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AN EXTENSION OF KRASNOSEL'SKII'S FIXED POINT THEOREM FOR CONTRACTIONS AND COMPACT MAPPINGS

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Abstract. Let X be a Banach space, Y a metric space, $A \subseteq X$, $C: A \to Y$ a compact operator and T an operator defined at least on the set $A \times C(A)$ with values in X. By assuming that the family $\{T(\,\cdot\,,y) : y \in C(A)\}$ is equicontractive we present two fixed point theorems for the operator of the form $Ex := T(x, C(x))$. Our results extend the well known Krasnosel'skiĭ's fixed point theorem for contractions and compact mappings. The results are used to prove the existence of (global) solutions of integral and integrodifferential equations.

1. Introduction

Contraction mappings shrink sets. Compact mappings send bounded sets into precompact sets. Making a combination of such two mappings on Banach spaces Krasnosel'skiĭ [14], [15] (see, also [11]) proved a fixed point theorem which states as follows:

Theorem 1.1 (Krasnosel'skiĭ). *Suppose A is a closed bounded convex subset of a Banach space X. If $T: A \to X$ is a contraction, $C: A \to X$ is compact and $T(A) + C(A) := \{z = T(x) + C(y),\ x, y \in A\} \subseteq A$, then $T + C$ has a fixed point in A.*

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Reinerman ([19]) extented this result for the case of A as a uniformly convex subset of the Banach space X. Also, Burton et al. ([3]) combined Theorem 1.1 with Schaefers' fixed point theorem ([23] or Smart [24, p. 29]) and gave an analogous fixed point theorem. Dhage et al. in [6] formulated a fixed point theorem of an operator of the form $T := L \cdot B + C$, where L is Lipschitz and B, C compact. Notice that although L is a Lipschitz operator, in the proof of the main theorem it is a contraction, see Remark 2.4 below.

In this paper we give an extension of Theorem 1.1 by assuming that the effect of the two operators involved (namely the contracting operator and the compact operator) on the resulting operator is implicit and not necessarily linear. More precisely, consider a (real) Banach space X with norm $\|\cdot\|$ and a complete metric space Y with metric $d(\,\cdot\,,\,\cdot\,)$. Take a set $A \subseteq X$, a mapping $C\colon A \to C(A) \subseteq Y$ and a mapping T defined at least on the set $A \times C(A)$ and having range in X. We are interested in the existence of a point $x \in A$ such that

$$x = T(x, C(x)). \tag{1.1}$$

Let us agree on the terminology: An operator $C\colon A \to X$ is said to be *compact* if it is continuous and maps bounded sets into precompact sets. (The old term *completely continuous* was brought about by the fact that in case of normed vector spaces a linear operator which maps bounded sets into precompact sets is continuous.) However, concerning the same terminology Krasnosel'skiĭ is in his survey [14, p. 370] using the convention of Smart [24, p. 25] to mean that the operator C is continuous and the set $C(A)$ belongs to a compact set, where A need not be bounded. We shall use this meaning in our first theorem.

We return to our subject. The basic assumption on the operator T is that for each y the operator $x \to T(x, y)$ is a contraction having the same contracting constant uniform for all y. We shall give two fixed point theorems which extend Theorem 1.1 and [6]. Our first theorem is proved by means of the classical Schauder fixed point theorem, while the second one uses the Darbo's theorem for k-set contractions involving the Kuratowski measure of noncompactness.

In an attempt to justify our results, we present two applications from the theory of nonlinear integral equations and integrodifferential equations.

2. The main results

Let us first recall the meaning of equicontractivity (see, also, [7, p. 497]):

Definition 2.1. The family $\{T(\,\cdot\,, y) : y\}$ is called *equicontractive*, if there is a $k \in [0, 1)$ such that

$$\|T(x_1, y) - T(x_2, y)\| \le k\|x_1 - x_2\|,$$

for all $(x_1, y), (x_2, y)$ in the domain of T.

