{R}$,

$$\begin{aligned}&|(v_k(t),w_k(t))-(v(t),w(t))|_{-\beta}\\ &\qquad\le\frac{M_{\varepsilon,-\beta}}{\varepsilon}\int_{-\infty}^t e^{-(t-s)\alpha_{\varepsilon,-\beta}}|g_k(s)-g(s)|_{-\beta}\,ds\to 0.\end{aligned}\tag{5.12}$$

By (5.8) and (5.12) we have for all $t\in\mathbb{R}$

$$\begin{aligned}|A(u_k(t)-u(t))|_{-\beta}=&\,|v_k(t)-v(t)|_{-\beta}+|\varepsilon w_k(t)-\varepsilon w(t)|_{-\beta}\\ &+|f(u_k(t))-f(u(t))|_{-\beta}\to 0.\end{aligned}\tag{5.13}$$

Now (5.12) and (5.13) prove the second part of the proposition. □

Let us recall the following imbedding result for interpolation spaces.

Proposition 5.5. *Assume that $X=L^2(\Omega)$ and X^1 is continuously included in $H^2(\Omega)$. Let $\alpha\in[0,1]$ be arbitrary. Then the following statements hold:*

(a) *If $q\ge 2$ and $2\alpha>(N/2)-(N/q)$ then $X^\alpha\subset L^q(\Omega)$.*
(b) *If $2\alpha>N/2$ then $X^\alpha\subset C(\overline{\Omega})$.*
(c) *If $2\alpha>(N/q)-(N/2)$ then $L^q(\Omega)\subset X^{-\alpha}$.*

The maps induced by the above inclusions are continuous.

Proof. Parts (a) and (b) follow from Theorem 1.6.1 in [11]. Part (c) is obtained from part (a) by passing to dual spaces. □

We also require the following essentially known results.

Proposition 5.6. *Assume the following hypotheses:*

(a) $r \in]0,\infty[$ *and* $p_i \in [1,\infty[$, $i \in [\![1,4]\!]$, *are given numbers such that* $p_2 = p_1/r$, $p_3 \leq p_1$ *and* $1/p_4 = 1/p_2 + 1/p_3$.

(b) $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$, $(x,s) \mapsto g(x,s)$, *is such that* $g(\,\cdot\,,0) \in L^{p_4}(\Omega)$, $g(\,\cdot\,,s)$ *is measurable for every* $s \in \mathbb{R}$ *and* $g(x,\,\cdot\,)$ *is of class* C^1 *for every* $x \in \Omega$.

(c) $h(x,s) \equiv g'_s(x,s)$ *satisfies the estimate* $|h(x,s)| \leq a(x) + b|s|^r$ *for all* $(x,s) \in \Omega \times \mathbb{R}$, *where* $a \in L^{p_2}(\Omega)$ *and* $b \in [0,\infty[$.

(d) $I \subset \mathbb{R}$, $u\colon I \to L^{p_1}(\Omega)$, $t \in I$, u *is continuous at* t *as a map into* $L^{p_1}(\Omega)$ *and* u *is differentiable* (*resp. continuously differentiable*) *at* t *as a map into* $L^{p_3}(\Omega)$.

Then the map $\widehat{g} \circ u$ *is defined and differentiable* (*resp. continuously differentiable*) *at* t *as a map into* $L^{p_4}(\Omega)$, *where* $1/p_4 := 1/p_2 + 1/p_3$. *Finally,*

$$\partial^{(L^{p_4})}(\widehat{g} \circ u)(t) = \widehat{h}(u(t))\partial^{(L^{p_3})}u(t). \tag{5.14}$$

Proof. We have the estimate

$$|g(x,s)| \leq |g(x,0)| + |a(x)||s| + (b/(r+1))|s|^{r+1}, \quad (x,s) \in \Omega \times \mathbb{R}.$$

If $w \in L^{p_1}(\Omega)$ then, by Hölder inequality, $aw \in L^{p_5}(\Omega)$ where $1/p_5 = 1/p_2 + 1/p_1 \leq 1/p_2 + 1/p_3 = 1/p_4$ so $p_5 \geq p_4$. Moreover, $|w|^{r+1} \in L^{p_6}(\Omega)$, where $p_6 = p_1/(r+1)$, so $1/p_6 = r/p_1 + 1/p_1 = 1/p_2 + 1/p_1 = 1/p_5$. Thus $p_6 = p_5 \geq p_4$. Altogether we see that $\widehat{g}(w) \in L^{p_4}(\Omega)$ for every $w \in L^{p_1}(\Omega)$. Suppose now that u is differentiable at t (into $L^{p_3}(\Omega)$). For every $x \in \Omega$ and ξ with $t + \xi \in I$ set

$$\alpha_\xi(x) := \int_0^1 (h(x,(1-\theta)u(t)(x) + \theta u(t+\xi)(x)) - h(x,u(t)(x))\,d\theta. \tag{5.15}$$

Moreover, write $v := \partial^{(L^{p_3})}u(t)$. Then we easily obtain

$$\widehat{g}(u(t+\xi)) - \widehat{g}(u(t)) - \xi\widehat{h}(u(t))v = (\alpha_\xi + \widehat{h}(u(t)))(u(t+\xi) - u(t) - \xi v) + \xi\alpha_\xi v.$$

Thus, by the Hölder inequality, in order to prove the differentiability claim and formula (5.15) we only need to show that $\alpha_\xi \in L^{p_2}(\Omega)$ and $|\alpha_\xi|_{L^{p_2}} \to 0$ as $\xi \to 0$. Now, for every $\theta \in [0,1]$ the integrand of (5.15) is easily seen to be a measurable function of $x \in \Omega$. Since for every $x \in \Omega$

$$\alpha_\xi(x) = \lim_{m\to\infty}(1/m)\sum_{j=1}^{m}(h(x,(1-j/m)u(t)(x)+(j/m)u(t+\xi)(x))-h(x,u(t)(x))),$$

it follows that α_ξ is measurable.

Now

$$\begin{aligned}|\alpha_\xi(x)| &\leq \sup_{\theta\in[0,1]}|h(x,(1-\theta)u(t)(x) + \theta u(t+\xi)(x)) - h(x,u(t)(x))| \\ &\leq 2|a(x)| + 2b|u(t)(x)|^r + b|u(t+\xi)(x)|^r.\end{aligned} \tag{5.16}$$

It follows that $\alpha_\xi \in L^{p_2}(\Omega)$. Suppose that $|\alpha_\xi|_{L^{p_2}} \not\to 0$ as $\xi \to 0$. Then there is a sequence (ξ_n) converging to 0 and there is a $\delta \in]0,\infty[$ with

$$|\alpha_{\xi_n}|_{L^{p_2}} \geq \delta, \quad n \in \mathbb{N}. \tag{5.17}$$

We may assume that $u(t+\xi_n)(x) \to u(t)(x)$ for almost every $x \in \Omega$. (5.16) thus implies that, for almost every $x \in \Omega$, $\alpha_{\xi_n}(x) \to 0$ and, moreover,

$$|\alpha_\xi(x)|^{p_2} \leq C(|a(x)|^{p_2} + |u(t)(x)|^{p_1} + |u(t+\xi)(x)|^{p_1}) =: \zeta_\xi(x) \tag{5.18}$$

for some constant $C \in [0,\infty[$ independent of $x \in \Omega$. Now we use some classical results on equi-integrability (cf. [2]). Set $\zeta(x) := C(|a(x)|^{p_2} + 2|u(t)(x)|^{p_1})$, $x \in \Omega$. Then $\zeta_{\xi_n}(x) \to \zeta(x)$ for almost every $x \in \Omega$ and so $\zeta_{\xi_n} \to \zeta$ stochastically (Theorem 20.5 in [2]). Since $u(t+\xi) \to u(t)$ in $L^{p_1}(\Omega)$ as $\xi \to 0$, we have that $|u(t+\xi)|_{L^{p_1}} \to |u(t)|_{L^{p_1}}$ as $\xi \to 0$ and so $\int_\Omega |\zeta_\xi|\,dx \to \int_\Omega |\zeta|\,dx$ as $\xi \to 0$. Theorem 21.7 in [2] now implies that $\zeta_{\xi_n} \to \zeta$ in $L^1(\Omega)$ as $n \to \infty$, so, by Theorem 21.4 in [2], we have that the set $\{\zeta_{\xi_n} \mid n \in \mathbb{N}\}$ is equi-integrable. Thus formula (5.18) and the definition of equi-integrability shows that the set $\{|\alpha_{\xi_n}|^{p_2} \mid n \in \mathbb{N}\}$ is equi-integrable and thus, by Theorem 21.7 in [2], we obtain that $|\alpha_{\xi_n}|_{L^{p_2}} \to 0$ as $n \to \infty$, a contradiction to (5.17).

If u is continuously differentiable at t into L^{p_3} then, by what has been proved so far, for all $t' \in I$ lying in a neighbourhood of t we see that $\partial^{(L^{p_4})}(\widehat{g} \circ u)(t')$ exists and

$$\partial^{(L^{p_4})}(\widehat{g} \circ u)(t') = \widehat{h}(u(t'))\partial^{(L^{p_3})}u(t'). \tag{5.19}$$

Since, for $t' \to t$, $\partial^{(L^{p_3})}u(t') \to \partial^{(L^{p_3})}u(t)$ in $L^{p_3}(\Omega)$ and $\widehat{h}(u(t')) \to \widehat{h}(u(t))$ in $L^{p_2}(\Omega)$, it follows from (5.19) and Hölder's inequality that $\partial^{(L^{p_4})}(\widehat{g} \circ u)(t') \to \partial^{(L^{p_4})}(\widehat{g} \circ u)(t)$ in $L^{p_4}(\Omega)$ as $t' \to t$. This proves the proposition. □

Proposition 5.7. *Let $g\colon \mathbb{R} \to \mathbb{R}$, be a C^1-function and $h := g'$. Let $p \in [1,\infty[$, $I \subset \mathbb{R}$, $t \in I$ and $u\colon I \to C(\overline{\Omega})$ be a map such that u is continuous at t as a map into $C(\overline{\Omega})$ and u is differentiable* (*resp. continuously differentiable*) *at t as a map into $L^p(\Omega)$. Then the map $\widehat{g} \circ u$ is defined and differentiable* (*resp. continuously differentiable*) *at t as a map into $L^p(\Omega)$. Finally,*

$$\partial^{(L^p)}(\widehat{g} \circ u)(t) = \widehat{h}(u(t))\partial^{(L^p)}u(t). \tag{5.20}$$

Proof. For $h(x,s) \equiv h(s)$ define the functions α_ξ as in the proof of Proposition 5.6. It is easily seen that $\alpha_\xi \in C(\overline{\Omega})$ and $|\alpha_\xi|_{C(\overline{\Omega})} \to 0$ as $\xi \to 0$. Thus the arguments from the proof of Proposition 5.6 complete the proof of the present proposition. □

For the rest of this section we assume the following Standing Hypothesis.

