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# FIXED POINT INDICES OF ITERATIONS OF PLANAR HOMEOMORPHISMS

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Abstract. Let $f$ be a local planar homeomorphism with an isolated fixed point at 0. We study the form of the sequence $\{\operatorname{ind}(f^n,0)\}_{n\neq 0}$, where $\operatorname{ind}(f,0)$ is a fixed point index at 0.

## 1. Introduction

A. Dold in 1983 (cf. [7]) proved that a sequence $\{\operatorname{ind}(f^n)\}_{n=1}^{\infty}$ of fixed point indices cannot take arbitrary values but must satisfy some congruences (Dold relations). Babenko and Bogatyi showed that every sequence of integers that fulfils this relations can be realized as $\{\operatorname{ind}(f^n,0)\}_{n=1}^{\infty}$, where $f\colon D^3\to D^3$ is a homeomorphism and $D^3$ is a unit disk in $\mathbb{R}^3$ (cf. [1]). Differentiability of $f$ impacts heavily on the form of $\{\operatorname{ind}(f^n,0)\}_{n=1}^{\infty}$, it must be periodic with the period determined by the derivative of $f$ (cf. [9], [6]). It is natural to ask about the shape of indices of iterations for non-differentiable maps in two dimensions. The class of maps under consideration in this note consists of planar homeomorphisms. For orientation preserving planar homeomorphism M. Brown found strong restrictions on $\{\operatorname{ind}(f^n,0)\}_{n\neq 0}$ (cf. [2]), which are a consequence of topological properties of the plane. We develop the work of M. Brown by a use of Dold relations and obtain additional bounds on the form of $\{\operatorname{ind}(f^n,0)\}_{n\neq 0}$.

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Considering the case of an orientation reversing planar homeomorphism we base on the recent result of M. Bonino (announced earlier by M. Brown in [2]), who proved that $\mathrm{ind}(f,0)$ may be only one of the three values: $-1$, $0$, $1$ (cf. [3]). Again by applying Dold relations we get the information about indices of iterations.

## 2. Dold relations

We will consider local homeomorphisms $f\colon U \to \mathbb{R}^2$ ($U$ is an open neigbourhood of 0) such that for each $n \neq 0$ the origin 0 is an isolated fixed point for $f^n$, though the neighbourhood of isolation may depend on $n$. Then the fixed point index $\mathrm{ind}(f^n,0)$ is defined for $f^n$ restricted to a small enough neighbourhood of 0.

Definition 2.1. The classical arithmetical Möbius function $\mu$ is defined by three properties:

- $\mu(1) = 1$,
- $\mu(k) = (-1)^r$ if $k$ is a product of $r$ different primes,
- $\mu(k) = 0$ otherwise.

Definition 2.2. For natural $n$ define integers $i_n(f,0)$ by the equality:

$$i_n(f,0) = \sum_{k|n} \mu(n/k)\mathrm{ind}(f^k,0).$$

Notice (cf. [5]) that an alternative representation of $i_n(f,0)$ is the following:

$$i_n(f,0) = \sum_{k|n} \mu(k)\mathrm{ind}(f^{n/k},0).$$

Theorem 2.3 (Dold relations, cf. [7]). *For each natural $n$ we have:*

$$i_n(f,0) \equiv 0 \pmod{n}.$$

One of the important consequences of Dold relations is the theorem below (cf. [1]):

Theorem 2.4. *The sequence $\{\mathrm{ind}(f^n,0)\}_{n=1}^{\infty}$ is bounded if and only if it is periodic.*

## 3. Orientation preserving homeomorphisms

Theorem 3.1 (cf. [2]). *Let $f$ be an orientation preserving local homeomorphism of the plane. Then there is an integer $p \neq 1$ such that, for each $n \neq 0$,*

$$\mathrm{ind}(f^n,0) = \begin{cases} p & \text{if } \mathrm{ind}(f,0) = p, \\ 1 \text{ or } p & \text{if } \mathrm{ind}(f,0) = 1. \end{cases} \tag{3.1}$$

Corollary 3.2. *If* $\operatorname{ind}(f^n,0) = p$*, then* $\operatorname{ind}(f^{kn},0) = p$ *for every* $k \in \mathbb{Z} \setminus \{0\}$*.*

By the above corollary we may consider only natural $n$, since $\operatorname{ind}(f^n,0) = \operatorname{ind}(f^{-n},0)$.

