Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 22, 2003, 115–138

ON DETECTING OF CHAOTIC DYNAMICS VIA ISOLATING CHAINS

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Abstract. We show in the paper the method of isolating chains in proof of chaotic dynamics. It is based on the earlier notion of an isolating segment but gives more powerful tool for exploring dynamics of periodic ODE's. As an application we show that the processes generated by the equations in the complex plane $\dot{z} = \overline{z}^k + e^{i\phi t}\overline{z}$, where $k \geq 3$ is an odd number and ϕ is close to 0, has chaotic behaviour.

Introduction

The notion of an isolating chain was defined in [5] as the generalization of the isolating segment. We use some theorems and properties proved there to show potential power of the method of isolating chains in theory of dynamical systems. The aim of the paper is to present the general Theorem 2.1 on calculation of fixed point index of iterations of Poincaré map. As a consequence we obtain results on the existence chaotic dynamics (Theorem 3.8).

Similar, but simpler, methods are also used to equations on the complex plane in [6], [3]. However this note seems to be the first dealing with such the equation and proving chaotic dynamics in a local process generated by equation $\dot{z} = \overline{z}^k + e^{i\phi t}\overline{z}$.

2000 *Mathematics Subject Classification*. Primary: 34C25; Secondary: 55M20, 58F22.

Key words and phrases. Periodic solutions, chaotic dynamics, fixed point index, isolating segment, isolating chain.

Supported by KBN, Grant P03A/028/17.

The isolating chain is a generalization of an isolating segment and it becomes useful in many situations where one can construct isolating segments which time sections have different Euler–Poincaré characteristic. In particular in [6], [7], [9], [3] authors use isolating segments to deal with equations $\dot{z} = f(t, |z|)\overline{z}^n$. However this approach does not give results of type obtained in the paper.

We prove the chaotic dynamics on more than 2 symbols through the construction of family of isolating chains which are not related in any way by inclusion. Our method is essentially different from [9] where authors also obtain dynamics semiconjugated to the shift on more than 2 symbols. We claim that our approach is new and gives results that are impossible to obtain using other methods known to the author.

1. Definitions

1.1. Local processes. Assume that X is a metric space and $D \subset \mathbb{R}\times X\times\mathbb{R}$ open subset. For given $\varphi\colon D \to X$ we will denote $\varphi(\sigma, \cdot\,, t)$ by $\varphi_{(\sigma,t)}$.

Continuous mapping $\varphi\colon D \to X$ is called a local process if the following conditions are satisfied

(a) for any $\sigma \in \mathbb{R}$, $x \in X$ the set $I_{(\sigma,x)} = \{t \in \mathbb{R} : (\sigma, x, t) \in D\}$ is an open interval containing 0,

(b) $\varphi_{(\sigma,0)} = \mathrm{id}_X$ for any $\sigma \in \mathbb{R}$,

(c) $\varphi_{(\sigma,s+t)}(x) = \varphi_{(\sigma+s,t)} \circ \varphi_{(\sigma,s)}(x)$, for any $x \in X$, $\sigma, s, t \in \mathbb{R}$ such that $s \in I_{(\sigma,x)}$, $t \in I_{(\sigma+s,\varphi_{(\sigma,s)}(x))}$, $s + t \in I_{(\sigma,x)}$.

Local process φ independent of first variable is called a local dynamical system or local flow on X and we will denote it by φ_s instead of $\varphi_{(\sigma,s)}$. Local process (local flow) defined on $D = \mathbb{R} \times X \times \mathbb{R}$ is called a process (flow).

For a given local process φ on X one can define a local flow $\widetilde{\varphi}$ on $\mathbb{R} \times X$ by

$$\widetilde{\varphi}_t(\sigma, x) = (\sigma + t, \varphi_{(\sigma,t)}(x)).$$

We will call X *a phase space* and $\mathbb{R} \times X$ *an extended phase space.*

Remark 1.1. The differential equation

$$\dot{x} = f(t, x)$$

with f regular enough to guarantee the uniqueness of solutions and continuous dependence on initial conditions generates local process

$$\varphi_{(t_0,\tau)}(x_0) = x(t_0, x_0; \tau),$$

where $x(t_0, x_0; \cdot\,)$ is a solution of Cauchy problem with $x(t_0, x_0; 0) = x_0$.

An equation with the right-hand side of equation independent of t generates in the same way a local flow. The local flow $\widetilde{\varphi}$ defined above is essentially the same as the one generated for equation

$$(\dot{t}, \dot{x}) = (1, f(t, x)).$$

In the extended phase space we define time translations $\tau_\sigma(t, x) = (t + \sigma, x)$ and projections on time and space subspaces $\pi_1 \colon \mathbb{R} \times X \to \mathbb{R}$ and $\pi_2 \colon \mathbb{R} \times X \to X$.

For any set $Z \subset \mathbb{R} \times X$ we define intersection of Z in time t by

$$Z_t = \{x \in X : (t, x) \in Z\}.$$

Local process φ is *T-periodic* if $\varphi_{(\sigma+T,t)} = \varphi_{(\sigma,t)}$ for any $\sigma, t \in \mathbb{R}$.

The map $\varphi_{(\sigma,T)}$ is called *a Poincaré map.* In the T-periodic local process there is one-to-one correspondence of T-periodic solutions of the process and fixed points of Poincaré map.

1.2. Isolating sets. Let A be a subset of a space X with a local flow ψ. We define *entry* and *exit* sets for A in flow ψ as

$$\begin{aligned} \operatorname{Exit}_\psi A &= \{x \in A \mid \text{there exists } \varepsilon_n > 0 : \varepsilon_n \to 0, \psi_{\varepsilon_n}(x) \notin A\}, \\ \operatorname{Entry}_\psi A &= \{x \in A \mid \text{there exists } \varepsilon_n < 0 : \varepsilon_n \to 0, \psi_{\varepsilon_n}(x) \notin A\}. \end{aligned}$$

Maximal invariant set in A is

$$\operatorname{Inv}_\psi A = \{x \in A \mid \text{for all } t \in \mathbb{R} : \psi_t(x) \in A\}.$$

A compact set $B \subset X$ is called an *isolating block* if $\operatorname{Exit}_\psi B$ and $\operatorname{Entry}_\psi B$ are closed and $\operatorname{Exit}_\psi B \cup \operatorname{Entry}_\psi B = \partial B$. The set $\operatorname{Inv}_\psi B$ is then compact and included in the interior of B.

Definition 1.2. Let $\psi = \widetilde{\varphi}$ be a local flow generated by a local process on $\mathbb{R} \times X$. A set $W \subset [a, b] \times X$ is called *isolating segment* over an interval $[a, b]$ if it is a compact ENR and there are $W^-, W^+ \subset W$ compact ENRs such that

- (a) there exists homeomorphism $h \colon [a, b] \times X \to [a, b] \times X$ such that $\pi_1 \circ h = \pi_1$ and $h([a, b] \times W_a) = W$, $h([a, b] \times W_a^\pm) = W^\pm$,
- (b) $\partial W_a = W_a^- \cup W_a^+$,
- (c) $\operatorname{Exit}_\psi W = W^- \cup (b \times W_b)$ and $\operatorname{Entry}_\psi W = W^+ \cup (a \times W_a)$.

Isolating segments are in particular isolating blocks. Later on we will show examples of the segments. Sets W^-, W^+ we will call *proper exit and entry sets* of W.

Definition 1.3. Let $a < b < c$, U, V be isolating segments over $[a, b]$ and $[b, c]$. Segments U and V are called *contiguous* if $U \cup V$ is an isolating block for φ.

Useful characterization of contiguity is given by

PROPOSITION 1.4 ([5]). *Isolating segments U, V over $[a,b]$ and $[b,c]$ are contiguous if and only if*

$$(\overline{U_b \setminus V_b} \cup U_b^-) \cap V_b \subset V_b^-, \tag{1}$$

$$(\overline{V_b \setminus U_b} \cup V_b^+) \cap U_b \subset U_b^+. \tag{2}$$

DEFINITION 1.5. Let $N > 0$, $a_0 < \ldots < a_N$ and U^i be isolating segments over $[a_{i-1}, a_i]$ (for $i = 1, \ldots, N$). If segments U^{i-1}, U^i are contiguous for $i = 1, \ldots, N$, the set $\bigcup_{i=1}^N U^i$ will be called *isolating chain* over the interval $[a_0, a_N]$.

In the case of $N = 1$ an isolating chain is in fact isolating segment, but such an identification will be useful.

We will use notations UV and $U^1 \ldots U^N$ for isolating chains consisting of segments U, V or $U^1, \ldots, U^N$.

Assume now, that local process is T-periodic. For every isolating segment (chain) U over interval $[a,b]$ its time translation $\tau_T(U)$ also is an isolating segment (chain) over $[a+T, b+T]$. T-periodic isolating chains $U^1 \ldots U^N$, $V^1 \ldots V^M$ over the same interval $[a, a+T]$ are called contiguous if U^N, $\tau_T(V^1)$ are contiguous. Isolating chain $U^1 \ldots U^N$ over $[a, a+T]$ is called T-periodic, if it is contiguous to itself.