Our first main result, where compact operators are considered in the sense of Krasnosel'skiĭ [14, p. 370], is the following:

THEOREM 2.2. *Suppose A is a closed convex subset of the Banach space X and $C: A \to Y$ a continuous operator such that $C(A)$ is a precompact subset of Y. Let*

$$T: A \times \overline{C(A)} \to A \tag{2.1}$$

be a continuous operator such that the family $\{T(\,\cdot\,, y) : y \in \overline{C(A)}\}$ is equicontractive. Then the operator equation (1.1) *admits a solution in A.*

PROOF. We start from an arbitrary point $y \in \overline{C(A)}$. Since the operator $x \to T(x, y): A \to A$ is a contraction, there is a unique point $x = F(y) \in A$ that satisfies the operator equation

$$T(F(y), y) = F(y).$$

We shall show that the mapping $y \to F(y): \overline{C(A)} \to A$ is continuous. To do this we let (y_n) be a sequence in $\overline{C(A)}$, with $\lim y_n = y_0 \in \overline{C(A)}$. We then observe that

$$\begin{aligned}
\|F(y_n) - F(y_0)\| &= \|T(F(y_n), y_n) - T(F(y_0), y_0)\| \\
&\le \|T(F(y_n), y_n) - T(F(y_0), y_n)\| \\
&\quad + \|T(F(y_0), y_n) - T(F(y_0), y_0)\| \\
&\le k\|F(y_n) - F(y_0)\| + \|T(F(y_0), y_n) - T(F(y_0), y_0)\|,
\end{aligned}$$

and therefore

$$\|F(y_n) - F(y_0)\| \le (1 - k)^{-1}\|T(F(y_0), y_n) - T(F(y_0), y_0)\|.$$

This proves the continuity of F.

We see that the operator $F \circ C$ maps the set A into itself and it is continuous. (The proof given by Kreyszig in [16, Problem 15, pp. 412, 656] is valid for metric spaces.) Let P be the closed convex hull of the set $F \circ C(A)$. Then P is a closed convex subset of A and moreover according to a classical theorem of Mazur (see e.g. [9, p. 416]) P is compact. We also have $F \circ C(P) \subseteq P$. Thus Schauder's fixed point theorem (or the Schauder's Principle) [13, p. 640] implies the existence of a point $\overline{x} \in P \subseteq A$ such that $F(C(\overline{x})) = \overline{x}$. This means that

$$T(\overline{x}, C(\overline{x})) = T(F(C(\overline{x})), C(\overline{x})) = F(C(\overline{x})) = \overline{x}. \qquad \square$$

REMARK 2.3. We emphasize that in Theorem 2.2 the set A is not necessarily bounded as in Theorem 1.1. All we need is that the image $C(A)$ is precompact.

REMARK 2.4. Instead of the closed convex set A, we can consider a complete absolute retract and C compact in the previous sense, i.e. mapping bounded sets into precompact sets. Then following the same procedure as in the proof of Theorem 2.2 one can prove the existence of a solution of equation (2.1), where Schauder's fixed point theorem concerning continuous images of convex compact sets [13, p. 640] is replaced by the generalized Schauder fixed point theorem concerning absolute retracts [8, Chapter 2].

In most of the cases when dealing with the existence of solutions of operator equations the basic condition (2.1) is satisfied. In what follows this condition is replaced by the following:

$$T(x, C(x)) \in A, \quad x \in A. \tag{2.2}$$

Condition (2.1) implies (2.2), but the inverse is not true. Indeed, take $A := [0, 1]$, $Y := \mathbb{R}$, $C(x) = 1 - 0.5x$ and $T(x, y) := 0.5x + y$. Then we have $T(x, C(x)) = 1 \in A$, for all $x \in A$, while $T(1, C(0)) = T(1, 1) = 1.5 \notin A$.

THEOREM 2.5. *Suppose A is a closed convex bounded subset of the Banach space X and $C\colon A \to Y$ a compact operator. Also suppose that $T\colon A \times C(A) \to A$ is an operator such that the family $\{T(\,\cdot\,, y) : y \in C(A)\}$ is equicontractive and for each $x \in A$ the function $T(x, \cdot\,)$ is uniformly continuous. If* (2.2) *holds, then the operator equation* (1.1) *admits a solution in A.*

To prove Theorem 2.5 we shall make use of Kuratowski's well known measure of noncompactness: Let B be a bounded subset of a Banach space X. *Kuratowski's measure $\alpha(B)$ of noncompactness of B* is defined by

$$\alpha(B) := \inf\{\varepsilon > 0 : B \text{ has a finite cover by sets of diameter smaller than } \varepsilon\}$$

(see [17, Vol. I, p. 318]).