(5.21) *$N \in \{1,2,3\}$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and such that $X = L^2(\Omega)$ and X^1 is continuously included in $H^2(\Omega)$;*

$\gamma \in L^2(\Omega)$ and $\phi\colon \mathbb{R} \to \mathbb{R}$ is a C^1-function such that, for $N \geq 2$, there are constants $\overline{r}$ and $\overline{C} \in [0,\infty[$ with $|\phi'(\xi)| \leq \overline{C}(1+|\xi|^{\overline{r}})$ for all $\xi \in \mathbb{R}$. If $N = 3$ then $\overline{r} < 2$.

Proposition 5.8. *Let $\Phi(s) := \int_0^s \phi(t)\,dt$, $t \in \mathbb{R}$. Let $\varepsilon \in]0,\infty[$, $\theta \in [0,1]$ and $n \in \mathbb{N}$ be arbitrary. Define $f = f_{n,\theta}\colon X^{1/2} \to X$ by*

$$f(u) := (1-\theta)(\widehat{\phi}(u) + \gamma) + \theta P_n(\widehat{\phi}(P_n u) + \gamma), \quad u \in X^{1/2}.$$

Then f is well-defined, Lipschitzian on bounded subsets of $X^{1/2}$ and compact. Moreover, let $F = F_{n,\theta}\colon X^{1/2} \to L^1(\Omega)$ be defined by

$$F(u) := (1-\theta)(\widehat{\Phi}(u) + \gamma u) + \theta(\widehat{\Phi}(P_n u) + \gamma P_n u), \quad u \in X^{1/2}.$$

Define the function $W_\varepsilon = W_{\varepsilon,n,\theta}\colon Z_0 \to \mathbb{R}$ by

$$W_\varepsilon(u,v) := \frac{1}{2}|u|_{1/2}^2 + \frac{1}{2}\varepsilon|v|_0^2 - \int_\Omega F(u(x))\,dx, \quad (u,v) \in Z_0.$$

Under these assumptions, whenever $J \subset \mathbb{R}$ is an interval and $z\colon J \to Z_0$, $z(t) = (u(t),v(t))$, $t \in J$, is a solution of $\pi_{\varepsilon,f}$ then the function $W_\varepsilon \circ z\colon J \to \mathbb{R}$ is continuously differentiable and

$$(W_\varepsilon \circ z)'(t) = -|v(t)|_0^2, \quad t \in J.$$

Proof. All statements of the proposition are known and easily proved. In particular, the assertions concerning f follow from our Standing Hypothesis (5.21), Proposition 5.5 and the fact that the inclusion $X^\beta \subset X^\alpha$ is compact whenever $0 \leq \alpha < \beta < 1$. □

We will now prove a generalization of a Haraux-Babin-Vishik smoothing result.

Theorem 5.9. *Let ε and f be as in Proposition 5.8. Then the following properties are satisfied:*

(a) *For every $r \in]0,\infty[$ there is a $C(r) \in]0,\infty[$ such that whenever (u,v) is a full bounded solution of $\pi_{\varepsilon,f}$ with $\sup_{t\in\mathbb{R}} |(u(t),v(t))|_{Z_0} \leq r$ then (u,v) lies in $Z_{1/2}$ and*

$$\sup_{t\in\mathbb{R}}(|(u(t),v(t))|_{Z_{1/2}} \leq C(r).$$

Moreover, the map $f \circ u$ is continuously differentiable from $\mathbb{R}$ into X. Setting $w := \partial^{(X^{-1/2})}v$ we have

$$\begin{aligned} \partial^{(X)}(f\circ u)(t) &= (1-\theta)\widehat{\phi'}(u(t))\cdot v(t) + \theta P_n(\widehat{\phi'}(P_n u(t))\cdot v(t)), && t \in \mathbb{R},\\ \varepsilon w(t) &= -v(t) - Au(t) + \partial^{(X)}(f\circ u)(t), && t \in \mathbb{R}. \end{aligned}$$

(b) *Whenever* (u,v) *and* (u_k,v_k), $k\in\mathbb{N}$, *are full bounded solutions of* $\pi_{\varepsilon,f}$ *such that* $\sup_{k\in\mathbb{N}}\sup_{t\in\mathbb{R}}|(u_k(t),v_k(t)|_{Z_0}<\infty$ *and* $(u_k(t),v_k(t))\to(u(t),v(t))$ *in* Z_0 *for every* $t\in\mathbb{R}$, *then* $(u_k(t),v_k(t))\to(u(t),v(t))$ *in* $Z_{1/2}$ *for every* $t\in\mathbb{R}$.

PROOF. We follow, in spirit, the proof method by Haraux ([10]). We first treat the case $N=1$. Then, by Proposition 5.5, $X^{1/2}\subset C(\overline{\Omega})$ with continuous inclusion. Let (u,v) be full bounded solution of $\pi_{\varepsilon,f}$. Then an application of Proposition 5.7 shows that $f\circ u$ is continuously differentiable into $X=L^2(\Omega)$ and

$$g(t)=(1-\theta)\widehat{\phi}'(u(t))\cdot v(t)+\theta P_n(\widehat{\phi}'(P_nu(t))\cdot v(t)),\quad t\in\mathbb{R} \tag{5.22}$$

where $g:=\partial^{(X)}(f\circ u)$. Thus hypothesis (5.7) of Proposition 5.4 is satisfied. Actually, (5.22) implies that for every $r\in\,]0,\infty[$ there is a $C_1(r)\in\,]0,\infty[$, $C_1(r)\ge r$, such that whenever $\sup_{t\in\mathbb{R}}|(u(t),v(t))|_{Z_0}\le r$ then

$$\sup_{t\in\mathbb{R}}|g(t)|_0\le C_1(r). \tag{5.23}$$

Let $C_2(r):=C(r',\beta)$, where $C(r',\beta)$ is as in Proposition 5.4, $r':=C_1(r)$ and $\beta:=0$. It follows from that proposition that (u,v) lies in $Z_{1/2}$ and

$$\sup_{t\in\mathbb{R}}(|u(t)|_1+|v(t)|_{1/2})\le C_2(r).$$

This clearly implies part (a) of the theorem. Let (u,v) and (u_k,v_k), $k\in\mathbb{N}$, satisfy the assumption of part (b) of this theorem. Then formula (5.22) implies that $\partial^{(X)}(f\circ u_k)(t)\to\partial^{(X)}(f\circ u)(t)$ in X for all $t\in\mathbb{R}$ and

$$\sup_{k\in\mathbb{N}}\sup_{t\in\mathbb{R}}|\partial^{(X)}(f\circ u_k)(t)|_0<\infty.$$

Now Proposition 5.4 implies that $(u_k(t),v_k(t))\to(u(t),v(t))$ in $Z_{1/2}$ for all $t\in\mathbb{R}$. Let us now consider the case $N=2$. Then, by Proposition 5.5, $X^{1/2}$ is continuously included in $L^p(\Omega)$ for every $p\in[2,\infty[$. Choose $p\in[2,\infty[$ so that $\overline{r}/p<(1/2)$. Then $(1/q):=(\overline{r}/p)+(1/2)<1$ and so Proposition 5.6 implies that $f\circ u$ is continuously differentiable into $L^q(\Omega)$ and

$$\partial^{(L^q(\Omega))}(f\circ u)(t)=(1-\theta)\widehat{\phi}'(u(t))\cdot v(t)+\theta P_n(\widehat{\phi}'(P_nu(t))\cdot v(t)), \tag{5.24}$$

for $t\in\mathbb{R}$. Since $N\overline{r}/p<1$ it is possible to choose $\beta\in[0,1]$ such that $N\overline{r}/p<2\beta<1$. This implies, by Proposition 5.5, that $L^q(\Omega)$ is continuously included in $X^{-\beta}$. Thus $f\circ u$ is continuously differentiable into $X^{-\beta}$ and (5.24) shows that $\sup_{t\in\mathbb{R}}|g(t)|_{-\beta}<\infty$ where $g:=\partial^{(X^{-\beta})}(f\circ u)$. In particular, hypothesis (5.7) of Proposition 5.4 holds and so u is defined and continuous as a map from $\mathbb{R}$ into X^α, where $\alpha:=-\beta+1$. Moreover, that proposition, together with formula (5.24) imply that for every $r\in\,]0,\infty[$ there is a $C_1(r)\in\,]0,\infty[$ such that whenever

$\sup_{t\in\mathbb{R}} |(u(t),v(t))|_{Z_0} \le r$ then $\sup_{t\in\mathbb{R}}(|u(t)|_\alpha + |v(t)|_0) \le C_1(r)$. Since $\alpha \in [0,1]$ and $2\alpha > 1 = N/2$ it follows that X^α is continuously included in $C(\overline{\Omega})$. Now proceeding as in the case $N=1$ we see that (u,v) lies in $Z_{1/2}$. Moreover, starting with $r' := C_1(r)$ we see that there is a $C_2(r') \in]0,\infty[$ such that whenever $\sup_{t\in\mathbb{R}}(|u(t)|_\alpha + |v(t)|_0) \le r'$ then $\sup_{t\in\mathbb{R}}(|(u(t),v(t))|_{Z_{1/2}} \le C_2(r')$. Setting $C(r) := C_2(r')$ we complete the proof of part (a) of the theorem. Let (u,v) and (u_k,v_k), $k\in\mathbb{N}$, satisfy the assumption of part (b) of this theorem. Then formula (5.24) implies that $\partial^{(X^{-\beta})}(f\circ u_k)(t) \to \partial^{(X^{-\beta})}(f\circ u)(t)$ in X for all $t\in\mathbb{R}$ and

$$\sup_{k\in\mathbb{N}}\sup_{t\in\mathbb{R}} |\partial^{(X^{-\beta})}(f\circ u_k)(t)|_0 < \infty.$$

Now Proposition 5.4 implies that $(u_k(t),v_k(t)) \to (u(t),v(t))$ in $X^\alpha\times X$ for all $t\in\mathbb{R}$. Proceeding as in case $N=1$ we now see that $(u_k(t),v_k(t)) \to (u(t),v(t))$ in $Z_{1/2}$ for all $t\in\mathbb{R}$. The proof of part (b) is complete.

