

**A DEFORMATION THEOREM
AND SOME CRITICAL POINT RESULTS
FOR NON-DIFFERENTIABLE FUNCTIONS**

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ABSTRACT. A deformation lemma for functionals which are the sum of a locally Lipschitz continuous function and of a concave, proper and upper semicontinuous function is established. Some critical point theorems are then deduced and an application to a class of elliptic variational-hemivariational inequalities is presented.

Introduction

It is by now well known that the Mountain Pass Theorem of Ambrosetti and Rabinowitz [2, Theorem 2.1] employs fruitfully in the study of various questions concerning differential equations. This result basically applies to each case when the solutions of the problem under consideration can be regarded as critical points of a continuously differentiable real-valued functional f on a Banach space $(X, \|\cdot\|)$, with the following property:

(f) *there exist $x_0, x_1 \in X$, $r > 0$, $a \in \mathbb{R}$ such that $\|x_1 - x_0\| > r$ and*

$$\max\{f(x_0), f(x_1)\} < a \leq f(x) \quad \text{for all } x \in \partial B(x_0, r),$$

where $\partial B(x_0, r) = \{x \in X : \|x - x_0\| = r\}$.

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An additional compactness condition of Palais–Smale type then ensures that f possesses a critical value $c \geq a$. Intuitively, this critical value occurs because x_0 and x_1 are low points on either side of the “mountain ring” $\partial B(x_0, r)$, so that between them there must be a lowest critical point, or “mountain pass”. Now, the question whether the conclusion is still true when the “mountain ring” separating x_0 and x_1 has “zero altitude”, namely in (f) one merely has

$$\max\{f(x_0), f(x_1)\} \leq a \leq f(x) \quad \text{for all } x \in \partial B(x_0, r),$$

spontaneously arises. Complete and satisfactory results in this direction have been obtained by Pucci and Serrin ([13, Theorem 1]), Rabinowitz ([15, Theorem 2.13]), Ghoussoub and Preiss ([9, Theorem 1.bis]), and, as regards the more general framework of linkings, Du ([7, Theorem 2.1]).

However, chiefly because of the regularity hypothesis on f , several problems one meets in important concrete situations cannot be treated directly through the Mountain Pass Theorem. As an example, let us mention both variational inequalities and elliptic equations with discontinuous nonlinearities. Indeed, concerning the first case the indicator function of some convex closed subset of X must appear in the expression of f , while in the second case f turns out locally Lipschitz continuous at most.

Starting from the seminal papers by Chang ([4]) and Szulkin ([17]), a version of the Mountain Pass Theorem which works for functions $f: X \rightarrow]-\infty, \infty]$ fulfilling the structure assumption

$$(H_f) \quad f = \Phi + \alpha, \text{ where } \Phi: X \rightarrow \mathbb{R} \text{ is locally Lipschitz continuous while } \alpha: X \rightarrow]-\infty, \infty] \text{ is convex, proper, besides lower semicontinuous,}$$

has recently been established; see [10, Theorem 3.2]. Critical points of f are defined as solutions to the problem

$$(*) \quad \text{Find } u \in X \text{ such that } \Phi^0(u; x - u) + \alpha(x) - \alpha(u) \geq 0 \text{ for all } x \in X,$$

$\Phi^0(u; x - u)$ being the generalized directional derivative [5, p. 25] of Φ at the point u along the direction $x - u$. The standard Palais–Smale condition becomes here:

(PS) $_{f,c}$ If $\{u_n\}$ is a sequence in X satisfying $f(u_n) \rightarrow c$, $c \in \mathbb{R}$, and

$$\Phi^0(u_n; x - u_n) + \alpha(x) - \alpha(u_n) \geq -\varepsilon_n \|x - u_n\| \quad \text{for all } n \in \mathbb{N}, x \in X,$$

where $\varepsilon_n \rightarrow 0^+$, then $\{u_n\}$ possesses a convergent subsequence.

When $\Phi \in C^1(X, \mathbb{R})$, problem $(*)$ reduces to a classical variational inequality, and the relevant critical point theory as well as meaningful applications are developed in [17]. If $\alpha \equiv 0$ then $(*)$ coincides with the problem treated by Chang ([4]), which also exploits the abstract results to study elliptic equations having discontinuous nonlinearities. Finally, when both $\Phi \in C^1(X, \mathbb{R})$ and $\alpha \equiv 0$,

problem (*) simplifies to the Euler equation $\Phi'(u) = 0$, and the theory is by now classical; vide for instance [1], [14].

Regarding this new setting, it makes sense – like before – to ask whether the standard strict inequality appearing in Theorem 3.2 of [10] can be weakened to allow also equality. A partial answer, namely for $\alpha \equiv 0$, is already known; see [11, Theorem 2.1]. The present paper continues such investigation by treating the general case. To do this, we first establish a deformation lemma (Theorem 2.1 below) for the function $-f$, which includes both [7, Lemma 2.1] and [11, Theorem 1.1]. From a technical point of view, it represents the most difficult part of the work and is presented in Section 2. After that, in Section 3, a version of [10, Theorem 3.2] where “less than or equal to” takes the place of “less than” is established (see Theorem 3.1) and some classical results (as the Mountain Pass Theorem) are reformulated in our framework. Finally, Section 4 contains an application (Theorem 4.1) to a class of elliptic variational-hemivariational inequalities in the sense of Panagiotopoulos ([12]). Let us point out that variational-hemivariational inequalities arise in the modelling of important mechanical and engineering problems, like for instance the behaviour of an adhesive material in the direction orthogonal to the interface.

1. Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space. If V is a subset of X , we write $\text{int}(V)$ for the interior of V , \overline{V} for the closure of V , ∂V for the boundary of V . When V is nonempty, $x \in X$, and $\delta > 0$, we define $B(x, \delta) = \{z \in X : \|z - x\| < \delta\}$ as well as

$$B_\delta = B(0, \delta), \quad \overline{B}_\delta = \overline{B(0, \delta)},$$

$$d(x, V) = \inf_{z \in V} \|x - z\|, \quad N_\delta(V) = \{z \in X : d(z, V) \leq \delta\}.$$

Given $x, z \in X$, the symbol $[x, z]$ indicates the line segment joining x to z , i.e.

$$[x, z] = \{(1 - t)x + tz : t \in [0, 1]\}.$$

We denote by X^* the dual space of X , while $\langle \cdot, \cdot \rangle$ stands for the duality pairing between X^* and X . A function $\Phi: X \rightarrow \mathbb{R}$ is called locally Lipschitz continuous when to every $x \in X$ there correspond a neighbourhood V_x of x and a constant $L_x \geq 0$ such that

$$|\Phi(z) - \Phi(w)| \leq L_x \|z - w\| \quad \text{for all } z, w \in V_x.$$

If $x, z \in X$, we write $\Phi^0(x; z)$ for the generalized directional derivative of Φ at the point x along the direction z , namely

$$\Phi^0(x; z) = \limsup_{w \rightarrow x, t \rightarrow 0^+} \frac{\Phi(w + tz) - \Phi(w)}{t}.$$

It is known ([5, Proposition 2.1.1]) that Φ^0 is upper semicontinuous on $X \times X$. The generalized gradient of the function Φ in x , denoted by $\partial\Phi(x)$, is the set

$$\partial\Phi(x) = \{x^* \in X^* : \langle x^*, z \rangle \leq \Phi^0(x; z) \text{ for all } z \in X\}.$$

Proposition 2.1.2 of [5] ensures that $\partial\Phi(x)$ turns out nonempty, convex, in addition to weak* compact.