Definition 3.3. Let us define $A$, the set of generators for $\{\operatorname{ind}(f^n,0)\}_{n=1}^{\infty}$, in the following way:

$$A = \{a \in \mathbb{N} : \operatorname{ind}(f^a,0) = p \text{ and for all } b|a \text{ such that } b \neq a \ \operatorname{ind}(f^b,0) = 1\}.$$

From Theorem 3.1 we have: if $A = \emptyset$, then $\operatorname{ind}(f^n,0) = 1$ for each $n$.

Theorem 3.4. *Let $f$ be an orientation preserving local homeomorphism of the plane, $A \neq \emptyset$. Then $A$ is finite and*

$$\operatorname{LCM}(A)|(p-1),$$

*where* $\operatorname{LCM}(A)$ *denotes the lowest common multiplicity of all elements in $A$.*

Proof.

$$\begin{aligned} i_n(f,0) &= \sum_{k:\exists a\in A\ a|k|n} \mu(n/k)\operatorname{ind}(f^k,0) + \sum_{k:\forall a\in A\ a\nmid k|n} \mu(n/k)\operatorname{ind}(f^k,0) \\ &= \sum_{k:\exists a\in A\ a|k|n} \mu(n/k)(p-1+1) + \sum_{k:\forall a\in A\ a\nmid k|n} \mu(n/k) \\ &= (p-1)\sum_{k:\exists a\in A\ a|k|n} \mu(n/k) + \sum_{k|n}\mu(n/k). \end{aligned}$$

For $n > 1$ by the well-known equalities (cf. [5]):

$$\sum_{k|n}\mu(n/k) = \sum_{k|n}\mu(k) = 0, \tag{3.2}$$

we obtain:

$$i_n(f,0) = \begin{cases} (p-1)\sum_{k:\exists a\in A\ a|k|n}\mu(n/k) & \text{if exists } a \in A\ a|n, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

For $A = \{1\}$ we get $i_1(f,0) = p$ and $i_n(f,0) = 0$ if $n > 1$. For $A \neq \{1\}$ and $n > 1$, substituting $n = a \in A$ into (3.3), we get:

$$i_a(f,0) = p-1. \tag{3.4}$$

It follows from Dold relations (Theorem 2.3) that for each $a \in A$ we have: $a|(p-1)$, hence $A$ is finite and $\operatorname{LCM}(A)|(p-1)$, which is the desired conclusion.$\square$

Remark 3.5. M. Brown stated that if $\operatorname{ind}(f,0) = 1$, then every integer $p$ may appear as an index of some iteration in the formula (3.1) of Theorem 3.1 (cf. [2], Remark after Theorem 4). He gave examples of homeomorphisms for all $p$ except for $p = 0$ and $p = 2$. However, by Theorem 3.4 in this two cases

$\mathrm{LCM}(A)| \pm 1$, which means that $A = \emptyset$ and therefore $\mathrm{ind}(f^n, 0) = 1$ for every $n$. This imply that $p = 0$ and $p = 2$ cannot occur as indices of any iteration if $\mathrm{ind}(f, 0) = 1$.

Notice that by Theorem 3.1 $\{\mathrm{ind}(f^n, 0)\}_{n=1}^{\infty}$ is bounded. Thus Theorem 2.4 gives the periodicity of $\{\mathrm{ind}(f^n, 0)\}_{n=1}^{\infty}$. On the other hand the same consequence gives Theorem 3.4. Theorem 3.4 is in fact equivalent to Dold relations in the class of sequences which come from a preserving orientation planar homeomorphism.

Theorem 3.6. *Let a sequence of integers* $\{\mathrm{ind}(f^n, 0)\}_{n=1}^{\infty}$ *has the form* (3.1) *from Theorem* 3.1*, satisfies Corollary* 3.2*,* $A \neq \emptyset$ *and* $\mathrm{LCM}(A)|(p-1)$*. Then* $\{\mathrm{ind}(f^n, 0)\}_{n=1}^{\infty}$ *satisfies Dold relations.*

Proof. Assume $n > 1$ and let us consider two cases.