For the basic properties of the measure of noncompactness one can consult e.g. [2], [5], [8], [12], [17], [19], [22]. A nice use of this (universal) function was given by Ambrosetti in [2], who presented a connection of the measure of noncompactness of a set of functions with the measure of noncompactness of their ranges. His result has generalized by Goebel et al. ([10]) and it found a wide use in the literature. A significant application was given by Rzepecki in [20]. Here we need to recall that for arbitrary bounded subsets B, B_1, B_2 of X the following properties hold:

(a) $\alpha(B) = 0$, if and only if B is a relatively compact set.
(b) $\alpha(B_1 \cup B_2) = \max\{\alpha(B_1), \alpha(B_2)\}$.
(c) $\alpha(\overline{B}) = \alpha(B)$.

The following result, which is the basis of the proof of Theorem 2.5, was originally proved by Darbo in [3] (see, also, Potter [18]) and refers to the so-called k-set contractions, or α-contractions (Hale et al. [12, p. 113]): Given a real number $k \in [0, 1)$, an operator $E\colon M \to N$ (M, N metric spaces) is said to be a *k-set contraction*, if E is continuous and satisfies $\alpha(E(B)) \le k\alpha(B)$ for all bounded subsets B of M.

Lemma 2.6 (Darbo's fixed point theorem for k-set contractions). *Let A be a closed convex bounded subset of a Banach space X and suppose $E\colon A \to A$ is a k-set contraction, $k < 1$. Then there exists an $x \in A$ such that $E(x) = x$.*

Proof of Theorem 2.5. We see that the operator

$$E(x) := T(x, C(x)), \quad x \in A$$

maps the set A into itself. We shall show that E has a fixed point in A.

Consider a set $B \subseteq A$ and an arbitrary real number $\varepsilon > 0$. The uniform continuity guarantees that there is a $\delta(\varepsilon) > 0$ such that for all $x \in A$ and $y_1, y_2 \in C(A)$ it holds $\|T(x, y_1) - T(x, y_2)\| < \varepsilon$, whenever we have $d(y_1, y_2) < \delta(\varepsilon)$.

From the definition of Kuratowski's measure of noncompactness it follows that there is a finite family of sets $B_1, \dots, B_n$ such that $B = \bigcup_{i=1,\dots,n} B_i$ and $\operatorname{diam}(B_i) \le \alpha(B) + \varepsilon$, $i = 1, \dots, n$.

Since $C(B)$ is precompact, we have $\alpha(C(B)) = \alpha(\overline{C(B)}) = 0$ and hence there is a finite family of sets $V_1, \dots, V_m$, such that

$$C(B) = \bigcup_{j=1,\dots,m} V_j \quad \text{and} \quad \operatorname{diam}(V_j) \le \delta(\varepsilon), \quad j = 1, \dots, m.$$

Thus $B \subseteq \bigcup_{j=1,\dots,m} C^{-1}(V_j)$ and therefore

$$B = \Big(\bigcup_{i=1,\dots,n} B_i\Big) \cap \Big(\bigcup_{j=1,\dots,m} C^{-1}(V_j)\Big) = \bigcup_{i=1,\dots,n} \bigcup_{j=1,\dots,m} (B_i \cap C^{-1}(V_j)).$$

The latter gives

$$E(B) = \bigcup_{i=1,\dots,n} \bigcup_{j=1,\dots,m} E(B_i \cap C^{-1}(V_j)).$$

Fix two indices i, j and take points $x_1, x_2 \in B_i \cap C^{-1}(V_j)$. Then we have

$$\|x_1 - x_2\| \le \operatorname{diam}(B_i) \le \alpha(B) + \varepsilon \quad \text{and} \quad d(C(x_1), C(x_2)) \le \operatorname{diam}(V_j) \le \delta(\varepsilon).$$