Let X be reflexive and let $\alpha: X \rightarrow]-\infty, \infty]$ be convex, proper and lower semicontinuous. The function α is continuous on $\text{int}(D_\alpha)$, where

$$D_\alpha = \{x \in X : \alpha(x) < \infty\},$$

vide for instance [6, Exercise 1, p. 296]. If $\partial\alpha(x)$ indicates the sub-differential of α at the point $x \in X$ while $D_{\partial\alpha} = \{x \in X : \partial\alpha(x) \neq \emptyset\}$ then Theorem 23.5 in the same reference gives $\text{int}(D_\alpha) = \text{int}(D_{\partial\alpha})$. Moreover, one has

PROPOSITION 1.1. *Suppose $x \in \text{int}(D_\alpha)$. Then for every $x_n \rightarrow x$ in X and every $z_n^* \in \partial\alpha(x_n)$, $n \in \mathbb{N}$, there exist $z^* \in \partial\alpha(x)$ as well as a subsequence $\{z_{r_n}^*\}$ of $\{z_n^*\}$ satisfying $z_{r_n}^* \rightarrow z^*$ in X^* .*

PROOF. Recall that the multifunction $\partial\alpha: \text{int}(D_{\partial\alpha}) \rightarrow 2^{X^*}$ takes convex weakly closed values, is locally bounded, and upper semicontinuous with respect to the strong topology of X and the weak one in X^* ; see [6, Section 23]. Hence, by the assumptions, $\{z_n^*\}$ turns out bounded. Since X is reflexive, we can find a subsequence $\{z_{r_n}^*\}$ of $\{z_n^*\}$ such that $z_{r_n}^* \rightarrow z^*$ in X^* . The upper semicontinuity of $\partial\alpha$ then implies $z^* \in \partial\alpha(x)$. \square

Finally, if $h: X \rightarrow]-\infty, \infty]$ and $a \in \mathbb{R}$, we write

$$h_a = \{x \in X : h(x) \leq a\}, \quad h^a = \{x \in X : a \leq h(x)\}.$$

2. A deformation result

Let $(X, \|\cdot\|)$ be a real reflexive Banach space and let g be a function on X fulfilling the structural hypothesis:

(H_g) $g = \Psi + \beta$, where $\Psi: X \rightarrow \mathbb{R}$ is locally Lipschitz continuous while $\beta: X \rightarrow]-\infty, \infty[$ is concave, proper and upper semicontinuous.

We say that $u \in X$ is a critical point of g when

$$\Psi^0(u; u - x) + \beta(u) - \beta(x) \geq 0 \quad \text{for all } x \in X.$$

Given a real number c , write

$$K_c(g) = \{u \in X : g(u) = c, u \text{ is a critical point of } g\}.$$

We denote by D_β the set $\{x \in X : \beta(x) > -\infty\}$ while

$$\partial\beta(x) = -\partial(-\beta)(x), \quad x \in X.$$

The following assumption will be posited in the sequel.

(g) *A and B are two nonempty closed subsets of X such that*

$$A \cap B = \emptyset, \quad A \subseteq g^c, \quad B \subseteq g_c, \quad K_c(g) \cap B = \emptyset.$$

Moreover, there exists $\varepsilon_0 > 0$ satisfying $N_{\varepsilon_0}(B) \subseteq \text{int}(D_\beta)$.

REMARK 2.1. Because of Proposition 1.1, the conditions $x \in N_{\varepsilon_0}(B)$, $x_n \rightarrow x$ in X , $z_n^* \in \partial\beta(x_n)$, $n \in \mathbb{N}$, yield a subsequence $\{z_{r_n}^*\}$ of $\{z_n^*\}$ weakly converging in X^* to some point $z^* \in \partial\beta(x)$.

We shall also suppose that the function g complies with the next Palais–Smale condition around B at the level c :

(PS) $_{g,B,c}$ *Each sequence $\{x_n\} \subseteq X$ such that $d(x_n, B) \rightarrow 0$, $g(x_n) \rightarrow c$, and*

$$\Psi^0(x_n; x_n - x) + \beta(x_n) - \beta(x) \geq -\varepsilon_n \|x_n - x\| \quad \text{for all } n \in \mathbb{N}, \quad x \in X,$$

where $\varepsilon_n \rightarrow 0^+$, possesses a convergent subsequence.

LEMMA 2.2. *Let (H_g) , (g), and (PS) $_{g,B,c}$ be fulfilled. Then there exist $\varepsilon_1 \in]0, \varepsilon_0[$, $\sigma > 0$ such that for every $x \in N_{\varepsilon_1}(B) \cap g^{c-\varepsilon_1} \cap g_{c+\varepsilon_1}$, $x^* \in \partial\Psi(x)$, $z^* \in \partial\beta(x)$ one has $\|x^* + z^*\|_{X^*} \geq \sigma$.*

PROOF. If the conclusion were false one could construct three sequences $\{x_n\} \subseteq X$, $\{x_n^*\}, \{z_n^*\} \subseteq X^*$ having the following properties:

$$(2.1) \quad d(x_n, B) \rightarrow 0,$$

$$(2.2) \quad g(x_n) \rightarrow c,$$

$$(2.3) \quad x_n^* \in \partial\Psi(x_n) \quad \text{and} \quad z_n^* \in \partial\beta(x_n) \quad \text{for all } n \in \mathbb{N},$$

$$(2.4) \quad \|x_n^* + z_n^*\|_{X^*} \rightarrow 0.$$

From (2.3) we obtain easily

$$(2.5) \quad \Psi^0(x_n; x_n - x) + \beta(x_n) - \beta(x) \geq -\|x_n^* + z_n^*\|_{X^*} \|x_n - x\|$$

for all $n \in \mathbb{N}$, $x \in X$. Setting $\varepsilon_n = \|x_n^* + z_n^*\|_{X^*}$ and using (PS) $_{g,B,c}$ as well as (2.1), (2.2), (2.4), inequality (2.5) produces $x_n \rightarrow u$ in X for some $u \in X$, where a subsequence is considered when necessary. Since Ψ^0 and β are upper semicontinuous, this forces

$$(2.6) \quad \Psi^0(u; u - x) + \beta(u) - \beta(x) \geq 0 \quad \text{for all } x \in X,$$

namely u is a critical point of g . By (2.1) and (g) we then infer $u \in \text{int}(D_\beta)$. Thus $\beta(x_n) \rightarrow \beta(u)$, which leads to $u \in K_c(g)$ on account of (2.2). However, via (2.1) we also have $u \in B$, against assumption (g). \square

LEMMA 2.3. *Suppose the function g satisfies (H_g) , (g) , and $(PS)_{g,B,c}$ while ε_1, σ are as in Lemma 2.2. Then to every $x \in N_{\varepsilon_1}(B) \cap g^{c-\varepsilon_1} \cap g_{c+\varepsilon_1}$ there corresponds a point $\xi_x \in X$ such that*

$$(2.7) \quad \|\xi_x\| = 1, \quad \langle x^* + z^*, \xi_x \rangle \geq \sigma \quad \text{for all } x^* \in \partial\Psi(x), z^* \in \partial\beta(x).$$

PROOF. Fix $x \in N_{\varepsilon_1}(B) \cap g^{c-\varepsilon_1} \cap g_{c+\varepsilon_1} \subseteq D_{\partial\beta}$. Since $\partial\Psi(x)$ and $\partial\beta(x)$ are nonempty and convex, the same holds for $\partial\Psi(x) + \partial\beta(x)$. Let us show that this set is also closed. Pick $\{x_n^*\} \subseteq \partial\Psi(x)$ and $\{z_n^*\} \subseteq \partial\beta(x)$ complying with $x_n^* + z_n^* \rightarrow u^*$ in X^* . The reflexivity of X as well as Proposition 2.1.2 in [5] yield $x^* \in \partial\Psi(x)$ such that, taking a subsequence if necessary, $x_n^* \rightharpoonup x^*$. Hence, $z_n^* \rightharpoonup u^* - x^*$. The choice of $\{z_n^*\}$ clearly forces $u^* - x^* \in \partial\beta(x)$, from which the assertion follows.