In our work we will construct a family of pairwise contiguous isolating chains over $[0,T]$ and then build of them isolating chains over intervals of the form $[mT, nT]$, $m, n \in \mathbb{Z}$.

Let $\alpha = (U^1, \ldots, U^N)$ be a sequence of isolating chains over an interval $[a, a+T]$ such that U^i and U^{i+1} are contiguous for $i = 1, \ldots, N$. By $|\alpha|$ we denote the chain $U^1 \ldots U^N$, $\#\alpha = N$ is the length of the sequence, and $V \sqsubset \alpha$ means that $V = U^j$ for some j.

Let W be an isolating segment for a local process on X. Homeomorphism h in Definition 1.2(a) induces homeomorphism m_W of pointed spaces

$$m_W \colon (W_a/W_a^-, [W_a^-]) \to (W_b/W_b^-, [W_b^-])$$

by

$$m_W([x]) = [\pi_2 h(b, \pi_2 h^{-1}(a,x))].$$

The map m_W is called *monodromy maps* of segment W.

Let U, V be contiguous isolating segments (or chains) over $[a,b]$ and $[b,c]$. Let us define

$$n_{UV} \colon (U_b/U_b^-, [U_b^-]) \to (V_b/V_b^-, [V_b^-])$$

as

$$n_{UV}([x]) = \begin{cases} [x] & \text{if } x \in U_b \cap V_b, \\ [V_b^-] & \text{if } x \in U_b \setminus V_b. \end{cases}$$

The property (1) guarantees correctness of definition, continuity of n_{UV} and equality $n_{UV}([U_b^-]) = [V_b^-]$. The map n_{UV} is called *a transfer map* of contiguous segments U and V.

The monodromy map m_W and the transfer map n_{UV} induces homomorphisms in reduced homologies over $\mathbb{Q}$

$$\begin{aligned}\mu_W &= \widetilde{H}(m_W)\colon \widetilde{H}(W_a/W_a^-) \to \widetilde{H}(W_b/W_b^-),\\ \nu_{UV} &= \widetilde{H}(n_{UV})\colon \widetilde{H}(U_b/U_b^-) \to \widetilde{H}(V_b/V_b^-).\end{aligned}$$

For convenience we will use the notation n_{UV}, ν_{UV} instead of $n_{U\tau_T(V)}$, $\nu_{U\tau_T(V)}$ in the situation of contiguous isolating chains U, V over the $[a, a+T]$.

In [5] was proved the following

PROPOSITION 1.6. *All monodromy maps of a given isolating segment are in the same pointed homotopy class.*

In the paper we will look for periodic solutions of equations, which correspond to fixed points of the Poincaré map. To proove existence of such points it is enough to show, that fixed point index of Poincaré map $\operatorname{ind}(\varphi_{(a,T)}, S)$ is nonzero in some set S.

The following theorem showing how to use Lefschetz number in the case of isolating chains will play the crucial role in the work. For definitions and properties of fixed point index and Lefschetz number see e.g. [2].

THEOREM 1.7 ([5]). *Let X be an* ENR *and let φ be a T-periodic local process on X. If $C = U^1 \dots U^n$ is a T-periodic isolating chain over $[a, a+T]$ then the set $F_C = \{x \in U_a^1 \mid \varphi_{(a,T)}(x) = x$ for all $t \in [0,T]$ such that $\varphi_{(a,t)}(x) \in C_{a+t}\}$ is compact and open subset of the set of fixed points of $\varphi_{(a,T)}$ and*

$$\operatorname{ind}(\varphi_{(a,T)}, F_C) = \operatorname{Lef}(\nu_{U^N\tau_T(U^1)} \circ \mu_{U^N} \circ \dots \circ \nu_{U^2U^3} \circ \mu_{U^2} \circ \nu_{U^1U^2} \circ \mu_{U^1}).$$

REMARK 1.8. The theorem will be used not only for T being base period of equation, but also for its multiplicities.

As a consequence we derive following

COROLLARY 1.9. *Let C and $D_1, \dots, D_m$ be isolating chains over $[a, a+kT]$ for T-periodic local process on X. Then the set*

$$F_C^{D_1\dots D_m} = F_C \setminus \bigcup_{i=1,\dots,m} F_{D_i}$$

is compact and open in the set of fixed points of $\varphi_{(a,kT)}$.

PROOF. From Theorem 1.7 follows closedness and openness of F_C, F_{D_i}. Thus $X \setminus \bigcup_{i=1,\dots,k} F_{D_i}$ is closed and open. But F_C is compact and open and so is $F_C^{D_1\dots D_k} = F_C \cap (X \setminus \bigcup_{i=1,\dots,k} F_{D_i})$. □

With more complicated problem of computing fixed point index we will deal in next sections.

For a sequence of contiguous isolating chains $\alpha = (U_1, \ldots, U_N)$ we will use notation

$$\mu_\alpha = \mu_{U^N} \circ \ldots \circ \nu_{U^2 U^3} \circ \mu_{U^2} \circ \nu_{U^1 U^2} \circ \mu_{U^1}.$$

Definition 1.10. Let $A = \{U^1, \ldots, U^k\}$ be a set of pairwise contiguous isolating chains over $[0,T]$. Let $\alpha = (U^{n_1}, \ldots, U^{n_p})$. $|\alpha|$ is an isolating chain of length pT. Let

$$F_A(\alpha) = F_{|\alpha|} \setminus \bigcup \{F_{|\beta|} : \beta = (U^{m_1} \ldots U^{m_p}),\ 1 \leq m_1, \ldots, m_p \leq k,\ |\beta| \subsetneq |\alpha|\}.$$

The set $F_A(\alpha)$ is a collection of all fixed points of Poincaré map $\varphi_{(0,pT)}$ with trajectories in $|\alpha|$, but there is no isolating chain $|\beta|$ build of the elements of A containing the trajectory and being proper subset of $|\alpha|$.

From finiteness of A and Corollary 1.9 one derives compactness and openness of $F_A(\alpha)$ in the set of fixed points of $\varphi_{(0,pT)}$.

1.3. Chaotic maps. Let $\Sigma_k = \{0, \ldots, k-1\}^{\mathbb{Z}}$ be the space of bidirectional sequences with the product topology. The shift map $\sigma\colon \Sigma_k \to \Sigma_k$ is defined as $\sigma(x)_i = x_{i+1}$ for any $x \in \Sigma_k$.

Definition 1.11. T-periodic local process φ on X will be called Σ_k-*chaotic* if there is a compact set $I \subset \mathrm{dom}(\varphi_{(0,T)}) \subset X$ invariant with respect to Poincaré map $\varphi_{(0,T)}$ and continouous suriection $g\colon I \to \Sigma_k$ such that

(a) $\sigma \circ g = g \circ \varphi_{(0,T)}$, i.e. $\varphi_{(0,T)}$ is semiconjugated on I with shift map,

(b) counterimage by g of any n-periodic sequence $x \in \Sigma_k$ contains at least one n-periodic point for $\varphi_{(0,T)}$.

Equation $x' = f(t,x)$ with T-periodic right-hand side is Σ_k-*chaotic*, if the generated T-periodic local process is Σ_k-chaotic.

2. Main theorems

In the Theorem 2.1 we have set of chains with some technical properties. The examples of such sets are those in the Figures 1, 5, 6. The theorem says that for any sequence of such chains we can compute fixed point index of Poincaré map in the set of points, which have trajectories in set obtained by "gluing" these chains (in fact it is also an isolating chain). The sets W_{ij} are introduced for technical reason, as they are used in proof of last case of (3).

Theorem 2.1. *Let $A = \{U, V_1, \ldots, V_s\}$ be a collection of T-periodic isolating chains over $[0,T]$ in T-periodic local process φ. Assume, that $W_{ij} = V_i \cap V_j$ are isolating chains such that $U \subset W_{ij}$ for $i, j = 1, \ldots, s$ and*

(a) $((Z_1)_T, (Z_1^-)_T) = \tau_T((Z_2)_0, (Z_2^-)_0)$ *for any Z_1, Z_2 taken from U, V_i, W_{ij},*

(b) $\mu_U = \mu_{W_{ij}} \neq \mathrm{id}$ *for* $i \neq j$,

(c) $\mu_U^2 = \mu_{V_1} = \mu_{V_2} = \dots = \mu_{V_s} = \mathrm{id}$.

Then for any sequence $\alpha = (Z_1, \dots, Z_N)$ *of chains* U, V_i, W_{ij} *the following equality holds*

$$(3)\quad \begin{aligned} I_\alpha &= \operatorname{ind}(\varphi_{(0,NT)}, F_A(\alpha)) \\ &= \begin{cases} \operatorname{Lef}\mu_U & \text{if } \alpha = (U,\dots,U),\ N \text{ odd}, \\ \chi(U_0, U_0^-) & \text{if } \alpha = (U,\dots,U),\ N \text{ even}, \\ 0 & \text{if } W_{ij} \sqsubset \alpha \text{ for some } i \neq j, \\ (-1)^N(-2)^{k_\alpha - 1}(\operatorname{Lef}\mu_U - \chi(U_0, U_0^-)) & \text{in other cases}, \end{cases} \end{aligned}$$

where $k_\alpha = \#\{i \in \{1,\dots,N\} \mid \text{exists } j \in \{1,\dots,s\} : Z_i = V_j\}$ *is the number of* V_i *in the sequence* α.