*Case 1.* If $n|\mathrm{LCM}(A)$, then by our assumption $n|(p-1)$. On the other hand by the equality (3.3) we have: $i_n(f, 0) = (p-1)\sum_{k:\exists a\in A\ a|k|n} \mu(n/k)$ or $i_n(f, 0) = 0$, thus $n|i_n(f, 0)$.

*Case 2.* If $n \nmid \mathrm{LCM}(A)$ then, there are: a prime number $r$ and $\alpha \in \mathbb{N}$ such that $r^\alpha|n$ and $\forall_{a\in A}\ r^\alpha \nmid a$. We express the formula for $i_n(f, 0)$ in another way:

$$\begin{aligned} i_n(f,0) &= \sum_{k|n} \mu(k)\mathrm{ind}(f^{n/k}, 0) \\ &= \sum_{k:r\nmid k|n} \mu(k)\mathrm{ind}(f^{n/k}, 0) + \sum_{k:r|k|n} \mu(k)\mathrm{ind}(f^{n/k}, 0). \end{aligned}$$

The last term we rewrite in the following way:

$$\sum_{k:r|k|n} \mu(k)\mathrm{ind}(f^{n/k}, 0) = \sum_{k:r|k|n;r^2\nmid k} \mu(k)\mathrm{ind}(f^{n/k}, 0).$$

Substituting $k' = k/r$ we get that the sum above is equal to:

$$\begin{aligned} &\sum_{k':r|k'r|n;r^2\nmid k'r} \mu(rk')\mathrm{ind}(f^{n/(rk')}, 0) \\ &\qquad = \sum_{k':r\nmid\, k'|n/r} \mu(rk')\mathrm{ind}(f^{n/(rk')}, 0) = \sum_{k':r\nmid\, k'|n} -\mu(k')\mathrm{ind}(f^{n/(rk')}, 0). \end{aligned}$$

We used the definition of $\mu$ (Definition 2.1) and the fact that $\mu$ is a multiplicative function (cf. [5]). Thus we obtain for a prime $r$ the following formula:

$$i_n(f,0) = \sum_{k:r\nmid k|n} \mu(k)[\mathrm{ind}(f^{n/k}, 0) - \mathrm{ind}(f^{n/(rk)}, 0)]. \tag{3.5}$$

Notice that by Corollary 3.2 and Definition 3.3, if $\mathrm{ind}(f^{n/(rk)}, 0) = p$ then, $\mathrm{ind}(f^{n/k}, 0) = p$ since $n/k$ is a multiplicity of $n/(rk)$; if $\mathrm{ind}(f^{n/k}, 0) = p$, then

$\mathrm{ind}(f^{n/(rk)},0)=p$ since for $a\in A$ we have: $a|(n/k)$ implies $a|n/(rk)$, which is a consequence of the fact that $r^\alpha|n$ but $r^\alpha\nmid a$ and $r\nmid k$. Finally,

$$i_n(f,0)=0 \quad \text{for } n\nmid \mathrm{LCM}(A), \tag{3.6}$$

which completes the proof. □

## 4. Orientation reversing homeomorphisms

Theorem 4.1 (cf. [3]). *Let $f$ be a planar orientation reversing local homeomorphism. Then* $\mathrm{ind}(f,0)\in\{-1,0,1\}$.

The theorem above and Dold relations determine the shape of indices of odd iterations of an orientation reversing homeomorphism.

Theorem 4.2. *Let $f$ be a planar orientation reversing local homeomorphism. Then for every $n$ odd* $\mathrm{ind}(f^n,0)\in\{-1,0,1\}$ *and*

$$\mathrm{ind}(f^n,0)=\begin{cases}\mathrm{ind}(f,0) & \text{if } n>0,\\ -\mathrm{ind}(f,0) & \text{if } n<0.\end{cases}$$