Hence $\|T(x_2, C(x_1)) - T(x_2, C(x_2))\| \le \varepsilon$ and therefore

$$\begin{aligned}\|E(x_1) - E(x_2)\| \le \|T(x_1, C(x_1)) - T(x_2, C(x_1))\| \\ + \|T(x_2, C(x_1)) - T(x_2, C(x_2))\| \le k\alpha(B) + k\varepsilon + \varepsilon.\end{aligned}$$

This means that $\operatorname{diam} E(B_i \cap C^{-1}(V_j)) \le k\alpha(B) + k\varepsilon + \varepsilon$ and so

$$\begin{aligned}\alpha(E(B)) = \max\{\operatorname{diam} E(B_i \cap C^{-1}(V_j)) : i = 1, \dots, n,\ j = 1, \dots, m\} \\ \le k\alpha(B) + k\varepsilon + \varepsilon.\end{aligned}$$

Since ε is arbitrary, we get

$$\alpha(E(B)) \leq k\alpha(B).$$

Now, Darbo's Lemma 2.3 applies and the operator E has therefore a fixed point in A. $\square$

Remark 2.7. Instead of the measure of noncompactness one can consider Hausdorff's measure of noncompactness χ as defined by

$$\chi(B) := \inf\{\delta > 0 : B \text{ has a } \delta - net\}$$

and work with balls in the same manner. (For a good background information on the various kinds of measures of noncompactness one can consult e.g. [1, Chapter 1].) Then Sadovskiĭ's fixed point theorem for the so-called dencifying or condensing operators (see [21]) applies instead of Darbo's Lemma.

Remark 2.8. It is not hard to see that our results extend Krasnosel'skiĭ's Theorem 1.1 as well as the results in [6]. Furthermore, we wish to elaborate a little in the case exhibited in [6]. Suppose X is a Banach Algebra with multiplication $\cdot$ and we let $Y := X^2$. For any $y := (y_1, y_2) \in X^2$ define $T(x,y) := L(x) \cdot y_1 + y_2$. Thus T can be written in the form $L \cdot B + C$. Now assume that there is a continuous function $\phi\colon [0,+\infty) \to [0,\infty)$ such that

$$m(r) := \sup\{|Bx| : x \in A\}\phi(r) < r,$$

for all $r > 0$. Let q be a bound of the set A and define $k := \sup\{m(r)/r : r \in (0, 2q]\}$. Then we have $k < 1$ and so the operator T is equicontractive in its domain. The uniform continuity is obvious and so Theorem 2.5 applies to T. This shows that Theorem 2.5 extends the fixed point theorem given in [6].

3. Applications

In this section we present two applications of Theorem 2.5 from the theory of integral equations, which are of theoretical interest. We emphasize that in these applications there has no been attempt to achieve the best possible results, since our main intention is to show how our theorems might be applied in suitable situations.

Assume that $\mathbb{R}$ is the real line and $\mathbb{R}^n$ the n-dimentional euclidean space. We shall use the symbol $|\cdot|$ to denote their euclidean norms. The same symbol will be used to denote an admissible norm in the set of $n \times n$-matrices.

Let I be an interval of the real line of the form $[0, \tau)$, where τ is a positive number, or ∞. Let X stands for the Banach space $C_b(I, \mathbb{R}^n)$ of all continuous and bounded functions $x\colon I \to \mathbb{R}^n$ endowed with the sup-norm $\|\cdot\|$. By $B(0, r)$ we shall denote the ball in X centered at zero and having radius $r(> 0)$.

We start with an equation of the form

$$x(t) = f(t) + \left[\int_0^t (d_s B(t,s))x(s) + \int_0^t H(t,s)x(s)\,ds \right]^{\gamma+1}, \quad t \in I, \tag{3.1}$$

where $\gamma > 0$ and the first integral is meant in the Riemann–Stieltjes sense.

