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MOUNTAIN PASS SOLUTIONS AND AN INDEFINITE SUPERLINEAR ELLIPTIC PROBLEM ON $\mathbb{R}^N$

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Abstract. We consider the elliptic problem

$$-\Delta u - \lambda u = a(x)g(u),$$

with $a(x)$ sign-changing and $g(u)$ behaving like u^p, $p > 1$. Under suitable conditions on $g(u)$ and $a(x)$, we extend the multiplicity, existence and nonexistence results known to hold for this equation on a bounded domain (with standard homogeneous boundary conditions) to the case that the bounded domain is replaced by the entire space $\mathbb{R}^N$. More precisely, we show that there exists $\Lambda > 0$ such that this equation on $\mathbb{R}^N$ has no positive solution for $\lambda > \Lambda$, at least two positive solutions for $\lambda \in (0, \Lambda)$, and at least one positive solution for $\lambda \in (-\infty, 0] \cup \{\Lambda\}$.

Our approach is based on some descriptions of mountain pass solutions of semilinear elliptic problems on bounded domains obtained by a special version of the mountain pass theorem. These results are of independent interests.

1. Introduction

This paper is a continuation of [13] where the elliptic problem

$$(1.1)\qquad -\Delta u - \lambda u = a(x)u^p, \quad x \in \mathbb{R}^N$$

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with $p > 1$ and $a(x)$ sign-changing is studied. This is known as an indefinite superlinear problem. The fact that $a(x)$ changes sign poses extra difficulties from the cases that $a(x)$ is always negative (the sublinear case) and $a(x)$ is always positive (the superlinear case).

The approach in [13] is based on bounded domain approximation and global bifurcation arguments. Here we replace the global bifurcation argument by some variational ones. This allows us to relax the restriction on the range of p made in [13] for the main result. A key ingredient for this improvement is some nonlinear Liouville theorems obtained in [23] for solutions of some limiting entire space problems that have finite Morse index. In order to use these Liouville theorems, we have to prove a variation of the well known mountain pass theorem and give some descriptions of its solutions; this may have other applications and seems to be of independent interest.

When (1.1) is considered on a bounded domain $\Omega \subset \mathbb{R}^N$ with standard homogeneous boundary conditions on $\partial\Omega$, it is known from recent results (see, for example, [1], [2], [4], [5], [25]) that, under suitable conditions on p and on the behaviour of $a(x)$ near its zero set, (1.1) has a positive solution for $\lambda = \lambda_1(\Omega)$ (the first eigenvalue of the Laplacian under the corresponding boundary conditions on $\partial\Omega$) if and only if

$$\int_\Omega a(x)\phi^{p+1}(x)\,dx < 0, \tag{1.2}$$

where ϕ denotes the (normalized) positive eigenfunction corresponding to $\lambda_1(\Omega)$. Moreover, when (1.2) is satisfied, there exists $\Lambda > 0$ such that (1.1) has at least two positive solutions for every $\lambda \in (\lambda_1(\Omega), \Lambda)$, at least one positive solution for $\lambda = \Lambda$ and for $\lambda = \lambda_1(\Omega)$, and no positive solution for $\lambda > \Lambda$. Under less restrictive conditions, (1.1) has at least one positive solution for each $\lambda < \lambda_1(\Omega)$.

We are interested in extending these results to the entire space problem (1.1). We were motivated by a recent work of Costa and Tehrani ([11]), where such an extension was partially achieved through a variational approach. To overcome the typical difficulties with entire space problems, such as loss of compactness, [11] considered a problem on $\mathbb{R}^N$ including (1.1) as a typical case, but with λ replaced by $\lambda h(x)$, where h is a nonnegative function belonging to the space $L^{N/2}(\mathbb{R}^N) \cap L^\alpha(\mathbb{R}^N)$ for some $\alpha > N/2$. This allows them to regain compactness for the variational approach. Moreover, the eigenvalue problem

$$-\Delta u = \lambda h(x)u, \quad u \in D^{1,2}(\mathbb{R}^N)$$

behaves similarly to the finite domain case, with a first eigenvalue $\lambda_1(h) > 0$. Under conditions on p and $a(x)$ similar to those for the bounded domain case, and furthermore,

$$\lim_{|x|\to\infty} a(x) = a_\infty < 0, \tag{1.3}$$

it is shown in [11] that the entire space problem has at least one positive solution for $\lambda \le \lambda_1(h)$, and at least two positive solutions for λ in a small right neighbourhood of $\lambda_1(h)$. The existence of a critical $\Lambda > 0$ as in the finite domain case was not considered in [11]. The introduction in [11] contains a fairly detailed account of other studies of entire space problems. We refer to that and the references therein for the interested reader. See also [10] for some more recent results.

In contrast with [11], we use a bounded domain approximation approach to study (1.1). This allows us to avoid replacing λ by $\lambda h(x)$ as in [11]. Under similar conditions on $p > 1$ and $a(x)$ as in the bounded domain case, and (1.3), a complete extension of the bounded domain result is obtained in [13], namely, there exists $\Lambda > 0$ such that (1.1) on $\mathbb{R}^N$ has no positive solution for $\lambda > \Lambda$, at least two positive solutions for $\lambda \in (0, \Lambda)$, and at least one positive solution for $\lambda \in (-\infty, 0] \cup \{\Lambda\}$. Note that (1.3) implies (1.2) for all "large" enough Ω.

However, due to the method used in [13] to obtain a priori bound for positive solutions on bounded domains, the optimal range of p is not reached in the main result there. In this paper, we replace the global bifurcation argument in [13] by a variational approach which yields, for the bounded domain problems, positive solutions with uniformly bounded Morse index. This enables us to use techniques of [23] to obtain a priori bound for these solutions with p reaching the optimal range, and hence the main result for such p.

In Section 2, we prove a variant of the mountain pass theorem and provide some descriptions of the mountain pass solutions so obtained, where upper and lower solutions and the order structure of some widely used spaces are employed. Similar considerations have been extensively used for various purposes; we refer to [3], [8], [12], [18], [20] for some examples of these. However, none of the existing results seems directly applicable to our situation here.

In Section 3, we apply the results of Section 2 to the indefinite superlinear problem (1.1). We mainly follow the lines of [13] but with the variational consideration replacing the global bifurcation arguments. While the global bifurcation argument allows us to replace u^p in (1.1) by a wide class of more general functions (as mentioned at the end of Section 1 in [13]), the variational approach seems more sensitive to such generalizations. This is caused mainly by checking the (PS) condition with a sign-changing $a(x)$ for the bounded domain problems. In order to demonstrate this point, we have replaced u^p by $g(u)$ with the requirements on $g(u)$ stated explicitly in each step of our proof towards the main result. Note that we have also replaced $a(x)$ by $-b(x)$ to match the notations in [13].

2. Mountain pass solutions

In this section, we will give some descriptions of certain solutions of a semilinear elliptic problem on a bounded domain obtained by a special version of the mountain pass theorem. These descriptions will become important in our study of the entire space problem in Section 3.

The results here are variants of some well known conclusions about mountain pass solutions. These variations seem necessary in order to be applicable to our entire space problem, and some of them cause nontrivial difficulties in the proof.

Consider the problem

$$-\Delta u = f(x,u) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, f is continuous in $x \in \overline{\Omega}$ and C^1 in $u \in (-\infty,\infty)$ uniformly for $x \in \overline{\Omega}$, and we assume the natural growth condition

$$|f(x,u)| \le c(1+|u|^q), \tag{2.2}$$

where c is a positive constant and $1 < q \le (N+2)/(N-2)$ when $N \ge 3$, $q > 1$ is arbitrary when $N = 1,2$.

Recall that (weak) solutions of (2.1) are critical points of

$$I(u) = \int_\Omega |\nabla u|^2/2 - F(x,u), \quad u \in H_0^1(\Omega),$$

where $F(x,u) = \int_0^u f(x,s)\,ds$.

$u_0 \in H_0^1(\Omega)$ is called a *lower solution* to (2.1) if

$$\int_\Omega \nabla u_0 \cdot \nabla\phi - f(x,u_0)\phi \le 0 \quad \text{for all } \phi \in C_0^\infty(\Omega),\ \phi \ge 0.$$

u_0 is an upper solution to (2.1) if

$$\int_\Omega \nabla u_0 \cdot \nabla\phi - f(x,u_0)\phi \ge 0 \quad \text{for all } \phi \in C_0^\infty(\Omega),\ \phi \ge 0.$$

Suppose that C is a subset of $H_0^1(\Omega)$. We say that I satisfies the (PS) condition in C if for any sequence $\{u_n\} \subset C$, $\{I(u_n)\}$ bounded and $I'(u_n) \to 0$ imply that $\{u_n\}$ has a convergent subsequence.

A solution u_0 to (2.1) is said to have Morse index k, written $m(u_0) = k$, if

$$J(\phi) = \int_\Omega |\nabla\phi|^2 - f_u(x,u_0(x))\phi^2$$

is negative definite on a k-dimensional subspace of $H_0^1(\Omega)$ but not on any $k+1$-dimensional subspace of $H_0^1(\Omega)$.

We will use the following notations. For $u \in H_0^1(\Omega)$,

$$[u,\infty) = \{w \in H_0^1(\Omega) : w \ge u \text{ a.e. in } \Omega\}.$$

For $u, v \in H_0^1(\Omega)$ satisfying $u \le v$ a.e. in Ω,

$$[u, v] = \{w \in H_0^1(\Omega) : u \le w \le v \text{ a.e. in } \Omega\}.$$

The following technical condition will be useful.

