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FIXED POINT THEORY AND GENERALIZED LERAY–SCHAUDER ALTERNATIVES FOR APPROXIMABLE MAPS IN TOPOLOGICAL VECTOR SPACES

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Abstract. Some new fixed point theorems for approximable maps are obtained in this paper. Homotopy results, via essential maps, are also presented for approximable maps.

1. Introduction

This paper presents new fixed point and homotopy results for approximable maps. Our theory extends and complements results in [3], [4], [6], [7] and relies only on Brouwer's fixed point theorem.

For the remainder of this section we present some preliminaries which will be needed in this paper. Let X and Y be subsets of Hausdorff topological vector spaces E_1 and E_2 respectively and $F: X \to K(Y)$; here $K(Y)$ denotes the family of nonempty compact subsets of Y.

Given two open neighbourhoods U and V of the origins in E_1 and E_2 respectively, a (U,V)-approximative continuous selection ([4], [5]) of $F: X \to K(Y)$ is a continuous function $s: X \to Y$ satisfying

$$s(x) \in (F[(x+U)\cap X] + V) \cap Y \quad \text{for every } x \in X.$$

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We say $F: X \to K(Y)$ is *approachable* if it has a (U,V)-approximative continuous selection for every open neighbourhood U and V of the origins in E_1 and E_2 respectively. We let $\mathcal{A}_0(X,Y)$ denote the class of approachable maps. We say $F: X \to K(Y)$ is *approximable* if its restriction $F|_\Omega$ to any compact subset Ω of X admits a (U,V)-approximative continuous selection for every open neighbourhood U and V of the origins in E_1 and E_2 respectively. We let $\mathcal{A}(X,Y)$ denote the class of approximable maps.

The following elementary result was established in [3] using Schauder projections.

Theorem 1.1. *Let E be a Hausdorff locally convex topological vector space, $X \subseteq E$ and $F: X \to K(E)$ a compact map. Assume $F \in \mathcal{A}_0(X,E)$ takes its values in a convex compact subset K of E. Then $F \in \mathcal{A}_0(X,K)$.*

A nonempty subset X of a Hausdorff topological vector space E is said to be *admissible* if for every compact subset Ω of X and every neighbourhood V of 0, there exists a continuous map $h: \Omega \to X$ with $x - h(x) \in V$ for all $x \in \Omega$ and $h(\Omega)$ is contained in a finite dimensional subspace of E. X is said to be *q-admissible* if any nonempty compact, convex subset Ω of X is admissible.

2. Fixed point theory

This section presents new fixed point results for approximable maps. All our results rely on Brouwer's fixed point theorem. In addition in this section we present an essential map theory for approximable maps .

Fixed point theory for approachable maps defined on topological vector spaces was first discussed in [4]. The following fixed point result can be found in [4, Corollary 7.3].

Theorem 2.1. *Let X be a convex subset of a (Hausdorff) locally convex topological vector space E and let $F \in \mathcal{A}_0(X,X)$ be a upper semicontinuous compact map. Then F has a fixed point.*

Theorem 2.1 automatically leads to a fixed point result for approximable maps (an alternate proof will be provided using Brouwer's fixed point theory later in this section; see Remark 2.6(a)).

Theorem 2.2. *Let X be a convex subset of a (Hausdorff) complete locally convex topological vector space E and let $F \in \mathcal{A}(X,X)$ be a closed compact map. Then F has a fixed point.*

Proof. Let $\Omega = \overline{\mathrm{co}}\,(\overline{F(X)}) \subseteq X$. Note [10, p. 67] guarantees that Ω is compact (and of course convex). Note also that $F(\Omega) \subseteq \Omega$. In addition guarantees that $F|_\Omega$ is upper semicontinuous (see [2, p. 465]). Now since Ω is compact we have $F|_\Omega \in \mathcal{A}_0(\Omega, X)$, so in particular $F|_\Omega \in \mathcal{A}_0(\Omega, E)$. Theorem 1.1 guarantees that $F|_\Omega \in \mathcal{A}_0(\Omega, \Omega)$. The result now follows from Theorem 2.1. □

REMARK 2.3. In Theorem 2.2, $F \in \mathcal{A}(X, X)$ could be replaced by $F\colon X \to 2^X$ (here 2^X denotes the family of nonempty subsets of X) with $F \in \mathcal{A}(X, E)$.