A great number of differential and integrodifferential equations reduce to equation (3.1). For instance consider the following Cauchy problem for the Riccati-like differential equation:

$$\begin{cases} \dfrac{dz}{dt} = H(t)z^{\gamma+1} + Z(t) & \text{for } t \geq 0, \\ z(0) = 0. \end{cases} \tag{3.2}$$

Assume that the n-vector valued function Z and the $n \times n$-matrix valued function H are at least continuous on the interval $[0,\infty)$ and for all t the matrix $H(t)$ is nonsingular. Set $f(t) := H(t)^{-1}Z(t)$ and $x(t) := f(t) + z(t)^{\gamma+1}$. Then obtain

$$x(t) = f(t) + \left[\int_0^t H(s)x(s)\,ds \right]^{\gamma+1},$$

which is of the form (3.1).

Coming now to equation (3.1), we establish the following conditions:

(a) The function f belongs to X.

(b) For each $t \in I$ the $n \times n$-matrix valued function $B(t,s)$, $0 \leq s \leq t$ is of bounded variation on $[0,t]$ and the function $t \to \mathrm{var}_{[0,t]} B(t,\,\cdot\,)$ is bounded. Let
$$b := \sup_{t \in I} \mathrm{var}_{[0,t]} B(t,\,\cdot\,).$$

(c) The $n \times n$-matrix valued function $H(t,s)$, $s \in [0,t]$ is integrable on $[0,t]$ and the mapping $t \to \int_0^t |H(t,s)| ds$ is bounded and such that
$$c := \sup_{t \in I} \int_0^t |H(t,s)|\,ds > 0.$$

To simplify our results we set

$$\delta := \gamma^{-1}, \quad K := \delta^\delta(\delta+1)^{-\delta}(b+c)^{-(\delta+1)} \quad \text{and} \quad M := K(\delta+1)^{-1}.$$

Theorem 3.1. *Suppose that the conditions* (a)–(c) *hold. Then equation* (3.1) *admits a solution $x \in B(0,K)$ provided that $f \in B(0,M)$.*

Proof. Let A be the closed ball $\overline{B(0,K)}$ in X. Then the operator $C\colon X \to X$ defined by the type

$$(Cx)(t) := \int_0^t H(t,s)x(s)\,ds$$

is compact and it maps the ball $B(0,K)$ into the ball $B(0,cK)$. We consider the operator $T\colon A\times B(0,cK)\to X$ defined by

$$T(x,y)(t):=f(t)+\bigg[\int_0^t (d_sB(t,s))x(s)+y(t)\bigg]^{\gamma+1},\quad t\in I$$

and we claim that it satisfies the conditions of Theorem 2.5.

We shall first show that the family $\{T(\,\cdot\,,y):y\in C(A)\}$ is equicontractive. Indeed, let $x_1,x_2\in A$ and $y\in C(A)$. Since the set A is convex, we observe that by using Gâteaux derivation it holds

$$\begin{aligned}&|T(x_1,y)(t)-T(x_2,y)(t)|\\&\quad\le\bigg|\bigg[\int_0^t (d_sB(t,s))x_1(s)+y(t)\bigg]^{\gamma+1}-\bigg[\int_0^t (d_sB(t,s))x_2(s)+y(t)\bigg]^{\gamma+1}\bigg|\\&\quad\le k\|x_1-x_2\|,\end{aligned}$$

for each $t\in I$, where

$$k:=(\gamma+1)(bK+cK)^{\gamma}b=(\gamma+1)b(b+c)^{\gamma}K^{\gamma}=\frac{b}{b+c}<1.$$

This shows equicontractivity.

The fact that the family $\{T(x,\,\cdot\,):x\in A\}$ is uniformly continuous on $C(A)$ is obvious.