For any given positive constant $K > 0$, there exists a strictly increasing C^1 function $m_K(u)$ with $m_K(0) = 0$ satisfying, for every $x \in \Omega$ and $|\alpha| \le K$,

$$\begin{cases} \text{(a) } |m_K(u)| \le c_K(1 + |u|^r), \\ \text{(b) } f(x, \alpha + u) + m_K(u) \text{ is strictly increasing in } u, \end{cases} \tag{2.3}$$

where c_K is a positive constant and $1 < r \le (N+2)/(N-2)$ when $N \ge 3$, $r > 1$ is arbitrary if $N = 1, 2$.

We are now ready to state our main results of this section.

Theorem 2.1. *Suppose that* (2.2) *and* (2.3) *hold and that* $\underline{u}, \overline{u} \in C_0^1(\overline{\Omega})$ *is a pair of lower and upper solutions of* (2.1) *satisfying* $\underline{u} \le \overline{u}$ *in* Ω. *Moreover, suppose that* $\overline{u}$ *is not a solution of* (2.1) *and there exists* $u_0 \in C_0^1(\overline{\Omega})$ *such that*

$$u_0 \ge \overline{u} \quad \text{in } \Omega, \quad I(u_0) \le \inf_{u \in [\underline{u}, \overline{u}]} I(u),$$

and I *satisfies the* (PS) *condition in* $[\underline{u}, \infty)$. *Then*

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) > \inf_{u \in [\underline{u}, \overline{u}]} I(u),$$

where $\Gamma = \{\gamma \in C([0,1], [u_*, \infty) \cap C_0^1(\overline{\Omega})) : \gamma(0) = u_*,\ \gamma(1) = u_0\}$, *and* u_* *is the maximal solution of* (2.1) *in* $[\underline{u}, \overline{u}]$. *Moreover,* (2.1) *has a solution* u^* *satisfying*

(a) $I(u^*) = c$,
(b) $u^* \in [\underline{u}, \infty) \setminus [\underline{u}, \overline{u}]$,
(c) $u^* > u_*$ *in* Ω,
(d) u^* *is either a local minimizer of* I *in the smaller space* $C_0^1(\overline{\Omega})$ *or is a critical point of* I *of mountain pass type in* $C_0^1(\overline{\Omega})$.

Here following Hofer ([19]), u^* is called a critical point of I of mountain pass type in $C_0^1(\overline{\Omega})$ if given any neighbourhood N of u^* in $C_0^1(\overline{\Omega})$, $N \cap I^c$ is not empty and not path-connected, where I^c denotes the set $\{u \in H_0^1(\Omega) : I(u) < c\}$ and $c = I(u^*)$.

Let us note that by [7], u^* is a local minimizer of I in $C_0^1(\overline{\Omega})$ implies that it is also a local minimizer of I in $H_0^1(\Omega)$. In our applications in Section 3, we will need an estimate of the Morse index of the solution u^* in Theorem 2.1. This is provided by the following result.

Theorem 2.2. *Suppose that u^* is a solution of* (2.1) *with f satisfying* (2.2). *If u^* is either a local minimizer of I in $C_0^1(\overline{\Omega})$ or is a critical point of I of mountain pass type in $C_0^1(\overline{\Omega})$, then $m(u^*) \leq 1$. Note that we do not assume that u^* is an isolated solution of* (2.1).

The rest of this section consists of the proofs for the above two theorems. A large part of the proofs consists of examination or variation of existing proofs of various facts from different sources. Therefore we will from time to time be rather brief in the proof. On the other hand, in order to make the ideas clear, efforts are made to provide necessary details for all the major steps.

Proof of Theorem 2.1. Choose a finite interval $[\alpha,\beta] \subset (-\infty,\infty)$ such that $[\alpha,\beta] \supset [\underline{u}(x), \overline{u}(x)]$ for $x \in \Omega$. Then we can find a large positive constant M such that $f(x,u) + Mu$ is strictly increasing in u for $u \in [\alpha,\beta]$. Let us define $v_0 = \overline{u}$ and for $n \geq 1$, let v_n be the unique weak solution to

$$-\Delta v_n + M v_n = f(x, v_{n-1}) + M v_{n-1}, \quad v_n|_{\partial\Omega} = 0.$$

It is well known that as $n \to \infty$, v_n decreases to a maximal solution u_* of (2.1) in the order interval $[\underline{u}, \overline{u}]$. Since $-\Delta u_* = f(\,\cdot\,, u_*) \in L^\infty(\Omega)$, a standard regularity consideration yields that $u_* \in C_0^1(\overline{\Omega})$. Since $\overline{u}$ is not a solution, by standard comparison argument, $u_* < \overline{u}$ in Ω.

Let $C = [u_*, \infty)$ or $[u_*, \overline{u}]$ or $[\underline{u}, \overline{u}]$. We will consider the restriction of I on C. In the case $C = [u_*, \overline{u}]$ or $[\underline{u}, \overline{u}]$, since $f(x,u) + Mu$ is increasing in u for $u \in [\alpha,\beta]$ which covers the range of u for $u \in C$, it is well known (see [18]) that by choosing a suitable equivalent norm in $H_0^1(\Omega)$, $I'(u)$ has the form

$$I'(u) = u - K(u),$$

where K is a nonlinear order-preserving operator that maps C into itself. This enables the construction of a flow which leaves C invariant and can be used to obtain a deformation lemma in C, and therefore a critical point theory on C can be established. Such an approach carries over to the case $C = [u_*, \infty)$ if there exists some $M_1 > 0$ such that $f(x,u) + M_1 u$ is increasing in u for $u \in [\alpha, \infty)$, which, however, can never be satisfied if $f(x,u) = \lambda u + a(x) u |u|^{p-1}$, with $p > 1$ and $a(x)$ sign-changing. To overcome this difficulty, we make use of (2.3) and adapt a trick in [20]. Denote $\widetilde{f}(x,u) = m(u) + f(x,u)$, where $m(u) = m_K(u)$ with K satisfying $\underline{u}(x), \overline{u}(x) \in [-K, K]$. Then (2.1) can be rewritten as

$$-\Delta u + m(u) = \widetilde{f}(x,u), \quad u|_{\partial\Omega} = 0.$$

Correspondingly, $I(u)$ can be rewritten in the form

$$I(u) = \int_\Omega |\nabla u|^2/2 + \int_\Omega M(u) - \int_\Omega \widetilde{F}(x,u),$$

where $M(u) = \int_0^u m(s)\,ds$, $\widetilde{F}(x,u) = \int_0^u \widetilde{f}(x,s)\,ds$. Now we can write $I'(u)$ in the form

$$I'(u) = A_0(u) - K(u),$$

where $A_0(u) = u + (-\Delta)^{-1}m(u)$ and $K(u) = (-\Delta)^{-1}\widetilde{f}(\,\cdot\,,u)$ are both order preserving in C, since both $m(u)$ and $\widetilde{f}(x,u)$ are increasing in u. Moreover, it is shown in [20, Section 4] that A_0 is C^1 and has a C^1 inverse A_0^{-1} which is order-preserving and satisfies, for some $a_0 > 0$,

$$\|A_0^{-1}(u)\| \geq a_0\|u\| \quad \text{if } \|A_0^{-1}(u)\| < 1,$$
$$\int_\Omega uA_0^{-1}(u) \geq \|A_0^{-1}(u)\|^2.$$

Here and in what follows, $\|\cdot\|$ denotes the norm in $H_0^1(\Omega)$.

By choosing a suitable locally Lipschitz continuous function $\psi(u)$ satisfying $\psi(u) \geq 0$, it is proved in [20] that the unique solution $\sigma(t,u)$ of

$$\frac{d}{dt}\sigma(t,u) = -\psi(\sigma)g(\sigma), \quad \sigma(0,u) = u,$$

with $g(u) = A_0^{-1}(I'(u))$, can be used to obtain a deformation lemma (see Lemma 2.2 in [20]) whose details are listed in Claim 1 below.

Claim 1. Let $S \subset E := H_0^1(\Omega)$, $c \in (-\infty,\infty)$, $\varepsilon > 0$ and $\delta > 0$ be such that

$$\|I'(u)\| \geq \frac{2\varepsilon}{\delta a_0} \quad \text{for all } u \in I^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap S_{2\delta}.$$

Then there exists $\eta \in C([0,1]\times E, E)$ satisfying

(a) $\eta(t,u) = u$, if $t = 0$ or if $u \notin I^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap S_{2\delta}$,
(b) $\eta(1, I^{c+\varepsilon} \cap S) \subset I^{c-\varepsilon}$,
(c) $\eta(t,\,\cdot\,)$ is a homeomorphism of E, for any fixed $t \in [0,1]$,
(d) $\|\eta(t,u) - u\| \leq \delta$ for all $u \in E$ and all $t \in [0,1]$,
(e) $I(\eta(t,u))$ is non-increasing in t for $t \in [0,1]$,
(f) $I(\eta(t,u)) < c$, for all $u \in I^c \cap S_\delta$ and all $t \in (0,1]$,
(g) $\eta(t,u) \in C$ for $t \in [0,1]$ if $u \in C$,
(h) $\eta(t,u) \in C \cap C_0^1(\overline{\Omega})$ for $t \in [0,1]$ if $u \in C \cap C_0^1(\overline{\Omega})$.

