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REMOVING COINCIDENCES OF MAPS BETWEEN MANIFOLDS OF DIFFERENT DIMENSIONS

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Abstract. We consider sufficient conditions of local removability of coincidences of maps $f,g\colon N\to M$, where M, N are manifolds with dimensions $\dim N\geq \dim M$. The coincidence index is the only obstruction to the removability for maps with fibers either acyclic or homeomorphic to spheres of certain dimensions. We also address the normalization property of the index and coincidence-producing maps.

1. Introduction

Let N^{n+m} and M^n be orientable compact smooth manifolds (possibly with boundaries ∂N, ∂M), $n\geq 2$, and suppose $f,g\colon N\to M$ are maps. We shall call m the *codimension*. The *coincidence set* is a compact subset of N defined by $\mathrm{Coin}(f,g)=\{x\in N : f(x)=g(x)\}$.

The Coincidence Problem asks what can be said about the coincidence set. When $m=0$, the main tools for studying the problem is the *Lefschetz number* $L(f,g)$ defined as the alternating sum of traces of certain endomorphism on the (co)homology group of M. The famous Lefschetz coincidence theorem provides a sufficient condition for the existence of coincidences: if $L(f,g)\neq 0$ then $\mathrm{Coin}(f,g)\neq\emptyset$. Under some circumstances the converse is also true (up to homotopy): $L(f,g)=0\to$ there are maps f',g' homotopic to f,g respectively

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such that $\mathrm{Coin}(f', g') = \emptyset$. Now the problem reads as follows: "Can we remove coincidences by a homotopy of f and g?"

Let $K = \mathrm{Coin}(f, g)$. By H_* (H^*) we denote the integral singular (co)homology. For any space Y we define the diagonal map $d: Y \to Y \times Y$ by $d(x) = (x, x)$. Let

$$Y^{\times} = (Y \times Y, Y \times Y \setminus d(Y)).$$

For codimension $m = 0$, the (cohomology) *coincidence index* I_{fg} of (f, g) is defined as follows. Since all coincidences lie in K, the map $(f, g): (N, N \setminus K) \to M^{\times}$ is well defined. Let τ be the generator of $H^n(M^{\times}) = \mathbb{Z}$ and O_N the fundamental class of N around K, then let

$$I_{fg} = \langle (f, g)^*(\tau), O_N \rangle \in \mathbb{Z}.$$

The coincidence index satisfies the following natural properties.

(1) Homotopy Invariance: the index is invariant under homotopies of f, g.
(2) Additivity: the index over a union of disjoint sets is equal to the sum of the indices over these sets.
(3) Existence of Coincidences: if the index is nonzero then there is a coincidence.
(4) Normalization: the index is equal to the Lefschetz number.
(5) Removability: if the index is zero then the coincidence set can be removed by a homotopy.

While the coincidence theory for codimension $m = 0$ is well developed (see [1, VI.14], [4], [13], [17, Chapter 7]), very little is known beyond this case. For $m > 0$, the vanishing of the coincidence index does not always guarantee removability. For codimension $m = 1$, the secondary obstruction to removability was considered by Fuller ([7], [8]) for M simply connected. In the context of Nielsen Theory the sufficient conditions of the local removability for $m = 1$ were studied by Dimovski and Geoghegan ([6]), Dimovski ([5]) for the projection $f: M \times [0, 1] \to M$, and by Jezierski ([12]) for M, N subsets of Euclidean spaces or M parallelizable. Necessary conditions of the global removability for arbitrary codimension were considered by Gonçalves, Jezierski, and Wong ([9, Section 5]) with N a torus and M a nilmanifold (see also [10]).

The main purpose of this note is to provide sufficient conditions of removability of coincidences for some codimensions higher than 1. Under a certain technical condition, the coincidence index defined below is the only obstruction to removability. This condition holds when

(1) M is a surface,
(2) fibers $f^{-1}(y)$ of f are acyclic, or
(2) fibers of f are m-spheres for $m = 4, 5, 12$ and n large.

The main theorem partially complements the results listed above. The proof follows and extends the one of Brown and Schirmer ([4, Theorem 3.1]) for codimension 0 (see also Vick [17, p. 194]).