Next observe that, by Lemma 2.1, $0 \notin \partial\Psi(x) + \partial\beta(x)$. Through Corollary III.20 in [3] we thus obtain $u^* \in \partial\Psi(x), v^* \in \partial\beta(x)$ fulfilling

$$B_{\delta^*} \cap (\partial\Psi(x) + \partial\beta(x)) = \emptyset, \quad \text{where } \delta^* = \|u^* + v^*\|_{X^*}.$$

Now, the Hahn–Banach Theorem [3, Theorem I.6] provides a point $\xi_x \in X$ with the properties $\|\xi_x\| = 1$ and, whenever $x^* \in \partial\Psi(x), z^* \in \partial\beta(x)$,

$$\langle x^* + z^*, \xi_x \rangle \geq \langle w^*, \xi_x \rangle \quad \text{for all } w^* \in B_{\delta^*}.$$

Since

$$\|u^* + v^*\|_{X^*} = \|u^* + v^*\|_{X^*} \|\xi_x\| = \max\{\langle w^*, \xi_x \rangle : w^* \in \overline{B_{\delta^*}}\},$$

the above inequality and Lemma 2.1 lead to

$$\langle x^* + z^*, \xi_x \rangle \geq \|u^* + v^*\|_{X^*} \geq \sigma \quad \text{for all } x^* \in \partial\Psi(x), z^* \in \partial\beta(x),$$

as claimed. \square

LEMMA 2.4. *Let (H_g) , (g) , $(PS)_{g,B,c}$ be satisfied and let ε_1, σ be like in Lemma 2.2. Then for every $x \in N_{\varepsilon_1}(B) \cap g^{c-\varepsilon_1} \cap g_{c+\varepsilon_1}$ there exists $\delta_x > 0$ such that*

$$\langle x^* + z^*, \xi_x \rangle > \frac{\sigma}{2} \quad \text{for all } x^* \in \partial\Psi(x'), z^* \in \partial\beta(x''), x', x'' \in B(x, \delta_x),$$

where ξ_x is given by Lemma 2.3.

PROOF. If the conclusion were false we could find $x \in N_{\varepsilon_1}(B) \cap g^{c-\varepsilon_1} \cap g_{c+\varepsilon_1}$, $\{x'_n\}, \{x''_n\} \subseteq X$, and $\{x_n^*\}, \{z_n^*\} \subseteq X^*$ fulfilling the following conditions:

$$(2.8) \quad x'_n \rightarrow x, \quad x_n^* \in \partial\Psi(x'_n) \quad \text{for all } n \in \mathbb{N},$$

$$(2.9) \quad x''_n \rightarrow x, \quad z_n^* \in \partial\beta(x''_n) \quad \text{for all } n \in \mathbb{N},$$

$$(2.10) \quad \langle x_n^* + z_n^*, \xi_x \rangle \leq \frac{\sigma}{2} \quad \text{for all } n \in \mathbb{N}.$$

Due to the reflexivity of X and (2.8), Proposition 2.1.2 in [5] yields $x^* \in X^*$ such that $x_n^* \rightharpoonup x^*$ in X^* , where a subsequence is considered when necessary, while Proposition 2.1.5 of the same reference forces $x^* \in \partial\Psi(x)$. From (2.10) we thus get

$$(2.11) \quad \limsup_{n \rightarrow \infty} \langle z_n^*, \xi_x \rangle \leq \frac{\sigma}{2} - \langle x^*, \xi_x \rangle.$$

Now, exploiting (2.9) and Remark 2.1 provides $z^* \in \partial\beta(x)$ and a subsequence $\{z_{r_n}^*\}$ of $\{z_n^*\}$ which comply with

$$\langle z^*, \xi_x \rangle = \lim_{n \rightarrow \infty} \langle z_{r_n}^*, \xi_x \rangle.$$

By (2.11) it implies $\langle x^* + z^*, \xi_x \rangle \leq \sigma/2$, against (2.7). □

The next deformation theorem represents the main result of this section. It extends [7, Lemma 1.1] and [11, Theorem 1.1] to the framework of the present paper.

THEOREM 2.5. *Assume the function g satisfies (H_g) , (g) , $(PS)_{g,B,c}$ and the set $N_{\varepsilon_1}(B) \cap g^{c-\varepsilon_1} \cap g_{c+\varepsilon_1}$, with ε_1 like in Lemma 2.1, is closed. Then there exist $\varepsilon > 0$ and a homeomorphism $\eta: X \rightarrow X$ having the following properties:*

- (a) $\eta(x) = x$ for every $x \in A$,
- (b) $\eta(B) \subseteq g_{c-\varepsilon}$.

PROOF. The family of balls $\mathcal{B} = \{B(x, \delta_x) : x \in N_{\varepsilon_1}(B) \cap g^{c-\varepsilon_1} \cap g_{c+\varepsilon_1}\}$ constructed through Lemma 2.4 represents an open covering of $N_{\varepsilon_1}(B) \cap g^{c-\varepsilon_1} \cap g_{c+\varepsilon_1}$. Since, by [8, Theorem VIII.2.4], this set is paracompact, \mathcal{B} possesses an open locally finite refinement $\mathcal{V} = \{V_i : i \in I\}$. Moreover, to each $i \in I$ there corresponds $\xi_i \in X$ such that $\|\xi_i\| = 1$ as well as

$$(2.12) \quad \langle x^* + z^*, \xi_i \rangle > \frac{\sigma}{2} \quad \text{for all } x^* \in \partial\Psi(x'), z^* \in \partial\beta(x''), x', x'' \in V_i.$$

Shrink \mathcal{V} to an open locally finite covering $\mathcal{W} = \{W_i : i \in I\}$ fulfilling $\text{cl}(W_i) \subseteq V_i$ for every $i \in I$ (vide [8, Theorems VIII.2.2 and VII.6.1]) and write

$$d_i(x) = d(x, X \setminus W_i), \quad x \in X.$$

Evidently, d_i is Lipschitz continuous while $d_i(x) = 0$ means $x \in X \setminus W_i$. Thus, defining

$$W = \bigcup_{i \in I} W_i, \quad \rho_i(x) = \frac{d_i(x)}{\sum_{j \in I} d_j(x)} \quad \text{for all } x \in W, i \in I,$$

we obtain a family of locally Lipschitz continuous functions $\rho_i: W \rightarrow [0, 1]$, $i \in I$, with the following properties:

$$\text{supp } \rho_i = \text{cl}(W_i) \subseteq V_i \quad \text{for all } i \in I, \quad \sum_{i \in I} \rho_i(x) = 1 \quad \text{for all } x \in W.$$

Now, let $\theta: W \rightarrow X$ be given by

$$\theta(x) = \sum_{i \in I} \rho_i(x) \xi_i, \quad x \in W.$$

Clearly, θ is locally Lipschitz continuous and $\|\theta(x)\| \leq 1$ in W . Put, for every $x \in X$,

$$(2.13) \quad \Theta(x) = \begin{cases} -\varepsilon_1 \theta(x) & \text{if } x \in W, \\ 0 & \text{otherwise.} \end{cases}$$

The function $\Theta: X \rightarrow X$ turns out locally Lipschitz continuous as well. To see this, we simply note that the set $\cup_{i \in I} \text{supp } \rho_i$ is closed, which comes from the local finiteness of \mathcal{V} , while $\Theta(x) = 0$ in $X \setminus \cup_{i \in I} \text{supp } \rho_i$. Moreover, one has $\|\Theta(x)\| \leq \varepsilon_1$ for all $x \in X$. Hence, the basic existence-uniqueness theorem for ordinary differential equations in Banach spaces provides a function $\gamma \in C^0(\mathbb{R} \times X, X)$ satisfying

$$(2.14) \quad \frac{d\gamma(t, x)}{dt} = \Theta(\gamma(t, x)), \quad \gamma(0, x) = x \quad \text{for all } (t, x) \in \mathbb{R} \times X.$$