Finally, we have to show that given any $x\in A$ the point $T(x,C(x))$ belongs to A. We observe that the previous arguments imply that for each $x\in X$ the function $T(x,C(x))$ is continuous. Now take $x\in A$ and set $y:=C(x)$. Then, for all $t\in I$ we have

$$\begin{aligned}|T(x,y)(t)|&\le\|f\|+(bK+cK)^{\gamma+1}=\|f\|+(b+c)^{\gamma+1}K^{\gamma+1}\\&\le\frac{\delta^{\delta}}{[(b+c)(\delta+1)]^{\delta+1}}+(b+c)^{\gamma+1}K^{\gamma+1}\\&=\frac{\gamma}{(\gamma+1)^{(\gamma+1)/\gamma}(b+c)^{(\gamma+1)/\gamma}}+\frac{1}{(\gamma+1)^{(\gamma+1)/\gamma}(b+c)^{(\gamma+1)/\gamma}}\\&=\frac{\gamma+1}{(\gamma+1)^{(\gamma+1)/\gamma}(b+c)^{(\gamma+1)/\gamma}}=\frac{1}{(\gamma+1)^{1/\gamma}(b+c)^{(\gamma+1)/\gamma}}=K.\end{aligned}$$

Here Theorem 2.5 applies and so we get the result. □

Remark 3.2. In Theorem 3.1 we proved the existence of a solution x of equation (3.1) defined on the entire interval I. It is clear that another way of proving existence results for (3.1) is to use standard arguments from the basic theory of differential equations. Namely seek first for the existence of a solution z of problem (3.2) (see, e.g. [4, p. 32]) and then transform it to a solution x of (3.1). But as we noticed above, such a procedure requires some regularities on the coefficient matrix-valued function H.

Next consider the initial value problem

$$\begin{cases} x'(t) = g(t) + F(t)x(t)\left[\displaystyle\int_0^t D(t-s)x(s)\,ds\right]^\eta, & t \in I := [0,\tau), \\ x(0) = 0, \end{cases} \tag{3.3}$$

in the real scalar case, where $\eta > 0$ and τ is a number as in equation (3.1). We will show the existence of a solution x in the space $X := C_b(I, \mathbb{R})$.

Assume that $g, F, D \in L_1(I, \mathbb{R})$ and let $\|\cdot\|_1$ stand for the usual L_1-(semi)-norm. Also assume that $\|F\|_1\|D\|_1 > 0$.

Fix any $K_1 \in (0, \|F\|_1^{-1/\eta}\|D\|_1^{-1})$ and set

$$N := \|F\|_1^\eta\|D\|_1^\eta \quad \text{and} \quad M_1 := K_1(1 - \|F\|_1 N).$$

Theorem 3.3. *Suppose that the above conditions hold. Then the Cauchy problem* (3.3) *admits a* (*global*) *solution* $x \in B(0, K_1)$ *provided that* $g \in L_1(I, \mathbb{R})$ *and satisfies* $\|g\|_1 \le M_1$.

Proof. Observe that the operator $C\colon X \to X$ defined by the type

$$(Cx)(t) := \left(\int_0^t D(t-s)x(s)\,ds\right)^\eta$$

is compact. Let A be the set of all x in X such that $\|x\| \le K_1$. Then C maps the ball $B(0, K_1)$ into the ball $B(0, N)$.

Consider the operator $T\colon A \times B(0, N) \to X$ as defined by

$$T(x, y)(t) := \int_0^t g(s)\,ds + \int_0^t F(s)x(s)y(s)\,ds, \quad t \in I.$$

We claim that it satisfies the conditions of Theorem 2.5.

Indeed, take $x_1, x_2 \in A$ and $y \in C(A)$. Then, for each $t \in I$, we get

$$|T(x_1, y)(t) - T(x_2, y)(t)| \le \int_0^t |F(s)||x_1(s) - x_2(s)||y(s)|\,ds \le k\|x_1 - x_2\|,$$

where $k := \|F\|_1 N < 1$. This proves equicontractivity.

The uniform continuity of the family $\{T(x, \cdot\,) : x \in A\}$ on the set $C(A)$ is evident.

Finally Theorem 2.5 applies provided that $T(x, C(x)) \in A$ holds for each $x \in A$. Indeed, we observe that the operator $x \to T(x, C(x))$ is continuous and if, in addition, $x \in A$ and $y := C(x)$, then, for all $t \in I$, it holds

$$|T(x, y)(t)| \le \|g\|_1 + K_1\|F\|_1 N \le K_1(1 - \|F\|_1 N) + K_1\|F\|_1 N = K_1.$$

This completes the proof. □

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