Proof. First notice, that assumptions about the chains V_i, W_{ij} imply that no V_i is contained in any other V_j or in any W_{jk}.

The chain U is included in any other chain of A, so if $\alpha = (U,\dots,U)$ then we have $F_A(\alpha) = F_{|\alpha|}$ and from Theorem 1.7 we obtain $I_\alpha = \operatorname{Lef}\mu_U^{\#\alpha}$. This gives first two cases of the formula (3).

Let $W = W_{i_0 j_0} \sqsubset \alpha$. We can assume, that $Z_1 = W$. Thus we have $|\alpha| = |W\beta|$.

$$(4)\quad \begin{aligned} \operatorname{ind}(\varphi_{(0,NT)}, F_{|\alpha|}) &= \operatorname{ind}(\varphi_{(0,NT)}, F_{|W\beta|}) \\ &= \sum_{|\beta'|\subset|\beta|} \Big(I_{W\beta'} + \sum_{W_{ij}\subsetneq W} I_{W_{ij}\beta'} + I_{U\beta'} \Big) \\ &= \sum_{|\beta'|\subsetneq|\beta|} I_{W\beta'} + I_{W\beta} + \sum_{|\beta'|\subset|\beta|, W_{ij}\subsetneq W} I_{W_{ij}\beta'} + \sum_{|\beta'|\subset|\beta|} I_{U\beta'} \\ &= \sum_{\substack{|\beta'|\subset|\beta|, W_{ij}\subset W \\ |W_{ij}\beta'|\neq|W\beta|}} I_{W_{ij}\beta'} + I_{W\beta} + \operatorname{ind}(\varphi_{(0,NT)}, F_{|U\beta|}). \end{aligned}$$

But $\operatorname{ind}(\varphi_{(0,NT)}, F_{|U\beta|}) = \operatorname{ind}(\varphi_{(0,NT)}, F_{|W\beta|})$ from assumption (b) and we thus

$$I_{W\beta} = -\sum_{\substack{|\beta'|\subset|\beta|, W_{ij}\subset W \\ |W_{ij}\beta'|\neq|W\beta|}} I_{W_{ij}\beta'}.$$

Now we can prove the third part of (3). Every term on right-hand side of the above equality can be written out using this equality or is equal to 0 if there is no its proper subchain of the form $|W_{ij}\beta'|$. Because of finiteness of the set $\{U, V_i, W_{ij}\}$ this operation is finite and as a result we will obtain equality $I_\alpha = I_{W\beta} = 0$.

Thus we can always omit fixed point indices in chains containing W_{ij} as one of "subchains".

Now let $W_{ij} \not\subset \alpha$. The proof is inductive with respect to k_α. For $k_\alpha = 1$ we have $\alpha = (U, \ldots, U, V_l, U, \ldots, U)$ for some l, and

$$\begin{aligned}\operatorname{ind}(\varphi_{(0,NT)}, F_{|\alpha|}) &= I_{U\ldots UV_lU\ldots U} + \sum_{W_{ij}\subset V_l} I_{U\ldots UW_{ij}U\ldots U} + I_{U\ldots UUU\ldots U} \\ &= I_{U\ldots UV_lU\ldots U} + I_{U\ldots UUU\ldots U}.\end{aligned}$$

Hence

$$\begin{aligned}I_{U\ldots UV_lU\ldots U} &= \operatorname{ind}(\varphi_{(0,NT)}, F_{U\ldots UV_lU\ldots U}) - \operatorname{ind}(\varphi_{(0,NT)}, F_{U\ldots U}) \\ &= (-1)^N(\operatorname{Lef}\mu_U - \chi(U_0, U_0^-)).\end{aligned}$$

Now fix α and assume that for any α' such that $k_{\alpha'} < k_\alpha$ the formula holds. Using the third part of formula (3) we obtain

$$\operatorname{ind}(\varphi_{(0,NT)}, F_{|\alpha|}) = \sum_{|\alpha'|\subset|\alpha|, W\not\subset\alpha'} I_{\alpha'}.$$

Every sequence α' containing only chains U and V_i and such that $|\alpha'| \subset |\alpha|$ can be obtain from α by exchanging $k_\alpha - k_{\alpha'}$ from k_α sets V_i with U. Thus there are $\binom{k_\alpha}{k_\alpha - k} = \binom{k_\alpha}{k}$ sequences α' with $k_{\alpha'} = k$. From the inductive assumption we obtain

$$\operatorname{ind}(\varphi_{(0,NT)}, F_{|\alpha|}) = I_\alpha + (-1)^N(\operatorname{Lef}\mu_U - \chi(U_0, U_0^-)) \sum_{i=1}^{k_\alpha - 1} \binom{k_\alpha}{i} (-2)^{(i-1)} + I_{U^N}.$$

But

$$\begin{aligned}\sum_{i=1}^{k-1} \binom{k}{i} (-2)^i &= \sum_{i=0}^{k} \binom{k}{i} (-2)^i - (-2)^k - 1 \\ &= (1-2)^k - (-2)^k - 1 = (-1)^k - (-2)^k - 1,\end{aligned}$$

and inserting this into previous formula we get

$$\begin{aligned}(5)\quad &\operatorname{ind}(\varphi_{(0,NT)}, F_{|\alpha|}) \\ &\quad = I_\alpha - (-1)^N(\operatorname{Lef}\mu_U - \chi(U_0, U_0^-))\left(\frac{1}{2} - \frac{(-1)^{k_\alpha}}{2} + (-2)^{k_\alpha - 1}\right) + I_{U^N}.\end{aligned}$$

Assume that N, k_α are even numbers. Notice that

$$\operatorname{ind}(\varphi_{(0,NT)}, F_{|\alpha|}) = \chi(U_0, U_0^-) = I_{U^N}$$

and equation (5) reduces to

$$0 = I_\alpha - (-1)^N(\operatorname{Lef}\mu_U - \chi(U_0, U_0^-))(-2)^{k_\alpha - 1}$$

which proves theorem in this case.

If N is even and k_α odd we get $\operatorname{ind}(\varphi_{(0,NT)}, F_{|\alpha|}) = \chi(U_0, U_0^-)$ i $I_{U^N} = \operatorname{Lef}\mu_U$ and putting it into (5) produces $\chi(U_0, U_0^-) = I_\alpha - (-1)^N(\operatorname{Lef}\mu_U - \chi(U_0, U_0^-))(1 + (-2)^{k_\alpha - 1}) + \operatorname{Lef}\mu_U$. Rearrangement of the terms again gives the theorem.

In the same way one can prove the formula (3) in the case of odd N. $\square$

Corollary 2.2. *Under assumptions of Theorem* 2.1 *suppose, that there are indices* i_0, i_T *and a number* $\varepsilon_0 > 0$ *such that for any* $i = 1, \dots, s$, $\varepsilon \in [0, \varepsilon_0]$ *inclusions* $(V_i)_\varepsilon \subset (V_{i_0})_\varepsilon$ *and* $(V_i)_{T-\varepsilon} \subset (V_{i_T})_{T-\varepsilon}$ *hold. If in addition* $0 \neq \operatorname{Lef}\mu_U \neq \chi(U_0, U_0^-) \neq 0$ *then local process is* Σ_{s+1}*-chaotic.*

Proof. Denote $\widetilde{V} = \bigcup_{i=1}^s V_i$.

Let I be the set of points of $\mathbb{C}$ which trajectories starting in time 0 stay in any time interval $[mT, (m+1)T]$ in the set $\tau_{mT}(\widetilde{V})$. The compactness of I follows from closedness of $\widetilde{V}$. Let $g\colon I \to \Sigma_{s+1}$ be defined as follows

$$g(x)_m = \begin{cases} i & \text{if for all } t \in [0,T] : \varphi_{(0,t+mT)}(x) \in (V_i)_t, \\ & \text{and for all } j \neq i \text{ exists } t \in [0,T] \text{ such that } \varphi_{(0,t+mT)}(x) \notin (V_j)_t, \\ 0 & \text{if there is no } i \text{ satisfying above conditions.} \end{cases}$$

The map g is continuous if and only if $g(\,\cdot\,)_m$ are continuous for any $m \in \mathbb{Z}$. Fix m.

If the trajectory of $x \in I$ leaves $\tau_{mT}(V_j)$ for some time in $[mT, (m+1)T]$ then any point y closed enough to x also leaves $\tau_{mT}(V_j)$.

We will prove, that if the trajectory of $x \in I$ stays in $\tau_{mT}(V_j)$ then trajectories of nearby points do so. Assume, on the contrary, that there is a sequence of points going to x and such that each of them leaves $\tau_{mT}(V_j)$ for some time in $[mT, (m+1)T]$. Thus there is $t_0 \in [0,T]$ such that $\phi_{(0,t_0+mT)}(x) \in \partial(V_j)_{t_0}$. But $\tau_{mT}(V_j)$ is an isolating block, so it is impossible that $t_0 \in (0,T)$.