Next, define $B_1 = \gamma([0, 1] \times B)$. If $x \in B$ then

$$\|\gamma(t, x) - x\| = \left\| \int_0^t \frac{d\gamma(\tau, x)}{d\tau} d\tau \right\| = \left\| \int_0^t \Theta(\gamma(\tau, x)) d\tau \right\| \leq \varepsilon_1 \quad \text{for all } t \in [0, 1],$$

namely

$$(2.15) \quad B_1 \subseteq N_{\varepsilon_1}(B).$$

Let us verify that the set B_1 is closed. To this end, pick a sequence $\{y_n\} \subseteq B_1$ converging to some $y \in X$. Since $y_n = \gamma(t_n, x_n)$ with $(t_n, x_n) \in [0, 1] \times B$, by possibly taking a subsequence we can suppose $t_n \rightarrow t$ in $[0, 1]$. Write $z_n = \gamma(t, x_n)$, $n \in \mathbb{N}$, and observe that

$$\|y_n - z_n\| = \left\| \int_t^{t_n} \frac{d\gamma(\tau, x_n)}{d\tau} d\tau \right\| \leq \varepsilon_1 |t_n - t| \quad \text{for all } n \in \mathbb{N}.$$

Therefore, $z_n \rightarrow y$. Through the properties of γ we thus achieve

$$x_n = \gamma(-t, z_n) \rightarrow \gamma(-t, y).$$

Setting $x = \gamma(-t, y)$ one has $x_n \rightarrow x$, the point x lies in B because B is closed, while $y = \gamma(t, x) \in \gamma([0, 1] \times B) = B_1$, which represents the desired conclusion.

Our next goal is to show that

$$(2.16) \quad \text{for all } x \in B \text{ the function } t \mapsto g(\gamma(t, x)) \text{ turns out decreasing on } [0, 1].$$

Obviously, the claim will be proved once we see that to each $x_0 \in B$, $t_0 \in [0, 1]$ it corresponds $\delta_0 > 0$ fulfilling

$$(2.17) \quad \frac{g(\gamma(t, x_0)) - g(\gamma(t_0, x_0))}{t - t_0} \leq 0 \quad \text{for all } t \in [0, 1] \cap B(t_0, \delta_0) \setminus \{t_0\}.$$

So, fix $(t_0, x_0) \in [0, 1] \times B$. If $\gamma(t_0, x_0) \notin \bigcup_{i \in I} \text{supp } \rho_i$ then one can easily find $\delta_0 > 0$ such that

$$\gamma(t, x_0) \notin \bigcup_{i \in I} \text{supp } \rho_i \quad \text{for all } t \in [0, 1] \cap B(t_0, \delta_0).$$

It implies $\Theta(\gamma(t, x_0)) = 0$ and hence $\gamma(t, x_0) = \gamma(t_0, x_0)$ in $[0, 1] \cap B(t_0, \delta_0)$, from which (2.17) follows at once. Suppose now $\gamma(t_0, x_0) \in \bigcup_{i \in I} \text{supp } \rho_i$. Since the family $\{\text{supp } \rho_i : i \in I\}$ is locally finite, there exists $\delta' > 0$ satisfying

$$\text{supp } \rho_i \cap B(\gamma(t_0, x_0), \delta') \neq \emptyset$$

for a finite number of $i \in I$, say i_1, \dots, i_p . Consequently,

$$(2.18) \quad \text{supp } \rho_i \cap B(\gamma(t_0, x_0), \delta') \begin{cases} \neq \emptyset & \text{if } i \in \{i_1, \dots, i_p\}, \\ = \emptyset & \text{otherwise.} \end{cases}$$

Let i'_1, \dots, i'_q be the elements in $\{i_1, \dots, i_p\}$ such that $\gamma(t_0, x_0) \in \text{supp } \rho_{i'_j}$ whenever $j = 1, \dots, q$. One clearly has

$$(2.19) \quad \begin{aligned} \gamma(t_0, x_0) \in \text{supp } \rho_{i'_j} & \quad \text{for all } j = 1, \dots, q, \\ \delta_i = d(\gamma(t_0, x_0), \text{supp } \rho_i) > 0 & \quad \text{for all } i \in \{i_1, \dots, i_p\} \setminus \{i'_1, \dots, i'_q\}. \end{aligned}$$

Pick $\delta'' \in]0, \delta'[$ with the following properties:

$$\begin{aligned} \delta'' < \delta_i & \quad \text{for all } i \in \{i_1, \dots, i_p\} \setminus \{i'_1, \dots, i'_q\}, \\ B(\gamma(t_0, x_0), \delta'') \subseteq V_{i'_j} & \quad \text{for all } j = 1, \dots, q. \end{aligned}$$

Thanks to (2.18) and (2.19) we get

$$(2.20) \quad \begin{aligned} \text{supp } \rho_i \cap B(\gamma(t_0, x_0), \delta'') = \emptyset & \quad \text{for all } i \in I \setminus \{i'_1, \dots, i'_q\}, \\ B(\gamma(t_0, x_0), \delta'') \subseteq V_{i'_j} & \quad \text{for all } j = 1, \dots, q. \end{aligned}$$

Finally, choose $\delta_0 > 0$ such that

$$(2.21) \quad \gamma(t, x_0) \in B(\gamma(t_0, x_0), \delta'') \quad \text{for all } t \in [0, 1] \cap B(t_0, \delta_0).$$

Let $t \in [0, 1] \cap B(t_0, \delta_0) \setminus \{t_0\}$. Suppose $t > t_0$. Since $x_0 \in B$, inclusion (2.15) and assumption (g) force

$$\gamma(\tau, x_0) \in N_{\varepsilon_1}(B) \subseteq D_{\partial\beta} \quad \text{for all } \tau \in [t_0, t].$$

Exploiting Theorem 2.3.7 in [5] as well as the definition of $\partial\beta$ we have, for suitable $x \in [\gamma(t_0, x_0), \gamma(t, x_0)]$, $x^* \in \partial\Psi(x)$, and $z^* \in \partial\beta(\gamma(t_0, x_0))$,

$$(2.22) \quad \begin{aligned} g(\gamma(t, x_0)) - g(\gamma(t_0, x_0)) &\leq \langle x^* + z^*, \gamma(t, x_0) - \gamma(t_0, x_0) \rangle \\ &= \left\langle x^* + z^*, \int_{t_0}^t \frac{d\gamma(\tau, x_0)}{d\tau} d\tau \right\rangle = \int_{t_0}^t \langle x^* + z^*, \Theta(\gamma(\tau, x_0)) \rangle d\tau. \end{aligned}$$

On account of (2.21), (2.20) it results

$$\Theta(\gamma(\tau, x_0)) = -\varepsilon_1 \sum_{j=1}^q \rho_{i'_j}(\gamma(\tau, x_0)) \xi_{i'_j} \quad \text{for all } \tau \in [t_0, t].$$

Bearing in mind that $x, \gamma(t_0, x_0) \in V_{i'_j}$, $j = 1, \dots, q$, inequality (2.12) can be applied and we obtain

$$(2.23) \quad \langle x^* + z^*, \Theta(\gamma(\tau, x_0)) \rangle \leq -\frac{\varepsilon_1 \sigma}{2} \sum_{j=1}^q \rho_{i'_j}(\gamma(\tau, x_0)) = -\frac{\varepsilon_1 \sigma}{2}$$

for every $\tau \in [t_0, t]$. Hence, by (2.22),

$$\frac{g(\gamma(t, x_0)) - g(\gamma(t_0, x_0))}{t - t_0} \leq -\frac{\varepsilon_1 \sigma}{2} < 0.$$

Now, suppose $t < t_0$. Gathering (2.22), with t_0 and t exchanged, and (2.23) yields

$$g(\gamma(t_0, x_0)) - g(\gamma(t, x_0)) \leq \int_t^{t_0} \langle x^* + z^*, \Theta(\gamma(\tau, x_0)) \rangle d\tau \leq -\frac{\varepsilon_1 \sigma}{2}(t_0 - t),$$

which leads to the conclusion (as for $t > t_0$) once more. Thus (2.17) is completely achieved.