Here we used the notation $S_\delta = \{u \in H_0^1(\Omega) : d(u,S) < \delta\}$.

We should remark that some modifications of the argument in [20] are needed in order to prove properties (g) and (h) above. These are proved in [20] only for $C = [0,\infty)$ or C is a finite order interval, and for this they require the extra assumption that $A_0(u)$ is linear when $\|u\|_{L^\infty(\Omega)} \leq M$. But such an extra assumption on A_0 does not seem to work when C is of the form $[u,\infty)$ with $u \neq 0$. This difficulty is circumvented here by making use of (2.3). For example, to show that for $v \geq 0$ one has $\underline{u} + v - A_0^{-1}(I'(\underline{u}+v)) \geq \underline{u}$, it suffices to show $A_0(v) \geq I'(\underline{u}+v)$, which is equivalent to $(-\Delta)^{-1}[f(\,\cdot\,,\underline{u}+v) + m(v)] \geq \underline{u}$. By (2.3),

$f(\,\cdot\,,\underline{u}+v)+m(v)\geq f(\,\cdot\,,\underline{u})$. Thus the required inequality is a consequence of the assumption that $\underline{u}$ is a lower solution of (2.1).

Claim 2. u_* is a strict local minimizer of I on $[u_*,\infty)$.

We consider first the restriction of I on $[u_*,\overline{u}]$. It is well known (see [8] or Theorem 2.4 in [24]) that the global minimizer of I on $[u_*,\overline{u}]$ is a weak solution of (2.1). Since u_* is the only solution of (2.1) in this order interval, necessarily, u_* is the global minimizer on $[u_*,\overline{u}]$. A simple variant of the argument in [7] now reveals that u_* is a local minimizer of I on $[u_*,\infty)$.

We show next that u_* is a strict local minimizer of I on $[u_*,\infty)$. Otherwise we can find $u_n\in[u_*,\infty)$ such that $u_n\to u_*$ in $H_0^1(\Omega)$ and $I(u_n)=I(u_*)$. Hence u_n are also local minimizers of I on $[u_*,\infty)$. The proof of Theorem 2.4 in [24] then infers that u_n are weak solutions of (2.1). By standard regularity considerations as in [7], we deduce that $u_n\to u_*$ in $C^1(\overline{\Omega})$. But this implies that $u_n\in[u_*,\overline{u}]$ for all large n, contradicting the fact that u_* is the only solution of (2.1) in that order interval. This proves Claim 2.

Claim 3. There exists $\delta_0>0$ such that

$$\inf\{I(u):u\in[u_*,\infty),\|u-u_*\|=\delta\}>I(u_*)\quad\text{for all }\delta\in(0,\delta_0].$$

Arguing indirectly we assume that for some $\delta_n\to0$, we have

$$\inf\{I(u):u\in[u_*,\infty),\ \|u-u_*\|=\delta_n\}\leq I(u_*).$$

By Claim 2, we can find some n_0 such that

$$I(u)>I(u_*)\text{ if }n\geq n_0\text{ and }u\in[u_*,\infty),\ 0<\|u-u_*\|\leq\delta_n. \tag{2.4}$$

Fix $m>n_0$ such that $0<\delta_m<\delta_{n_0}$. Then we can find $\{u_k\}\subset[u_*,\infty)$ such that

$$\|u_k-u_*\|=\delta_m,\quad I(u_*)<I(u_k)\to I(u_*).$$

If $I'(u_k)\to0$, then by the (PS) condition, we can conclude that, subject to a subsequence, $u_k\to w$ in $H_0^1(\Omega)$ and w is a solution of (2.1) satisfying $\|w-u_*\|=\delta_m$ and $I(w)=I(u_*)$. But this contradicts (2.4). Therefore, by passing to a subsequence, we may assume that $\|I'(u_k)\|\geq\varepsilon_0>0$ for all k. But then by the deformation lemma in Claim 1, we can find $t_0>0$ such that $v_k:=\eta(t_0,u_k)\in[u_*,\infty)$, $\|v_k-u_*\|<\delta_{n_0}$ and $I(v_k)\leq I(u_k)-\xi_0$ for some $\xi_0>0$. It follows that for all large k, $I(v_k)<I(u_*)$, again contradicting (2.4). This completes the proof of Claim 3.

Claim 4. (2.1) has a solution u^* satisfying (a)–(d) in the statement of Theorem 2.1.

Consider the restriction of I on $[u_*, \infty)$. Clearly it satisfies the (PS) condition on this set. By our assumption, there exists $u_0 \in [u_*, \infty)$ such that

$$I(u_0) \le \inf_{u \in [\underline{u}, \overline{u}]} I(u) \le I(u_*).$$

In view of Claim 3, we find that the mountain pass conditions are satisfied by I on the convex set $[u_*, \infty)$. Using the deformation lemma in Claim 1, we conclude that I has at least one critical point u^* which satisfies $I(u^*) = c$ with

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) > I(u_*), \tag{2.5}$$

where $\Gamma = \{\gamma \in C([0,1], [u_*, \infty) \cap C_0^1(\overline{\Omega})) : \gamma(0) = u_*, \gamma(1) = u_0\}$.

By the (PS) condition, the set $\Sigma := \{u \in [u_*, \infty) : I'(u) = 0, I(u) = c\}$ is compact in $H_0^1(\Omega)$. Since Σ consists of solutions of (2.1), by (2.2) and standard regularity results (see [7]), Σ is bounded in $C^{1,\mu}(\overline{\Omega})$, for all $\mu \in (0,1)$. Hence Σ is compact in $C_0^1(\overline{\Omega})$.

Now a careful examination of the proof of the main theorem in [19] shows that when the deformation lemma in [19] is replaced by our deformation lemma in Claim 1, everything carries over to our present case, namely, the Banach space F there can be replaced by $[u^*, \infty) \cap C_0^1(\overline{\Omega})$. Therefore, by [19], I has a critical point u^* in $[u_*, \infty)$ with $I(u^*) = c$, c given by (2.5), which is either a local minimizer or is of mountain pass type in $[u^*, \infty) \cap C_0^1(\overline{\Omega})$. Since u_* is the only solution of (2.1) in $[u_*, \overline{u}]$ and $I(u^*) = c > I(u_*)$, we necessarily have $u^* > u_*$ in Ω and $u^* \notin [u_*, \overline{u}]$. As $u^* \in [u_*, \infty)$, this implies that $u^* \in [\underline{u}, \infty) \setminus [\underline{u}, \overline{u}]$. Hence u^* satisfies (a)–(c). This also implies that u^* is an interior point in the set $[u^*, \infty) \cap C_0^1(\overline{\Omega})$ under the $C^1(\overline{\Omega})$ topology. Thus u^* is either a local minimizer or a mountain pass type critical point of I in $C_0^1(\overline{\Omega})$, and (d) is satisfied. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Let us first recall that it follows from (2.2) and a regularity consideration (see [7]) that $u^* \in C_0^1(\overline{\Omega})$.

The proof for the case that u^* is a local minimizer of I is rather trivial. Indeed, in this case we must have $m(u^*) = 0$ for if $m(u^*) \ge 1$, then the first eigenvalue $\lambda_1(\Omega)$ of the problem

$$-\Delta u = f_u(x, u^*)u + \lambda u, \quad u|_{\partial\Omega} = 0 \tag{2.6}$$

is negative. Let ϕ_1 denote a corresponding positive eigenfunction. Then $\phi_1 \in C_0^1(\overline{\Omega})$ and it is easily checked that $I(u^* + t\phi_1) < I(u^*)$ for all small nonzero t. This contradicts the assumption that u^* is a local minimizer of I in $C_0^1(\overline{\Omega})$.

Suppose from now on that u^* is a mountain pass type critical point of I in $C_0^1(\overline{\Omega})$. Regarding u^* as a critical point of I in the Hilbert space $H_0^1(\Omega)$ (possibly not isolated), we can apply the generalized Morse lemma (see Theorem 8.3

in [21]) to conclude that there exists a small ball $B_\delta(0)$ of radius δ with center 0 in $H_0^1(\Omega)$, and a local homeomorphism h defined on $B_\delta(0)$ and a C^2-mapping $\widehat{\phi}$ defined in a neighbourhood of zero of $\ker(I''(u^*))$, such that $h(0)=0$, $\widehat{\phi}(0)=0$, $\widehat{\phi}'(0)=0$, $\widehat{\phi}''(0)=0$ and

$$I(u^*+h(u)) = I(u^*) + (I''(u^*)v,v)/2 + \widehat{\phi}(w) \tag{2.7}$$

for $u=v+w\in B_\delta(0)$, where $v\in N^\perp, w\in N$, and we understand that $w=0$, $\widehat{\phi}(w)=0$ when $\ker(I''(u^*))=\{0\}$.

A crucial fact that we will use is that h can be chosen such that it maps $C_0^1(\overline{\Omega})$ functions into $C_0^1(\overline{\Omega})$ functions. This observation is due to K. C. Chang (see [9]) where a proof is given for the local homeomorphism constructed in Theorem 5.1 of [8]. Here we show that the local homeomorphism given in Theorem 8.3 of [21] also possesses this property.