Let us now consider the case $N=3$. Let us first assume that $\overline{r}<1$. Then $X^{1/2}$ is continuously included in $L^p(\Omega)$ for all $p\in[2,6[$. Since $(\overline{r}/6) < (1/6)$, we may choose $p\in[2,6[$ satisfying $(\overline{r}/p) < (1/6)$. Then $(1/q) := (\overline{r}/p) + (1/2) < 1$. Since $(N\overline{r}/p) = (3\overline{r}/p) < (1/2)$, it is possible to choose $\beta\in[0,1]$ such that $(N\overline{r}/p) < 2\beta < (1/2)$.

It follows that with $\alpha := -\beta+1$ we have $\alpha\in[0,1]$ and $2\alpha = -2\beta+2 > -(1/2)+2 = 3/2 = N/2$ so X^α is continuously included in $C(\overline{\Omega})$. Now the proof in the present case proceeds exactly as in the case $N=2$. Let us now turn to the proof of case $\overline{r}\ge 1$. We will show that there is a $\mu\in\mathbb{N}$ and there are finite sequences (α_m), $m\in[\![1,\mu+1]\!]$, in $[0,1]$, (p_m), $m\in[\![1,\mu]\!]$, in $[2,\infty[$ and (β_m), $m\in[\![1,\mu]\!]$, in $[0,1]$ such that $\alpha_1 = (1/2)$, $2\alpha_{\mu+1} > (N/2)$, $2\alpha_m > (N/2) - (N/p_m)$, $(\overline{r}/p_m) + (1/2) \le 1$, $2\beta_m > (N\overline{r}/p_m)$, $m\in[\![1,\mu]\!]$, and $\alpha_m = -\beta_{m-1}+1$ for $m\in[\![2,\mu+1]\!]$. Then a finite number of applications of the arguments from the preceding cases completes the proof of the theorem. Let $\delta\in]0,\infty[$ be arbitrary, to be specified later. Define the sequence (α_m), $m\in\mathbb{N}$, by induction, setting $\alpha_1 := (1/2)$ and $\alpha_{m+1} := \overline{r}\alpha_m + b$ where $b := 1-(N\overline{r}/4)-(\overline{r}\delta/2)-(\delta/2)$. Moreover, define the sequence (s_m), $m\in\mathbb{N}$ so that $2\alpha_m = (N/2) - Ns_m + \delta$, $m\in\mathbb{N}$. Finally, let (β_m), $m\in\mathbb{N}$ be defined so that $2\beta_m = N\overline{r}s_m + \delta$. We thus see that $\alpha_{m+1} = -\beta_m + 1$ for all $m\in\mathbb{N}$. It also follows that, for $\overline{r}=1$

$$\alpha_{m+1} = (1/2) + m((1/4)-\delta), \quad m\in\mathbb{N} \tag{5.25}$$

and, for $\overline{r}>1$,

$$\alpha_{m+1} = (1/2)\overline{r}^m + b(\overline{r}^m - 1)/(\overline{r}-1), \quad m\in\mathbb{N}. \tag{5.26}$$

Since $\overline{r}<2$ we see that, for all $\delta<(1/4)$ small enough, $(1/2)+(b/(\overline{r}-1)) > 0$ so, in this case the sequence (α_m) is increasing by (5.25) and (5.26). Thus the sequences (s_m) and (β_m) are decreasing. Since $\alpha_1 = (1/2)$ and $\overline{r} < 2$ we may arrange, by choosing δ even smaller, that $\overline{r}s_1 + (1/2) \le 1$ and $\beta_1 \in [0,(1/2)]$. It follows that $\overline{r}s_m + (1/2) \le 1$, $m \in \mathbb{N}$ and so, in particular, $s_m \le (1/2)$, $m \in \mathbb{N}$ (as $\overline{r} \ge 1$). Our choice of δ also implies that $\alpha_m \to \infty$ as $m \to \infty$. Thus there is a smallest $\mu \in \mathbb{N}$ such that $2\alpha_{\mu+1} > (N/2)$. Therefore $2\alpha_m \le (N/2)$ for all $m \in [\![1,\mu]\!]$. In particular $\alpha_m \in [0,1]$ and $s_m > 0$ for all $m \in [\![1,\mu]\!]$. Therefore $\beta_m \in [0,(1/2)]$ for $m \in [\![1,\mu]\!]$. Set $p_m := (1/s_m)$, $m \in [\![1,\mu]\!]$. It follows that $p_m \ge 2$ and $(\overline{r}/p_m) + (1/2) \le 1$, $m \in [\![1,\mu]\!]$. The theorem is proved. □

Corollary 5.10. *Let ε and f be as in Proposition 5.4. If $K \subset Z_0$ is compact in Z_0 and invariant relative to $\pi_{\varepsilon,f}$ then $K \subset Z_{1/2}$ and K is compact in $Z_{1/2}$. Moreover, if (u,v) is a full bounded solution of $\pi_{\varepsilon,f}$ and (s_k) is a sequence in $\mathbb{R}$ such that $(u(s_k),v(s_k)) \to (\overline{u},\overline{v})$ in Z_0 for some $(\overline{u},\overline{v}) \in Z_0$, then $(u(s_k),v(s_k)) \to (\overline{u},\overline{v})$ in $Z_{1/2}$.*

Proof. There is an $r \in \,]0,\infty[$ such that $|(\overline{u},\overline{v})|_{Z_0} \le r$ for all $(\overline{u},\overline{v}) \in K$. Let $(\overline{u}_k,\overline{v}_k)$, $k \in \mathbb{N}$, be an arbitrary sequence in K. Since K is compact in Z_0 and invariant relative to $\pi_{\varepsilon,f}$, it follows that K is strongly $\pi_{\varepsilon,f}$-admissible. Thus we may assume, by taking a subsequence, if necessary, that there is a sequence (u_k,v_k), $k \in \mathbb{N}$, of full solutions of $\pi_{\varepsilon,f}$ lying in K with $(u_k(0),v_k(0)) = (\overline{u}_k,\overline{v}_k)$, $k \in \mathbb{N}$ and $(u_k(t),v_k(t)) \to (u(t),v(t))$ in Z_0 for every $t \in \mathbb{N}$, where (u,v) is a full solution of $\pi_{\varepsilon,f}$ lying in K. It follows from Theorem 5.9 that (u,v) and (u_k,v_k), $k \in \mathbb{N}$, lie in $Z_{1/2}$ and $(u_k(t),v_k(t)) \to (u(t),v(t))$ in $Z_{1/2}$ for $t \in \mathbb{R}$. In particular, $(\overline{u}_k,\overline{v}_k) \to (u(0),v(0)) \in K$ in $Z_{1/2}$. This proves the first assertion.

The second assertion follows from the first one, noting that, by the compactness of f, every closed bounded subset of Z_0 is $\pi_{\varepsilon,f}$-admissible (cf. Theorem 5.3 in [5] and its proof) so the closure in Z_0 of a full bounded orbit of $\pi_{\varepsilon,f}$ is compact in Z_0 and invariant relative to $\pi_{\varepsilon,f}$. The corollary is proved. □

We will now prove an extension of a ε-uniform boundedness and smoothing result by Hale and Raugel.

Theorem 5.11. *Let $\varepsilon_0 \in \,]0,\infty[$ be arbitrary. Then, for every $r \in \,]0,\infty[$ there is a $C(r) = C(r,\varepsilon_0) \in \,]0,\infty[$ such that*

$$\sup_{t\in\mathbb{R}}(|u(t)|_1^2 + |v(t)|_{1/2}^2 + \varepsilon|w(t)|_0^2)^{1/2} \le C(r)$$

for all $\varepsilon \in \,]0,\varepsilon_0[$, $n \in \mathbb{N}$ and $\theta \in [0,1]$ and every full solution (u,v) of $\pi_{\varepsilon,f}$ with $\sup_{t\in\mathbb{R}}|(u(t),v(t))|_{Z_0} \le r$. Here, $w := \partial^{(X^{-(1/2)})}v$ and $f = f_{n,\theta}\colon X^{1/2} \to X$ is defined by $f(u) = (1-\theta)(\widehat{\phi}(u)+\gamma) + \theta P_n(\widehat{\phi}(P_n u)+\gamma)$.

In the proof we use arguments from the proof of Theorem 2.5 in [9]. We first need a lemma.

Lemma 5.12. *For every $r \in]0,\infty[$ there is a $C_1(r) \in]0,\infty[$ such that*

$$|\widehat{\phi}'(u)|^2_{L^\infty} \leq C_1(r)(1+|u|_1^2)$$

for all $u \in X^1$ with $|u|_{1/2} \leq r$.

Proof. Let $r \in]0,\infty[$ and $u \in X^1$ with $|u|_{1/2} \leq r$ be arbitrary.