We next claim that

$$(2.24) \quad A \cap B_1 = \emptyset.$$

Indeed, if (2.24) were false one could find $(t_0, x_0) \in]0, 1] \times B$ fulfilling $\gamma(t_0, x_0) \in A$. Because of assumption (g) and (2.16) this easily implies

$$(2.25) \quad g(\gamma(t, x_0)) = c \quad \text{for all } t \in [0, t_0].$$

Hence, due to (2.15), $\gamma(t, x_0) \in N_{\varepsilon_1}(B) \cap g^{c-\varepsilon_1} \cap g_{c+\varepsilon_1}$ for all $t \in [0, t_0]$ and, in particular, $\gamma(t_0, x_0) \in \bigcup_{i \in I} \text{supp } \rho_i$. Arguing as before gives $\delta_0 > 0$ such that

$$\frac{g(\gamma(t, x_0)) - g(\gamma(t_0, x_0))}{t - t_0} \leq -\frac{\varepsilon_1 \sigma}{2} \quad \text{for all } t \in [0, 1] \cap B(t_0, \delta_0) \setminus \{t_0\}.$$

Through (2.25) we then obtain, whenever $t \in]t_0 - \delta_0, t_0[\cap [0, 1]$,

$$\begin{aligned} g(\gamma(t_0, x_0)) &= \frac{g(\gamma(t_0, x_0)) - g(\gamma(t, x_0))}{t_0 - t} (t_0 - t) + g(\gamma(t, x_0)) \\ &\leq -\frac{\varepsilon_1 \sigma}{2} (t_0 - t) + c < c, \end{aligned}$$

which contradicts (2.25) written for $t = t_0$.

Note that from (2.24) it follows $d(x, A) + d(x, B_1) > 0$ at each point $x \in X$. Let $A_1 = \{x \in X : \zeta_1(x) \leq 1/2\}$, where

$$\zeta_1(x) = \frac{d(x, A)}{d(x, A) + d(x, B_1)}, \quad x \in X.$$

Since the function ζ_1 is evidently continuous, the set A_1 turns out closed. Moreover, one has $A \subseteq \text{int}(A_1)$ as well as $A_1 \cap B_1 = \emptyset$. Putting

$$\zeta(x) = \frac{d(x, A_1)}{d(x, A_1) + d(x, B_1)} \quad \text{for all } x \in X$$

provides a locally Lipschitz continuous function $\zeta: X \rightarrow [0, 1]$ such that

$$(2.26) \quad \zeta|_{A_1} \equiv 0, \quad \zeta|_{B_1} \equiv 1.$$

Thanks to the properties of Θ the function $\Lambda: X \rightarrow X$ given by

$$(2.27) \quad \Lambda(x) = \zeta(x)\Theta(x), \quad x \in X,$$

comes bounded and locally Lipschitz continuous. Indicate with $\chi: \mathbb{R} \times X \rightarrow X$ the solution of the Cauchy problem

$$\frac{d\chi(t, x)}{dt} = \Lambda(\chi(t, x)), \quad \chi(0, x) = x,$$

and define

$$(2.28) \quad \varepsilon = \varepsilon_1 \min \left\{ \frac{\sigma}{2}, 1 \right\}, \quad \eta(x) = \chi(1, x) \quad \text{for all } x \in X.$$

Classical results concerning ordinary differential equations in Banach spaces ensure that $\eta: X \rightarrow X$ is a homeomorphism. If $x \in A$ then $x \in \text{int}(A_1)$ and, because of (2.26), $\Lambda \equiv 0$ on some neighbourhood of x . This implies immediately $\eta(x) = x$, thus showing assertion (a).

Finally, the proof is accomplished once we verify (b). Suppose on the contrary that there exists $x_0 \in B$ satisfying

$$(2.29) \quad g(\eta(x_0)) > c - \varepsilon.$$

Through (2.27) and (2.26) we obtain

$$\Lambda(\gamma(t, x_0)) = \Theta(\gamma(t, x_0)) \quad \text{for all } t \in [0, 1],$$

from which it follows, bearing in mind (2.14),

$$\frac{d\gamma(t, x_0)}{dt} = \Lambda(\gamma(t, x_0)) \quad \text{in } [0, 1], \quad \gamma(0, x_0) = x_0.$$

By uniqueness we thus have

$$(2.30) \quad \gamma(t, x_0) = \chi(t, x_0) \quad \text{for all } t \in [0, 1].$$

Fix $t_0 \in [0, 1]$. Since $x_0 \in B$, (2.29), (2.28), (2.30), (2.16), and (g) lead to

$$c - \varepsilon < g(\gamma(t, x_0)) = g(\chi(t, x_0)) < c + \varepsilon, \quad t \in [0, 1],$$

while gathering (2.27) and (2.13) together yields

$$\|\chi(t, x_0) - x_0\| = \left\| \int_0^t \frac{d\chi(\tau, x_0)}{d\tau} d\tau \right\| = \left\| \int_0^t \Lambda(\chi(\tau, x_0)) d\tau \right\| \leq \varepsilon_1 t \leq \varepsilon_1$$

for all $t \in [0, 1]$. Therefore,

$$\gamma(t, x_0) = \chi(t, x_0) \in N_{\varepsilon_1}(B) \cap g^{c-\varepsilon_1} \cap g_{c+\varepsilon_1} \quad \text{for all } t \in [0, 1].$$

Using the compactness of $[0, 1]$ and the fact that \mathcal{W} is a locally finite covering of $N_{\varepsilon_1}(B) \cap g^{c-\varepsilon_1} \cap g_{c+\varepsilon_1}$ we can find a decomposition $0 = t_0 < t_1 < \dots < t_{p-1} < t_p = 1$ of $[0, 1]$ such that to every $j \in \{1, \dots, p\}$ there corresponds a finite family $I_j \subseteq I$ for which

$$\begin{aligned} [\gamma(t_{j-1}, x_0), \gamma(t_j, x_0)] &\subseteq W_i \subseteq \text{supp } \rho_i \subseteq V_i, \\ \gamma(\tau, x_0) &\in W_i \quad \text{for all } \tau \in [t_{j-1}, t_j] \end{aligned}$$

whenever $i \in I_j$. By [5, Theorem 2.3.7] and the definition of $\partial\beta$ there exist $x_j \in [\gamma(t_{j-1}, x_0), \gamma(t_j, x_0)]$, $x_j^* \in \partial\Psi(x_j)$, and $z_j^* \in \partial\beta(\gamma(t_{j-1}, x_0))$ fulfilling

$$\begin{aligned} g(\gamma(t_j, x_0)) - g(\gamma(t_{j-1}, x_0)) &\leq \langle x_j^* + z_j^*, \gamma(t_j, x_0) - \gamma(t_{j-1}, x_0) \rangle \\ &= \left\langle x_j^* + z_j^*, \int_{t_{j-1}}^{t_j} \frac{d\gamma(\tau, x_0)}{d\tau} d\tau \right\rangle = \int_{t_{j-1}}^{t_j} \langle x_j^* + z_j^*, \Theta(\gamma(\tau, x_0)) \rangle d\tau. \end{aligned}$$

Due to (2.13) this inequality becomes

$$g(\gamma(t_j, x_0)) - g(\gamma(t_{j-1}, x_0)) \leq -\varepsilon_1 \sum_{i \in I_j} \langle x_j^* + z_j^*, \xi_i \rangle \int_{t_{j-1}}^{t_j} \rho_i(\gamma(\tau, x_0)) d\tau.$$

Now, since $x_j, \gamma(t_{j-1}, x_0) \in V_i$ for all $i \in I_j$, using (2.12) we get

$$g(\gamma(t_j, x_0)) - g(\gamma(t_{j-1}, x_0)) \leq -\frac{\varepsilon_1 \sigma}{2} \int_{t_{j-1}}^{t_j} \sum_{i \in I_j} \rho_i(\gamma(\tau, x_0)) d\tau = -\frac{\varepsilon_1 \sigma}{2} (t_j - t_{j-1}).$$

Hence, as j was arbitrary,

$$g(\gamma(1, x_0)) - g(\gamma(0, x_0)) = \sum_{j=1}^p [g(\gamma(t_j, x_0)) - g(\gamma(t_{j-1}, x_0))] \leq -\frac{\varepsilon_1 \sigma}{2}.$$

Taking account of (2.28), (2.30), and (g) one finally has

$$g(\eta(x_0)) \leq -\frac{\varepsilon_1 \sigma}{2} + g(x_0) \leq c - \varepsilon,$$

which contradicts (2.29). The proof is thus complete. \square

3. Existence of critical points

In this section we establish a version of the minimax principle by Motreanu and Panagiotopoulos ([10, Theorem 3.2]) where the usual strict inequality is weakened to allow also equality; see Theorem 3.1 below. It can be considered as a further contribution to the study initiated in [7] for the smooth case and then continued in [11] regarding the locally Lipschitz continuous setting. Classical results on the same subject are those by Pucci and Serrin ([13, Theorem 1]), Rabinowitz ([15, Theorem 2.13]), Ghoussoub and Preiss ([9, Theorem 1.bis]).

