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A NOTE ON ADDITIONAL PROPERTIES OF SIGN CHANGING SOLUTIONS TO SUPERLINEAR ELLIPTIC EQUATIONS

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Abstract. We obtain upper bounds for the number of nodal domains of sign changing solutions of semilinear elliptic Dirichlet problems using suitable min-max descriptions. These are consequences of a generalization of Courant's nodal domain theorem. The solutions need not to be isolated. We also obtain information on the Morse index of solutions and the location of sub- and supersolutions.

1. Introduction

In this paper we are concerned with sign changing solutions of the semilinear Dirichlet problem

$$\begin{cases} -\Delta u = f(x,u) & \text{for } x \in \Omega, \\ u = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{1.1}$$

on a smooth, bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$. We make the following assumptions:

(f_1) $f \in \mathcal{C}^1(\Omega \times \mathbb{R}, \mathbb{R})$, $f(x,0) = 0$ for all $x \in \Omega$.

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(f_2) There exists $p \in (2, 2N/(N-2))$, resp. $p \in (2,\infty)$ in case $N = 2$, such that

$$|f'(x,t)| \le C(1 + |t|^{p-2}) \quad \text{for all } x \in \Omega,\ t \in \mathbb{R},$$

where $f' := \partial f/\partial t$.

(f_3) $f'(x,t) > f(x,t)/t$ for all $x \in \Omega,\ t \ne 0$.

(f_4) There exist $R > 0$ and $\theta > 2$ such that $0 < \theta F(x,t) \le tf(x,t)$ for all $x \in \Omega,\ |t| \ge R$.

Here $F(x,t) := \int_0^t f(x,s)\,ds$ is a primitive of f. Clearly these assumptions hold for

$$f(x,t) = \sum_{i=1}^{d} a_i(x)|t|^{p_i-2}t, \tag{1.2}$$

where $2 \le p_1 < \ldots < p_d < 2N/(N-2)$, $a_1,\ldots,a_d$ are bounded nonnegative $\mathcal{C}^1$-functions and a_d is bounded from below by a positive constant. In fact, $a_i \in L^\infty$ is sufficient, since the differentiability of f with respect to x is not necessary in (f_1).

Problem (1.1) has been studied extensively, and much progress has been made recently concerning the existence of sign changing solutions, see [1]–[4], [8], [10], [11], [14], [15]. The aim of this paper is to gain further information on sign changing solutions, in particular on the nodal structure, extremality properties and the Morse index with respect to the energy functional

$$\Phi\colon E := H_0^1(\Omega) \to \mathbb{R}, \quad \Phi(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2\,dx - \int_\Omega F(x,u(x))\,dx.$$

It is a well known consequence of (f_1) and (f_2) that $\Phi \in \mathcal{C}^2(E)$ and that critical points of Φ are weak solutions of (1.1). Let $\lambda_1 < \lambda_2 \le \ldots$ be the Dirichlet eigenvalues of the operator $-\Delta - f'(x,0)$ on Ω. Our main results are the following:

Theorem 1.1. *Suppose* (f_1)–(f_4) *hold and* $\lambda_2 > 0$. *Then* (1.1) *has a sign changing solution* $\overline{u}$ *with the following properties:*

(a) $\overline{u}$ *has precisely two nodal domains.*
(b) $\overline{u}$ *has Morse index* 2.
(c) *If* $u < \overline{u}$ *is a subsolution of* (1.1), *then* $u \le 0$.
(d) *If* $u > \overline{u}$ *is a supersolution of* (1.1), *then* $u \ge 0$.

Here and in the following we write $u < v$ if $u \le v$ but $u \ne v$. Moreover, $u \in \mathcal{C}^2(\overline{\Omega}) \cap E$ is called a subsolution if $-\Delta u \le f(x,u)$, and a supersolution if $-\Delta u \ge f(x,u)$.

Theorem 1.1 improves a result of the first author ([1], see also Bartsch *et al.* [2]) as well as one of Castro, Cossio and Neuberger ([8]). In [8] no extremality properties of the form (c) and (d) are considered. Note also that the approach of [8] requires $\lambda_1 > 0$ which we do not need. Moreover, the statement on the Morse index of $\overline{u}$ has been proved in [8] only under the condition that $\overline{u}$ is isolated. In [1] properties (c) and (d) are established for $\overline{u}$, whereas (a) and (b) could only be shown under the hypothesis that all sign changing solutions of (1.1) are isolated – a hypothesis which is generic (see [7]) but can almost never be checked. We do not need such a hypothesis here, not even for the calculation of the Morse index of $\overline{u}$.

Theorem 1.2. *Suppose* (f_1)–(f_4) *hold and f is odd in u. Then there exists a sequence of distinct solutions $\pm u_k$, $k \geq \min\{l : \lambda_l > 0\}$, of* (1.1) *with the properties:*

(a) $\|u_k\|_E \to \infty$ *as* $k \to \infty$.
(b) u_k *changes sign for* $k \geq 2$.
(c) u_k *has at most k nodal domains.*
(d) *If $u < u_k$ is a subsolution of* (1.1), *then* $u \leq 0$.
(e) *If $u > u_k$ is a supersolution of* (1.1), *then* $u \geq 0$.