Suppose, that $t_0 = T$. It means, that $\phi_{(0,T+mT)}(x) \in (V_j^-)_T$. From the assumption (a) of Theorem 2.1 there is $\phi_{(0,T+mT)}(x) \in (V_{i_0}^-)_0$. But that means, that there is $\varepsilon \in [0, \varepsilon_0]$ such that $\phi_{(0,T+mT+\varepsilon)}(x) \notin (V_{i_0})_\varepsilon$. Assumptions of the corollary imply that also $\phi_{(0,T+mT+\varepsilon)}(x) \notin (V_i)_\varepsilon$ for any other i and finally, that $\phi_{(0,T+mT+\varepsilon)}(x) \notin (\widetilde{V})_\varepsilon$ which contradicts the fact, that trajectory of x stays in $\tau_{(m+1)T}(\widetilde{V})$ for time in $[(m+1)T, (m+2)T]$.

Similarly one can prove that t cannot be 0.

Above facts jusify the continuity of $g(\,\cdot\,)_m$. Let $x \in I$.

(1) If $g(x)_m = k \neq 0$ then the trajectory of x for time in $t \in [mT, (m+1)T]$ stays in $\tau_{mT}(V_k)$, but it leaves every other $\tau_{mT}(V_j)$. We proved, that all nearby points should also stay in $\tau_{mT}(V_k)$ and leave $\tau_{mT}(V_j)$ and it means that $g(y)_m = k$.
(2) If $g(x)_m = 0$ and the trajectory of x in time interval $[mT, (m+1)T]$ lays in intersection of some $\tau_{mT}(V_k)$, $\tau_{mT}(V_l)$, then also nearby trajectories

also stay for this time in that intersection and from the definition of g there is $g(y)_m = 0$.

(3) If $g(x)_m = 0$ and the trajectory of x leaves every $\tau_{mT}(V_j)$ in time $[mT, (m+1)T]$ then all close enough points also have trajectories leaving all $\tau_{mT}(V_j)$ and again $g(y)_m = 0$.

The semiconjugacy of g and the shift map follows easily from the definition of g. Assume, that $c = (\ldots, c_1, \ldots, c_k, c_1, \ldots, c_k, \ldots)$, $c_i \in \{0, \ldots, s\}$ is a k-periodic sequence in Σ_{s+1}. Let $\alpha = (V_{c_1}, \ldots, V_{c_k})$. (For convenience denote U by V_0.) Under the assumptions of corollary we have $\operatorname{ind}(\varphi_{(0,kT)}, F_A(\alpha)) \neq 0$, so the set of fixed points of Poincaré map $\phi_{(0,kT)}$ with trajectories described by the sequence c is nonempty and for each such a point there is $g(x) = c$.

We proved, that the image of g contains all periodic sequences in Σ_{s+1}, dense subspace of Σ_{s+1}. If $c \in \Sigma_{s+1}$ is any sequence, it can be approximated by periodic sequences $\{c^n\}$. There are $x^n \in I$ such that $g(x^n) = c^n$ and we can assume that $x^n \to x \in I$. Continuity of g guarantees equality $g(x) = c$. $\square$

3. Applications

In this section we will show construction of isolating segments and chains for the local process generated by equations

$$\dot{z} = \overline{z}^k + e^{i\phi t}\overline{z} \tag{6}$$

for odd k and small enough $|\phi|$. In the case of $k = 5$ it is enough to assure that $|\phi| < 0.015$.

The period of equation (6) is equal to $2\pi/\phi$ and by T we will always understand this number.

3.1. Isolating segments $U_r^{[a,b]}$, $D_R^{[a,b]}$. In extended phase space the vector field generated by equation (6) has a form

$$F(t, z) = \begin{pmatrix} 1 \\ \overline{z}^k + e^{i\phi t}\overline{z} \end{pmatrix}. \tag{7}$$

The first isolating segment $U_r^{[a,b]}$ will be a square-based twisted prism. Its intersection in time $t \in [a, b]$ is a square with sides of length $2r$, centered in origin, and rotated at the angle $\phi t/2$ (see Figure 1, shaded part of boundary is its exit set). If we have omitted $\overline{z}^k$ term, that set would be an isolating segment for small enough $|\phi|$. Thus it is also an isolating segment for small r when we add $\overline{z}^k$ term, as near 0 this term is dominated by linear part. Lemma 3.1 shows conditions on k, ϕ, r ensuring U_r is an isolating segment.

Formally we introduce 4 functions

$$\Lambda^j(t, z) = \Re(z e^{i(j\pi - t\phi)/2}), \quad j = 0, \ldots, 3.$$

Now let

$$\begin{aligned} U_r &= \{(t,z) \in \mathbb{R}\times\mathbb{C} : \Lambda^j(t,z) \le r,\ j = 0,\dots,3\},\\ U_r^j &= \{(t,z) \in U_r : \Lambda^j(t,z) = r\}, \end{aligned}$$

and $U_r^{[a,b]} = U_r \cap ([a,b]\times\mathbb{C})$.

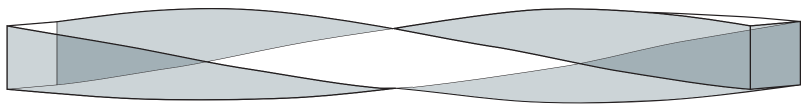

FIGURE 1. Isolating segment $U_r^{[0,T]}$

Next lemma shows conditions under which $U_r^{[a,b]}$ is an isolating segment and exit and entry sets have a form

$$\begin{aligned} U_r^- &= U_r^0 \cup U_r^2, \quad U_r^+ = U_r^1 \cup U_r^3,\\ \mathrm{Exit}_\varphi U_r^{[a,b]} &= (U_r^- \cap U_r^{[a,b]}) \cup (\{b\}\times (U_r)_b), \\ \mathrm{Entry}_\varphi U_r^{[a,b]} &= (U_r^+ \cap U_r^{[a,b]}) \cup (\{a\}\times (U_r)_a). \end{aligned} \tag{8}$$

LEMMA 3.1. *If $k \ge 2$, $\phi, r > 0$ satisfy inequality $1 - \phi/\sqrt{2} - \sqrt{2}^k r^{k-1} > 0$ then*

$$F(t,z)\cdot\nabla\Lambda^j(t,z) > 0 \quad ((t,z)\in U_r^j, j = 0,2), \tag{9}$$

$$F(t,z)\cdot\nabla\Lambda^j(t,z) < 0 \quad ((t,z)\in U_r^j, j = 1,3). \tag{10}$$

PROOF. The vector field orthogonal to the boundary of U_r has a form

$$\nabla\Lambda^j(t,z) = \begin{pmatrix} -\phi\Re(ze^{i(j\pi - t\phi)/2})/2 \\ e^{-i(j\pi-t\phi)/2} \end{pmatrix}.$$

Let $(t,z)\in U_r^j$. Then

$$\begin{aligned} F(t,z)\cdot\nabla\Lambda^j(t,z) &= -\frac{\phi}{2}\Re(ze^{i(j\pi-t\phi)/2}) + \Re((\overline{z}^k + e^{i\phi t}\overline{z})e^{i(j\pi-t\phi)/2}) \\ &= -\frac{\phi}{2}\Re(ze^{i(j\pi-t\phi)/2}) + \Re(\overline{z}^k e^{i(j\pi-t\phi)/2}) + \Re((-1)^j\overline{z}e^{i(\phi t - j\pi)}e^{i(j\pi-t\phi)/2}) \\ &= -\frac{\phi}{2}\Re(ze^{i(j\pi-t\phi)/2}) + \Re(\overline{z}^k e^{i(j\pi-t\phi)/2}) + (-1)^j\Re(\overline{z}e^{-i(j\pi-t\phi)/2}). \end{aligned} \tag{11}$$

The modulus of two first summands can be estimated by, respectively $\phi|z|/2 \le \phi r\sqrt{2}/2$ and $|z|^k \le (r\sqrt{2})^k$ and from the definition of U_r^j we obtain

$$\Re(\overline{z}e^{-i(j\pi-t\phi)/2}) = \Re(ze^{i(j\pi-t\phi)/2}) = r.$$

Hence

$$\begin{aligned} F(t,z)\cdot\nabla\Lambda^j(t,z) &\geq r-\frac{\phi}{2}r\sqrt{2}-(r\sqrt{2})^k \quad \text{for } (t,z)\in U_r^j,\ j=0,2,\\ F(t,z)\cdot\nabla\Lambda^j(t,z) &\leq -r+\frac{\phi}{2}r\sqrt{2}+(r\sqrt{2})^k \quad \text{for } (t,z)\in U_r^j,\ j=1,3. \end{aligned}$$

This and the assumption of the lemma give needed inequalities. $\square$

Figure 2. Isolating segment $D_R^{[a,b]}$

The second isolating segment $D_R^{[a,b]}$ is a prism with base of regular polygon with $2(k+1)$ sides circumscribed on the circle of radius R. It is an isolating segment for equation $G(t,z)=\binom{1}{\bar{z}^k}$ with any R. The Lemma 3.2 will show that it remains the segment for equation (7) for large enough R.