REMARK 2.4. It is possible to consider a more general space E in Theorem 2.2 if one works in a larger class of maps (see [8], [9]).

Recently in [6] another type of fixed point result for approximable maps defined on Hausdorff topological vector spaces was presented. One of the main reasons for examining this approximable type map was that the proof of the fixed point result is elementary and just relies on Brouwer's fixed point theorem. For completeness we present the proof here.

THEOREM 2.5. *Let X be an admissible convex set in a Hausdorff topological vector space E and suppose $F\colon X \to 2^X$ is a closed compact map with $F \in \mathcal{A}(X, \overline{F(X)})$. Then F has a fixed point.*

PROOF. Let $\mathcal{N}$ be a fundamental system of neighbourhoods of the origin 0 in E and $V \in \mathcal{N}$. Let $C = \overline{F(X)}$. Now there exists a continuous function $h\colon C \to X$ and a finite dimensional subspace L of E with

$$y - h(y) \in V \text{ for all } y \in C \text{ and } h(C) \subseteq L. \tag{2.1}$$

Let $M = h(C)$ and $\Omega = co(M)$. Since M is a compact subset of $L \cap X$ it follows that Ω is a compact convex subset of $L \cap X$. Since $F|_\Omega\colon \Omega \to 2^C$ admits a (U, V)-approximative continuous selection, there exists a continuous function $s\colon \Omega \to C$ with

$$s(x) \in (F[(x + V) \cap \Omega] + V) \cap C \quad \text{for all } x \in \Omega. \tag{2.2}$$

Lets look at $h \circ s\colon \Omega \to \Omega$. Brouwer's fixed point theorem implies there exists $x_V \in \Omega$ with $h \circ s(x_V) = x_V$. Let $y_V = s(x_V)$. Now (2.1) and (2.2) imply

$$y_V - h(s(x_V)) = y_V - h(y_V) \in V$$

$$\text{and } s(x_V) - w_V \in V \text{ for some } w_V \in F(z_V) \text{ such that } z_V \in (x_V + V) \cap \Omega.$$

Since Ω is compact we may suppose $h(s(x_V)) = x_V \to x$ for some $x \in \Omega$. Thus $z_V \to x$ and $s(x_V) = y_V \to x$ and so $w_V \to x$. Since F is closed and $w_V \in F(z_V)$ we have $x \in F(x)$. □

REMARK 2.6. (a) In Theorem 2.3 if X is compact then $F \in \mathcal{A}(X, \overline{F(X)})$ could be replaced by $F \in \mathcal{A}(X, X)$. This is immediate since we can take $C = X$ in the proof of Theorem 2.5 and notice (2.2) is true since $F \in \mathcal{A}(X, X)$.

(b) Suppose E is a (Hausdorff) locally convex topological vector space and X a convex subset of E. Also suppose $F \in \mathcal{A}(X, X)$ is a closed compact map.

Notice X is admissible. Now let $C = \overline{\mathrm{co}}\overline{F(X)}$ in the proof of Theorem 2.5 and notice (2.2) is true from Theorem 1.1 (note $F \in \mathcal{A}(X,X)$). As a result we have a proof of Theorem 2.2 using Brouwer's fixed point theorem. Similarly the proof of Remark 2.3 follows if $F \in \mathcal{A}(X,X)$ is replaced by $F\colon X \to 2^X$ with $F \in \mathcal{A}(X,E)$.

Our next result removes the compactness assumption on the map F in Theorem 2.2.

Theorem 2.7. *Let X be a closed convex subset of a (Hausdorff) complete locally convex topological vector space E with $x_0 \in X$. Suppose $F \in \mathcal{A}(X,X)$ is a closed map with the following condition holding:*

$$A \subseteq X, A = \overline{\mathrm{co}}(\{x_0\} \cup F(A)) \text{ implies } A \text{ is compact.} \tag{2.3}$$

Then F has a fixed point.