Theorem 1.2 is an improvement of [1, Theorem 7.3, see also Theorem 1.1]; again we do not require the sign changing solutions of (1.1) to be isolated. However, this condition is needed to calculate the Morse index of u_k (cf. [1]).

The proofs of the above theorems are motivated by the approach used in [1]. In particular, we recover the extremality properties using the same idea. However, we construct the critical value $\Phi(\overline{u})$ of the solution $\overline{u}$ in Theorem 1.1 in a somewhat different way compared to [1]. This new version is important in order to obtain the additional properties (a) and (b) which will be deduced by a closer investigation of the construction of $\Phi(\overline{u})$. To be more precise, consider the set

$$M := \{u \in E : u^+ \neq 0, u^- \neq 0, \Phi'(u)u^+ = \Phi'(u)u^- = 0\}.$$

Here we set $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$. The set M is not a manifold, and we do not expect it to be a complete metric space if $\lambda_1 < 0$. The condition $\lambda_1 > 0$ required in [8] implies that M is a closed subset of E, hence a complete metric space. Obviously, it contains all sign changing solutions of (1.1). Now put

$$\beta := \inf_{u \in M} \Phi(u). \tag{1.3}$$

We will show that $\Phi(\overline{u}) = \beta$, i.e. $\overline{u}$ is a least energy sign changing solution. Thus we also obtain that the infimum of Φ on M is achieved by a critical point of Φ. We then conclude by the following result.

Theorem 1.3. *Suppose* (f_1)–(f_4) *hold. Then every weak solution $u \in M$ with $\Phi(u) = \beta$ has Morse index* 2 *and has precisely* 2 *nodal domains.*

In fact, the nodal property is not difficult to prove, and it has already been used in [8] in a special case. The statement on the Morse index is new.

Property (c) in Theorem 1.2 also follows from an investigation of critical levels. We need the following nodal estimate which is valid for odd nonlinearities.

Theorem 1.4. *Suppose* (f_1)–(f_4) *hold, and that f is odd in u. For $n \in \mathbb{N}$ put*

$$\beta_n = \inf_{\substack{V \le E \\ \dim V \ge n}} \sup \Phi(V). \tag{1.4}$$

Then every weak solution $u \in E$ of (1.1) *with $\Phi(u) \le \beta_n$ has at most n nodal domains.*

This estimate holds under considerably weaker conditions than (f_1)–(f_4), see Section 2 below. It seems to be more useful then the well-known nodal estimates in terms of the Morse index, cf. [5], [2] and [1]. Note in particular that Theorem 1.4 immediately provides the nodal properties proven in [5]. Note also that (1.4) bears a resemblance to the variational characterization of the n-th eigenvalue of $-\Delta - g$ in case that $f(x,u) = g(x)u$, i.e. in case that f is linear in u. Due to this similarity Theorem 1.4 may be viewed as a nonlinear version of Courant's nodal domain theorem; cf. [9].

The paper is organized as follows. In Section 2 we prove the nodal estimates contained in Theorem 1.3 and Theorem 1.4. In Section 3 we calculate the Morse index of least energy sign changing solutions. Section 4 contains the proofs of Theorem 1.1 and Theorem 1.2.

2. Estimates for the number of nodal domains

In this section we replace the assumptions (f_1)–(f_4) by the following weaker hypotheses.

- (A_1) $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function and $f(x,0) = 0$ for all $x \in \Omega$.
- (A_2) There exist $p \in (2, 2N/(N-2)]$, resp. $p \in (2,\infty)$ in case $N = 2$, and $C > 0$ such that $|f(x,t)| \le C(|t| + |t|^{p-1})$ for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$.
- (A_3) The function $t \mapsto f(x,t)/|t|$ is nondecreasing on $\mathbb{R} \setminus \{0\}$ for a.e. $x \in \Omega$.

We will sometimes also need the following stronger variant of (A_3).

- $(\widetilde{\mathrm{A}}_3)$ The function $t \mapsto f(x,t)/|t|$ is strictly increasing on $\mathbb{R} \setminus \{0\}$ for a.e. $x \in \Omega$.

Note that for nonlinearities of the form (1.2) condition (A_3) is a consequence of the sign condition

$$a_i(x) \ge 0 \quad \text{for } x \in \Omega,\ i = 1, \dots, d.$$

In view of (A_2) the nonlinearity f may have critical growth at infinity. Nevertheless (A_2) ensures that every weak solution u of (1.1) is at least continuous in Ω. Indeed, combining (A_2) with Sobolev embeddings, we deduce that

$$\frac{f(\,\cdot\,,u(\,\cdot\,))}{u} \in L^{N/2}(\Omega).$$

Hence the Brézis–Kato Theorem ([6]) yields $u \in L^q_{\mathrm{loc}}(\Omega)$ for every $2 \le q < \infty$. In particular there is a number $s > N/2$ such that $f(\,\cdot\,,u(\,\cdot\,)) \in L^s_{\mathrm{loc}}(\Omega)$, thus u is continuous by elliptic regularity.