Case 1. Let $N = 1$. Then $X^{1/2} \subset C(\overline{\Omega})$ with some embedding constant C. Thus $\widehat{\phi}'(u) \in C(\overline{\Omega}) \subset L^\infty(\Omega)$ and

$$|\widehat{\phi}'(u)|_{L^\infty} \leq \sup_{|s|\leq Cr} |\phi'(s)| =: C_2(r)$$

so

$$|\widehat{\phi}'(u)|^2_{L^\infty} \leq C_1(r)(1+|u|_1^2)$$

with $C_1(r) := C_2(r)^2$.

If $N \in \{2,3\}$ then the Gagliardo–Nirenberg inequality implies that, given p and θ with

$$p \in [1,\infty[,\ \theta \in [0,1] \text{ and } 0 < \theta(2-(N/2)) - (1-\theta)(N/p) \tag{5.27}$$

there is some constant $C(p,\theta) \in]0,\infty[$, independent of u, such that

$$|u|_{L^\infty} \leq C(p,\theta)|u|^\theta_{H^2}|u|^{(1-\theta)}_{L^p}. \tag{5.28}$$

Case 2. Let $N = 2$. Choose $\theta \in [0,1]$ with $\theta\overline{r} \leq 1$ and $p \in [2,\infty[$ with $\theta - (1-\theta)(2/p) > 0$. Then (5.27) is satisfied so estimate (5.28) implies that, for almost all $x \in \Omega$,

$$|u(x)|^{\overline{r}} \leq C(p,\theta)^{\overline{r}}|u|^{\theta\overline{r}}_{H^2}|u|^{(1-\theta)\overline{r}}_{L^p} \leq C(p,\theta)^{\overline{r}}|u|^{\theta\overline{r}}_{H^2}C_p^{(1-\theta)\overline{r}}|u|^{(1-\theta)\overline{r}}_{1/2} \leq C_2(r)|u|_1^{\theta\overline{r}}$$

where C' and C_p are embedding constants for the embeddings $X^1 \subset H^2(\Omega)$ and $X^{1/2} \subset L^p(\Omega)$, and $C_2(r) := C(p,\theta)^{\overline{r}}(C')^{\theta\overline{r}}(C_p r)^{(1-\theta)\overline{r}}$. Therefore, again for almost all $x \in \Omega$,

$$\begin{aligned}|\widehat{\phi}'(u)(x)|^2 &\leq |\overline{C}(1+|u(x)|^{\overline{r}}|^2 \leq 2\overline{C}^2(1+|u(x)|^{2\overline{r}}) \leq 2\overline{C}^2(1+C_2(r)|u|_1^{2\theta\overline{r}}) \\ &\leq 2\overline{C}^2(1+C_2(r)^2(1+|u|_1^2) \leq C_1(r)(1+|u|_1^2)\end{aligned}$$

where $C_1(r) := 2\overline{C}^2(1+C_2(r)^2)$.

Case 3. Let $N = 3$. Since $\overline{r} < 2$, it follows that there is a $\theta \in](1/2),1]$ with $\theta\overline{r} \leq 1$. Thus $\theta > 1-\theta$ so there is a $p \in [2,6[$ with $0 < \theta(1/2) - (1-\theta)(3/p)$. Using the fact that $X^{1/2}$ is continuously imbedded in $L^p(\Omega)$ we now complete the proof exactly as in Case 2. □

Proof of Theorem 5.11. Fix $\varepsilon_0 \in]0,\infty[$ arbitrarily and let $r \in]0,\infty[$, $\varepsilon \in]0,\varepsilon_0]$, $n \in \mathbb{N}$ and $\theta \in [0,1]$ be arbitrary. Let (u,v) be an arbitrary full solution of $\pi_{\varepsilon,f}$ with

$$\sup_{t\in\mathbb{R}} |(u(t),v(t))|_{Z_0} \leq r. \tag{5.29}$$

In the course of this proof we denote by $C_i(r)$, resp. $C_i(r,\varepsilon_0)$, $i\in\mathbb{N}$, real positive constants which depend on r, resp. on r and ε_0, but are independent of the choice of $\varepsilon\in\,]0,\varepsilon_0]$, $n\in\mathbb{N}$, $\theta\in[0,1]$ or the solution (u,v). Set $g:=\partial^{(X^{-1})}(f\circ u)$. It follows from Theorem 5.9 that

$$g(t)=(1-\theta)\widehat{\phi}'(u(t))v(t)+\theta P_n(\widehat{\phi}'(P_n u)P_n v(t)),\quad t\in\mathbb{R}. \tag{5.30}$$

Then, by Theorem 5.9, g is continuous into X, and $z:=(v,w)\colon\mathbb{R}\to Z_0$ is well-defined and differentiable into $Z:=Z_{-(1/2)}$ with

$$\partial^{(Z)}z(t)=-B_{\varepsilon,-(1/2)}z(t)+(0,(1/\varepsilon)g(t)),\quad t\in\mathbb{R}. \tag{5.31}$$

Now (5.30) implies that, for all $t\in\mathbb{R}$,

$$|g(t)|_0\le(|\widehat{\phi}'(u(t))|_{L^\infty}+|\widehat{\phi}'(P_n u(t))|_{L^\infty})|v(t)|_0$$

so, by Lemma 5.12,

$$\begin{aligned}|g(t)|_0^2&\le 2(|\widehat{\phi}'(u(t))|_{L^\infty}^2+|\widehat{\phi}'(P_n u(t))|_{L^\infty}^2)|v(t)|_0^2\\&\le 4C_1(r)(1+|u(t)|_1^2)|v(t)|_0^2.\end{aligned} \tag{5.32}$$

Since, by (5.31), $\varepsilon w(t)=-v(t)-Au(t)+f(u(t))$ for $t\in\mathbb{R}$, we see that

$$|u(t)|_1=|Au(t)|_0\le\varepsilon|w(t)|_0+|v(t)|_0+|f(u(t))|_0,\quad t\in\mathbb{R}$$

and

$$|\varepsilon w(t)|_0\le|v(t)|_0+|u(t)|_1+|f(u)|_0,\quad t\in\mathbb{R}.$$

Therefore, using (5.29) and the fact that $\sup_{a\in X^{1/2},\,|a|_{1/2}\le r}|f(a)|_0$ is independent of ε, n and θ, we finally obtain that

$$|u(t)|_1^2\le 2\varepsilon^2|w(t)|_0^2+C_2(r),\quad t\in\mathbb{R}, \tag{5.33}$$

$$\varepsilon^2|w(t)|_0^2\le 2|u(t)|_1^2+C_2(r),\quad t\in\mathbb{R}. \tag{5.34}$$

Thus, by (5.32) and (5.33), we have

$$\begin{aligned}|g(t)|_0^2&\le 4C_1(r)(1+2\varepsilon^2|w(t)|_0^2+C_2(r))|v(t)|_0^2\\&\le C_3(r,\varepsilon_0)\varepsilon|w(t)|_0^2|v(t)|_0^2+C_3(r,\varepsilon_0)|v(t)|_0^2.\end{aligned} \tag{5.35}$$

Let $c\in\,]0,(1/2)[$ be arbitrary and $V:=V_{\varepsilon,c}$ be defined as in Proposition 5.2. Since, for all $t\in\mathbb{R}$,

$$V(z(t))=\frac{1}{2}|v(t)|_{1/2}^2+\frac{1}{2}\varepsilon|w(t)|_0^2+\varepsilon c\langle v(t),w(t)\rangle_0$$

we have

$$V(z(t))\le\left(\frac{1}{2}+\frac{c}{2}\right)\varepsilon|w(t)|_0^2+\frac{1}{2}|v(t)|_{1/2}^2+\frac{c\varepsilon}{2}|v(t)|_0^2,\quad t\in\mathbb{R}, \tag{5.36}$$

$$V(z(t))\ge\left(\frac{1}{2}-\frac{c}{2}\right)\varepsilon|w(t)|_0^2+\frac{1}{2}|v(t)|_{1/2}^2-\frac{c\varepsilon}{2}|v(t)|_0^2,\quad t\in\mathbb{R}. \tag{5.37}$$

It follows from (5.37) that

$$\left(\frac{1}{2}-\frac{c}{2}\right)\varepsilon|w(t)|_0^2 \le V(z(t)) + \frac{c\varepsilon}{2}|v(t)|_0^2, \quad t\in\mathbb{R}$$

so (5.29) and our choice of c imply that

$$\varepsilon|w(t)|_0^2 \le 4V(z(t)) + \varepsilon r^2, \quad t\in\mathbb{R}.$$

Thus, by (5.35),

$$\begin{aligned}(5.38)\qquad |g(t)|_0^2 &\le C_3(r,\varepsilon_0)(4V(z(t))+\varepsilon r^2)|v(t)|_0^2 + C_3(r,\varepsilon_0)|v(t)|_0^2\\ &\le C_4(r,\varepsilon_0)(V(z(t))+1)|v(t)|_0^2.\end{aligned}$$

An application of (5.38) and Proposition 5.2 shows that $V\circ z$ is continuously differentiable and, for every $t\in\mathbb{R}$,

$$\begin{aligned}(5.39)\quad (V\circ z)'(t) =& -(1-\varepsilon c)|w(t)|_0^2 - c|v(t)|_{1/2}^2 - c\langle v(t),w(t)\rangle_0\\ &+\langle g(t),w(t)\rangle_0 + c\langle g(t),v(t)\rangle_0\\ \le& -(1-\varepsilon c)|w(t)|_0^2 - c|v(t)|_{1/2}^2 + \frac{c}{2}|v(t)|_0^2 + \frac{c}{2}|w(t)|_0^2\\ &+\frac{1}{2}|g(t)|_0^2 + \frac{1}{2}|w(t)|_0^2 + \frac{c}{2}|g(t)|_0^2 + \frac{c}{2}|v(t)|_0^2\\ \le& \left(-1+\varepsilon c+\frac{c}{2}+\frac{1}{2}\right)|w(t)|_0^2 - c|v(t)|_{1/2}^2 + c|v(t)|_0^2\\ &+\left(\frac{1}{2}+\frac{c}{2}\right)(C_4(r,\varepsilon_0)(V(z(t))+1)|v(t)|_0^2).\end{aligned}$$