Proof. Consider $\mathcal{F}$ the family of all closed, convex subsets C of X with $x_0 \in C$ and $F(x) \subseteq C$ for all $x \in C$. Note $\mathcal{F} \neq \emptyset$ since $X \in \mathcal{F}$. Let

$$C_0 = \bigcap_{C \in \mathcal{F}} C.$$

Notice C_0 is nonempty, closed and convex and $F\colon C_0 \to 2^{C_0}$ since if $x \in C_0$ then $F(x) \subseteq C$ for all $C \in \mathcal{F}$. Let

$$C_1 = \overline{\mathrm{co}}(\{x_0\} \cup F(C_0)). \tag{2.4}$$

Notice $F\colon C_0 \to 2^{C_0}$ together with C_0 closed and convex implies $C_1 \subseteq C_0$. Also $F(C_1) \subseteq F(C_0) \subseteq C_1$ from (2.4). Thus C_1 is closed and convex with $F(C_1) \subseteq C_1$. As a result $C_1 \in \mathcal{F}$, so $C_0 \subseteq C_1$. Consequently

$$C_0 = \overline{\mathrm{co}}(\{x_0\} \cup F(C_0)). \tag{2.5}$$

Now (2.3) implies C_0 is compact and notice (2.5) implies $F(C_0) \subseteq C_0$. Also [2, p. 465] guarantees that $F|_{C_0}$ is upper semicontinuous and in addition we have (note C_0 is compact) that $F|_{C_0} : C_0 \to 2^{C_0}$ is a compact map. Now since C_0 is compact we have $F|_{C_0} \in \mathcal{A}_0(C_0, X)$, so in particular $F|_{C_0} \in \mathcal{A}_0(C_0, E)$. Theorem 1.1 guarantees that $F|_{C_0} \in \mathcal{A}_0(C_0, C_0)$ and the result follows from Theorem 2.1. □

Remark 2.8. Theorem 2.7 extends Theorem 6 in [7]. It is also possible to consider a more general space E in Theorem 2.7 (see [1]) if one works in a larger class of maps.

Remark 2.9. In Theorem 2.7, $F \in \mathcal{A}(X,X)$ could be replaced by $F\colon X \to 2^X$ with $F \in \mathcal{A}(X,E)$.

It is also possible to obtain a result in the Hausdorff topological vector space setting using Theorem 2.5.

Theorem 2.10. *Let X be a q-admissible closed convex set in a Hausdorff topological vector space E with $x_0 \in X$. Suppose $F: X \to 2^X$ is a closed map with* (2.3) *holding. Also assume $F|_\Omega \in \mathcal{A}(\Omega, \overline{F(\Omega)})$ for any convex compact set Ω of X with $F(\Omega) \subseteq \Omega$. Then F has a fixed point.*

Proof. Let C_0 be as in Theorem 2.7. Notice C_0 is convex and compact with $F(C_0) \subseteq C_0$. Also since X is q-admissible we have that C_0 is admissible. In addition $F|_{C_0} \in \mathcal{A}(C_0, \overline{F(C_0)})$ and $F|_{C_0}$ is a closed map. Now apply Theorem 2.5. □

Next we present an essential map approach for compact approximable maps. Assume E is a Hausdorff topological vector space, C a closed convex admissible subset of E and U an open subset of C with $0 \in U$.

Definition 2.11. We say $F \in GA(\overline{U}, C)$ if $F: \overline{U} \to K(C)$ is a closed compact map with $F \in \mathcal{A}(\overline{U}, \overline{F(\overline{U})})$; here $\overline{U}$ denotes the closure of U in C.

Definition 2.12. We say $F \in GA_{\partial U}(\overline{U}, C)$ if $F \in GA(\overline{U}, C)$ with $x \notin F(x)$ for $x \in \partial U$; here ∂U denotes the boundary of U in C.

Definition 2.13. A map $F \in GA_{\partial U}(\overline{U}, C)$ is essential in $GA_{\partial U}(\overline{U}, C)$ if for every $G \in GA_{\partial U}(\overline{U}, C)$ with $G|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $x \in G(x)$.