Let us first recall the construction of the local homeomorphism h in the proof of Theorem 8.3 in [21]. We denote $L=I''(u^*)$ and will assume that $\dim(\ker(L))\geq 1$ as the proof for the case $\ker(L)=\{0\}$ is similar and simpler. Without loss of generality, we also assume that $u^*=0$.

It is well known that L is a Fredholm operator of index zero in the space $V:=H_0^1(\Omega)$. Therefore V is the orthogonal direct sum of $\ker(L)$ and its range $R(L)$. Let $v+w$ denote the decomposition of $u\in V$ with $v\in R(L)$ and $w\in\ker(L)$, and $Q\colon V\to V$ be the orthogonal projection onto $R(L)$. By the implicit function theorem, we can find $r>0$ and a C^1-mapping $g\colon B_r(0)\cap\ker(L)\to R(L)$ such that

$$g(0)=0,\quad g'(0)=0,\quad QI'(w+g(w))=0,\quad \text{for all } w\in B_r(0)\cap\ker(L). \tag{2.8}$$

Define $\widehat{\phi}$ on $B_r(0)\cap\ker(L)$ by $\widehat{\phi}(w)=I(w+g(w))$. By direct computation one finds $\widehat{\phi}\in C^2(B_r(0)\cap\ker(L))$ and $\widehat{\phi}'(0)=0, \widehat{\phi}''(0)=0$. Define

$$\Psi(c,w) = I(v+w+g(w)) - \widehat{\phi}(w) - (Lv,v)/2.$$

Then $\Psi(0,w)=0, \Psi_v(0,w)=0, \Psi_{vv}(0,w)=0$ and it follows that for each $\varepsilon>0$, there exists $\delta(\varepsilon)\in(0,r)$ such that

$$|\Psi(v,w)|\leq\varepsilon\|v\|^2,\quad \|\Psi_v(v,w)\|\leq\varepsilon\|v\|^2 \quad\text{whenever } \|v+w\|\leq\delta(\varepsilon).$$

Since $L\colon R(L)\to R(L)$ is continuous and invertible, there exists $c>0$ such that

$$c^{-1}\|v\|\leq\|Lv\|\leq c\|v\| \quad\text{for all } v\in R(L).$$

Define

$$f(t,v,w) = -\Psi(v,w)\|Lv+t\Psi_v(v,w)\|^{-2}(Lv+t\Psi_v(v,w))$$

when $v\neq 0$ and $f(t,0,w)=0$. Then the Cauchy problem

$$\eta_t = f(t,\eta,w),\quad \eta(0)=v$$

has a unique solution $\eta(t) = \eta(t, v, w)$ which is well-defined and continuous on $[0, 1] \times B_{r_1}(0)$, where $B_{r_1}(0)$ is a small ball in V centered at 0. The homeomorphism h is given by

$$h(u) = h(v + w) = w + g(w) + \eta(1, v, w).$$

We are now ready to show that $u \in C_0^1(\overline{\Omega}) \cap B_\delta(0)$ implies $h(u) \in C_0^1(\overline{\Omega})$. To start, let us observe that $w \in \ker(L)$ if and only if w solves (2.6) with $\lambda = 0$. Hence $w \in \ker(L)$ implies that $w \in C_0^1(\overline{\Omega})$. By (2.8), $z := w + g(w)$ satisfies

$$I'(z) = h, \quad h \in \ker(L).$$

Hence

$$\int_\Omega \nabla z \cdot \nabla\phi - f(x, z)\phi = \int_\Omega \nabla h \cdot \nabla\phi \quad \text{for all } \psi \in H_0^1(\Omega).$$

It follows that

$$-\Delta(z - h) = f(x, z), \quad z|_{\partial\Omega} = 0$$

in the weak sense. Hence by (2.2) and a regularity consideration (see [7]) we have $z - h \in C_0^1(\overline{\Omega})$. As $w, h \in \ker(L) \subset C_0^1(\overline{\Omega})$, we conclude that $g(w) \in C_0^1(\overline{\Omega})$.

It remains to show that $\eta(1, v, w) \in C_0^1(\overline{\Omega})$ when $v \in C_0^1(\overline{\Omega})$. To this end, we note that

$$Lv = v - K(f_u(\,\cdot\,, 0)v) \quad \text{with } K = (-\Delta)^{-1},$$

and

$$\begin{aligned} Lv + t\Psi_v(v, w) &= Lv + t(I'(v + w + g(w)) - Lv) \\ &= Lv + t(v + w + g(w) - K \circ f(\,\cdot\,, v + w + g(w)) - Lv) \\ &= v + (1 - t)K(f_u(\,\cdot\,, 0)v) + t(w + g(w)) \\ &\quad - tK \circ f(\,\cdot\,, v + w + g(w)). \end{aligned}$$

Denote $\eta(t) = \eta(t, v, w)$ and

$$\sigma(t) = \begin{cases} \Psi(\eta(t), w)\|L\eta(t) + t\Psi_v(\eta(t), w)\|^{-2} & \text{if } \eta(t) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We find that $\sigma(t)$ is continuous on $[0, 1]$ (when $\|v + w\|$ is small enough) and

$$\begin{aligned} \eta'(t) &= -\sigma(t)(L\eta(t) + t\Psi_v(\eta(t), w)) \\ &= -\sigma(t)\eta(t) - \sigma(t)t(w + g(w)) - \sigma(t)(1 - t)K(f_u(\,\cdot\,, 0)\eta(t)) \\ &\quad + \sigma(t)tK \circ f(\,\cdot\,, \eta(t) + w + g(w)). \end{aligned}$$

Hence, denoting

$$\widehat{\sigma}(t) = \int_0^1 \sigma(s)\, ds,$$

we obtain

$$\begin{aligned}\eta(t) = {}& e^{-\widehat{\sigma}(t)}\eta(0) - e^{-\widehat{\sigma}(t)}\int_0^1 e^{\widehat{\sigma}(s)}\sigma(s)s\,ds(w+g(w)) \\ & - e^{-\widehat{\sigma}(t)}K\bigg(\int_0^t e^{\widehat{\sigma}(s)}\sigma(s)(1-s)f_u(\,\cdot\,,0)\eta(s)\,ds\bigg) \\ & + e^{-\widehat{\sigma}(t)}K\bigg(\int_0^t e^{\widehat{\sigma}(s)}\sigma(s)sf(\,\cdot\,,\eta(s)+w+g(w))\,ds\bigg).\end{aligned}$$

We have already proved that $w+g(w)\in C_0^1(\overline{\Omega})$. Thus the second term on the right side of the above identity is in $C_0^1(\overline{\Omega})$. Due to the regularity of K and (2.2), we find that the third and fourth terms on the right side also belong to $C_0^1(\overline{\Omega})$. Hence, whenever $\eta(0)=v\in C_0^1(\overline{\Omega})$ (with $\|v+w\|$ small), $\eta(t)\in C_0^1(\overline{\Omega})$ for all $t\in[0,1]$. In particular $\eta(1)\in C_0^1(\overline{\Omega})$, as we wanted.

We can now use (2.7) to prove $m(u^*)\le 1$. Recall that by assumption, for any neighbourhood N of u^* in $C_0^1(\overline{\Omega})$, $N\cap I^c$ is not empty and not path-connected, where $c=I(u^*)$. Arguing indirectly, we assume that $m(u^*)\ge 2$. We then decompose $V=H_0^1(\Omega)$ as $V=V_-\oplus V_0\oplus V_+$, where (Lu,u) is negative definite on V_-, positive definite on V_+, and $V_0=\ker(L)$. Hence $\dim(V_-)=m(u^*)\ge 2$ and there exists $\delta_0>0$ such that

$$\begin{aligned}(Lu,u)&\le -\delta_0\|u\|^2 \quad \text{for all } u\in V_-,\\ (Lu,u)&\ge \delta_0\|u\|^2 \quad \text{for all } u\in V_+.\end{aligned}$$

We now fix $\delta_1>0$ small enough such that

$$N_0:=\{u=v_-+v_++w\in V_1\oplus V_+\oplus V_0:\|v_-\|'<\delta_1,\|v_+\|'<\delta_1,\ \|w\|'<\delta_1\}$$

is contained in $B_\delta(0)$ in which (2.7) holds, where $\|\cdot\|'$ denotes the $C_0^1(\overline{\Omega})$-norm.

Let us observe that N_0 given above is a neighbourhood of 0 in $C_0^1(\overline{\Omega})$. Indeed, V_- is of finite dimension spanned by the eigenfunctions of (2.6) corresponding to negative eigenvalues. Since the eigenfunctions are in $C_0^1(\overline{\Omega})$, one easily sees that the component v_- of u is in $C_0^1(\overline{\Omega})$ with small $C_0^1(\overline{\Omega})$-norm if u is so. A similar reasoning shows that the same holds for the component w. It then follows from these conclusions on v_- and w that the same holds for v_+. Therefore N_0 contains all u with small $C_0^1(\overline{\Omega})$-norm, and is a neighbourhood of 0 in $C_0^1(\overline{\Omega})$.