Now let $k\in\,]0,\infty[$ be arbitrary. Then (5.36) and (5.39) imply

$$\begin{aligned}(5.40)\quad (V\circ z)'(t) \le& -kV(z(t)) + \left(-1+\varepsilon c+\frac{c}{2}+\frac{1}{2}+\frac{\varepsilon k}{2}+\frac{\varepsilon kc}{2}\right)|w(t)|_0^2\\ &+\left(-c+\frac{k}{2}\right)|v(t)|_{1/2}^2\\ &+\left(c+\left(\frac{1}{2}+\frac{c}{2}\right)C_4(r,\varepsilon_0)+\frac{\varepsilon kc}{2}\right)|v(t)|_0^2\\ &+\left(\frac{1}{2}+\frac{c}{2}\right)C_4(r,\varepsilon_0)V(z(t))|v(t)|_0^2,\end{aligned}$$

for $t\in\mathbb{R}$. We can choose the constants c and k, depending only on ε_0, such that the coefficients of the terms $|w(t)|_0^2$ and $|v(t)|_{1/2}^2$ in (5.40) are nonpositive. With this choice of c and k we have, for $t\in\mathbb{R}$,

$$(5.41)\qquad (V\circ z)'(t) \le -kV(z(t)) + C_5(r,\varepsilon_0)|v(t)|_0^2 + C_5(r,\varepsilon_0)V(z(t))|v(t)|_0^2.$$

By an elementary differential inequality we thus see that, for all t_0 and $t \in \mathbb{R}$ with $t_0 \le t$,

$$\begin{aligned}
V(z(t)) \le \bigg(&\exp\bigg(\int_{t_0}^t (-k + C_5(r,\varepsilon_0)|v(\rho)|_0^2)\,d\rho\bigg)\bigg) V(z(t_0)) \\
&+ \int_{t_0}^t \bigg(\exp\bigg(\int_s^t (-k + C_5(r,\varepsilon_0)|v(\rho)|_0^2)\,d\rho\bigg)\bigg) C_5(r,\varepsilon_0)|v(s)|_0^2\,ds \\
\le{}& \exp(-k(t-t_0) + C_5(r,\varepsilon_0)\int_{t_0}^\infty |v(s)|_0^2\,ds) V(z(t_0)) \\
&+ C_5(r,\varepsilon_0)\exp\bigg(C_5(r,\varepsilon_0)\int_{t_0}^\infty |v(s)|_0^2\,ds\bigg)\bigg(\int_{t_0}^\infty |v(s)|_0^2\,ds\bigg).
\end{aligned}$$

Let W_ε be as in Proposition 5.8. Then, by that proposition,

$$\int_{t_0}^t |v(s)|_0^2\,ds = W_\varepsilon((u(t_0),v(t_0)) - W_\varepsilon((u(t),v(t))) \le C_6(r,\varepsilon_0),$$

for $t_0, t \in \mathbb{R}$, $t_0 \le t$ so

$$\int_{t_0}^\infty |v(s)|_0^2\,ds \le C_6(r,\varepsilon_0), \quad t_0 \in \mathbb{R}.$$

Thus, for $t_0, t \in \mathbb{R}$, $t_0 \le t$,

$$V(z(t)) \le C_7(r,\varepsilon_0)\exp(-k(t-t_0))V(z(t_0)) + C_7(r,\varepsilon_0). \tag{5.42}$$

Now (5.36), (5.37) and (5.42) imply that

$$\begin{aligned}
\varepsilon|w(t)|_0^2 + |v(t)|_{1/2}^2 \le{}& C_8(r,\varepsilon_0)\exp(-k(t-t_0))(\varepsilon|w(t_0)|_0^2 + |v(t_0)|_{1/2}^2) \\
&+ C_8(r,\varepsilon_0), \quad t_0, t \in \mathbb{R},\ t_0 \le t.
\end{aligned}\tag{5.43}$$

By (5.33), (5.34) and (5.43) we see that

$$\begin{aligned}
\varepsilon|w(t)|_0^2 + {}&|v(t)|_{1/2}^2 + |u(t)|_1^2 \\
&\le (1+2\varepsilon_0)(\varepsilon|w(t)|_0^2 + |v(t)|_{1/2}^2) + C_2(r) \\
&\le C_9(r,\varepsilon_0)\exp(-k(t-t_0))(\varepsilon|w(t_0)|_0^2 + |v(t_0)|_{1/2}^2) + C_9(r,\varepsilon_0) \\
&\le C_9(r,\varepsilon_0)\exp(-k(t-t_0))((1/\varepsilon)(2|u(t_0)|_1^2 + C_2(r)) \\
&\quad + |v(t_0)|_{1/2}^2) + C_9(r,\varepsilon_0), \quad t_0, t \in \mathbb{R},\ t_0 \le t.
\end{aligned}\tag{5.44}$$

Let $t \in \mathbb{R}$ be arbitrary. Since $\pi_{\varepsilon,f}$ is gradient-like, there are an equilibrium $(\overline{u},\overline{v})$ of $\pi_{\varepsilon,f}$ and a sequence $(s_\nu)_\nu$ with $s_\nu \to \infty$ such that $(u(-s_\nu), v(-s_\nu)) \to (\overline{u},\overline{v})$ in Z_0. An application of Corollary 5.10 shows that $(u(-s_\nu), v(-s_\nu)) \to (\overline{u},\overline{v})$ in $Z_{1/2}$ so there is a $\nu \in \mathbb{N}$ such that, setting $t_0 := -s_\nu$, we have

$$t_0 \le t,\ e^{-k(t-t_0)}(1/\varepsilon) \le 1 \text{ and } |(u(t_0),v(t_0)) - (\overline{u},\overline{v})|_{Z_{1/2}} \le 1,$$

so that, by (5.44),

$$\begin{aligned}(5.45)\quad \varepsilon|w(t)|_0^2+|v(t)|_{1/2}^2+|u(t)|_1^2 &\leq C_{10}(r,\varepsilon_0)|(\overline{u},\overline{v})|_{Z_{1/2}}^2+C_{10}(r,\varepsilon_0)\\ &=C_{10}(r,\varepsilon_0)||f(\overline{u})|_0^2+C_{10}(r,\varepsilon_0)\leq C_{11}(r,\varepsilon_0).\end{aligned}$$

Here, we used the fact that $\overline{v}=0$ and $A\overline{u}=f(\overline{u})$ so that $|\overline{u}|_{1/2}=|(\overline{u},\overline{v})|_{Z_0}\leq r$ and so $|(\overline{u},\overline{v})|_{Z_{1/2}}=|\overline{u}|_1=|A\overline{u}|_0=|f(\overline{u})|_0\leq C_{12}(r)$.

Setting $C(r,\varepsilon_0):=C_{11}(r,\varepsilon_0)^{1/2}$ we complete the proof. □

We can now state the following important singular compactness result.

Theorem 5.13. *Define the maps f and $f_{n,\theta}$, $n\in\mathbb{N}$ and $\theta\in[0,1]$, from $X^{1/2}$ to X by*

$$f(u)=\widehat{\phi}(u)+\gamma,\quad u\in X^{1/2}$$

and

$$f_{n,\theta}(u)=(1-\theta)(\widehat{\phi}(u)+\gamma)+\theta P_n(\widehat{\phi}(P_nu)+\gamma),\quad u\in X^{1/2}.$$

Let $(\varepsilon_\kappa)_\kappa$, $(n_\kappa)_\kappa$ and $(\theta_\kappa)_\kappa$ be sequences in $]0,\infty[$, $\mathbb{N}$ and $[0,1]$, respectively. Suppose that $\varepsilon_\kappa\to 0$ and $n_\kappa\to\infty$. For each $\kappa\in\mathbb{N}$ let (u_κ,v_κ) be a full solution of $\pi_{\varepsilon_\kappa,f_{n_\kappa,\theta_\kappa}}$ such that

$$\sup_{\kappa\in\mathbb{N}}\sup_{t\in\mathbb{R}}|(u_\kappa(t),v_\kappa(t))|_{Z_0}=:r<\infty.$$

Then there is a subsequence of $((u_\kappa,v_\kappa))_k$, denoted $((u_\kappa,v_\kappa))_k$ again, and there is a full bounded solution u of π'_f such that $(u_\kappa,v_\kappa)\to(u,v)$ in Z_0, uniformly on compact subsets of $\mathbb{R}$. Here, $(u,v)=\Gamma\circ u$.