Theorem 2.14. *Let E be a Hausdorff topological vector space, C a closed convex admissible subset of E and U an open subset of C with $0 \in U$. Then the zero map is essential in $GA_{\partial U}(\overline{U}, C)$.*

Proof. Let $\theta \in GA_{\partial U}(\overline{U}, C)$ with $\theta|_{\partial U} = \{0\}$. We must show there exists $x \in U$ with $x \in \theta(x)$. Let

$$J(x) = \begin{cases} \theta(x) & \text{for } x \in \overline{U}, \\ \{0\} & \text{for } x \in C \backslash \overline{U}. \end{cases}$$

Clearly $J: C \to K(C)$ is a closed compact map. We next show $J \in \mathcal{A}(C, \overline{J(C)})$. By the definition of J it suffices to show for any compact set Ω of $\overline{U}$ that $J|_\Omega: \Omega \to K(\overline{J(C)})$ is approachable. Let U_1 and V_1 be two open neighbourhoods of the origin and let $s: \Omega \to \overline{\theta(\overline{U})}$ be a (U_1, V_1) approximative continuous selection of $\theta|_\Omega$ i.e. $s(x) \in \overline{\theta(\overline{U})}$ and

$$s(x) \in \theta[(x + U_1) \cap \Omega] + V_1 = J[(x + U_1) \cap \Omega] + V_1 \qquad \text{for all } x \in \Omega.$$

Now for $x \in \Omega$ we have

$$s(x) \in \overline{\{y : y \in \theta(x) \text{ and } x \in \overline{U}\}} = \overline{J(C)},$$

so $J|_\Omega: \Omega \to K(\overline{J(C)})$ is approachable. Theorem 2.5 guarantees that there exists $x \in C$ with $x \in J(x)$. Note $x \in U$ since $0 \in U$. Thus $x \in \theta(x)$ and we are finished. $\square$

We now present a generalized Leray–Schauder alternative.

Theorem 2.15. *Let E be a Hausdorff topological vector space, C a closed convex admissible subset of E and U an open subset of C with $0 \in U$. Suppose $F \in GA(\overline{U}, C)$ with*

$$x \notin \lambda Fx \quad \text{for every } x \in \partial U \text{ and } \lambda \in (0,1]. \tag{2.6}$$

Then F is essential in $GA_{\partial U}(\overline{U}, C)$ (in particular F has a fixed point in U).

Proof. Let $H \in GA_{\partial U}(\overline{U}, C)$ with $H|_{\partial U} = F|_{\partial U}$. We must show H has a fixed point in U. Consider

$$B = \{x \in \overline{U} : x \in \lambda H(x) \text{ for some } \lambda \in [0,1]\}.$$

Now $B \neq \emptyset$ since $0 \in U$. Also B is closed (in C) and in fact compact since H is a closed compact map (note $\overline{U}$ is closed in C). In addition $B \cap \partial U = \emptyset$ since (2.6) holds and $H|_{\partial U} = F|_{\partial U}$ and $0 \in U$. Since C is a subset of a Hausdorff topological vector spaces it is completely regular and so there exists a continuous function $\mu: \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(B) = 1$. Define a map R_μ by $R_\mu(x) = \mu(x)H(x)$. Now $R_\mu: \overline{U} \to K(C)$ is closed and compact. In fact $R_\mu \in \mathcal{A}(\overline{U}, \overline{R_\mu(\overline{U})})$. To see this let U_1 and V_1 be two open neighbourhoods of the origin and without loss of generality assume V_1 is balanced. Let Ω be a compact subset of $\overline{U}$ and let $s: \Omega \to \overline{H(\overline{U})}$ be a (U_1, V_1) approximative continuous selection of $H|_\Omega$ i.e. $s(x) \in \overline{H(\overline{U})}$ and

$$s(x) \in H[(x + U_1) \cap \Omega] + V_1 \quad \text{for each } x \in \Omega.$$

Notice for each $x \in \Omega$ that

$$\mu(x)s(x) \in \mu(x)(H[(x + U_1) \cap \Omega] + V_1) \subseteq R_\mu[(x + U_1) \cap \Omega] + V_1$$