Let the following $2k+2$ functions be given

$$\Xi^j(t,z)=\Re(ze^{-ij\pi/(k+1)}),\quad j=0,\dots,2k+1.$$

Define

$$\begin{aligned} D_R &= \{(t,z)\in\mathbb{R}\times\mathbb{C}:\Xi^j(t,z)\leq R,\ j=0,\dots,2k+1\},\\ D_R^j &= \{(t,z)\in D_R:\Xi^j(t,z)=R\},\\ D_R^{[a,b]} &= D_R\cap([a,b]\times\mathbb{C}). \end{aligned}$$

Next lemma shows conditions under which $D_r^{[a,b]}$ is an isolating segment and exit and entry sets have a form

$$\begin{aligned} D_R^- &= \bigcup\{D_R^j : j \text{ even}\},\\ D_R^+ &= \bigcup\{D_R^j : j \text{ odd}\},\\ \mathrm{Exit}_\varphi D_R^{[a,b]} &= (D_R^-\cap D_R^{[a,b]})\cup(\{b\}\times(D_R^{[a,b]})_b),\\ \mathrm{Entry}_\varphi D_R^{[a,b]} &= (D_R^+\cap D_R^{[a,b]})\cup(\{a\}\times(D_R^{[a,b]})_a). \end{aligned}$$

On the Figure 2 one can see isolating segment $D_R^{[a,b]}$ and its exit set (shaded part of boundary) for $k=5$.

Lemma 3.2. *If $k \geq 2$, $R > \sqrt{[}k-1]2(k+1)/3$ then*

$$F(t,z)\cdot\nabla\Xi^j(t,z) > 0 \quad \text{for } (t,z)\in D_R^j,\ j \text{ even}, \tag{12}$$
$$F(t,z)\cdot\nabla\Xi^j(t,z) < 0 \quad \text{for } (t,z)\in D_R^j,\ j \text{ odd}. \tag{13}$$

Proof. Fix even j and let $(t,z)\in D_R^j$. Then $\nabla\Xi^j(t,z) = (0, e^{ij\pi/(k+1)})$ and $z = \rho e^{i\alpha}$, where $\alpha\in[j\pi/(k+1)-\pi/2(k+1), j\pi/(k+1)+\pi/2(k+1)]$, $\rho\in[R, R/\cos(\pi/2(k+1))]$. Thus

$$\begin{aligned} F(t,z)\cdot\nabla\Xi^j(t,z) &= \Re((\overline{z}^k + e^{i\phi t}\overline{z})e^{-ij\pi/(k+1)}) \\ &= \Re(\rho^k e^{i(-k\alpha - j\pi/(k+1))} + \rho e^{i(\phi t-\alpha-j\pi/(k+1))}) \\ &\geq \Re(\rho^k e^{i(-k\alpha-j\pi/(k+1))}) - |\rho e^{i(\phi t-\alpha-j\pi/(k+1))}|. \end{aligned}$$

But $-k\alpha - j\pi/(k+1)\in[-j\pi - k\pi/2(k+1), -j\pi + k\pi/2(k+1)]$ and j is even, so

$$\Re(e^{i(-k\alpha-j\pi/(k+1))}) \geq \cos\frac{k\pi}{2(k+1)} \tag{14}$$

and

$$F_k(t,x,y)\cdot\nabla\Xi^j(t,x,y) \geq \rho^k\cos\frac{k\pi}{2(k+1)} - \rho = \rho\biggl(\rho^{k-1}\cos\frac{k\pi}{2(k+1)} - 1\biggr). \tag{15}$$

The function $\rho^{k-1}\cos(k\pi/2(k+1)) - 1$ is increasing for positive ρ, so

$$\rho^{k-1}\cos\frac{k\pi}{2(k+1)} - 1 \geq R^{k-1}\cos\frac{k\pi}{2(k+1)} - 1.$$

In addition the inequality

$$\sin\frac{\pi}{l} \geq \frac{\pi}{l}\frac{\sin(\pi/6)}{\pi/6} = \frac{3}{l}$$

holds for $l\geq 6$, so substituting $l = 2(k+1)$ we derive

$$\cos\frac{k\pi}{2(k+1)} = \sin\frac{\pi}{2(k+1)} \geq \frac{3}{2(k+1)}$$

and thus

$$\rho^{k-1}\cos\frac{k\pi}{2(k+1)} - 1 \geq R^{k-1}\frac{3}{2(k+1)} - 1.$$

Assumptions of lemma imply that $3R^{k-1}/2(k+1) - 1 > 0$ and thus right-hand side of (15) is positive.

For odd j the proof is, up to change of signs, the same. □

3.2. Isolating segments P_θ, Q_θ. In this section we show the construction of some segments for the local process generated by equation (6) for odd $k \geq 3$. First we will construct one of them, P_0 and then the rest will be obtained as images of P_0 in some isometries.

Let $f(t,z)$ be the right-hand side of equation (6), and $f_0(t,z) = f(0,z) = \overline{z}^k + \overline{z}$. Firstly we construct an isolating segment for that autonomous equation. It will be also the the segment for f.

In some neighbourhood of the origin the linear part of that equation dominates the nonlinear part, and origin is a saddle point with axes as stable and unstable manifolds. There is also neighbourhood of x-axis such that any solution starting in it goes to infinity closing to x-axis (see vector field). Thus we can find the value r such that if $|y_0| < r$, then absolute value of y-coordinate of the solution starting from (x_0, y_0) does not exceed r. Moreover if $|x_0| > r$, then solutions goes in x-coordinate in direction opposite to origin.

Thus we can construct an isolating segment (see Figure 4) with "horizontal" proper entrance set and "vertical" proper exit set (in the sense that time sections of the entrance (exit) set are horizontal (vertical) segments).

We give detailed description of the construction below.

Choose p such that $p^{k-1} < 1/\sqrt{2}^k$. As $k \geq 3$ we get $1/\sqrt{2}^k < \tan(\pi/2k)$, and also $p^{k-1} < \tan(\pi/2k)$.

Let $r = p\tan(\pi/2k)$, $s = r/\tan(\pi/k) = p\tan(\pi/2k)/\tan(\pi/k)$ (see Figure 3).

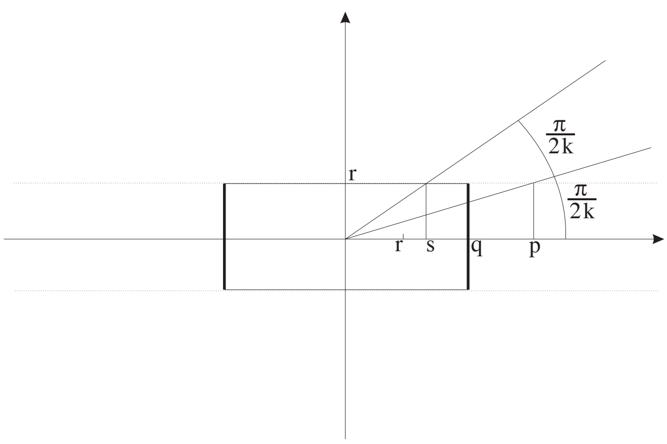

Figure 3. Rectangle Π_{qr}

Firstly we will show transversality of the vector field f_0 to the boundary of the rectangle $\Pi_{qr} = \{z \in \mathbb{C} : |\Re z| \leq q,\ |\Im z| \leq r\}$ for $q \geq r$ and that f_0 points outward of its vertical edges and inward horizontal ones. Let $z_0 = (x,y) \in \partial\Pi_{qr}$.

(1) $|x| = q$. Assume, that $x = q$ (for negative x proof is similar). To show that the vector field is directed in this point outward of Π_{qr} we should prove that $\Re f_0(t, z_0) > 0$.

(a) $q \geq p$. One has $\Re \overline{z_0}^k > 0$, so $\Re f_0(t, z_0) > \Re z = q > 0$.

(b) $q < p$. $\Re f_0(t, z_0) = x + \Re \overline{z}^k \geq x - |z_0|^k$. But $|x| \geq |y|$, so

$$\begin{aligned}\Re f_0(t, z_0) &\geq x - (x\sqrt{2})^k = x(1 - \sqrt{2}^k x^{k-1}) \\ &= q(1 - \sqrt{2}^k q^{k-1}) > q(1 - \sqrt{2}^k p^{k-1}) > 0.\end{aligned}$$

(2) $|y| = r$. Assume, that $y = r$. We should prove that $\Im f_0(t, z_0) < 0$.