An area of possible applications is discrete dynamical systems. A *dynamical system* on a manifold M is determined by a map $f\colon M \to M$. Then the next position, or state, $f(x)$ depends only on the current one, $x \in M$. Suppose we have a fiber bundle $F \to N \xrightarrow{g} M$ and a map $f\colon N \to M$. Then this is a *parametrized dynamical system*, where the next position $f(x,s)$ depends not only on the current one, $x \in M$, but also the "input", $s \in F$. Then the Coincidence Problem asks whether there are a position and an input such that the former remains unchanged, $f(x,s) = x$. A parametrized dynamical system can also be a model for a non-autonomous ordinary differential equation: M is the space, F is the time, and N is the space-time.

2. Normalization property

For nonzero codimension the homology coincidence index $I'_{fg} = (f,g)_*(O_N)$ is replaced with the *homology coincidence homomorphism* (see [3])

$$I'_{fg} = (f,g)_*\colon H_*(N, N \setminus V) \to H_*(M^\times),$$

where V is a neighbourhood of $\mathrm{Coin}(f,g)$. Let $\pi\colon M \times M \to M$ be the projection on the first factor, then $\zeta = (M, \pi, M \times M, d)$ is the tangent microbundle of M and the Thom isomorphism $\varphi\colon H_*(M^\times) \to H_*(M)$ is given by $\varphi(x) = \pi_*(\tau \frown x)$, where $\tau \in H^n(M^\times)$ is the Thom class of ζ. The Lefschetz number is replaced with the *Lefschetz homomorphism* $\Lambda_{fg}\colon H_*(N, N \setminus V; \mathbb{Q}) \to H_*(M; \mathbb{Q})$ of degree $(-n)$ (see [15]) defined as follows. Suppose $f(N \setminus V) \subset \partial M$. For each $z \in H_*(N, N \setminus V)$, let

$$f_!^z = (f^* D^{-1}) \frown z,$$

where $D\colon H^*(M, \partial M; \mathbb{Q}) \to H_{n-*}(M; \mathbb{Q})$ is the Poincaré–Lefschetz duality isomorphism $D(x) = x \frown O_M$. Now let

$$\Lambda_{fg}(z) = \sum_k (-1)^{k(k+m)} \sum_j x_j^k \frown g_* f_!^z(a_j^k),$$

where $\{a_1^k, \dots, a_{m_k}^k\}$ is a basis for $H_k(M)$ and $\{x_1^k, \dots, x_{m_k}^k\}$ the corresponding dual basis for $H^k(M)$. Then the Lefschetz-type coincidence theorem ([15, Theorem 6.1]) states that $\varphi I'_{fg} = \Lambda_{fg}$. This is the Normalization Property, which makes the coincidence homomorphism computable by algebraic means.

Since obstructions to removability of coincidences lie in certain cohomology groups, we need a cohomological analogue of the theory outlined above. Just as in the homology case, the cohomology coincidence index can be replaced with the cohomology coincidence homomorphism.

Definition 2.1. Let C be an isolated subset of $\mathrm{Coin}(f,g)$, W,V neighbourhoods of C, $C\subset V\subset \overline{V}\subset W\subset N$, and $W\cap \mathrm{Coin}(f,g)=C$. Then let

$$I_{fg}=(f,g)^*\colon H^*(M^\times)\to H^*(W,W\setminus V).$$

However in this paper we consider only the restriction of I_{fg} to $H^n(M^\times)=\mathbb{Z}$. Therefore the only thing that matters is the class $I_{fg}(\tau)\in H^n(W,W\setminus V)$, where τ is the generator of $H^n(M^\times)=\mathbb{Z}$, which will still be called the (*cohomology*) *coincidence index*. This index satisfies the properties of additivity, existence of coincidences and homotopy invariance proven similarly to Lemmas 7.1, 7.2, 7.4 in [17, p. 190–191], respectively.

We will state the Normalization Property under assumptions similar to the ones in [14, Section 2], [15, Section 5]. Assume that $f(W\setminus V)\subset\partial M$.