Proof. For $n>0$ we prove the theorem by induction. For $n=1$ the thesis is true by Theorem 4.1. Inductively assume that it is true for all odd $k<n$. Using the inductive hypothesis and equality (3.2) we obtain:

$$\begin{aligned} i_n(f,0) &= \mathrm{ind}(f^n,0)+\sum_{k|n;k\neq n}\mu(n/k)\mathrm{ind}(f^k,0)\\ &= \mathrm{ind}(f^n,0)+\mathrm{ind}(f,0)\Big(\sum_{k|n}\mu(n/k)-1\Big)\\ &= \mathrm{ind}(f^n,0)-\mathrm{ind}(f,0). \end{aligned}$$

As a result, by Dold relations, $i_n(f,0)=\mathrm{ind}(f^n,0)-\mathrm{ind}(f,0)\equiv 0 \pmod n$. Since $n>2$ and $|\mathrm{ind}(f^n,0)|\le 1$, we get $i_n(f,0)=0$ which gives the thesis for $n>0$. For $n<0$ we use the equality $\mathrm{ind}(f^{-1},0)=-\mathrm{ind}(f,0)$, which is valid for orientation reversing planar homeomorphisms. □

Our goal now is to characterize the elements of $\{\mathrm{ind}(f^n,0)\}_{n=1}^\infty$ for $n$ even and $n>0$ (for $n<0$ the situation is the same, since $\mathrm{ind}(f^{2n},0)=\mathrm{ind}(f^{-2n},0)$). First we will investigate some special cases. For the rest of the paper we assume that $\widetilde{A}$ is a set of generators for $\{\mathrm{ind}((f^2)^n,0)\}_{n=1}^\infty$ and $\widetilde{p}$ an integer from the formula (3.1) of Theorem 3.1 for $f^2$.

Remark 4.3. Let $\mathrm{ind}(f,0)=0$. Then for every natural $n$ $\mathrm{ind}(f^{2n},0)=2l$, where $l$ is an integer. This is a straightforward consequence of Dold relations (Theorem 2.3) for $n=2$: $i_2(f,0)=\mathrm{ind}(f,0)-\mathrm{ind}(f^2,0)\equiv 0 \pmod 2$ and Theorem 3.1. Let $\mathrm{ind}(f,0)=\pm1$. By the definition of $\widetilde{A}$ (cf. Definition 3.3) and Theorem 3.1 we obtain: if $\widetilde{A}=\emptyset$, then $\mathrm{ind}(f^{2n},0)=1$ for each $n$.

Theorem 4.4. *Let $f$ be an orientation reversing local homeomorphism, $\widetilde{A} \neq \emptyset$ and* $\operatorname{ind}(f,0) = \pm 1$. *Then* $2\mathrm{LCM}(\widetilde{A})|(\widetilde{p}-1)$.

In the sequel we will need the following lemma:

Lemma 4.5. *Let $f$ be an orientation reversing local homeomorphism and $n > 2$ an even number. Then $i_n(f,0) = i_{n/2}(f^2,0)$.*

Proof. Consider two cases. If $n/2$ is even, then:

$$\begin{aligned} i_n(f,0) &= \sum_{k|n} \mu(k)\operatorname{ind}(f^{n/k},0) \\ &= \sum_{k|(n/2)} \mu(k)\operatorname{ind}((f^2)^{(n/2)/k},0) = i_{n/2}(f^2,0). \end{aligned}$$

If $n/2$ is odd, then taking $r = 2$ in the formula (3.5) we obtain:

$$\begin{aligned} i_n(f,0) &= \sum_{k:2\nmid k|n} \mu(k)[\operatorname{ind}(f^{n/k},0) - \operatorname{ind}(f^{(n/2)/k)},0)] \\ &= \sum_{k|(n/2)} \mu(k)\operatorname{ind}((f^2)^{(n/2)/k},0) - \sum_{k|(n/2)} \mu(k)\operatorname{ind}(f^{(n/2)/k},0) \\ &= i_{n/2}(f^2,0). \end{aligned}$$

The last equality results from the fact that $(n/2)/k$ are odd numbers, so by Theorem 4.2 $\sum_{k|(n/2)} \mu(k)\operatorname{ind}(f^{(n/2)/k},0) = \operatorname{ind}(f,0)\sum_{k|(n/2)} \mu(k)$, which is equal to zero for $n/2 > 1$ by (3.2). □