Proof. Fix $\varepsilon_0\in\,]0,\infty[$ with $\sup_{\kappa\in\mathbb{N}}\varepsilon_\kappa\leq\varepsilon_0$. Set $w_\kappa:=\partial^{(X^{-1/2})}v_\kappa$, $\kappa\in\mathbb{N}$. Then, by Theorem 5.11,

$$(5.46)\qquad \sup_{\kappa\in\mathbb{N}}\sup_{t\in\mathbb{R}}(|u_\kappa(t)|_1^2+|v_\kappa(t)|_{1/2}^2+\varepsilon_\kappa|w_\kappa(t)|_0^2)^{1/2}\leq C(r,\varepsilon_0).$$

Theorem 5.9 implies that, for every $\kappa\in\mathbb{N}$, (u_κ,v_κ) is continuous into $Z_{1/2}$ and so, since $v_\kappa=\partial^{(X)}u_\kappa$, we obtain that u_κ is continuously differentiable into $X^{1/2}$ with $v_\kappa=\partial^{(X^{1/2})}u_\kappa$. Since X^1 is compactly included in $X^{1/2}$ we thus obtain from (5.46) and the Arzelà–Ascoli theorem that there is a subsequence of $((u_\kappa,v_\kappa))_\kappa$, denoted $((u_\kappa,v_\kappa))_\kappa$ again, and a $u\in C(\mathbb{R}\to X^{1/2})$ such that $u_\kappa\to u$ in $X^{1/2}$, uniformly on compact subsets of $\mathbb{R}$. Since

$$\varepsilon_\kappa w_\kappa(t)=-v_\kappa(t)-Au_\kappa(t)+f_{n_\kappa,\theta_\kappa}(u_\kappa(t)),\quad \kappa\in\mathbb{N},\ t\in\mathbb{R}$$

and $f_{n_\kappa,\theta_\kappa}(a)\to f(a)$ in X, uniformly for a lying in compact subsets of $X^{1/2}$, we thus obtain from (5.46) that $v_\kappa\to v$ in $X^{-1/2}$, uniformly on compact subsets of $\mathbb{R}$, where $v\colon\mathbb{R}\to X^{-1/2}$ is defined by $v(t):=-Au(t)+f(u(t))$ for all $t\in\mathbb{R}$. It follows that u is differentiable into $X^{-1/2}$ and $\partial^{(X^{-1/2})}u=v$. We thus obtain

$$\partial^{(X^{-1/2})}u(t)=-Au(t)+f(u(t)),\quad t\in\mathbb{R}$$

and so, by a result analogous to Proposition 5.1,

$$u(t) = e^{-A(t-t_0)}u(t_0) + \int_{t_0}^{t} e^{-A(t-s)} f(u(s))\, ds, \quad t_0, t \in \mathbb{R},\ t_0 \le t.$$

Thus, by Lemma 3.3.2 in [11], u is a full solution of π'_f and so, in particular, $(u, v) = \Gamma \circ u$. □

6. The main result

In this section, we again assume our Standing Hypothesis (5.21). Define the map $f: X^{1/2} \to X$ by

$$f(u) = \widehat{\phi}(u) + \gamma, \quad u \in X^{1/2}.$$

Moreover, for $n \in \mathbb{N}$ and $\theta \in [0,1]$, let $f_{n,\theta}: X^{1/2} \to X$ be defined by

$$f_{n,\theta} = (1-\theta) f(u) + \theta P_n f(P_n u) = (1-\theta)(\widehat{\phi}(u) + \gamma) + \theta P_n(\widehat{\phi}(P_n u) + \gamma),$$

where $u \in X^{1/2}$. For $\varepsilon \in\,]0,\infty[$, $n \in \mathbb{N}$ and $\theta \in [0,1]$ set $\pi' := \pi'_f$, $\Gamma := \Gamma_f$, $\mathcal{T} := \mathcal{T}_f$, $\pi_\varepsilon := \pi_{\varepsilon,f}$, $\pi'_{n,\theta} := \pi'_{f_{n,\theta}}$, $\pi'_n := \pi'_{n,1}$, $\Gamma_n := \Gamma_{f_{n,\theta}}$, $\mathcal{T}_n := \mathcal{T}_{f_{n,1}}$, $\pi_{\varepsilon,n,\theta} := \pi_{\varepsilon,f_{n,\theta}}$ and $\pi_{\varepsilon,n} := \pi_{\varepsilon,n,1}$.

We can now state the first main result of this paper.

Theorem 6.1. *Let $K' \subset X^{1/2}$ be a compact isolated invariant set relative to π'. Then $K' \subset X^1$ and $K := \Gamma(K')$ is compact in Z_0 and K is a $\mathcal{T}$-isolated invariant set. Let $N \subset Z_0$ be any bounded $\mathcal{T}$-isolating neighbourhood of K. Then there is an $\varepsilon_0 > 0$ such that for all $\varepsilon \in\,]0,\varepsilon_0]$ the set N is an isolating neighbourhood relative to π_ε of an isolated invariant set K_ε, the Conley index $h(\pi_\varepsilon, K_\varepsilon)$ is defined and*

$$h(\pi_\varepsilon, K_\varepsilon) = h(\pi', K').$$

The family $(K_\varepsilon)_{\varepsilon \in [0,\varepsilon_0]}$, where $K_0 := K = \Gamma(K')$, is upper semicontinuous at $\varepsilon = 0$ in Z_0, i.e.

$$\lim_{\varepsilon \to 0} \sup_{y \in K_\varepsilon} \inf_{z \in K_0} |y - z|_{Z_0} = 0.$$

The family $(K_\varepsilon)_{\varepsilon \in [0,\varepsilon_0]}$ is asymptotically independent of N in the sense that whenever N_1 and N_2 are two $\mathcal{T}$-isolating neighbourhoods of K and

$$K^j_\varepsilon := \mathrm{Inv}_{\pi_\varepsilon}(N_j), \quad \varepsilon \in\,]0, \varepsilon_j]\,,\ j = 1, 2,$$

then there is an $\varepsilon' \in\,]0,\infty[$, $\varepsilon' \le \min\{\varepsilon_1, \varepsilon_2\}$, such that $K^1_\varepsilon = K^2_\varepsilon$, $\varepsilon \in\,]0,\varepsilon']$.

The proof of Theorem 6.1 follows from a series of lemmas.

Lemma 6.2. *K' is compact in X^1 and K is compact in Z_0 and is a $\mathcal{T}$-isolated invariant set.*

Proof. This follows from Theorem 4.10. □

LEMMA 6.3. *Let N' be a bounded isolating neighbourhood of K' relative to π'. Then there is an $n_1 \in \mathbb{N}$ such that for every $n \geq n_1$ and every $\theta \in [0,1]$ the set N' is an isolating neighbourhood relative to $\pi'_{n,\theta}$.*

PROOF. This follows by an application of Corollary 4.9. □

LEMMA 6.4. *There is an $n_2 \in \mathbb{N}$ and an $\varepsilon_2 > 0$ such that for all $n \geq n_2$, $\varepsilon \in]0, \varepsilon_2]$ and $\theta \in [0,1]$ the set N is an isolating neighbourhood relative to $\pi_{\varepsilon,n,\theta}$.*

PROOF. This follows by an application of Theorem 5.13. □

LEMMA 6.5. *There is an $n_3 \in \mathbb{N}$ such that for every $n \geq n_3$ the set N is $\mathcal{T}_n$-isolating neighbourhood of $K_n := \Gamma_n(K'_n)$, where N' is as in Lemma 6.3 and $K'_n := \mathrm{Inv}_{\pi'_n}(N')$.*

PROOF. This follows from Theorem 4.11. □

LEMMA 6.6. *For every $n \geq n_1$ the set $Y_n := P_n(X^{1/2}) = P_n(X)$ is positively invariant relative to π'_n, the set $N' \cap Y_n$ is an isolating neighbourhood relative to the restriction $\pi'_n|Y_n$ of π'_n to Y_n and*

$$h(\pi'_n, N') = h(\pi'_n|Y_n, N' \cap Y_n).$$

(Here N' is as in Lemma 6.3.) Moreover, in the notation of Section 3, $\pi'_n = \pi'_g$, where $g = g_n\colon Y_n \to Y_n$ is defined by

$$g(u) = -Au + P_n(\widehat{\phi}(u) + \gamma), \quad u \in Y_n.$$

PROOF. This follows from Proposition 4.2. □

LEMMA 6.7. *For every $n \geq n_2$ and every $\varepsilon \in]0, \varepsilon_2]$ the set $X_n := P_n(X^{1/2}) \times P_n(X) = P_n(X) \times P_n(X)$ is positively invariant relative to $\pi_{\varepsilon,n}$, the set $N \cap X_n$ is an isolating neighbourhood relative to the restriction $\pi_{\varepsilon,n}|X_n$ of $\pi_{\varepsilon,n}$ to X_n and*

$$h(\pi_{\varepsilon,n}, N) = h(\pi_{\varepsilon,n}|X_n, N \cap X_n).$$

Moreover, in the notation of Section 3, $\pi_{\varepsilon,n} = \pi_{\varepsilon,g}$, where g is defined as in Lemma 6.6.

PROOF. This follows from Proposition 5.3. □

PROOF OF THEOREM 6.1. Let $n_0 := \max\{n_1, n_2, n_3\}$. Fix $n \geq n_0$ arbitrarily. Since $K'_n \subset \mathrm{Int}_{Y_n}(N' \cap Y_n)$, $K_n \subset \mathrm{Int}_{X_n}(N \cap X_n)$, $\mathrm{Inv}_{\pi'_n|Y_n}(N' \cap Y_n) = \mathrm{Inv}_{\pi'_n}(N')$ and, in the notation of Section 3, $\mathrm{Inv}_{\mathcal{T}_g}(N \cap X_n) = \mathrm{Inv}_{\mathcal{T}_n}(N)$, an application of Theorem 3.9 shows that there is an $\varepsilon_0 \in]0, \varepsilon_2]$ such that

$$h(\pi_{\varepsilon,n}|X_n, N \cap X_n) = h(\pi'_n|Y_n, N' \cap Y_n), \quad \varepsilon \in]0, \varepsilon_0]. \tag{6.1}$$

Let $\varepsilon \in \,]0, \varepsilon_0]$ be arbitrary. Since $\pi' = \pi'_{n,0}$ and $\pi'_n = \pi'_{n,1}$, we obtain from Lemma 6.3 and the homotopy invariance of the Conley index (see [13] or [14]) that

$$h(\pi', N') = h(\pi'_n, N'). \tag{6.2}$$

Since $\pi_\varepsilon = \pi_{\varepsilon,n,0}$ and $\pi_{\varepsilon,n} = \pi_{\varepsilon,n,1}$, we obtain from Lemma 6.4 and the homotopy invariance of the Conley index that

$$h(\pi_\varepsilon, N) = h(\pi_{\varepsilon,n}, N). \tag{6.3}$$

(The applicability of the Conley index continuation theorem from [13] or [14] is easily justified. In particular, the admissibility conditions follow from the compactness of the map $f\colon X^{1/2} \to X$, cf Theorems 5.3 and 5.5 in [5].) By Lemma 6.7 we have

$$h(\pi_{\varepsilon,n}, N) = h(\pi_{\varepsilon,n}|X_n, N \cap X_n). \tag{6.4}$$