Since $V_-\oplus V_0$ if of finite dimension, the norms $\|\cdot\|$ and $\|\cdot\|'$ on $V_-\oplus V_0$ are equivalent, that is, there exists $c_0>1$ such that

$$c_0^{-1}\|v\|\le\|v\|'\le c_0\|v\| \quad \text{for all } v\in V_-\oplus V_0.$$

By the properties of $\widehat{\phi}$ given before (2.7), we can find $\widehat{\delta}>0$ small enough such that

$$|\widehat{\phi}(w)|\le\frac{\delta_0}{16c_0^4}\|w\|^2 \quad \text{for all } w\in V_0,\ \|w\|\le\widehat{\delta}. \tag{2.9}$$

We may assume that δ_1 in the definition of N_0 has been chosen such that $\delta_1 \le \widehat{\delta}/c_0$; hence (2.9) holds if $\|w\|' < \delta_1$.

We show next that $\dim(V_-) \ge 2$ implies that $(u^* + h(N_0)) \cap I^c$ is path-connected. This contradiction would finish our proof that $m(u^*) \le 1$. To this end, for $u, \widetilde{u} \in N_0$ with $u^* + h(u), u^* + h(\widetilde{u}) \in I^c$, we write $u \sim \widetilde{u}$ if and only if $h(u)$ and $h(\widetilde{u})$ are in the same path-connected component of $(u^* + h(N_0)) \cap I^c$. For an arbitrary $u = v_- + v_+ + w \in N_0$ with $I(u^* + h(u)) < c$, we have, by (2.7),

$$I(u^* + h(u)) = I(u^*) + (Lv_-, v_-)/2 + (Lv_+, v_+)/2 + \widehat{\phi}(w). \tag{2.10}$$

Consider the curve $\gamma(t) = v_- + tv_+ + w$, $t \in [0,1]$. Since $v_-, v_+, w \in C_0^1(\overline{\Omega})$, we find that $\gamma(t) \in N_0$ for all $t \in [0,1]$. By (2.10)

$$I(u^* + h(\gamma(t))) \le I(u^* + h(\gamma(1))) = I(u^* + h(u)) < c \quad \text{for all } t \in [0,1].$$

Therefore, $u = \gamma(1) \sim \gamma(0) = v_- + w$.

We now choose $\widehat{v}_- \in V_-$ as follows.

$$\widehat{v}_- = \begin{cases} v_- & \text{if } \|v_-\| \ge \dfrac{\delta_1}{2c_0}, \\ \dfrac{\delta_1}{2c_0}\dfrac{v_-}{\|v_-\|} & \text{if } 0 < \|v_-\| < \dfrac{\delta_1}{2c_0}, \end{cases}$$

$\widehat{v}_- \in V_-$ is arbitrary with $\|\widehat{v}_-\| = \delta_1/(2c_0)$ if $v_- = 0$. Clearly $\widehat{v}_- \in N_0$ and $\|\widehat{v}_-\| \ge \delta_1/(2c_0)$. Moreover,

$$\|t\widehat{v}_- + (1-t)v_-\| \ge \|v_-\| \quad \text{for all } t \in [0,1].$$

Thus, by (2.10), $v_- + w \sim \widehat{v}_- + w$. Using (2.19) and (2.10) we find

$$\begin{aligned} I(u^* + h(\widehat{v}_- + tw)) &= I(u^*) + \left(L\widehat{v}_-, \frac{1}{2}\widehat{v}_-\right) + \widehat{\phi}(tw) \\ &\le I(u^*) - \frac{1}{2}\delta_0\|\widehat{v}_-\|^2 + \frac{\delta_0}{16c_0^4}\|tw\|^2 \\ &\le I(u^*) - \frac{\delta_0}{2}\left(\frac{\delta_1}{2c_0}\right)^2 + \frac{\delta_0}{16c_0^4}(c_0\delta_1)^2 \\ &= I(u^*) - \frac{\delta_0\delta_1^2}{16c_0^2} < I(u^*), \end{aligned}$$

for all $t \in [0,1]$. Hence $\widehat{v}_- + w \sim \widehat{v}_-$. Thus we must have $u \sim \widehat{v}_-$. That is to say, any given $u \in N_0$ with $u^* + h(u) \in I^c$ can be connected by a path in $(u^* + h(N_0)) \cap I^c \subset C_0^1(\overline{\Omega})$ to some $\widetilde{u} \in N_- := \{z \in V_- : 0 < \|z\|' < \delta_1\}$. Since $\dim(V_-) \ge 2$, the set N_- is path connected, and by (2.10), we find that $u^* + h(N_-) \subset I^c$. Thus $u^* + h(N_-)$ is path-connected in $(u^* + h(N_0)) \cap I^c$. It follows that $(u^* + h(N_0)) \cap I^c$ is path-connected. This contradiction finishes our proof of Theorem 2.2. $\square$

3. An indefinite superlinear problem on $\mathbb{R}^N$

In this section, we use the results obtained in Section 2 to study the following elliptic problem

$$-\Delta u = \lambda u - b(x)g(u), \quad x \in \mathbb{R}^N, \tag{3.1}$$

where λ is a real parameter and $g(u)$ is a $C^1([0,\infty))$ function to be specified later; it will include u^p, $p > 1$, as a special case.

We largely follow the strategy used in [13], that is, firstly analyze (3.1) over bounded domains and then letting the domains converge to $\mathbb{R}^N$. The difficulty of this approach lies in controlling the solutions as the domains enlarging to $\mathbb{R}^N$.

By a positive solution of equation (3.1) we mean a function $u \in C^1(\mathbb{R}^N)$ such that $u > 0$ on $\mathbb{R}^N$ and

$$\int_{\mathbb{R}^N} (\nabla u \cdot \nabla v - (\lambda u - b(x)g(u))v \, dx = 0 \quad \text{for all } v \in C_0^\infty(\mathbb{R}^N).$$

By classical theory on elliptic equations, we know that $u \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^N)$ for any $q > 1$ (a weak solution) and if further the function $b(x)$ is Hölder continuous on $\mathbb{R}^N$, then u belongs to $C^2(\mathbb{R}^N)$ (a classical solution).

In the following, we denote by $B_{r_0}(x_0)$ the open ball in $\mathbb{R}^N$ with center x_0 and radius r_0. In particular we denote $B_R := B_R(0)$.

Our first result is the following.

THEOREM 3.1. *Suppose that $b(x)$ is a continuous function such that*

$$b(x) < 0 \quad \text{in a ball } B_{r_0}(x_0), \qquad b(x) \geq \delta > 0 \quad \text{for } |x| \geq R_0, \tag{3.2}$$

and $g(u)$ is a C^1-function for $u \geq 0$ satisfying

(G$_1$) $g(u) > 0$ *for all* $u > 0$,

(G$_2$) $g(0) = g'(0) = 0$ *and, for some* $p > 1$, $\lim_{u \to 0^+} g(u)/u^p = l_0 \in (0, \infty)$.

Then there exists $\Lambda > 0$ such that (3.1) has at least one positive solution for each $\lambda \in (0, \Lambda)$, and it has no positive solution for $\lambda > \Lambda$.

The above result can be regarded as an extension to the entire space $\mathbb{R}^N$ of Theorem 2 in [4] where (3.1) was considered on a bounded domain. A proof for Theorem 3.1 for the special case $g(u) = u^p$ has been given in [13], by a bounded domain approximation argument. This proof carries easily to our present case here. We sketch it below for completeness and also for easy reference later.

We start with some results for (3.1) on bounded domains. By (3.2) and the proof of Proposition 2.2 in [13], we find that Theorem 2 of [4] can be used to conclude the following.

PROPOSITION 3.2. *Suppose that $b(x)$ satisfies* (3.2) *and $g(u)$ satisfies* (G_1) *and* (G_2). *Then one can find $R_* > R_0$ so that for any ball B_R with $R \geq R_*$ there exists $\Lambda_R \in (\lambda_1(B_R), \infty)$ such that the problem*

$$-\Delta u = \lambda u - b(x)g(u), \quad x \in B_R, \ u|_{\partial B_R} = 0 \tag{3.3}$$

has at least one positive solution for $\lambda \in (\lambda_1(B_R), \Lambda_R)$, and no positive solution for $\lambda > \Lambda_R$. Here $\lambda_1(B_R)$ denotes the first eigenvalue of $-\Delta$ on B_R with Dirichlet boundary conditions.

Moreover, by standard comparison arguments as in the proof of Proposition 2.3 in [13], we have the following result.

PROPOSITION 3.3. *Under the conditions of Proposition* 3.2, *for each $R \geq R_*$ and $\lambda \in (\lambda_1(B_R), \Lambda_R)$,* (3.3) *has a minimal positive solution u_λ^R in the sense that any positive solution u of* (3.3) *satisfies $u \geq u_\lambda^R$ in B_R. Moreover,*

$$\Lambda_{R_1} \geq \Lambda_{R_2} \quad \textit{whenever } R_* \leq R_1 \leq R_2 \tag{3.4}$$

and

$$u_{\lambda_1}^{R_1}(x) \leq u_{\lambda_2}^{R_2}(x) \textit{ whenever both sides are defined and } \lambda_1 \leq \lambda_2, \ R_1 \leq R_2. \tag{3.5}$$

A key step in the proof of Theorem 3.1 is the construction of an upper solution u_0 to (3.3) which is independent of R. This is done in the proof of Lemma 2.5 in [13] for the case $g(u) = u^p$, which can be easily adapted.