(a) $|x| \geq s$. $\Im f_0(t, z_0) = -y + \Im \overline{z_0}^k$. There is $\arg(z_0) \in (0, \pi/k]$ so

$$\Im f_0(t, z_0) < -y < 0.$$

(b) $|x| < s$.

$$\begin{aligned}\Im f_0(t, z_0) &\leq -y + |\overline{z_0}|^k = -y + \sqrt{x^2 + y^2}^k \\ &\leq -r + \sqrt{s^2 + r^2}^k \leq -r + \sqrt{s^2 + (\sqrt{3}s)^2}^k \\ &= -r + (2s)^k \leq -r + p^k = p(p^{k-1} - \tan(\pi/2k)) < 0.\end{aligned}$$

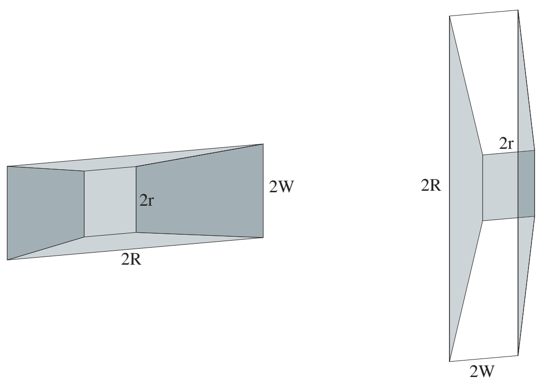

Figure 4. Isolating segments P_0, Q_T

Now let P_0 be a polyhedron contained in $[0, \tau] \times \mathbb{C}$ for some $\tau > 0$ such that its intersections have a form $(P_0)_t = \Pi_{q(t)r(t)}$, where $q, r : \mathbb{R} \to \mathbb{R}$ are affine functions (Figure 4). One can assume, that additionally

(1) $R = q(\tau)$ satisfies assumptions of Lemma 3.2,

(2) $\widetilde{p} < R$ is such that $\widetilde{p}^{k-1} < 1/\sqrt{2}^k$,

(3) $r(\tau) = \widetilde{p}\tan(\pi/2k)$,

(4) $0 < q(0) = r(0) = r_0 < r(\tau)$.

One can check, that $\Pi_{q(t)r(t)}$ for $t \in [0,\tau]$ satisfies conditions guaranteeing transversality of f_0 on the boundary of $\Pi_{q(t)r(t)}$. Taking large enough τ will assure also that f_0 is transversal to the boundary of P_0 and. $(P_0^-)_t = \{z \in \Pi_{q(t)r(t)} : |\Re z| = q(t)\}$. The difference $f(t,z) - f_0(t,z) = (e^{i\phi t} - 1)\overline{z}$ for bounded z and $t \in [0,\tau]$ can be arbitrarily small taking small enough $|\phi|$. In particular for $|\phi| \in [0,\phi_0]$ the transversality conditions for P_0 are kept for equation (6) and thus P_0 is an isolating segment.

Conditions (1), (3) mean, that P_0 is contiguous to $D_R^{[\tau,b]}$. From (2)–(4) we conclude that, possibly taking smaller ϕ_0, r_0 will satisfy assumptions of Lemma 3.1. As a result $U_{r_0}^{[a,0]}$ will be an isolating segment contiguous to P_0.

Last conditions we need is an inclusion $U_{r_0}^{[0,\tau]} \subset P_0$ and the technical inequality $\tau < T/2(k+1)$. This again can be obtained by taking smaller ϕ.

Now we can show construction of the rest of segments P_θ and Q_θ. Next lemma is a consequence of some symmetries of equation (6). Namely changes of variables (a)–(c) do not change vector field (or reverse it) and in the same time sets U_r and D_R are not altered and their exit and entry sets are exchanged in the case of reversed process.

LEMMA 3.3. *Suppose, that equation $\dot z = \overline{z}^k + e^{i\phi t}\overline{z}$ for odd $k \geq 3$ and some ϕ admits an isolating segment P_0 over $[0,\tau]$, $\tau < T/2(k+1)$ such that $U_{r_0}^{[a,0]}$ is contiguous to P_0 and P_0 contiguous to $D_R^{[\tau,b]}$. Then there exist for integer n isolating segments $P_{2nT/(k+1)}$ over $[2nT/(k+1), 2nT/(k+1)+\tau]$ contiguous to $U_{r_0}^{[a,2nT/(k+1)]}$ and $D_R^{[2nT/(k+1)+\tau,b]}$. There are also $Q_{mT/(k+1)}$, isolating segments over $[mT/(k+1)-\tau, mT/(k+1)]$ contiguous to $D_R^{[a,mTk+1-\tau]}$ and $U_{r_0}^{[mT/k+1,b]}$. Here m are even numbers if $k \in 4\mathbb{N}+1$ and odd numbers if $k \in 4\mathbb{N}+3$.*

PROOF. Choose $\theta = 2nT/(k+1)$. Changing coordinates by formulas

$$s = t - \theta, \quad \zeta = e^{-i\phi\theta/2} z, \tag{a}$$

we obtain equation

$$\begin{aligned}\frac{d\zeta}{ds} &= e^{-i\phi\theta/2}(\overline{z}^k + e^{i\phi t}\overline{z}) = e^{-i\phi\theta/2}((e^{-i\phi\theta/2}\overline{\zeta})^k + e^{i\phi s}e^{i\phi\theta}e^{-i\phi\theta/2}\overline{\zeta})\\ &= (e^{-i\phi nT/(k+1)})^{k+1}\overline{\zeta}^k + e^{i\phi s}\overline{\zeta} = \overline{\zeta}^k + e^{i\phi s}\overline{\zeta}.\end{aligned}$$

It is identical to (6) so the construction of an isolating segment P_0 can be done. But in coordinates (t,z) this segment is a segment over $[\theta, \theta+\tau]$. Denote it by P_θ. In addition, contiguous to P_0 segments $U_{r_0}^{[a',0]}$ and $D_R^{[\tau,b']}$ in primary coordinates have a form $U_{r_0}^{[a,\theta]}$ and $D_R^{[\theta+\tau,b]}$. Obviously, they are contiguous to P_θ.

Having defined segments $P_{2nT/(k+1)}$ we can construct isolating segments $Q_{nT/(k+1)}$. Let $k \in 4\mathbb{N}+1$. We change coordinates in phase space

$$s = T - t, \quad \zeta = e^{i\pi/2}\overline{z}. \tag{b}$$

In this coordinates we obtain

$$\begin{aligned} \frac{d\zeta}{ds} &= -e^{i\pi/2}(z^k + e^{-i\phi t} z) = -(e^{i\pi/2})^{k+1}\overline{\zeta}^k - e^{i\phi s}e^{-i\phi T}e^{-i\pi}\overline{\zeta} \\ &= -e^{i(k+1)\pi/2}\overline{\zeta}^k - e^{i\phi s}e^{-i(\phi T+\pi)}\overline{\zeta}. \end{aligned}$$

But $e^{i(k+1)\pi/2} = e^{-i(\phi T+\pi)} = -1$ and again we have primary equation. The segment $P_{T-2nT/(k+1)}$ in coordinates (s,ζ) will be denoted in (t,z) coordinates by $Q_{2nT/(k+1)}$. Similarly, contiguous segments U_{r_0} and D_R will remain contiguous in changed coordinates.

Notice, that here change of variables changes the direction of motion along trajectories and also exchanges exit points with entry points.

Similarly for $k \in 4\mathbb{N}+3$ we introduce

$$s = \frac{kT}{k+1} - t, \quad \zeta = e^{i(\pi/2+\phi T/2(k+1))}\overline{z}. \tag{c}$$

In this coordinates differential equation transforms into

$$\frac{d\zeta}{ds} = -e^{i((k+1)\pi/2+\phi T/2)}\overline{\zeta}^k - e^{i\phi s}e^{-i(\phi T+\pi)}\overline{\zeta}.$$

But $T = 2\pi/\phi$ and thus $e^{i((k+1)\pi/2+\phi T/2)} = e^{-i(\phi T+\pi)} = -1$ and again we have equation (6). The same as above reasoning completes the proof. $\square$

Remark 3.4. In case $k = 5$ it can be checked, that choice of $\tau = 7$, $R = r(\tau) = 1.6$, $q(\tau) = 1.6\tan(\pi/12)$, $r(0) = q(0) = r_0 = 0.4$ and any $\phi \in (0, 0.015]$ ensures all conditions listed above for sets U_{r_0}, D_R, P_θ, Q_θ to be isolating segments. We will not present here detailed calculations.

3.3. Isolatin chains.

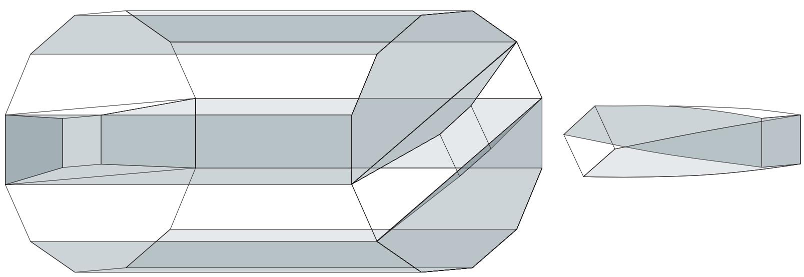

Figure 5. Isolating chain $V_1 = Y_{04}$

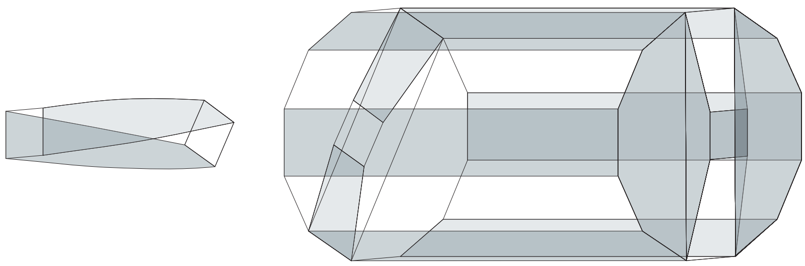

FIGURE 6. Isolating chain $V_2 = Y_{26}$

DEFINITION 3.5. For odd k, τ, R, r and ϕ such that above constructions are valid define the isolating chains

$$Y_{ij} = U_r^{[0,iT/(k+1)]} P_{iT/(k+1)} D_R^{[iT/(k+1)+\tau, jT/(k+1)-\tau]} Q_{jT/(k+1)} U_r^{[jT/(k+1),T]} \tag{16}$$

for $i = 0, 2, \ldots, k-1$, $j > i$ even for $k \in 4\mathbb{N}+1$, and odd for $k \in 4\mathbb{N}+3$.