By Lemma 6.6 we have

$$h(\pi'_n, N') = h(\pi'_n|Y_n, N' \cap Y_n). \tag{6.5}$$

Now formulas (6.1), (6.3), (6.4), (6.5) and (6.2) imply that

$$h(\pi_\varepsilon, N) = h(\pi', N').$$

Since $\varepsilon \in \,]0, \varepsilon_0]$ is arbitrary, the first assertion assertion of the theorem is proved. If the family $(K_\varepsilon)_{\varepsilon \in [0,\varepsilon_0]}$ is not upper-semicontinuous at $\varepsilon = 0$ in Z_0 then there are a $\delta \in \,]0, \infty[$, a sequence (ε_k) in $]0, \varepsilon_0[$ with $\varepsilon_k \to 0$ and a sequence (y_k) with $y_k \in K_{\varepsilon_k}$ for every $k \in \mathbb{N}$ such that $\inf_{z \in K_0} |y_k - z|_{Z_0} \geq \delta$ for all $k \in \mathbb{N}$. Thus, for every $k \in \mathbb{N}$ there is a full solution (u_k, v_k) of π_{ε_k} lying in N such that

$$\inf_{z \in K_0} |(u_k(0), v_k(0)) - z|_{Z_0} \geq \delta, \quad k \in \mathbb{N}. \tag{6.6}$$

Since $\pi_\varepsilon = \pi_{\varepsilon,k,0}$ for every $\varepsilon \in \,]0, \varepsilon_0]$ and every $k \in \mathbb{N}$, an application of Theorem 5.13 shows that a subsequence of $((u_k, v_k))$, denoted $((u_k, v_k))$ converges in Z_0 to (u, v), uniformly on compact subsets of $\mathbb{R}$, where u is a full bounded solution of π' and $(u, v) = \Gamma \circ u$. It follows that $(u, v) \in \mathcal{T}$ and (u, v) lies in N. This implies that, in particular, $(u(0), v(0)) \in \operatorname{Inv}_{\mathcal{T}}(N) = K_0$, a contradiction to (6.6), proving the second assertion of the theorem.

If the third assertion of the theorem is not true, then there are two $\mathcal{T}$-isolating neighbourhoods N_1 and N_2 of K and a sequence (ε_k) in $]0, \infty[$ with $\varepsilon_k \to 0$ such that

$$\operatorname{Inv}_{\pi_{\varepsilon_k}}(N_1) \not\subset \operatorname{Inv}_{\pi_{\varepsilon_k}}(N_2), \quad k \in \mathbb{N}.$$

Thus, for every $k \in \mathbb{N}$ there is a full solution (u_k, v_k) of π_{ε_k} lying in N_1 such that

$$(u_k(0), v_k(0)) \notin N_2, \quad k \in \mathbb{N}. \tag{6.7}$$

Again an application of Theorem 5.13 shows that a subsequence of $((u_k, v_k))$, denoted $((u_k, v_k))$ again, converges in Z_0 to (u, v), uniformly on compact subsets of $\mathbb{R}$, where u is a full bounded solution of π' and $(u, v) = \Gamma \circ u$. It follows that $(u, v) \in \mathcal{T}$ and (u, v) lies in N_1. This implies that, in particular, $(u(0), v(0)) \in \mathrm{Inv}_{\mathcal{T}}(N_1) = K_0 \subset \mathrm{Int}_{Z_0}(N_2)$, so $(u_k(0), v_k(0)) \in N_2$ for all $k \in \mathbb{N}$ large enough, a contradiction to (6.7), proving the third assertion of the theorem. The proof is complete. $\square$

Specializing, in Theorem 6.1, to the Dirichlet problem (cf Example 4.1) we obtain, in particular, Theorem A from the Introduction.

7. Continuation of Morse decompositions

In this section we again assume the Standing Hypothesis (5.21). Let the map $f\colon X^{1/2} \to X$ again be defined by

$$f(u) = \widehat{\phi}(u) + \gamma, \quad u \in X^{1/2}.$$

We will prove that Morse decompositions of the invariant set K', relative to π'_f, continue to Morse decompositions of the invariant sets K_ε, relative to $\pi_{\varepsilon,f}$, for $\varepsilon > 0$ small.

We will first recall some relevant concepts. For details, see [6] and [7].

Let P be a finite set and $\prec$ be a strict order relation on P. A subset I of P is called a *$\prec$-interval* if i, $k \in I$, $j \in P$ and $i \prec j \prec k$ imply $j \in I$. By $I(\prec)$ we denote the set of all $\prec$-intervals.

Let (Y, d) be a metric space. Similarly as in [6] we endow the set $\mathcal{C} := C(\mathbb{R} \to Y)$ of continuous functions from $\mathbb{R}$ to Y with the topology of uniform convergence on compact subsets of $\mathbb{R}$. If π is a local semiflow on Y and $N \subset Y$ then we denote by $\mathcal{T}_{\pi,N}$ the set of all full solutions of π lying in N.

Recall the following definition.

Definition 7.1 ([8]). Let π be a local semiflow on Y and S be a compact invariant set relative to π. A family $(M_i)_{i\in P}$ of subsets of S is called a *$\prec$-ordered Morse decomposition of S (relative to π)* if the following properties hold:

(a) The sets M_i, $i \in P$, are closed, π-invariant and pairwise disjoint.
(b) For every full solution σ of π lying in S either $\sigma(\mathbb{R}) \subset M_k$ for some $k \in P$ or else there are k, $l \in P$ with $k \prec l$, $\alpha(\sigma) \subset M_l$ and $\omega(\sigma) \subset M_k$.

This concept can be generalized as follows:

Definition 7.2 ([7]). Let $\mathcal{T}$ be a subset of $\mathcal{C}$. A family $(M_i)_{i\in P}$ of subsets of Y is called a *$\prec$-ordered $\mathcal{T}$-Morse decomposition* if the following properties hold:

(a) The sets M_i, $i \in P$, are closed, $\mathcal{T}$-invariant and pairwise disjoint.

(b) For every $\sigma \in \mathcal{T}$ either $\sigma(\mathbb{R}) \subset M_k$ for some $k \in P$ or else there are k, $l \in P$ with $k \prec l$, $\alpha(\sigma) \subset M_l$ and $\omega(\sigma) \subset M_k$.

It is easily proved that, for π and S as in Definition 7.1, a family $(M_i)_{i\in P}$ of subsets of S is a $\prec$-ordered Morse decomposition of S (relative to π) if and only if $(M_i)_{i\in P}$ is a $\prec$-ordered $\mathcal{T}$-Morse decomposition, where $\mathcal{T} := \mathcal{T}_{\pi,S}$.

If A, $B \subset Y$ then the *$\mathcal{T}$-connection set* $\mathrm{CS}_{\mathcal{T}}(A,B)$ *from A to B* is the set of all points $y \in Y$ for which there is a $\sigma \in \mathcal{T}$ with $\sigma(0) = y$, $\alpha(\sigma) \subset A$ and $\omega(\sigma) \subset B$. If π, S are as in Definition 7.1 and $\mathcal{T} := \mathcal{T}_{\pi,S}$, then we write

$$\mathrm{CS}_{\pi,S}(A,B) := \mathrm{CS}_{\mathcal{T}}(A,B).$$

Definition 7.3. Let $(\mathcal{T}_\kappa)_\kappa$ be a sequence of subsets of $\mathcal{C}$ and $\mathcal{T} \subset \mathcal{C}$ be arbitrary. We say that $(\mathcal{T}_\kappa)_\kappa$ *converges to $\mathcal{T}$ in Y*, if for every sequence $(\kappa_n)_n$ in $\mathbb{N}$ with $\kappa_n \to \infty$ as $n \to \infty$ and every sequence $(\sigma_n)_n$ such that $\sigma_n \in \mathcal{T}_{\kappa_n}$ for all $n \in \mathbb{N}$ there is a subsequence $(\sigma_{n_m})_m$ and a $\sigma \in \mathcal{T}$ such that $\sigma_{n_m}(t) \to \sigma(t)$ in Y as $m \to \infty$, uniformly for t lying in compact subsets of $\mathbb{R}$.

We can now state the second main result of this paper.

Theorem 7.4. *For $\varepsilon \in\,]0,\infty[$ set $\pi' := \pi'_f$, $\pi_\varepsilon := \pi_{\varepsilon,f}$ and $\Gamma := \Gamma_f$. Let K' be a compact isolated invariant set relative to π' and $K := \Gamma(K')$. Let $\mathcal{T}$ be the set of all $(u,v) \in \mathcal{T}_f$ such that $(u(t),v(t)) \in K$ for all $t \in \mathbb{R}$. Moreover, let $(M'_i)_{i\in P}$ be a family of subsets of K' which is a Morse decomposition of K' relative to π' and let $M_i := \Gamma(M'_i)$, $i \in P$. For every $I \in I(\prec)$ set*

$$M'(I) := \bigcup_{i,j\in I} \mathrm{CS}_{\pi',K'}(M'_i, M'_j)$$

and $M(I) := \Gamma(M'(I))$. Then $(M_i)_{i\in P}$ is a $\prec$-ordered $\mathcal{T}$-Morse decomposition and

$$M(I) = \bigcup_{i,j\in I} \mathrm{CS}_{\mathcal{T}}(M_i, M_j), \quad I \in I(\prec). \tag{7.1}$$

Moreover, the sets K, M_i, $i \in P$ and $M(I)$, $I \in I(\prec)$, are $\mathcal{T}_f$-isolated invariant sets. Let N be a bounded $\mathcal{T}_f$-isolating neighbourhood of K, $N_i \subset N$ be a $\mathcal{T}_f$-isolating neighbourhood of M_i, $i \in P$, and $N_I \subset N$ be a $\mathcal{T}_f$-isolating neighbourhood of $M(I)$, $I \in I(\prec)$.