Definition 2.2. For each $z\in H_n(W,W\setminus V;\mathbb{Q})$, define homomorphisms $\Theta_q\colon H^q(M,\partial M;\mathbb{Q})\to H^q(M,\partial M;\mathbb{Q})$ by

$$\Theta_q=D^{-1}g_*(f^*\frown z).$$

Then

$$L_z(f,g)=\sum_q(-1)^q Tr\Theta_q$$

is called the (cohomology) *Lefschetz number* with respect to z of the pair (f,g).

Theorem 2.3 (Normalization). *Suppose that $f(W\setminus V)\subset\partial M$. Then for each $z\in H_n(W,W\setminus V;\mathbb{Q})$,*

$$\langle I_{fg}(\tau),z\rangle=(-1)^n L_z(f,g).$$

Therefore, if $L_z(f,g)\neq 0$ then $\mathrm{Coin}(f,g)\neq\emptyset$.

Proof. The proof repeats the computation in the proof of Theorem 7.12 in [17, p. 197] with Lemmas 7.10 and 7.11 replaced with their generalizations, Lemmas 3.1 and 3.2 in [14]. □

The theorem is true even when N is not a manifold.

3. Local removability

Let $f\colon\mathbb{S}^3\to\mathbb{S}^2$ be the Hopf map. Then f is onto, in other words, it has a coincidence with any constant map c. However the coincidence homomorphism $I_{fc}\colon H^*((\mathbb{S}^2)^\times)\to H^*(\mathbb{S}^3)$ is zero. Therefore Theorem 2.3 fails to detect coincidences. In fact, f has a coincidence with any map homotopic to c (see [2]), therefore the converse of the Lefschetz coincidence theorem for spaces of different dimensions fails in general. Our main result below is a partial converse.

Theorem 3.1 (Local Removability). *Suppose $f(C) = g(C) = \{u\}$, $u \in M \setminus \partial M$, and the following condition is satisfied:*

(A) $H^{k+1}(W, W \setminus V; \pi_k(\mathbb{S}^{n-1})) = 0$ *for* $k \geq n+1$.

Then $I_{fg}(\tau) = 0$ implies that C can be removed via a local homotopy of f; specifically, there exists a map $f' \colon N \to M$ homotopic to f relative $N \setminus W$ such that

$$W \cap \mathrm{Coin}(f', g) = \emptyset.$$

The proof uses the classical obstruction theory. Condition (A) guarantees that only the primary obstruction to the local removability, i.e. the coincidence index, may be nonzero.

Proof. We can assume that $U = \mathbb{D}^n$ is a neighbourhood of u in M such that $f(W) = U$ and $g(W) \subset U$. Define $Q \colon \mathbb{D}^n \times \mathbb{D}^n \setminus d(\mathbb{D}^n) \to \mathbb{D}^n \setminus \{0\}$ by $Q(x, y) = 1/2(y - x)$. Consider the following commutative diagram

$$\begin{tikzcd}
H^{n-1}(\mathbb{S}^{n-1}) \arrow[d, "p^*", "\simeq"'] & & \\
H^{n-1}(\mathbb{D}^n \setminus \{0\}) \arrow[r, "\delta^*", "\simeq"'] \arrow[d, "Q^*"] & H^n(\mathbb{D}^n, \mathbb{D}^n \setminus \{0\}) \arrow[d, "Q^*", "\simeq"'] & \\
H^{n-1}(\mathbb{D}^n \times \mathbb{D}^n \setminus d(\mathbb{D}^n)) \arrow[r, "\delta^*"] \arrow[d, "(f,g)^*"] & H^n((\mathbb{D}^n)^\times) \arrow[d, "(f,g)^*"] & H^n(M^\times) \arrow[l, "k^*"', "\simeq"] \arrow[dl, "I_{fg}"] \\
H^{n-1}(W \setminus V) \arrow[r, "\delta^*"'] & H^n(W, W \setminus V). &
\end{tikzcd}$$

Here δ^* is the connecting homomorphism, k the inclusion, p the radial projection. Let $q = pQ(f, g) \colon W \setminus V \to \mathbb{S}^{n-1}$. Then q^* is given in the first column of the diagram.