The next definition of linking is adopted here; vide [10, Definition 3.3]. Let $(X, \|\cdot\|)$ be a real reflexive Banach space, let Q be a compact topological manifold in X with nonempty boundary (according to [16, p. 297]) ∂Q , and let S be a nonempty closed subset of X . Write

$$\Gamma = \{\gamma \in C^0(Q, X) : \gamma|_{\partial Q} = \text{id}|_{\partial Q}\}.$$

We say that Q links with S provided $\partial Q \cap S = \emptyset$ and for every $\gamma \in \Gamma$ one has $\gamma(Q) \cap S \neq \emptyset$. Now, let f be a function on X fulfilling the structure hypothesis

$$(H_f) \quad f = \Phi + \alpha, \text{ where } \Phi: X \rightarrow \mathbb{R} \text{ is locally Lipschitz continuous while } \alpha: X \rightarrow]-\infty, \infty] \text{ is convex, proper and lower semicontinuous.}$$

A critical point of f is a point $u \in X$ at which

$$\Phi^0(u; x - u) + \alpha(x) - \alpha(u) \geq 0 \quad \text{for all } x \in X.$$

Given a real number c , we put

$$K_c(f) = \{u \in X : f(u) = c, u \text{ is a critical point of } f\}.$$

The Palais–Smale condition around the set S at the level c takes the form

(PS) $_{f,S,c}$ Each sequence $\{x_n\} \subseteq X$ such that $d(x_n, S) \rightarrow 0$, $f(x_n) \rightarrow c$, and

$$\Phi^0(x_n; x - x_n) + \alpha(x) - \alpha(x_n) \geq -\varepsilon_n \|x_n - x\| \quad \text{for all } n \in \mathbb{N}, x \in X,$$

where $\varepsilon_n \rightarrow 0^+$, possesses a convergent subsequence.

(PS) $_{f,c}$ will denote the above condition without the request $d(x_n, S) \rightarrow 0$.

THEOREM 3.1. *Suppose Q and S link while the function f satisfies the following assumptions in addition to (H_f) .*

- (f₁) $\sup_{x \in Q} f(x) < \infty$.
- (f₂) $\partial Q \subseteq f_a$ and $S \subseteq f^a$ for some $a \in \mathbb{R}$.
- (f₃) *Setting*

$$c = \inf_{\gamma \in \Gamma} \sup_{z \in \gamma(Q)} f(z)$$

either (PS) $_{f,c}$ or (PS) $_{f,S,c}$ holds according to whether $c > a$ or $c = a$. Further, there exists $\varepsilon_0 > 0$ such that

- (f_{3.1}) $N_{\varepsilon_0}(S) \subseteq \text{int}(D_\alpha)$, and
 (f_{3.2}) the set $N_\delta(S) \cap f^{c-\delta} \cap f_{c+\delta}$, $\delta \in]0, \varepsilon_0[$, is closed.

Then one has

- (i₁) $c \geq a$,
 (i₂) $K_c(f) \setminus \partial Q \neq \emptyset$, and
 (i₃) $K_c(f) \cap S \neq \emptyset$ if $c = a$.

PROOF. We first note that $c < \infty$ because the function $\gamma = \text{id}|_Q$ lies in Γ while (f₁) gives $\sup_{z \in \gamma(Q)} f(z) < \infty$. Let us show (i₁). Since Q is linking with S , for every $\gamma \in \Gamma$ there exists $x \in Q$ such that $\gamma(x) \in S$. Thanks to (f₂) this forces $\sup_{z \in \gamma(Q)} f(z) \geq a$. As γ was arbitrary, we actually have

$$c = \inf_{\gamma \in \Gamma} \sup_{z \in \gamma(Q)} f(z) \geq a.$$

When $c > a$, by (f₂) again, it results $K_c(f) \setminus \partial Q = K_c(f)$. The same technique used to establish Theorem 3.2 of [10] ensures that $K_c(f) \neq \emptyset$, which yields (i₂) and completes the proof. So, let $c = a$. The conclusion will be achieved once we verify (i₃), because $\partial Q \cap S = \emptyset$. Suppose on the contrary that $K_c(f) \cap S = \emptyset$ and define $A = \partial Q$, $B = S$, $g = -f$. Then, bearing in mind the assumptions, the function g fulfils condition (PS) _{$g, B, -c$} while

$$A \cap B = \emptyset, \quad A \subseteq g^{-c}, \quad B \subseteq g_{-c}, \quad K_{-c}(g) \cap B = \emptyset.$$

Observe that for any $\delta > 0$ we have

$$N_\delta(B) \cap f^{c-\delta} \cap f_{c+\delta} = N_\delta(B) \cap g^{-c-\delta} \cap g_{-c+\delta}.$$

Consequently, by (f₃), both g and $-c$ satisfy all the hypotheses of Theorem 2.1. Thus, there exist $\varepsilon > 0$ as well as a homeomorphism $\eta: X \rightarrow X$ such that

$$(3.1) \quad \eta(x) = x \quad \text{for all } x \in \partial Q, \quad c + \varepsilon \leq f(\eta(x)) \quad \text{for all } x \in S.$$

Exploiting the definition of c produces, for some $\gamma_\varepsilon \in \Gamma$,

$$(3.2) \quad f(\gamma_\varepsilon(x)) < c + \varepsilon, \quad x \in Q.$$

Since Q links with S while $\eta^{-1} \circ \gamma_\varepsilon \in \Gamma$ we can find a point x_ε in Q fulfilling $\eta^{-1}(\gamma_\varepsilon(x_\varepsilon)) \in S$. So, due to (3.1) and (3.2),

$$c + \varepsilon \leq f(\eta(\eta^{-1}(\gamma_\varepsilon(x_\varepsilon)))) = f(\gamma_\varepsilon(x_\varepsilon)) < c + \varepsilon,$$

which is clearly impossible. \square

REMARK 3.2. When $\alpha \equiv 0$ the preceding result gives Theorem 2.1 by Motreanu and Varga ([11]), but with the above-mentioned definition of linking. To see this we simply note that, in view of [10, Proposition 3.1], the Palais–Smale condition around S at the level c adopted in [11] implies our (PS) _{f, S, c} .

Several classical results can be reformulated in the framework of the present paper through Theorem 3.1. As an example we state here the following versions of the Mountain Pass Theorem ([2, Theorem 2.1]) and the Saddle Point Theorem ([14, Theorem 4.6]).

THEOREM 3.3. *Let f be like in (H_f) . Suppose that*

(f₄) *there exist $x_1 \in X$, $r > 0$, and $a \in \mathbb{R}$ satisfying $\|x_1\| > r$ in addition to*

$$\max\{f(0), f(x_1)\} \leq a \leq f(x) \quad \text{for all } x \in \partial B_r,$$

(f₅) *assumption (f₃) of Theorem 3.1 holds with $Q = [0, x_1]$, $S = \partial B_r$.*

Then $c \geq a$ and $K_c(f) \setminus \{0, x_1\} \neq \emptyset$.