Examples of chains Y_{ij} for $k = 5$ are drawn on Figures 5–7. Now we will show how to compute transfer maps for chains Y_{ij}. One can look at Figure 8 as an illustration of the proof in the case of $k = 5$.

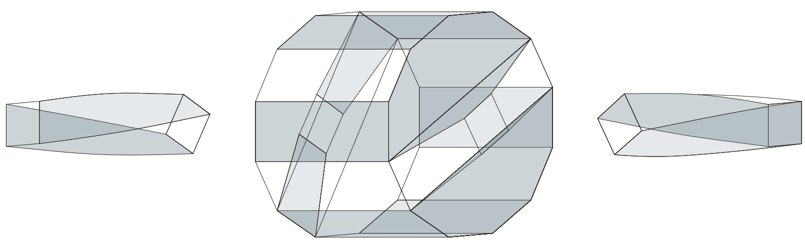

FIGURE 7. Isolating chain $W = V_1 \cap V_2 = Y_{24}$

LEMMA 3.6. *Let* $k = 4n+1$,

$$Y_{ij} = U_r^{[0,iT/(4n+2)]} P_{iT/(4n+2)} \\ D_R^{[iT/(4n+2)+\tau, jT/(4n+2)-\tau]} Q_{jT/(4n+2)} U_r^{[jT/(4n+2),T]},$$

and assume, that $V_p = Y_{2p-2,2p+2n}$ *are isolating chains for* $p = 1, \dots, n+1$, $W_{ij} = V_i \cap V_j$ *and let* $U = U_r^{[0,T]}$. *Then*

$$\begin{aligned} ((Z_1)_T, (Z_1^-)_T) &= \tau_T((Z_2)_0, (Z_2^-)_0) && \text{for any } Z_1, Z_2 \text{ taken from } U, V_i, W_{ij}, \\ \mu_U &= \mu_{W_{ij}} = -\mathrm{id} && \text{for } i, j = 0, \dots, n, \\ \mu_U^2 &= \mu_{V_i} = \mathrm{id} && \text{for } i = 0, \dots, n, \\ \operatorname{Lef} \mu_U &= 1, \\ \chi(U_0, U_0^-) &= -1. \end{aligned}$$

Proof. Equality of starting and ending faces of chains is obvious. Only the first homology group of $((Z)_0, (Z^-)_0)$ is nontrivial, so it is enough to study the maps on this level. Later on by id_p we denote the identity map on p-dimensional vector space.

Monodromy map m_U is homotopic to rotation at angle π, so $(\mu_U)_1 = -\mathrm{id}_1$. Thus $(\mu_U^2)_1 = \mathrm{id}_1$, $\operatorname{Lef}(\mu_U) = -\operatorname{tr}(\mu_U)_1 = 1$, $\chi(U_0, U_0^-) = \chi(U_0) - \chi(U_0^-) = 1 - 2 = -1$.

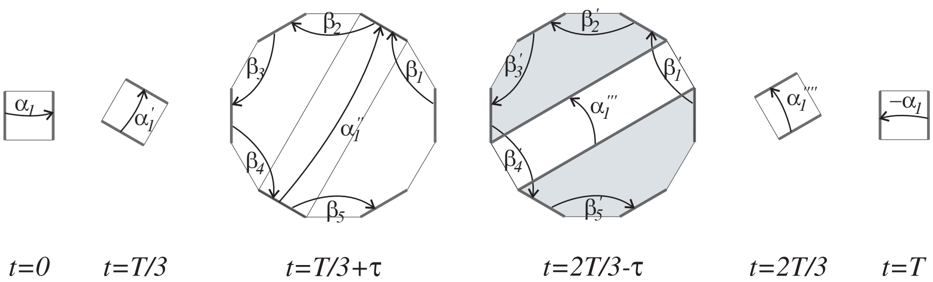

Figure 8. Transfer maps for $k = 5$

Let basis vector of each $H_1((U_r)_t, (U_r^-)_t) \cong H_1((U_r)_t/(U_r^-)_t, [(U_r^-)_t])$ be such that monodromy map of segment $U_r^{[0,t]}$ is identity map. Take basis in $H_1((D_R)_t, (D_R^-)_t)$ in the way as on the Figure 8 for $k = 5$ ($(\mu_{D_R^{[t_1,t_2]}})_1 = \mathrm{id}_k$).

Take also basis in the spaces

$$\begin{aligned} &H_1((P_\theta)_\theta, (P_\theta^-)_\theta), && H_1((P_\theta)_{\theta+\tau}, (P_\theta^-)_{\theta+\tau}), \\ &H_1((Q_\theta)_{\theta-\tau}, (Q_\theta^-)_{\theta-\tau}), && H_1((Q_\theta)_\theta, (Q_\theta^-)_\theta) \end{aligned}$$

in such a way, that monodromy maps of segments P_θ, Q_θ in homology are identity and transfer maps $\nu_{U_r^{[0,\theta]} P_\theta} = \mathrm{id}_1$, $\nu_{Q_\theta U_r^{[\theta,T]}} = \mathrm{id}_1$.

In that situation we have from the Theorem 1.7

$$(17)\quad (\mu_{Y_{2p,2q}})_1 = (-\mathrm{id}_1) \circ (\nu_{D_R^{[2pT/(4n+2)+\tau,2qT/(4n+2)-\tau]}Q_{2qT/(4n+2)}})_1 \circ (\nu_{P_{2pT/(4n+2)}D_R^{[2pT/(4n+2)+\tau,2qT/(4n+2)-\tau]}})_1.$$

The map $-\mathrm{id}_1$ must be used to correct the fact, that we chose basis of

$$H_1((Y_{2p,2q})_0, \quad (Y^-_{2p,2q})_0) \quad \text{and} \quad H_1((Y_{2p,2q})_T, (Y^-_{2p,2q})_T)$$

reversely oriented. Now we only need to find transfer maps

$$(\nu_{D_R^{[2pT/(4n+2)+\tau,2qT/(4n+2)-\tau]}Q_{2qT}/(4n+2)})_1, (\nu_{P_{2pT/(4n+2)}D_R^{[2pT/(4n+2)+\tau,2qT/(4n+2)-\tau]}})_1$$

and their compositions.

We have equalities

$$(\nu_{P_{2pT/(4n+2)}D_R^{[2pT/(4n+2)+\tau,2qT/(4n+2)-\tau]}})_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ \vdots \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} p \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} 2n+1 \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} 2n-p \end{matrix},$$

and

$$(\nu_{D_R^{[2pT/(4n+2)+\tau,2qT/(4n+2)-\tau]}Q_{2qT/(4n+2)}})_1 = \begin{cases} [\underbrace{0\ldots0}_{n+q}\ -1\ \underbrace{0\ldots0}_{2n}\ 1\,\underbrace{0\ldots0}_{n-q-1}] & \text{for } q<n, \\ [\underbrace{0\ldots0}_{2n}\ -1\ \underbrace{0\ldots0}_{2n}] & \text{for } q=n, \\ [\underbrace{0\ldots0}_{q-n-1}\ 1\ \underbrace{0\ldots0}_{2n}\ -1\ \underbrace{0\ldots0}_{3n-q}] & \text{for } q>n. \end{cases}$$

We are interested in sets of the form $V_p = Y_{2p-2,2p+2n}$ and their intersections, that is $Y_{2p,2q}$, $p=0,\ldots,n$, $q=n+1,\ldots,2n+1$. From equality (17) we obtain

$$(\mu_{Y_{2p,2q}})_1 = [-1] \circ [\underbrace{0\ldots0}_{q-n-1}\ 1\ \underbrace{0\ldots0}_{2n}\ -1\ \underbrace{0\ldots0}_{3n-q}] \circ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ \vdots \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} p \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} 2n+1 \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} 2n-p \end{matrix}.$$

Therefore

$$(\mu_{Y_{2p,2q}})_1 = \begin{cases} [1] & \text{if } q \geq n+p+1, \\ [-1] & \text{if } q \leq n+p, \end{cases}$$

and in particular for the sets $V_p = Y_{2p-2,2p+2n}$, $W_{pr} = Y_{2r-2,2p+2n}$ $(p<r)$ that gives the thesis of lemma. □

Lemma 3.7. *Let $k = 4n+3$,*

$$Y_{ij} = U_r^{[0,iT/4(n+1)]} P_{iT/4(n+1)} D_R^{[iT/4(n+1)+\tau, jT/4(n+1)-\tau]} Q_{jT/4(n+1)} U_r^{[jT/4(n+1),T]},$$