since $\mu(x) \in \mu([x + U_1] \cap \Omega)$ and V_1 is balanced. As a result

$$R_\mu(x) = \mu(x)s(x) \in R_\mu[(x + U_1) \cap \Omega] + V_1 \quad \text{for } x \in \Omega$$

and also we have for $x \in \Omega$ that

$$R_\mu(x) = \mu(x)s(x) \in \overline{\{\mu(x)y : y \in H(x) \text{ and } x \in \overline{U}\}} = \overline{R_\mu(\overline{U})}.$$

As a result $R_\mu: \Omega \to K(\overline{R_\mu(\overline{U})})$ is approachable, so $R_\mu \in \mathcal{A}(\overline{U}, \overline{R_\mu(\overline{U})})$. Thus $R_\mu \in GA(\overline{U}, C)$ with $R_\mu|_{\partial U} = \{0\}$. This together with Theorem 2.14 implies that there exists $x \in U$ with $x \in R_\mu(x)$. Thus $x \in B$ and so $\mu(x) = 1$, i.e. $x \in H(x)$. $\square$

For our next results we assume E is a complete locally convex topological vector space, C a closed convex subset of E and U an open subset of C with $0 \in U$.

Definition 2.16. We say $F \in GAA(\overline{U}, C)$ if $F\colon \overline{U} \to K(C)$ is a upper semicontinuous map with $F \in \mathcal{A}(\overline{U}, E)$ and which satisfies the following condition:

$$A \subseteq \overline{U},\ A \subseteq \overline{\mathrm{co}}(\{0\} \cup F(A)) \text{ implies } \overline{A} \text{ is compact.}$$

Remark 2.17. In the theory below, $F \in \mathcal{A}(\overline{U}, E)$ in Definition 2.16 could be replaced by $F \in \mathcal{A}(\overline{U}, C)$.

Definition 2.18. We say $F \in GAA_{\partial U}(\overline{U}, C)$ if $F \in GAA(\overline{U}, C)$ with $x \notin F(x)$ for $x \in \partial U$.

Definition 2.19. A map $F \in GAA_{\partial U}(\overline{U}, C)$ is essential in $GAA_{\partial U}(\overline{U}, C)$ if for every $G \in GAA_{\partial U}(\overline{U}, C)$ with $G|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $x \in G(x)$.

Theorem 2.20. *Let E be a complete locally convex topological vector space, C a closed convex subset of E and U an open subset of C with $0 \in U$. Then the zero map is essential in $GAA_{\partial U}(\overline{U}, C)$.*

Proof. Let $\theta \in GAA_{\partial U}(\overline{U}, C)$ with $\theta|_{\partial U} = \{0\}$. Let J be as in Theorem 2.14. Clearly $J\colon C \to K(C)$ is a upper semicontinuous map with $J \in \mathcal{A}(C, E)$. Next we claim the following holds:

$$\text{if } A \subseteq C \text{ with } A = \overline{\mathrm{co}}(\{0\} \cup J(A)) \text{ then } A \text{ is compact.} \tag{2.7}$$

If (2.7) holds then Theorem 2.7 (with Remark 2.9) guarantees that there exists $x \in C$ with $x \in Jx$. As in Theorem 2.14 we have $x \in U$ and we are finished.

It remains to show (2.7). To see this let $A \subseteq C$ with $A = \overline{\mathrm{co}}(\{0\} \cup J(A))$. Then

$$A \subseteq \overline{\mathrm{co}}(\{0\} \cup \theta(A \cap U)) \tag{2.8}$$

and so

$$A \cap U \subseteq \overline{\mathrm{co}}(\{0\} \cup \theta(A \cap U)).$$

Since $\theta \in GAA(\overline{U}, C)$ we have that $\overline{A \cap U}$ is compact, and since θ is upper semicontinuous we deduce that $\theta(\overline{A \cap U})$ is compact (see [2, p. 464]). This together with [10, p. 67] implies $\overline{\mathrm{co}}(\{0\} \cup \theta(\overline{A \cap U}))$ is compact, so (2.8) implies $\overline{A}(= A)$ is compact. Thus (2.7) holds. $\square$

Next we obtain a generalized Leray–Schauder alternative.