PROOF OF THEOREM 3.1. We choose a continuous radially symmetric function $\widetilde{b}(x) = \widetilde{b}(r)$, $r = |x|$, such that

$$\begin{aligned} \widetilde{b}(x) &\leq b(x), \quad \text{for all } x \in B_{R_0}, \\ \widetilde{b}(x) &= \delta, \qquad\quad |x| \geq R_0. \end{aligned} \tag{3.6}$$

This is possible due to our assumption (3.2). Then we can apply Theorem 2 of [4] (see [13, Proposition 2.4]) to conclude that, for some large $R_1 > R_0$, there exists $\widetilde{\lambda} > 0$ such that the following Neumann problem

$$-\Delta u = \widetilde{\lambda} u - \widetilde{b}(x) u^p, \quad x \in B_{R_1}, \quad \partial_\nu u|_{\partial B_{R_1}} = 0,$$

has a minimal positive solution $\widetilde{u}$. By the Hopf boundary lemma, we see that $\widetilde{u} > 0$ on $\overline{B}_{R_1}$. Since the equation is invariant under rotations around the origin, the minimality of $\widetilde{u}$ implies that $\widetilde{u}$ is radially symmetric.

Let $\eta = \widetilde{u}(R_1)$ and let $\lambda_0 \in (0, \widetilde{\lambda})$ be such that $\lambda_0 \eta - \delta g(\eta) \leq 0$, and define

$$u_0(x) = \begin{cases} \widetilde{u}(x) & \text{for } x \in B_{R_1}, \\ \eta & \text{for } |x| \geq R_1. \end{cases}$$

Then it is easy to check that u_0 is a weak upper solution of (3.3) for $\lambda = \lambda_0$ and every $R > R_1$. Now choose $R_2 > R_1$ so that the first Dirichlet eigenvalue

$\lambda_1(B_R) < \lambda_0$ for all $R \ge R_2$, and denote by ϕ_R the normalized positive eigenfunction corresponding to $\lambda_1(B_R)$. Then for all small $\varepsilon > 0, \varepsilon\phi_R < u_0$ in B_R and is a lower solution to (3.3) with $\lambda = \lambda_0$. Hence (3.3) has a minimal positive solution $u^R_{\lambda_0}$. Moreover,

$$u^R_{\lambda_0}(x) \le u_0(x), \quad \text{for all } x \in B_R \text{ and all } R \ge R_2. \tag{3.7}$$

Now by (3.5) and (3.7), we find that for any $x \in \mathbb{R}^N$, $U_0(x) := \lim_{R\to\infty} u^R_{\lambda_0}(x)$ exists and $U_0(x) \le u_0(x)$. Consequently, by regularity theory and a standard compactness argument, U_0 is a nonnegative solution of (3.1) with $\lambda = \lambda_0$. Moreover, since $U_0 \ge u^R_{\lambda_0}$ for every $R \ge R_2$, U_0 is a positive solution of (3.1) for $\lambda = \lambda_0$. Define

$$\Lambda := \sup\{\mu > 0 : (3.1) \text{ has a positive solution for } \quad \lambda = \mu\}.$$

Obviously $\Lambda \ge \lambda_0$.

We claim that $\Lambda \le \lambda_1(B_{r_0}(x_0))$. Indeed, if $\Lambda > \lambda_1(B_{r_0}(x_0))$, then we can find $\lambda > \lambda_1(B_{r_0}(x_0))$ such that (3.1) has a positive solution u with such a λ. By (3.2), we have

$$-\Delta u = \lambda u - b(x)g(u) \ge \lambda u \quad \text{on } B_{r_0}(x_0).$$

Let ϕ denote the normalized positive eigenfunction corresponding to $\lambda_1(B_{r_0}(x_0))$. We deduce

$$\begin{aligned}\lambda \int_{B_{r_0}(x_0)} u\phi &\le \int_{B_{r_0}(x_0)} (-\Delta u)\phi \\ &= \int_{B_{r_0}(x_0)} (-\Delta \phi)u + \int_{\partial B_{r_0}(x_0)} \partial_\nu \phi u < \lambda_1(B_{r_0}(x_0)) \int_{B_{r_0}(x_0)} \phi u.\end{aligned}$$

Hence $\lambda < \lambda_1(B_{r_0}(x_0))$, contradicting our assumption that $\lambda > \lambda_1(B_{r_0}(x_0))$.

It remains to show that (3.1) has a positive solution for every $\lambda \in (0, \Lambda)$. Let $\lambda \in (0, \Lambda)$ be fixed. By the definition of Λ, we can find $\lambda^* \in (\lambda, \Lambda]$ such that (3.1) has a positive solution u^* with $\lambda = \lambda^*$. Then u^* is an upper solution of (3.3) with the above fixed λ on any B_R. Let R^* be large enough so that $\lambda_1(B_R) < \lambda$ for all $R > R^*$. Then for any fixed $R > R^*$ and all small $\varepsilon > 0, \varepsilon\phi_R < u^*$ in B_R and are lower solutions to (3.3) with these given λ and R. Hence (3.3) has a positive solution and $\Lambda_R \ge \lambda$. It follows from Proposition 3.3 that u^R_λ exists for all $R > R^*$, and $u^R_\lambda \le u^*$. Now by the same arguments as before, $U^* := \lim_{n\to\infty} u^R_\lambda$ is a positive solution of (3.1) with the given λ. This completes the proof of Theorem 3.1. □

In the next, we will show that under suitable conditions on $g(u)$ and $b(x)$, a priori estimates (independent of R) for positive solutions of (3.3) with bounded Morse index can be established. This, together with our results in Section 2, will eventually enable us to show that equation (3.1) has at least one positive solution for $\lambda = \Lambda$, at least two positive solutions for $\lambda \in (0, \Lambda)$, and at least one positive solution for $\lambda \le 0$. Let us denote

$$\begin{aligned} \Omega_R^- &:= \{x \in B_R : b(x) < 0\}, \\ \Omega_R^+ &:= \{x \in B_R : b(x) > 0\}, \\ \Omega_R^0 &:= \{x \in B_R : b(x) = 0\}. \end{aligned}$$

Since $b(x)$ is continuous, clearly, under assumption (3.2), for all $R > R_0$, each of these three sets is nonempty. Moreover, Ω_R^- and Ω_R^0 are independent of such R. So we may denote $\Omega^- = \Omega_R^-$ and $\Omega^0 = \Omega_R^0$ when $R > R_0$. To establish the required a priori estimates, we will need the following assumption:

$$\nabla b(x) \neq 0 \quad \text{for all } x \in \Omega^0. \tag{3.8}$$

Theorem 3.4. *Suppose that $b(x)$ is C^2 in $\mathbb{R}^N$ and satisfies* (3.2) *and* (3.8). *Moreover, suppose that $g(u)$ is C^1 on $[0, \infty)$ and satisfies*

(G$_3$) $\lim_{u\to\infty} g'(u)/(qu^{q-1}) = l_\infty \in (0, \infty)$ *for some $q \in (1, N^*)$,*

where $N^* = (N+2)/(N-2)$ when $N \ge 3$, $N^* = \infty$ when $N = 1, 2$. Then for any given constant $M > R_0$ and integer $m \ge 0$, there exists a constant $C = C(M, m)$ such that any positive solution u of (3.3) *with Morse index $m(u) \le m$ and $R > M$, $|\lambda| \le M$ satisfies*

$$\|u\|_{L^\infty(B_R)} \le C. \tag{3.9}$$

Proof. This follows from a combination of the techniques in [4] and [23].

For any $R > M$, since we assume that $b(x) \in C^2(B_R)$, by elliptic regularity, the solutions of (3.3) belong to $C^2(\overline{B}_R)$. If (3.9) does not hold, then we can find sequences $R_n > M$ and $\lambda_n \in [-M, M]$ such that (3.3) with $R = R_n$ and $\lambda = \lambda_n$ has a positive solution u_n satisfying $m(u_n) \le m$ and $\|u_n\|_{L^\infty(B_{R_n})} \to \infty$. Thus there exists a sequence $\{x_n\} \subset B_{R_n}$ such that

$$M_n := \|u_n\|_{L^\infty(B_{R_n})} = u_n(x_n) \to \infty.$$

Since $\Delta u_n(x_n) \le 0$, the equation in (3.3) yields

$$b^+(x_n) g(M_n) \le M M_n + b^-(x_n) g(M_n), \tag{3.10}$$

where $b^+ := \max\{b, 0\}$, $b^- := b^+ - b$.

Note that (G$_3$) implies $\lim_{u\to\infty} g(u)/u^q = l_\infty$. By (3.2) and (3.10), we see that $\{x_n\}$ is bounded. Thus by passing to a subsequence, we can assume that $x_n \to x_0$. Using (3.10) again, we find that necessarily $b(x_0) \le 0$ and hence $x_0 \in B_{R_0}$.