THEOREM 3.4. *Let $X = V \oplus E$, where $V \neq \{0\}$ is finite dimensional. If*

(f₆) *there are two real numbers $r > 0$ and a such that*

$$\sup_{x \in V \cap \overline{B}_r} f(x) < \infty, \quad \partial(V \cap \overline{B}_r) \subseteq f_a, \quad E \subseteq f^a,$$

(f₇) *hypothesis (f₃) of Theorem 3.1 holds for $S = E$,*

then $c \geq a$ and $K_c(f) \neq \emptyset$.

4. An application

In this section we exploit Theorem 3.1 to solve an elliptic variational-hemi-variational inequality in the sense of Panagiotopoulos (see [12]).

Let Ω be a nonempty, bounded, open subset of the real Euclidean N -space $(\mathbb{R}^N, |\cdot|)$, $N \geq 3$, having a smooth boundary $\partial\Omega$ and let $H_0^1(\Omega)$ be the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| := \left(\int_{\Omega} |\nabla u(x)|^2 dx \right)^{1/2}.$$

Denote by 2^* the critical exponent for the Sobolev embedding $H_0^1(\Omega) \subseteq L^p(\Omega)$. Recall that $2^* = 2N/(N - 2)$, if $p \in [1, 2^*]$ then there exists a constant $c_p > 0$ fulfilling $\|u\|_{L^p(\Omega)} \leq c_p \|u\|$ for all $u \in H_0^1(\Omega)$, and the embedding is compact whenever $p \in [1, 2^*[$; see for instance Proposition B.7 in [14].

Now, let $\{\lambda_n\}$ be the sequence of eigenvalues of the operator $-\Delta$ in $H_0^1(\Omega)$, with $0 < \lambda_1 < \dots \leq \lambda_n \leq \dots$, and let $\{\varphi_n\}$ be a corresponding sequence of eigenfunctions normalized as follows:

$$(4.1) \quad \begin{aligned} \|\varphi_n\|^2 &= 1 = \lambda_n \|\varphi_n\|_{L^2(\Omega)}^2, \quad n \in \mathbb{N}, \\ \int_{\Omega} \nabla \varphi_m(x) \cdot \nabla \varphi_n(x) dx &= \int_{\Omega} \varphi_m(x) \varphi_n(x) dx = 0 \quad \text{provided } m \neq n. \end{aligned}$$

If $j, k: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy the conditions

(a₁) *j, k are measurable with respect to each variable separately, and*

(a₂) there exist $c > 0$, $p \in [1, 2^*[$ such that

$$\max\{|j(x, t)|, |k(x, t)|\} \leq c(1 + |t|^{p-1}) \quad \text{for all } (x, t) \in \Omega \times \mathbb{R},$$

then the functions $J, K: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ given by

$$J(x, \xi) = \int_0^\xi j(x, t) dt, \quad K(x, \xi) = \int_0^\xi k(x, t) dt, \quad (x, \xi) \in \Omega \times \mathbb{R},$$

turn out well defined, $J(\cdot, \xi)$, $K(\cdot, \xi)$ are measurable, while $J(x, \cdot)$, $K(x, \cdot)$ are locally Lipschitz continuous. So it makes sense to consider their generalized directional derivatives J_x^0 , K_x^0 with respect to the variable ξ .

Let q be a positive integer such that $\lambda_q < \lambda_{q+1}$ and let $\lambda \in [\lambda_q, \lambda_{q+1}]$. Setting

$$V = \text{span}\{\varphi_1, \dots, \varphi_q\}, \quad W = \text{span}\{\varphi_1, \dots, \varphi_q, \varphi_{q+1}\}$$

one clearly has $H_0^1(\Omega) = V \oplus \mathbb{R}\varphi_{q+1} \oplus W^\perp$. Consequently, each $u \in H_0^1(\Omega)$ can be written as $u = u_1 + u_2 + u_3$ where $u_1 \in V$, $u_2 \in \mathbb{R}\varphi_{q+1}$, $u_3 \in W^\perp$.

We will also assume that

$$(a_3) \quad J(x, \xi) \leq (1/2)(\lambda/\lambda_q - 1)\lambda_1\xi^2 \quad \text{for all } (x, \xi) \in \Omega \times \mathbb{R},$$

$$(a_4) \quad K(x, \xi) \geq -(1/2)(1 - \lambda/\lambda_{q+1})\lambda_{q+2}\xi^2 \quad \text{for all } (x, \xi) \in \Omega \times \mathbb{R}.$$

Given a positive real number r_0 and a convex closed subset U of $H_0^1(\Omega)$ fulfilling

$$(4.2) \quad W \oplus \{u \in W^\perp : \|u\| \leq r_0\} \subseteq U,$$

we have the following elliptic variational-hemivariational inequality problem:

(P) Find $u \in U$, $u = u_1 + u_2 + u_3$, such that

$$\begin{aligned} & - \int_\Omega \nabla u_1(x) \cdot \nabla (v_1 - u_1)(x) dx - \frac{\lambda}{\lambda_{q+1}} \int_\Omega \nabla u_2(x) \cdot \nabla (v_2 - u_2)(x) dx \\ & \quad - \int_\Omega \nabla u_3(x) \cdot \nabla (v_3 - u_3)(x) dx + \lambda \int_\Omega u(x)(v - u)(x) dx \\ & \leq \int_\Omega J_x^0(u_1(x); v_1(x) - u_1(x)) dx + \int_\Omega K_x^0(u_3(x); v_3(x) - u_3(x)) dx \end{aligned}$$

for all $v \in U$, $v = v_1 + v_2 + v_3$.

THEOREM 4.1. *Suppose (a₁)–(a₄) hold. Then (P) possesses a nontrivial solution $u \in B_{r_0} \cap V^\perp$.*

PROOF. Pick $X = H_0^1(\Omega)$ and define

$$\begin{aligned} \Phi(u) = & \frac{1}{2}(\|u\|^2 - \lambda\|u\|_{L^2(\Omega)}^2) + \int_\Omega J(x, u_1(x)) dx \\ & - \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_{q+1}}\right) \|u_2\|^2 + \int_\Omega K(x, u_3(x)) dx, \end{aligned}$$

for $u \in H_0^1(\Omega)$. Owing to (a₂) the function $\Phi: H_0^1(\Omega) \rightarrow \mathbb{R}$ turns out locally Lipschitz continuous. Moreover, if for any $u \in H_0^1(\Omega)$ we put

$$\alpha(u) = \begin{cases} 0 & \text{when } u \in U, \\ \infty & \text{otherwise,} \end{cases} \quad \text{and} \quad f(u) = \Phi(u) + \alpha(u),$$

then hypothesis (H_f) is satisfied. Now, write

$$Q = (V \cap \overline{B}_\rho) \oplus [0, \rho \varphi_{q+1}], \quad S = \partial B_r \cap V^\perp,$$

where $r \in]0, r_0[$ while $\rho > r$. In view of [1, Lemma 4.1] (vide also [14, Proposition 5.9]) the compact topological manifold Q links with the closed set S . Since, by (4.2),

$$(4.3) \quad Q \subseteq W \subseteq U,$$

we also have $f|_Q = \Phi|_Q$, which evidently implies (f₁).