Theorem 2.21. *Let E be a complete locally convex topological vector space, C a closed convex subset of E and U an open subset of C with $0 \in U$. Suppose $F \in GAA(\overline{U}, C)$ with* (2.6) *holding. Then F is essential in $GAA_{\partial U}(\overline{U}, C)$.*

Proof. Let $H \in GAA_{\partial U}(\overline{U}, C)$ with $H|_{\partial U} = F|_{\partial U}$, and let B be as in Theorem 2.15. Notice $B \neq \emptyset$ is closed and in fact compact (since $B \subseteq \mathrm{co}(\{0\} \cup H(B)))$. Next let R_μ be as in Theorem 2.15. It is immediate that $R_\mu : \overline{U} \to K(C)$ is an upper semicontinuous map with $R_\mu \in \mathcal{A}(\overline{U}, E)$. We now claim

$$R_\mu \in GAA(\overline{U}, C). \tag{2.9}$$

If (2.9) is true then $R_\mu|_{\partial U} = \{0\}$ together with Theorem 2.20 implies that there exists $x \in U$ with $x \in R_\mu(x)$. Thus $x \in B$ so $\mu(x) = 1$ and we are finished.

It remains to show (2.9). Suppose $A \subseteq \overline{U}$ with $A \subseteq \overline{\mathrm{co}}(\{0\} \cup R_\mu(A))$. Then $R_\mu(A) \subseteq \mathrm{co}(\{0\} \cup H(A))$ together with $\{0\} \cup \mathrm{co}(\{0\} \cup H(A)) = \mathrm{co}(\{0\} \cup H(A))$ yields

$$A \subseteq \overline{\mathrm{co}}(\{0\} \cup R_\mu(A)) \subseteq \overline{\mathrm{co}}(\mathrm{co}(\{0\} \cup H(A))) = \overline{\mathrm{co}}(\{0\} \cup H(A)).$$

Since $H \in GAA(\overline{U}, C)$ we know $\overline{A}$ is compact, so (2.9) is true. □

Next we present a Leray–Schauder alternative for noncompact maps defined on Hausdorff topological vector spaces. One could derive a theory (similar to the one above) using Theorem 2.10. However here we present a different approach.

Theorem 2.22. *Let E be a Hausdorff topological vector space, C a closed convex subset of E and U an open subset of C with $0 \in U$. Suppose $F : \overline{U} \to K(C)$ is a upper semicontinuous map with $F \in \mathcal{A}(\overline{U}, \overline{F(\overline{U})})$. Let $C_0 = \bigcap_{D \in \mathcal{F}} D$ where $\mathcal{F}$ is the family of all closed, convex subsets D of X with $0 \in D$ and $F(D \cap \overline{U}) \subseteq D$. Assume the following conditions hold:*

$$A \subseteq E, A = \overline{\mathrm{co}}(\{0\} \cup F(A \cap \overline{U})) \text{ implies } A \cap \overline{U} \text{ is compact}, \tag{2.10}$$

$$C_0 \text{ is admissible}, \tag{2.11}$$

$$x \notin \lambda F x \text{ for every } x \in \partial_{C_0}(\mathrm{int}_{C_0}(C_0 \cap \overline{U})) \text{ and } \lambda \in (0, 1]. \tag{2.12}$$

Then F has a fixed point.

Proof. Consider $\mathcal{F}$ the family of all closed, convex subsets D of X with $0 \in D$ and $F(D \cap \overline{U}) \subseteq D$. Let $C_0 = \bigcap_{D \in \mathcal{F}} D$ and

$$C_1 = \overline{\mathrm{co}}(\{0\} \cup F(C_0 \cap \overline{U})). \tag{2.13}$$

Notice C_0 is closed and convex, $0 \in C_0$ and $F(C_0 \cap \overline{U}) \subseteq F(D \cap \overline{U}) \subseteq D$ for all $D \in \mathcal{F}$. Thus $C_0 \in \mathcal{F}$ and so $C_1 \subseteq C_0$ since $F(C_0 \cap \overline{U}) \subseteq C_0$. Also