Proof of Theorem 4.4. If $\widetilde{A} = \{1\}$, then $i_2(f) = \widetilde{p} \pm 1$, so by Dold relations $2|(\widetilde{p}-1)$. If $\widetilde{A} \neq \{1\}$, then by Lemma 4.5 and formula (3.4) for $f^2$, we have: $i_{2a}(f,0) = i_a(f^2,0) = \widetilde{p}-1$. From Dold relations we deduce that $2a|(\widetilde{p}-1)$ for each $a \in \widetilde{A}$, thus $2\mathrm{LCM}(\widetilde{A})|(\widetilde{p}-1)$. □

Again, the information about $\{\operatorname{ind}(f^n,0)\}_{n=1}^\infty$ which we gathered in Theorems 4.2 and 4.4 is equivalent to Dold relations.

Theorem 4.6. *Let a sequence of integers $\{\operatorname{ind}(f^{2n},0)\}_{n=1}^\infty$ has the form* (3.1)*, satisfies Corollary* 3.2*, $\widetilde{A} \neq \emptyset$, $2\mathrm{LCM}(\widetilde{A})|(\widetilde{p}-1)$ and $\{\operatorname{ind}(f^{2n-1},0)\}_{n=1}^\infty$ is constantly equal one of the two values: $-1$, $1$. Then $\{\operatorname{ind}(f^n,0)\}_{n=1}^\infty$ satisfies Dold relations.*

Proof. Let us consider the following possibilities:

*Case 1.* $n > 1$ is odd. Then $n/k$ for $k|n$ is also odd, thus by Theorem 4.2 and equalities (3.2) we have:

$$i_n(f,0) = \sum_{k|n} \mu(k)\operatorname{ind}(f^{n/k},0) = \operatorname{ind}(f,0)\sum_{k|n} \mu(k) = 0.$$

*Case 2a.* $n$ is even and $n|2\mathrm{LCM}(\widetilde{A})$. If $n = 2$, then there are 4 possible values of $i_2(f,0)$: $0, 2, \widetilde{p} - 1, \widetilde{p} + 1$. By the assumption $2|(\widetilde{p} - 1)$, so Dold relation for $n = 2$ is always satisfied. Let $n > 2$. By the formula (3.3)

$$i_{n/2}(f^2,0) = (\widetilde{p} - 1) \sum_{k:\exists a\in\widetilde{A}\ a|k|(n/2)} \mu((n/2)/k)$$

or $i_{n/2}(f^2,0) = 0$. On the other hand by our assumption: $n|2\mathrm{LCM}(\widetilde{A})|(\widetilde{p} - 1)$, so $n|i_{n/2}(f^2,0)$ and Lemma 4.5 proves the theorem.

*Case 2b.* $n$ is even and $n \nmid 2\mathrm{LCM}(\widetilde{A})$. Then $(n/2) \nmid \mathrm{LCM}(\widetilde{A})$, which implies by the formula (3.6) that $i_{n/2}(f^2,0) = 0$. Lemma 4.5 completes the proof. $\square$

It is easy to observe by Lemma 4.5 that the sequence of integers from Remark 4.3: $\mathrm{ind}(f^n,0) = 0$ for $n$ odd and $\mathrm{ind}(f^n,0) = 2l$ ($l \in \mathbb{Z}$) for $n$ even, satisfies Dold relations.

## 5. Final remarks

One question that we leave unanswered is whether the restrictions on the sequence $\{\mathrm{ind}(f^n,0)\}_{n=1}^{\infty}$ obtained in this paper are maximal, i.e. is it true that every sequence of integers which satisfies them can be obtained as $\{\mathrm{ind}(f^n,0)\}_{n=1}^{\infty}$, where $f$ is a local planar homeomorphism.

For some classes of homeomorphisms (cf. [4], [8]) there is no room on the plane for more then one element in the set of generators $A$. This suggests that indices of an iterated planar homeomorphism may behave in the same way as indices of a planar differentiable map, for which the set $A$ is always either empty or has one element (cf. [1]). For orientation reversing planar homeomorphisms bounds on $\widetilde{A}$ may be even more restrictive, namely $\widetilde{A} \subset \{1\}$.

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