For $\varepsilon \in\,]0,\infty[$ set $K_\varepsilon := \mathrm{Inv}_{\pi_\varepsilon}(N)$, $M_{\varepsilon,i} := \mathrm{Inv}_{\pi_\varepsilon}(N_i)$, $i \in P$ and $M_\varepsilon(I) := \mathrm{Inv}_{\pi_\varepsilon}(N_I)$, $I \in I(\prec)$.

Then there is an $\varepsilon_0 \in\,]0,\infty[$ such that, for every $\varepsilon \in\,]0,\varepsilon_0]$, N (resp. N_i, resp. N_I) is an isolating neighbourhood of K_ε (resp. $M_{\varepsilon,i}$, resp. $M_\varepsilon(I)$) relative to

π_ε, for all $i \in P$ and all $I \in I(\prec)$. Moreover, the family $(M_{\varepsilon,i})_{i\in P}$ is a $\prec$-ordered Morse decomposition of K_ε and

$$M_\varepsilon(I) = \bigcup_{i,j\in I} \mathrm{CS}_{\pi_\varepsilon,K_\varepsilon}(M_{\varepsilon,i}, M_{\varepsilon,j}), \quad I \in I(\prec). \tag{7.2}$$

Proof. We see either directly or using Corollaries 3.5 and 3.6 in [7] that M'_i, $i \in P$ and $M'(I)$, $I \in I(\prec)$, are compact isolated invariant sets relative to π'. Thus, by Theorem 4.10, the sets K, M_i, $i \in P$, and $M(I)$, $I \in I(\prec)$, are $\mathcal{T}_f$-isolated invariant sets. Moreover, the sets M_i are closed in Z_0 (being compact in Z_0) and pairwise disjoint (Γ being one-to-one). If $i \in P$ and $(\overline{u}, \overline{v}) \in M_i$ are arbitrary, then $\overline{u} \in M'_i$ so there is a full solution u of π' lying in M'_i with $\overline{u} = u(0)$. Thus u lies in K' and so $z := \Gamma \circ u$ lies in $M_i \subset K$ and is an element of $\mathcal{T}_f$. Hence $z \in \mathcal{T}$, z lies in M_i and $z(0) = (\overline{u}, \overline{v})$. It follows that M_i is $\mathcal{T}$-invariant.

Let $(u, v) \in \mathcal{T}$ be arbitrary. Then u is a full solution of π' lying in K'. Thus either u lies in M'_k for some $k \in P$, which implies that (u, v) lies in M_k or else there are k, $l \in P$ with $k \prec l$, $\alpha(u) \subset M'_l$ and $\omega(u) \subset M'_k$. In the latter case it is clear from the continuity of Γ that $\alpha(u, v) \subset M_l$ and $\omega(u, v) \subset M_k$.

We have proved that $(M_i)_{i\in P}$ is a $\prec$-ordered Morse decomposition.

Now let i and $j \in P$ be arbitrary. We show that

$$\Gamma(\mathrm{CS}_{\pi',K'}(M'_i, M'_j)) = \mathrm{CS}_{\mathcal{T}}(M_i, M_j). \tag{7.3}$$

This immediately implies (7.1). If u is a full solution of π' lying in K' with $\alpha(u) \subset M'_i$ and $\omega(u) \subset M'_j$ then, clearly, $z := \Gamma \circ u \in \mathcal{T}_f$, z lies in K and $\alpha(z) \subset M_i$ and $\omega(z) \subset M_j$. Thus $z \in \mathrm{CS}_{\mathcal{T}}(M_i, M_j)$. Conversely, if $z = (u, v) \in \mathrm{CS}_{\mathcal{T}}(M_i, M_j)$, then u is a full solution of π' lying in K'. Clearly, $\alpha(u) \subset M'_i$ and $\omega(u) \subset M'_j$. Hence $z \in \Gamma(\mathrm{CS}_{\pi',K'}(M'_i, M'_j))$. This implies (7.1).

Now let N be a bounded $\mathcal{T}_f$-isolating neighbourhood of K, $N_i \subset N$ be a $\mathcal{T}_f$-isolating neighbourhood of M_i, $i \in P$, and $N_I \subset N$ be a $\mathcal{T}_f$-isolating neighbourhood of $M(I)$, $I \in I(\prec)$. Notice that

$$W \subset N \Rightarrow \mathrm{Inv}_{\mathcal{T}_f}(W) = \mathrm{Inv}_{\mathcal{T}}(W). \tag{7.4}$$

For $\varepsilon \in \,]0, \infty[$ set $K_\varepsilon := \mathrm{Inv}_{\pi_\varepsilon}(N)$, $M_{\varepsilon,i} := \mathrm{Inv}_{\pi_\varepsilon}(N_i)$, $i \in P$, and $M_{\varepsilon,I} := \mathrm{Inv}_{\pi_\varepsilon}(N_I)$, $I \in I(\prec)$; moreover, let $\mathcal{T}_\varepsilon := \mathcal{T}_{\pi_\varepsilon,N}$. It follows that

$$M_{\varepsilon,i} = \mathrm{Inv}_{\mathcal{T}_\varepsilon}(N_i), \quad i \in P, \tag{7.5}$$

$$M_{\varepsilon,I} = \mathrm{Inv}_{\mathcal{T}_\varepsilon}(N_I), \quad I \in I(\prec). \tag{7.6}$$

Now we claim that,

(7.7) whenever (ε_κ) is a sequence in $]0, \infty[$ with $\varepsilon_\kappa \to 0$ then $\mathcal{T}_{\varepsilon_k} \to \mathcal{T}$ in Z_0.

In fact, let (κ_n) be an arbitrary sequence in $\mathbb{N}$ with $\kappa_n \to \infty$ and, for every $n \in \mathbb{N}$ let (u_n, v_n) be a full solution of $\pi_{\varepsilon_{\kappa_n}}$ lying in N. Thus

$$\sup_{n\in\mathbb{N}} \sup_{t\in\mathbb{R}} |(u_n(t), v_n(t))|_{Z_0} < \infty$$

so, by Theorem 5.13, there is a subsequence $((u_{n_m}, v_{n_m}))_m$ of $((u_n, v_n))_n$ and there is a full bounded solution u of π' such that $(u_{n_m}, v_{n_m}) \to (u, v)$ in Z_0, uniformly on compact subsets of $\mathbb{R}$. Here, $(u, v) = \Gamma \circ u$. Since N is closed in Z_0, we see that (u, v) lies in N, so that (u, v) actually lies in K. Thus $(u, v) \in \mathcal{T}$ and (7.7) is proved.

We also claim that

(7.8) both $\mathcal{T}$ and $\mathcal{T}_\varepsilon$, $\varepsilon \in]0, \infty[$, are compact in $C(\mathbb{R} \to Z_0)$, translation and cut-and-glue invariant.

Let us assume (7.8) for a moment. Then an application of Theorem 6.1 and Theorem 3.3 in [7] together with (7.4), (7.7) and (7.8) shows that there is an $\varepsilon_0 \in]0, \infty[$ such that, for every $\varepsilon \in]0, \varepsilon_0]$, N is an isolating neighbourhood of K_ε (relative to π_ε) and N_i, resp. N_I, is a $\mathcal{T}_\varepsilon$-isolating neighbourhood of $M_{\varepsilon,i}$, resp. $M_\varepsilon(I)$, for all $i \in P$ and all $I \in I(\prec)$. Moreover, the family $(M_{\varepsilon,i})_{i\in P}$ is a $\prec$-ordered $\mathcal{T}_\varepsilon$-Morse decomposition and

$$M_{\varepsilon,I} = \bigcup_{i,j\in I} \mathrm{CS}_{\mathcal{T}_\varepsilon}(M_{\varepsilon,i}, M_{\varepsilon,j}), \quad I \in I(\prec). \tag{7.9}$$

Since

$$\mathcal{T}_\varepsilon = \mathcal{T}_{\pi,K_\varepsilon}, \quad \varepsilon \in]0, \infty[\,, \tag{7.10}$$

this completes the proof except for (7.8). Now (7.10) and Proposition 2.7 in [7] implies that, for $\varepsilon \in]0, \infty[$, the set $\mathcal{T}_\varepsilon$ is compact in $C(\mathbb{R} \to Z_0)$, translation and cut-and-glue invariant.

In order to prove the compactness of $\mathcal{T}$, let $((u_\kappa, v_\kappa))_\kappa$ be an arbitrary sequence in $\mathcal{T}$. Thus, for every $\kappa \in \mathbb{N}$, u_κ is a full solution of π' lying in K'. Now an application of Theorem 4.6 (with $f_\kappa \equiv f$) shows that there is a sequence (κ_n) in $\mathbb{N}$ with $\kappa_n \to \infty$ and there is a full solution u of π' lying in K' such that $u_{\kappa_n} \to u$ in X^1, uniformly on compact subsets of $\mathbb{R}$. Thus $(u_{\kappa_n}, v_{\kappa_n}) \to (u, v)$ in Z_0, uniformly on compact subsets of $\mathbb{R}$, where $(u, v) = \Gamma \circ u$. It follows that $(u, v) \in \mathcal{T}$ and so $\mathcal{T}$ is compact in $C(\mathbb{R} \to Z_0)$, as claimed.

The translation and cut-and-glue invariance of $\mathcal{T}$ is obvious. The proof is complete. $\square$

REMARK 7.5. Setting $M_{0,i} := M_i$ and $M_0(I) := M(I)$, $i \in P$ and $I \in I(\prec)$, we see, using the arguments from the proof of Theorem 6.1, that, for $i \in P$ and $I \in I(\prec)$, the families $(M_{\varepsilon,i})_{\varepsilon\in[0,\varepsilon_0]}$ and $(M_\varepsilon(I))_{\varepsilon\in[0,\varepsilon_0]}$ are upper-semicontinuous at $\varepsilon = 0$ in Z_0 and asymptotically independent of the choice of the isolating neighbourhoods N_i and N_I.

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