Now we apply the Extension Theorem, Corollary VII.13.13 in [1, p. 509]. Suppose $I_{fg} = 0$. Then from the commutativity of the diagram, $\delta^* q^* = 0$. Thus the primary obstruction to extending q to $q' \colon W \to \mathbb{S}^{n-1}$, $c^{n+1}(q) = \delta^* q^*$, vanishes. By condition (A) the other obstructions $c^{k+1}(q)$, $k \geq n$, also vanish.

Next, q has the form

$$q(x) = \frac{g(x) - f(x)}{||g(x) - f(x)||}.$$

Define a map $f' \colon W \to \mathbb{D}^n$ by $f'(x) = g(x) - a(x) q'(x)$, where $a \colon W \to (0, \infty)$ satisfies the following:

(1) a is small enough so that $f'(x) \in \mathbb{D}^n$ for all $x \in W$,
(2) $a(x) = ||g(x) - f(x)||$ for all $x \in W \setminus V$.

Then $\operatorname{Coin}(f', g) = \emptyset$ since $q'(x) \neq 0$.

To complete the proof observe that f' is homotopic to $f|_W$ relative $W \setminus V$ because $\mathbb{D}^n$ is convex. □

The implications of this result for Nielsen theory will be addressed in a forthcoming paper.

4. Further results

Suppose $C = f^{-1}(y)$, where $y \in M \setminus \partial M$ is a regular value for both f and $f|_{\partial N}$. Then C is a neat submanifold of N and it has a tubular neighbourhood T. Now T can be treated as a disk bundle $(\mathbb{D}^m, \mathbb{S}^{m-1}) \to (T, T') \to C$. Therefore condition (A) takes the form

(A') $H^{k+1}(T, T'; \pi_k(\mathbb{S}^{n-1})) = 0$ for $k \geq n+1$.

In case C is a boundaryless m-submanifold of N, we have $H^{n+m}(T, T'; G) = H^{n+m}(T, \partial T; G) = G \oplus \ldots \oplus G$. Therefore if we let $k = n+m-1$, then condition (A') implies the following:

(A*) $\pi_{n+m-1}(\mathbb{S}^{n-1}) = 0$.

This restriction cannot be relaxed, in the following sense. Suppose

$$[h] \in \pi_{n+m-1}(\mathbb{S}^{n-1}) \setminus \{0\}.$$

Then h can be extended to a map $f\colon \mathbb{D}^{n+m} \to \mathbb{D}^n \subset M$ by setting $f(0) = 0$ and $f(x) = ||x|| h(x/||x||)$ for $x \in \mathbb{D}^{n+m} \setminus \{0\}$. Hence any map homotopic to f relative $\mathbb{S}^{n+m-1}$ is onto [1, Theorem VII.5.8, p. 448]. Therefore coincidences of f and g, where g is constant, cannot be locally removed.

Below we treat condition (A) as a restriction on an arbitrary fiber $C = f^{-1}(y), y \in M \setminus \partial M$ of f.

LEMMA 4.1. *Suppose for $1 \leq p \leq m$, the submanifold C satisfies*

(a) $H^p(C) \otimes \pi_{n+p-1}(\mathbb{S}^{n-1}) = 0$, *and*
(b) $\operatorname{Tor}(H^{p+1}(C), \pi_{n+p-1}(\mathbb{S}^{n-1})) = 0$.

Then C satisfies condition (A).

PROOF. By the Thom Isomorphism Theorem (see [1, Section VI.11]), we have

$$H^{k+1}(T, T'; \pi_k(\mathbb{S}^{n-1})) = H^{k+1-n}(C; \pi_k(\mathbb{S}^{n-1})).$$

By condition (b) and the Universal Coefficient Theorem, Corollary 25.14 in [11, p. 263], we have also

$$H^p(C; \pi_k(\mathbb{S}^{n-1})) = H^p(C) \otimes \pi_k(\mathbb{S}^{n-1}).$$ □

It is known (see [16]) that $\pi_{n+m-1}(\mathbb{S}^{n-1}) = 0$, for the following values of m and n:

(1) $m = 4$ and $n \geq 6$,
(2) $m = 5$ and $n \geq 7$,
(3) $m = 12$ and $n = 7, 8, 9, 14, 15, 16, \dots$

Corollary 4.2. *The conclusion of Theorem* 3.1 *holds when*

(a) *M is a surface,*
(b) *fibers of f are acyclic, or*
(c) *fibers of f are unions of homology m-spheres for the above values of m and n.*

Proof. (a) $n = 2$ and $\pi_{n+p-1}(\mathbb{S}^{n-1}) = 0$ for all $p > 0$.