Now, the blow up argument used in the proof of Theorem 3.1 of [4] can be applied. More precisely, for some suitable sequence $\mu_n \to 0$, define

$$v_n(x) := \frac{u(\mu_n x + x_n)}{M_n}, \quad x \in \Omega_n := \frac{1}{\mu_n}(B_{R_0} - x_n).$$

Then depending upon the location of x_0 in Ω_n, we can determine μ_n and show that v_n has a subsequence converging to a positive function v satisfying a limiting problem on $\mathbb{R}^N$ which, up to a translation and rescaling of v, is of the form

$$-\Delta v = v^q, \qquad x \in \mathbb{R}^N, \tag{3.11}$$

or

$$-\Delta v = x_N v^q, \quad x \in \mathbb{R}^N. \tag{3.12}$$

In the case of (3.11), due to the restriction on q, it is well known (see [16]) that its only nonnegative solution is $v \equiv 0$, which yields a contradiction. In the case of (3.12), it is shown in [4] that its only nonnegative solution is $v \equiv 0$ provided that $1 < q < (N+2)/(N-1)$. However, the proof of Lemma 6 in [23] shows that $m(u_n) \le m$ implies v has finite index (in the sense defined in [23, p. 613]), which in turn implies that $v = 0$ if $1 < q < N^*$ (see Proposition 12 in [23]). Therefore, we arrive at a contradiction in either case. This proves the theorem. $\square$

Remark 3.5. An inspection of the above proof of Theorem 3.4 reveals that, if we replace $g(u)$ by a parameter dependent function $g_\xi(u)$ with ξ belonging to some index set A, and assume that (G_3) is satisfied by $g_\xi(u)$ uniformly in $\xi \in A$, then (3.9) holds with C independent of $\xi \in A$. This observation will be useful later.

An immediate consequence of Theorem 3.4 is the following existence result.

Theorem 3.6. *Suppose that the assumptions in Theorems* 3.1 *and* 3.4 *hold, and let Λ be as in Theorem* 3.1. *Then* (3.1) *has a positive solution for $\lambda = \Lambda$.*

Proof. From the proof of Theorem 3.1 we see that for every $\lambda \in (0, \Lambda)$ and $R > R^*$, we have $\lambda_1(B_R) < \Lambda$, $\Lambda_R \ge \lambda$. It follows that $\Lambda_R \ge \Lambda$. Therefore (3.3) with $\lambda = \Lambda$ and $R > R^*$ has a minimal positive solution u_Λ^R.

We claim that u_Λ^R has Morse index 0. Otherwise, the first eigenvalue μ_1 of the problem

$$-\Delta u = \Lambda u - b(x) g'(u_\Lambda^R) u + \mu u, \quad u|_{\partial B_R} = 0$$

is negative. Let ψ_1 denote the corresponding normalized positive eigenfunction. Then it is easily checked that for sufficiently small $\delta > 0$, $\overline{u} := u_\Lambda^R - \delta \psi_1$ is a positive upper solution of (3.3) with $\lambda = \Lambda$. Since $\Lambda > \lambda_1(B_R)$, for small enough $\varepsilon > 0$, $\underline{u} := \varepsilon \phi_R$ is a lower solution satisfying $\underline{u} \le \overline{u}$, where ϕ_R denotes the normalized positive eigenfunction corresponding to $\lambda_1(B_R)$. Therefore, (3.3) with $\lambda = \Lambda$ has a positive solution satisfying $\underline{u} \le u \le \overline{u}$. But this contradicts the fact that u_Λ^R is the minimal positive solution. This proves our claim.

We can now apply Theorem 3.4 to conclude that $u_\Lambda^R \le C$ for some C independent of R. Hence $U := \lim_{R\to\infty} u_\Lambda^R$ is a nonnegative solution of (3.1) with $\lambda = \Lambda$. From (3.5) we find that $U \ge u_\Lambda^R$ on B_R for every R. Therefore U is a positive solution. □

Now we apply Theorems 2.1 and 2.2 to prove a crucial step towards our main multiplicity result for (3.1).

Lemma 3.7. *Let the conditions of Theorem* 3.6 *be satisfied and let* Λ_R, u_λ^R *be as in Propositions* 3.2 *and* 3.3, *and* Λ *as in Theorem* 3.1. *Then there exists* $\widehat{R} > 0$ *such that for each* $R > \widehat{R}$ *and* $\lambda < \Lambda$, (3.3) *has a positive solution* $\widehat{u}_\lambda^R$ *with the following properties:*

(a) $\widehat{u}_\lambda^R(\widehat{x}) > u_\Lambda^R(\widehat{x})$ *for some* $\widehat{x} \in B_R$,
(b) *The Morse index* $m(\widehat{u}_\lambda^R) \le 1$.

Proof. Let us choose $\widehat{R}$ such that $\lambda_1(B_{\widehat{R}}) < \Lambda$. As recalled in the proof of Theorem 3.6 above, we have $\Lambda_R \ge \Lambda > \lambda_1(B_R)$ when $R > \widehat{R}$. Therefore u_Λ^R exists for every $R > \widehat{R}$.

Let $\lambda < \Lambda$ be fixed. Then clearly $\overline{u} := u_\Lambda^R$ is an upper solution of (3.3) but it is not a solution. On the other hand, $\underline{u} := 0$ is a solution and hence a lower solution to (3.3). Let $g(u)$ be extended to $u < 0$ by $g(u) = |u|^{q-1}u$ and let

$$f(x,u) = \lambda u + b(x)g(u).$$

Then our conditions on $g(u)$ guarantee that (2.2) and (2.3) are satisfied. If we can show that the corresponding functional $I(u)$ of (3.3) satisfies the (PS) condition on the order interval $[\underline{u},\infty) = [0,\infty)$ and there exists $u_0 \ge \overline{u} = u_\Lambda^R$ such that $I(u_0) \le \inf_{u\in[0,u_\Lambda^R]} I(u)$, then we can apply Theorems 2.1 and 2.2 to conclude.

As is widely known, due to the fact that $b(x)$ changes sign, the (PS) condition is difficult to verify directly. We adapt a truncation trick introduced in [23] to overcome this difficulty.

Let a_j be an increasing sequence of positive numbers such that $a_j \to \infty$ and $a_1 > \|u_\Lambda^R\|_{L^\infty(B_R)}$. Then define

$$g_j(u) = \begin{cases} A_j u^q + B_j & \text{for } u \ge a_j, \\ g(u) & \text{for } u \le a_j. \end{cases}$$

The coefficients are chosen in such a way that g_j is C^1.

We consider the modified problem

$$-\Delta u = \lambda u + b(x)g_j(u), \quad u \in H_0^1(B_R). \tag{3.13}$$

The corresponding energy functional is given by

$$I_j(u) = \frac{1}{2}\int_{B_R} (|\nabla u|^2 - \lambda u^2) - \int_{B_R} bG_j(u),$$

where $G_j(u) := \int_0^u g_j(s)\,ds$.

By our assumptions on a_j, we find that $\underline{u} = 0$ and $\overline{u} = u_\Lambda^R$ are still lower and upper solutions of (3.13). Moreover, for each j, the righthand side of (3.13) satisfies (2.2) and (2.3). Let us check that I_j satisfies the (PS) condition. Indeed, $g_j(u)$ clearly satisfies condition (b) in Lemma 1.5 of [1]. Moreover, (3.8) implies that Ω^0 is a C^2-manifold of dimension $N-1$. Therefore we can apply Lemma 1.5 of [1] to conclude that I_j satisfies the (PS) condition on $H_0^1(B_R)$.

Next we prove that there exists $u_0 \in H_0^1(B_R)$ such that

$$u_0 \geq \overline{u} \quad \text{in } B_R, \quad I_j(u_0) \leq \inf_{u\in[\underline{u},\overline{u}]} I_j(u). \tag{3.14}$$

To this end, let ϕ be a nonnegative function in $C_0^\infty(B_R)$ with nonempty support contained in some compact set $S \subset \Omega^-$. Then there exist positive constants σ_1, σ_2 and σ_3 such that

$$\begin{aligned} b(x) &\leq -\sigma_1 && \text{for all } x \in S, \\ G_j(u) &\geq \sigma_2 u^{q+1} - \sigma_3 && \text{for all } u \geq 0. \end{aligned}$$

It follows that,

$$\begin{aligned} I_j(\overline{u}+t\phi) &= \frac{1}{2}\int_{B_R} [|\nabla\overline{u} + t\nabla\phi|^2 - \lambda(\overline{u}+t\phi)^2] + \int_{B_R} b(x)G_j(\overline{u}+t\phi) \\ &= \frac{1}{2}\int_{B_R} [|\nabla\overline{u} + t\nabla\phi|^2 - \lambda(\overline{u}+t\phi)^2] + \int_{B_R\setminus S} b(x)G_j(\overline{u}) \\ &\quad + \int_S b(x)G_j(\overline{u}+t\phi) \\ &\leq \frac{1}{2}\int_{B_R} [|\nabla\overline{u} + t\nabla\phi|^2 - \lambda(\overline{u}+t\phi)^2] + \int_{B_R\setminus S} b(x)G_j(\overline{u}) \\ &\quad - \int_S \sigma_1[\sigma_2(\overline{u}+t\phi)^{q+1} - \sigma_3] \to -\infty \end{aligned}$$

as $t \to \infty$. Therefore, we can choose $t_0 > 0$ large enough such that $u_0 := \overline{u} + t_0\phi$ satisfies (3.14). According to Theorems 2.1 and 2.2, I_j admits a critical point u_j with the properties

$$u_j(x_j) > \overline{u}(x_j) \quad \text{for some } x_j \in B_R, \quad \text{and} \quad m(u_j) \leq 1.$$

Since $g(u)$ satisfies (G_3), it is easily checked that $g_j(u)$ satisfies (G_3) uniformly in $j \geq 1$. Therefore we can use Remark 3.5 (with fixed R and λ) to see that there exists C_0 independent of j such that $\|u_j\|_{L^\infty(B_R)} \leq C_0$ for all $j \geq 1$. We now fix j large enough so that $a_j > C_0$. Then clearly u_j solves the original problem (3.3), and $\widehat{u}_\lambda^R := u_j$ meets the requirements (a) and (b) of our lemma. This completes the proof. □

We are now ready to use Theorem 3.4 and techniques introduced in [13] to prove our main existence and multiplicity result for (3.1).