$F(C_1 \cap \overline{U}) \subseteq F(C_0 \cap \overline{U}) \subseteq \overline{\text{co}}(\{0\} \cup F(C_0 \cap \overline{U})) = C_1$ so $C_1 \in \mathcal{F}$. As a result $C_0 \subseteq C_1$ so

$$C_0 = \overline{\text{co}}(\{0\} \cup F(C_0 \cap \overline{U})). \tag{2.14}$$

Now (2.10) implies

$$C_0 \cap \overline{U} \text{ is compact and } F(C_0 \cap \overline{U}) \subseteq C_0. \tag{2.15}$$

Also notice since $F \in \mathcal{A}(\overline{U}, \overline{F(\overline{U})})$ that $F|_{C_0 \cap \overline{U}}: C_0 \cap \overline{U} \to K(\overline{F(\overline{U})})$ is approachable. Consider the set

$$B = \{x \in C_0 \cap \overline{U} : x \in \lambda F x \text{ for some } \lambda \in [0,1]\}.$$

Note $C_0 \cap \overline{U}$ is compact so $B \neq \emptyset$ is a compact convex subset of C_0. For later, notice since F is upper semicontinuous we have [2, p. 464] that $F(C_0 \cap \overline{U})$ is compact. Also we have $B \cap \partial_{C_0}(\text{int}_{C_0}(C_0 \cap \overline{U})) = \emptyset$, so there exists a continuous function $\mu: C_0 \to [0,1]$ with $\mu(\partial_{C_0}(\text{int}_{C_0}(C_0 \cap \overline{U}))) = 0$ and $\mu(B) = 1$. Notice $\text{int}_{C_0}(C_0 \cap \overline{U}) = C_0 \cap \text{int}_{C_0}(\overline{U})$. Let $S: C_0 \to K(C_0)$ (note (2.14) implies $F(C_0 \cap \overline{U}) \subseteq C_0$) be defined by

$$S(x) = \begin{cases} \mu(x)F(x) & \text{for } x \in C_0 \cap \text{int}_{C_0}(\overline{U}) = \text{int}_{C_0}(C_0 \cap \overline{U}), \\ \{0\} & \text{for } x \in C_0 \setminus \text{int}_{C_0}(C_0 \cap \overline{U}). \end{cases}$$

Since $F(C_0 \cap \overline{U})$ is compact we have that S is a closed compact map. We now show $S \in \mathcal{A}(C_0, \overline{S(C_0)})$. Note $\overline{\text{int}_{C_0}(C_0 \cap \overline{U})}^{C_0} = C_0 \cap \overline{U}$. By the definition of S it suffices to show that $S|_{C_0 \cap \overline{U}}: C_0 \cap \overline{U} \to K(\overline{S(C_0)})$ is approachable. Now since $F|_{C_0 \cap \overline{U}}: C_0 \cap \overline{U} \to K(\overline{F(\overline{U})})$ is approachable then essentially the same reasoning as in Theorem 2.15 guarantees that $S|_{C_0 \cap \overline{U}}: C_0 \cap \overline{U} \to K(\overline{S(C_0)})$ is approachable. As a result $S \in \mathcal{A}(C_0, \overline{S(C_0)})$. Now Theorem 2.5 (see (2.11)) guarantees that S has a fixed point $x_0 \in C_0$. It is immediate that $x_0 \in \text{int}_{C_0}(C_0 \cap \overline{U})$ so $x_0 \in \mu(x_0)F(x_0)$. Thus $x_0 \in B$ so $\mu(x_0) = 1$, i.e. $x_0 \in F(x_0)$. □

REMARK 2.23. If (2.10) is changed to

$$A \subseteq E, A = \overline{\text{co}}(\{0\} \cup F(A \cap \overline{U})) \text{ implies } F(A \cap \overline{U}) \text{ is compact}, \tag{2.16}$$

then it is easy to see that F upper semicontinuous in Theorem 2.21 can be replaced by F a closed map.

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