(b) $H^p(C) = 0$ for all $p > 0$.

(c) Either $\pi_{n+p-1}(\mathbb{S}^{n-1}) = 0$ or $H^p(C) = H^{p+1}(C) = 0$ for all $p > 0$. Thus the two conditions of Lemma 4.1 are satisfied. Now the conclusion follows from Theorem 3.1. □

Corollary 4.3. *Let $F \to N^{n+m} \xrightarrow{g} M^n$ be an m-sphere bundle with the above values of m and n, or an m-disk bundle. Then the set C of stationary points of the parametrized dynamical system $F \to N^{n+m} \xrightarrow{f,g} M^n$ can be removed via a local homotopy of f provided $I_{fg} = 0$.*

5. Coincidence-producing maps

A boundary preserving map $f\colon (N, \partial N) \to (M, \partial M)$ is called *coincidence-producing* if every map $g\colon N \to M$ has a coincidence with f. Brown and Schirmer in [4, Theorem 7.1] showed that if M is acyclic, $\dim N = \dim M = n \geq 2$, then f is coincidence-producing if and only if $f_*\colon H_n(N, \partial N) \to H_n(M, \partial M)$ is nonzero. Based on the Normalization and Removability Properties we prove a generalization of this theorem. We call a map $f\colon (N, \partial N) \to (M, \partial M)$ *weakly coincidence-producing* ([14, Section 5]) if every map $g\colon N \to M$ with $g_* = 0$ (in reduced homology) has a coincidence with f. In particular every weakly coincidence-producing map is onto.

A corollary to the Lefschetz type coincidence theorem ([14, Corollary 5.1]) states that if $f_*\colon H_n(N, \partial N) \to H_n(M, \partial M)$ is nonzero then the appropriate Lefschetz homomorphism is nontrivial and, therefore, f is weakly coincidence-producing. For the converse we need condition (A) as an additional assumption.

Theorem 5.1. *Suppose f is boundary preserving and suppose that each fiber C of f satisfies condition* (A)*. Then the following are equivalent:*

(a) *f is weakly coincidence-producing,*
(b) *$f_*\colon H_n(N, \partial N) \to H_n(M, \partial M)$ is nonzero.*

Proof. Suppose $f_*\colon H_n(N,\partial N) \to H_n(M,\partial M)$ is zero. Choose g to be identically equal to $y \in M \setminus \partial M$. Then $C = \mathrm{Coin}(f,g) = f^{-1}(y) \subset N \setminus \partial N$. Recall $I_{fg} = (f,g)^*\colon H^*(M^\times) \to H^*(N,\partial N)$. Then for all $z \in H_n(N,\partial N)$, we have the following.

$$\begin{aligned}
\langle I^N_{fg}(\tau), z\rangle &= (-1)^n L_z(f,g) && \text{by Theorem 2.3}\\
&= (-1)^n \mathrm{Tr}\,\Theta_n && \text{because } g_* = 0\\
&= (-1)^n \langle f^*(\overline{O}_M), z\rangle && \text{where } \overline{O}_M \text{ is the dual of } O_M\\
&= (-1)^n \langle \overline{O}_M, f_*(z)\rangle = 0.
\end{aligned}$$

Hence $I_{fg}(\tau) = 0$. Therefore by Theorem 3.1 the coincidence set can be removed. Thus f is not weakly coincidence-producing □

Condition (A) is clearly satisfied for $m = 0$. Therefore Brown and Schirmer's Theorem ([4, Theorem 7.1]) follows. Our theorem also includes the well-known fact that *a map has degree* 0 *if and only if it can be deformed into a map which is not onto.*

Examples of maps satisfying condition (a) of the theorem can be found in [4, Section 7], see also [15, Section 6].

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