We first recall two preliminary results in [13] in a slightly more general setting.

Lemma 3.8. *Suppose that R_n is an increasing sequence converging to ∞ and $B_n = B_{R_n}(0)$. Let $\lambda > 0$ be fixed and ξ_n be a sequence of positive numbers converging to $\xi > 0$ as $n \to \infty$. Suppose that $g(u)$ is continuous and $g(u)/u$ is strictly increasing over $[0,\infty)$ with*

$$\lim_{u\to 0^+} \frac{g(u)}{u} = 0, \quad \lim_{u\to\infty} \frac{g(u)}{u^q} = l_\infty \in (0,\infty) \quad \textit{for some } q > 1.$$

Then, for all large n, the problem

$$-\Delta u = \lambda u - \xi_n g(u) \quad \textit{in } B_n, \quad u|_{\partial B_n} = 0 \tag{3.15}$$

and the problem

$$-\Delta v = \lambda v - \xi_n g(v) \quad \textit{in } B_n, \quad v|_{\partial B_n} = \infty \tag{3.16}$$

have unique positive solutions u_n and v_n, respectively. Moreover,

$$u_n(x) \to \sigma^*, \quad v_n(x) \to \sigma^*,$$

uniformly on any bounded set of $\mathbb{R}^N$ as $n \to \infty$, where σ^* is the unique solution to $\lambda\sigma^* - \xi g(\sigma^*) = 0$. Here and in what follows, by $v|_{\partial B_n} = \infty$, we mean $v(x) \to \infty$ as $d(x, \partial B_n) \to 0$.

Proof. The proof is the same as that for Lemma 4.3 of [13], but we need to replace u^p by $g(u)$ in Lemmas 2.2 and 2.3 of [15]. By our assumption on $g(u)$, this change does not cause any difficulties in the proof of these lemmas in [15]. So we omit the details. □

Lemma 3.9. *Let $\lambda > 0$ be fixed, and R_n, ξ_n, $g(u)$, σ^* as in Lemma 3.8. Denote by A_n the annulus $\{x \in \mathbb{R}^N : R_n/2 < |x| < R_n\}$. Then for all large n, the problem*

$$-\Delta u = \lambda u - \xi_n g(u) \quad \textit{in } A_n, \quad u|_{\partial A_n} = 0 \tag{3.17}$$

and the problem

$$-\Delta v = \lambda v - \xi_n g(v) \quad \textit{in } A_n, \quad v|_{\{|x|=R_n/2\}} = \infty, \quad v|_{\{|x|=R_n\}} = 0 \tag{3.18}$$

have unique positive solutions u_n and v_n, respectively. Moreover, $u_n(x) = u_n(|x|)$, $v_n(x) = v_n(|x|)$, and if we define $U_n(r) = u_n(R_n + r)$ and $V_n(r) = v_n(R_n + r)$ for $r \in (-R_n/2, 0]$, then as $n \to \infty$,

$$U_n \to \Phi, \quad V_n \to \Phi \quad \textit{in } C^1([-T,0]), \textit{ for all } T > 0, \tag{3.19}$$

where Φ *is the unique positive solution to*

$$-\Phi'' = \lambda\Phi - \xi g(\Phi), \quad \Phi(-\infty) = \sigma^*, \quad \Phi(0) = 0. \tag{3.20}$$

PROOF. This follows the proof of Lemma 4.4 in [13]; the only difference is that we replace u^p by $g(u)$. By our assumption on $g(u)$, this change causes no extra difficulty. We omit the details. $\square$

The argument in Remark 4.5 of [13] leads now to the following.

REMARK 3.10. If $\lambda \le 0$, then in Lemma 3.9, v_n still exists and is unique. Moreover, V_n converges to 0 in $C^1([-T,0])$ for all $T > 0$.

We can now state and prove our main result for (3.1).

THEOREM 3.11. *Suppose that the conditions of Theorem* 3.6 *are satisfied, that is,* $b(x)$ *is* C^2 *and satisfies* (3.2) *and* (3.8), $g(u)$ *is* C^1 *for* $u \ge 0$ *and satisfies* (G_1), (G_2) *and* (G_3). *Moreover, we assume that*

$$\lim_{|x|\to\infty} b(x) = b_\infty \in (0,\infty),$$

and $g(u)/u$ *is strictly increasing over* $[0,\infty)$. *Then,* (3.1) *has at least two positive solutions for each* $\lambda \in (0,\Lambda)$, *and at least one positive solution for each* $\lambda \le 0$.

PROOF. Let $\lambda \in (0,\Lambda)$ be fixed. By the last part of the proof of Theorem 3.1 we see that the minimal positive solution u_λ^R of (3.3) exists for all large R, and $\lim_{R\to\infty} u_\lambda^R$ is a positive solution of (3.1). Let us denote this solution by U_λ. Then it follows from (3.5) that $U_{\lambda_1} \le U_{\lambda_2}$ when $\lambda_1 < \lambda_2$. Moreover, it is easily checked by using the strong maximum principle that actually $U_{\lambda_1}(x) < U_{\lambda_2}(x)$, for all $x \in \mathbb{R}^N$, when $\lambda_1 < \lambda_2$. Though it is not needed in the discussions to follow, let us also mention that U_λ is the minimal positive solution of (3.1), due to the properties of u_λ^R.

We now set to find a second positive solution for such λ. Let $R_n > \widehat{R}$ be an increasing sequence converging to ∞, where $\widehat{R}$ is as in Lemma 3.7. By Lemma 3.7, for each $n \ge 1$, we can find $\widehat{u}_n := \widehat{u}_\lambda^{R_n}$ which solves (3.3) with $R = R_n$ and has the properties (a) and (b) in Lemma 3.7. By Theorem 3.4, we can find $C > 0$ independent of n such that $\|\widehat{u}_n\|_{L^\infty(B_{R_n})} \le C$. By standard argument, this implies that subject to a subsequence, $\widehat{u}_n \to \widehat{U}_\lambda$ uniformly on compact subsets of $\mathbb{R}^N$, and $\widehat{U}_\lambda$ solves (3.1). Since $\widehat{u}_n \ge u_\lambda^{R_n}$, we necessarily have $\widehat{U}_\lambda \ge U_\lambda$, and hence $\widehat{U}_\lambda$ is a positive solution of (3.1).

It remains to show that $\widehat{U}_\lambda \not\equiv U_\lambda$. We will make use of property (a) in Lemma 3.7 for $\widehat{u}_n$ and proceed as in [13]. So the argument below will be rather sketchy. Let $x_n \in B_{R_n}$ be such that

$$\widehat{u}_n(x_n) > u_\Lambda^{R_n}(x_n). \tag{3.21}$$

Then as in the proof of Theorem 4.6 in [13], we can use Lemmas 3.8 and 3.9 to conclude that $\{x_n\}$ is bounded. Let B be a finite closed ball in $\mathbb{R}^N$ that contains all x_n and suppose by contradiction that $\widehat{U}_\lambda = U_\lambda$. Then we have

$$\widehat{u}_n(x) \to U_\lambda(x) \tag{3.22}$$

uniformly on any bounded set of $\mathbb{R}^N$.

On the other hand, since $\Lambda > \lambda > 0$ and B is compact, there exists $\varepsilon > 0$ such that, $U_\Lambda(x) \geq U_\lambda(x) + \varepsilon$ for all $x \in B$. Therefore, for all large n, $u_\Lambda^{R_n}(x) \geq U_\lambda(x) + (1/2)\varepsilon$ for all $x \in B$. It follows from (3.21) that, $\widehat{u}_n(x_n) > u_\Lambda^{R_n}(x_n) \geq U_\lambda(x_n) + (1/2)\varepsilon$ for all $n \geq 1$. But this contradicts (3.22). This proves $\widehat{U}_\lambda \not\equiv U_\lambda$.

It remains to consider the case $\lambda \leq 0$. Fix $\lambda \leq 0$ and consider $\widehat{v}_n := \widehat{u}_\lambda^{R_n}$, whose existence is guaranteed by Lemma 3.7. As in the previous case, we can apply Theorem 3.4 to conclude that $\|\widehat{v}_n\|_{L^\infty(B_{R_n})}$ has a bound independent of n. It follows as before that, up to a subsequence, $\widehat{v}_n$ converges to some $\widehat{V}_\lambda$ uniformly on compact subsets of $\mathbb{R}^N$. Moreover, $\widehat{V}_\lambda$ is a nonnegative solution of (3.1). As in the proof of Theorem 4.6 in [13], due to property (a) (as in Lemma 3.7) of $\widehat{v}_n$, we can make use of Lemmas 3.8, 3.9 and Remark 3.10 to show that $\widehat{V}_\lambda \not\equiv 0$. By the strong maximum principle, we must have $\widehat{V}_\lambda > 0$ in $\mathbb{R}^N$. Hence it is a positive solution of (3.1). This completes the proof of Theorem 3.11. □

Remark 3.12. The alternative proof presented at the end of the proof of Theorem 4.6 in [13] for the case $\lambda < 0$ can be easily adapted to our situation here. Therefore, the conditions in Theorem 3.6 are enough for the existence of a positive solution for (3.1) with $\lambda < 0$.

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