$V_p = Y_{2p-2,2p+2n+1}$ be for $p = 1,\ldots,n+1$ isolating chains, $W_{ij} = V_i \cap V_j$, $U = U_r^{[0,T]}$. Then

$$\begin{aligned} ((Z_1)_T, (Z_1^-)_T) &= \tau_T((Z_2)_0, (Z_2^-)_0) && \textit{for any } Z_1, Z_2 \textit{ taken from } U, V_i, W_{ij}, \\ \mu_U &= \mu_{W_{ij}} = -\mathrm{id} && \textit{for } i,j = 0,\ldots,n+1, \\ \mu_U^2 &= \mu_{V_i} = \mathrm{id} && \textit{for } i = 0,\ldots,n+1, \\ \operatorname{Lef} \mu_U &= 1, \\ \chi(U_0, U_0^-) &= -1. \end{aligned}$$

Proof. Like in the proof of the previous lemma we choose basis of homology to obtain

$$(\mu_{Y_{2p,2q+1}})_1 = (-\mathrm{id}_1) \circ (\nu_{D_R^{[2pT/4(n+1)+\tau,(2q+1)T/4(n+1)-\tau]} Q_{(2q+1)T/4(n+1)}})_1 \circ (\nu_{P_{2pT/4(n+1)} D_R^{[2pT/4(n+1)+\tau,(2q+1)T/4(n+1)-\tau]}})_1, \tag{18}$$

where

$$(\nu_{P_{2pT/4(n+1)} D_R^{[2pT/4(n+1)+\tau,(2q+1)T/(4n+2)-\tau]}})_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ \vdots \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} p \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} 2n+2 \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} 2n-p+1 \end{array},$$

$$(\nu_{D_R^{[2pT/4(n+1)+\tau,(2q+1)T/4(n+1)-\tau]}Q_{(2q+1)T/4(n+1)}})_1$$
$$= \begin{cases} [\underbrace{0\dots0}_{n+q+1}\,-1\,\underbrace{0\dots0}_{2n+1}\,1\,\underbrace{0\dots0}_{n-q-1}] & \text{for } q<n, \\ [\underbrace{0\dots0}_{2n+1}\,-1\,\underbrace{0\dots0}_{2n+1}] & \text{for } q=n, \\ [\underbrace{0\dots0}_{q-n-1}\,1\,\underbrace{0\dots0}_{2n+1}\,-1\,\underbrace{0\dots0}_{3n-q+1}] & \text{for } q>n. \end{cases}$$

For interesting sets $V_p = Y_{2p-2,2p+2n+1}$ and their intersections $Y_{2p,2q+1}$, $p = 0,\dots,n$, $q = n+1,\dots,2n+1$, from (18) we have

$$(\mu_{Y_{2p,2q+1}})_1 = [-1] \circ [\underbrace{0\dots0}_{q-n-1}\,1\,\underbrace{0\dots0}_{2n+1}\,-1\,\underbrace{0\dots0}_{3n-q+1}] \circ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ \vdots \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} p \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} 2n+2 \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} 2n-p+1 \end{matrix}$$
$$= \begin{cases} [1] & \text{if } q \ge n+p+1, \\ [-1] & \text{if } q \le n+p, \end{cases}$$

what gives the proof. □

3.4. Chaotic dynamics in equation $\dot z = \overline{z}^k + e^{i\phi t}\overline{z}$. As a consequence of Corollary 2.2 and Lemmas 3.6, 3.7 we obtain the following

THEOREM 3.8. *Equation $\dot z = \overline{z}^k + e^{i\phi t}\overline{z}$ for odd $k \ge 3$, $|\phi|$ small enough is $\Sigma_{\lfloor (k+7)/4 \rfloor}$-chaotic.*

PROOF. Assume, that $|\phi|$ is small enough to admit isolating chains from Definition 3.5. In Lemmas 3.6, 3.7 we proved that if $k = 4n+1$ or $k = 4n+3$, then there are $n+1$ isolating chains V_j and a chain U satisfying assumption of Corollary 2.2. The Corollary gives Σ_{n+2}-chaotic dynamics. But $\lfloor (k+7)/4 \rfloor = n+2$ in both cases, so theorem holds. □

From Remark 3.4 it follows that for $k = 5$ any $\phi \in (0, 0.015]$ are good for the theorem.

4. Further results

Change of variables can generalise obtained results on wider class of equations. The following lemma deals with more general situation then previous sections.

Lemma 4.1. *Equation* $\dot z = a\overline{z}^k + be^{i\phi t}\overline{z}^m$ *for nonzero* $a, b \in \mathbb{C}$ *and* $k \neq m$ *can be rewritten in distinct system of coordinates as* $\dot z = \overline{z}^k + e^{i\widetilde{\phi}t}\overline{z}^m$.

Proof. First assume that $b^{k+1}/a^{m+1} \in \mathbb{R}$. Denote $\gamma = (a/b)^{1/(k-m)}$ and let $\zeta = \overline{\gamma} z$. We have

$$\begin{aligned}\dot\zeta &= \overline{\gamma}\dot z = \overline{\gamma}(a\gamma^{-k}\overline{\zeta}^k + b\gamma^{-m}e^{i\phi t}\overline{\zeta}^m)\\ &= \overline{\gamma}\left(\frac{b^k}{a^m}\right)^{1/(k-m)}(\overline{\zeta}^k + e^{i\phi t}\overline{\zeta}^m) = \alpha(\overline{\zeta}^k + e^{i\phi t}\overline{\zeta}^m).\end{aligned}$$

Under our assumptions α is nonzero real number. Now let $\xi(t) = \zeta(t/\alpha)$.

$$\dot\xi(t) = \frac{1}{\alpha}\dot\zeta\left(\frac{t}{\alpha}\right) = \overline{\zeta}\left(\frac{t}{\alpha}\right)^k + e^{i\phi t/\alpha}\overline{\zeta}\left(\frac{t}{\alpha}\right)^m = \overline{\xi}(t)^k + e^{i\widetilde{\phi}t}\overline{\xi}(t)^m,$$

where $\widetilde{\phi} = \phi/\alpha$

If in contrast $b^{k+1}/a^{m+1} \notin \mathbb{R}$ then there exists $t_0 \in \mathbb{R}$ such that

$$(be^{i\phi t_0})^{k+1}/a^{m+1} \in \mathbb{R}.$$

Changing time variable by formula $\tau = t - t_0$ the coefficient of $e^{i\phi t}\overline{z}^m$ is equal to $\widetilde{b} = be^{i\phi t_0}$ and the second one remains unchanged, so the conditon $\widetilde{b}^{k+1}/a^{m+1} \in \mathbb{R}$ is satisfied. In the first part of the proof we showed, that it implies the thesis of the lemma. □

Notice, that constructed isolating segments and chains for small enough perturbation of right-hand side of equation remains isolating sets. (By small perturbation we mean such a function g, that $\|g\|_\rho = \sup\{|f(x)| : |x| \leq \rho\}$ is small and ρ is such a number, that for every point (t, z) of any isolating set the inequality $|z| < \rho$ holds. In our situation we can choose $\rho = R/\cos(\pi/2(k+1))$.)

Using this fact we proove following

Lemma 4.2. *Equation* $\dot z = e^{i\phi t}\overline{z}^k + \overline{z}$ *is* $\Sigma_{\lfloor (k+7)/4 \rfloor}$*-chaotic for odd* k *and* $|\phi|$ *small enough.*

Proof. Change variables in equation $\dot z = e^{i\phi t}\overline{z}^k + \overline{z}$ by the formula $\zeta(t) = e^{-i\phi t/(k+1)}z(t)$. We get

$$\begin{aligned}\dot\zeta &= e^{-i\phi t/(k+1)}\dot z - i\frac{\phi}{k+1}e^{-i\phi t/(k+1)}z\\ &= e^{ik\phi t/(k+1)}\overline{z}^k + e^{i\phi t/(k+1)}\overline{z} - i\frac{\phi}{k+1}e^{-i\phi t/(k+1)}z\\ &= \overline{\zeta}^k + e^{2i\phi t/(k+1)}\overline{\zeta} - i\frac{\phi}{k+1}\zeta = \overline{\zeta}^k + e^{i\widetilde{\phi}t}\overline{\zeta} - i\frac{\widetilde{\phi}}{k+1}\zeta.\end{aligned} \tag{19}$$

We constructed isolating chains for the equation $\dot\zeta = \overline{\zeta}^k + e^{i\widetilde{\phi}t}\overline{\zeta}$ with small $\widetilde{\phi}$, and they remain isolating chains for small perturbation, in particular for

$i\widetilde{\phi}\zeta/(k+1)$ with small enough $\widetilde{\phi}$. Thus the equation from lemma is $\Sigma_{\lfloor (k+7)/4 \rfloor}$-chaotic. □

REMARK 4.3. It is possible to prove the same results as in Theorem 3.8 and Lemma 4.2 for even k. However the construction of isolating chains is more complicated and proofs are slightly different.

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Manuscript received June 28, 2002

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