Let us next verify assumption (f₂) for $a = 0$. Each $u \in W$ can be written as $u = u_1 + u_2$ where $u_1 = \sum_{i=1}^q t_i \varphi_i$, $u_2 = t_{q+1} \varphi_{q+1}$, $t_1, \dots, t_{q+1} \in \mathbb{R}$. Through (4.3), (4.1), and (a₃) we obtain

$$\begin{aligned} (4.4) \quad f(u) &= \Phi(u) \\ &= \frac{1}{2} \sum_{i=1}^{q+1} \left(1 - \frac{\lambda}{\lambda_i}\right) t_i^2 + \int_\Omega J(x, u_1(x)) dx - \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_{q+1}}\right) t_{q+1}^2 \\ &\leq \frac{1}{2} \sum_{i=1}^q \left(1 - \frac{\lambda}{\lambda_i}\right) t_i^2 + \frac{1}{2} \left(\frac{\lambda}{\lambda_q} - 1\right) \lambda_1 \|u_1\|_{L^2(\Omega)}^2 \\ &\leq \frac{1}{2} \sum_{i=1}^q \left(1 - \frac{\lambda}{\lambda_q}\right) t_i^2 + \frac{1}{2} \left(\frac{\lambda}{\lambda_q} - 1\right) \lambda_1 \sum_{i=1}^q \frac{1}{\lambda_i} t_i^2 \\ &= \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_q}\right) \sum_{i=1}^q \left(1 - \frac{\lambda_1}{\lambda_i}\right) t_i^2 \leq 0 \end{aligned}$$

for all $u \in W$. Hence, on account of (4.3), $\partial Q \subseteq f_0$. Taking now $u \in V^\perp$ it results $u = u_2 + u_3$, where $u_2 = t_{q+1} \varphi_{q+1}$, $u_3 = \sum_{i=q+2}^\infty t_i \varphi_i$, $t_{q+1}, t_{q+2}, \dots \in \mathbb{R}$. Thanks to (a₄) and (4.1) we thus achieve

$$\begin{aligned} f(u) &\geq \Phi(u) = \frac{1}{2} \sum_{i=q+1}^\infty \left(1 - \frac{\lambda}{\lambda_i}\right) t_i^2 - \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_{q+1}}\right) t_{q+1}^2 + \int_\Omega K(x, u_3(x)) dx \\ &\geq \frac{1}{2} \sum_{i=q+2}^\infty \left(1 - \frac{\lambda}{\lambda_i}\right) t_i^2 - \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_{q+1}}\right) \lambda_{q+2} \|u_3\|_{L^2(\Omega)}^2 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{i=q+2}^{\infty} \left(1 - \frac{\lambda}{\lambda_i}\right) t_i^2 - \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_{q+1}}\right) \lambda_{q+2} \sum_{i=q+2}^{\infty} \frac{1}{\lambda_i} t_i^2 \\
&\geq \frac{1}{2} \lambda \sum_{i=q+2}^{\infty} \left(\frac{1}{\lambda_{q+1}} - \frac{1}{\lambda_i}\right) t_i^2 \geq 0
\end{aligned}$$

for all $u \in V^\perp$. Since $S \subseteq V^\perp$ this forces $S \subseteq f^0$, and (f₂) follows.

It is worth noting that $f(t\varphi_{q+1}) = 0$ whenever $t\varphi_{q+1} \in U$, as an elementary computation shows. Therefore,

$$0 = \min_{u \in S} f(u) = \max_{u \in \partial Q} f(u) = a.$$

Let us finally verify hypothesis (f₃). By virtue of the linking property, the inclusion $S \subseteq f^0$, and (4.4) one has

$$0 \leq \inf_{\gamma \in \Gamma} \sup_{z \in \gamma(Q)} f(z) \leq \sup_{x \in Q} f(x) \leq 0,$$

namely $c = a = 0$. Consequently, our first task will be to prove (PS)_{f,S,0}. Pick a sequence $\{u_n\} \subseteq X$ such that $d(u_n, S) \rightarrow 0$, $f(u_n) \rightarrow 0$, and

$$(4.5) \quad \Phi^0(u_n; v - u_n) + \alpha(v) - \alpha(u_n) \geq -\varepsilon_n \|v - u_n\| \quad \text{for all } n \in \mathbb{N}, v \in X,$$

where $\varepsilon_n \rightarrow 0^+$. Evidently, $\{u_n\}$ turns out bounded because so is the set S . Thus, passing to a subsequence if necessary, we may suppose $u_n \rightharpoonup u$ in X , $u_n \rightarrow u$ in $L^2(\Omega)$, $u_n(x) \rightarrow u(x)$ at almost all $x \in \Omega$. Inequality (4.5) can be equivalently written as

$$(4.6) \quad u_n \in U, \quad \Phi^0(u_n; v - u_n) \geq -\varepsilon_n \|v - u_n\| \quad \text{for all } n \in \mathbb{N}, v \in U.$$

Since U is convex and closed, we get $u \in U$. Exploiting (4.6) with $v = u$ and taking account of formula (2) at p. 77 in [5] yields

$$\begin{aligned}
&\int_{\Omega} \nabla u_n(x) \cdot \nabla u(x) dx - \lambda \int_{\Omega} u_n(x) \left(u(x) - u_n(x)\right) dx \\
&- \left(1 - \frac{\lambda}{\lambda_{q+1}}\right) \int_{\Omega} \nabla u_{n,2}(x) \cdot \nabla u_2(x) dx + \int_{\Omega} J_x^0(u_{n,1}(x); u_1(x) - u_{n,1}(x)) dx \\
&\quad + \int_{\Omega} K_x^0(u_{n,3}(x); u_3(x) - u_{n,3}(x)) dx \\
&\geq -\varepsilon_n \|u_n - u\| + \int_{\Omega} |\nabla u_n(x)|^2 dx - \left(1 - \frac{\lambda}{\lambda_{q+1}}\right) \int_{\Omega} |\nabla u_{n,2}(x)|^2 dx,
\end{aligned}$$

for $n \in \mathbb{N}$, where $u_n = u_{n,1} + u_{n,2} + u_{n,3}$ while $u = u_1 + u_2 + u_3$. By the upper semicontinuity of J_x^0 and K_x^0 we then achieve

$$\|u\|^2 - \left(1 - \frac{\lambda}{\lambda_{q+1}}\right) \|u_2\|^2 \geq \limsup_{n \rightarrow \infty} \left[\|u_n\|^2 - \left(1 - \frac{\lambda}{\lambda_{q+1}}\right) \|u_{n,2}\|^2 \right].$$

As $\mathbb{R}\varphi_{q+1}$ is finite dimensional and $\{u_{n,2}\} \subseteq \mathbb{R}\varphi_{q+1}$, the condition $u_{n,2} \rightarrow u_2$ in $\mathbb{R}\varphi_{k+1}$ actually means $u_{n,2} \rightarrow u_2$ in $\mathbb{R}\varphi_{k+1}$. Hence, the preceding inequality immediately leads to $u_n \rightarrow u$ in X , i.e. $(PS)_{f,S,0}$ holds. Observe next that

$$S \subseteq W \oplus \{u \in W^\perp : \|u\| \leq r_0\} \subseteq U$$

because $0 < r < r_0$. Choosing any $\varepsilon \in]0, r_0 - r[$ produces

$$\{u \in H_0^1(\Omega) : d(u, S) < \varepsilon\} \subseteq U.$$

Thus, $(f_{3.1})$ is satisfied whenever $\varepsilon_0 < \varepsilon$. Since for every $\delta \in]0, \varepsilon_0[$ it results

$$N_\delta(S) \cap f^{c-\delta} \cap f_{c+\delta} = N_\delta(S) \cap \{x \in U : c - \delta \leq \Phi(x) \leq c + \delta\},$$

we see at once that assertion $(f_{3.2})$ turns out true too.

Now, Theorem 3.1 can be applied, and we obtain a point $u \in \partial B_r \cap V^\perp$ such that

$$\Phi^0(u; v - u) + \alpha(v) - \alpha(u) \geq 0 \quad \text{for all } v \in H_0^1(\Omega).$$

The choice of α forces both $u \in U$ and $\Phi^0(u; v - u) \geq 0$ provided $v \in U$. Using formula (2) at p. 77 in [5] we realize that the function u is a nontrivial solution to problem (P). \square

REMARK 4.2. The above proof ensures that if $r \in]0, r_0[$ then there exists a solution of (P) lying in $\partial B_r \cap V^\perp$. Therefore, this problem really possesses infinitely many nontrivial solutions inside $B_{r_0} \cap V^\perp$.

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