Our nodal estimates are based on the following lemma.

LEMMA 2.1. *Suppose* (A_1)–(A_3) *hold, and consider* $u \in E \setminus \{0\}$ *with* $\Phi'(u)u = 0$. *Then*

$$0 \le \Phi(u) = \sup_{t \ge 0} \Phi(tu). \tag{2.1}$$

If in addition $(\widetilde{\mathrm{A}}_3)$ *is satisfied, then* $\Phi(u) > 0$.

PROOF. For $t \ge 0$ we define $h(t) := \Phi(tu)$. Then $h(0) = 0$, and

$$h'(t) = \Phi'(tu)u = t \int_\Omega \left(|\nabla u|^2 - \frac{f(x,tu)}{tu} u^2 \right)$$

for $t > 0$. Hence (A_3) implies that $t \mapsto h'(t)/t$ is nonincreasing on $(0,\infty)$, and thus the set $S := \{t > 0 : h'(t) = 0\}$ is a subinterval of $(0,\infty)$ which contains $t = 1$ by assumption. Let $b \le \infty$ be the right endpoint of S. Then h is strictly decreasing on (b,∞), whereas

$$0 \le \max_{t \in [0,b]} h(t) \le \max_{t \in S} h(t) = h(1).$$

This yields (2.1). If in addition $(\widetilde{\mathrm{A}}_3)$ holds, then $t \mapsto h'(t)/t$ is strictly decreasing on $(0,\infty)$. Hence $S = \{1\}$, and $h'(t) > 0$ for $0 < t < 1$. We conclude that $\Phi(u) = h(1) > h(0) = 0$, as claimed. $\square$

Now we consider the set M and the values β, β_n, $n \in \mathbb{N}$, as defined in the introduction.

THEOREM 2.2. *If* (A_1)–(A_3) *hold and* f *is odd in* u, *then every weak solution* $u \in E$ *of* (1.1) *with* $0 < \Phi(u) \le \beta_n$ *has at most* n *nodal domains.*

THEOREM 2.3. *Suppose* (A_1), (A_2) *and* $(\widetilde{\mathrm{A}}_3)$ *hold. Then every weak solution* $u \in M$ *of* (1.1) *with* $0 < \Phi(u) \le \beta$ *has precisely* 2 *nodal domains.*

It is easily seen that (f_1)–(f_3) imply (A_1), (A_2) and $(\widetilde{\mathrm{A}}_3)$, hence the nodal property asserted in Theorem 1.3 immediately follows from Theorem 2.3. Moreover, Theorem 1.4 follows from Theorem 2.2.

Proof of Theorem 2.2. Suppose in contradiction that u has more than n nodal domains. Being given a choice $\Omega_1, \dots, \Omega_n$ of such domains, we define functions $v_i \in E$, $v_i \neq 0$, $i = 1, \dots, n$, by

$$v_i(x) := \begin{cases} u(x) & \text{for } x \in \Omega_i, \\ 0 & \text{for } x \in \Omega \setminus \Omega_i. \end{cases} \tag{2.2}$$

It has been shown in [13, Lemma 1] that this defines elements of E. For $v := u - \sum_{i=1}^n v_i$ we have

$$0 < \Phi(u) = \Phi(v) + \sum_{i=1}^n \Phi(v_i).$$

Fixing an appropriate choice of $\Omega_1, \dots, \Omega_n$, we may assume that $\Phi(v) > 0$. Note also that

$$\Phi'(v_i)v_i = \Phi'(u)v_i = 0,$$

and since $\Phi(-v_i) = \Phi(v_i)$ we have

$$\Phi(v_i) = \sup_{t \in \mathbb{R}} \Phi(tv_i)$$

by Lemma 2.1. Hence, with V denoting the span of $v_1, \dots, v_n$, we obtain

$$\beta_n \le \sup_{w \in V} \Phi(w) = \sum_{i=1}^n \Phi(v_i) = \Phi(u) - \Phi(v) < \Phi(u).$$

This however contradicts the assumption $\Phi(u) \le \beta_n$. $\square$

Proof of Theorem 2.3. Suppose in contradiction that u has at least three nodal domains. We choose nodal domains Ω_1, Ω_2 such that $v_1 \ge 0$, $v_2 \le 0$ for the associated functions $v_1, v_2 \in H_0^1$ defined as in (2.2). Clearly $v_1 + v_2 \in M$, and the function $v := u - v_1 - v_2$ satisfies $\Phi'(v)v = 0$. This implies $\Phi(v) > 0$ by Lemma 2.1, hence $\beta \le \Phi(v_1 + v_2) < \Phi(u)$, contrary to the assumption. $\square$

3. The Morse index of sign changing solutions with least energy

We assume that (f_1)–(f_4) are in force throughout this section. Consider the Hilbert space $H := E \cap H^2(\Omega)$, endowed with the scalar product from $H^2(\Omega)$. Moreover, denote by $\|\cdot\|_H$ the induced norm. We need the following technical lemma concerning the functionals

$$Q_\pm : E \to \mathbb{R}, \quad Q_\pm(u) = \int_\Omega |\nabla u^\pm|^2 \, dx = \int_\Omega \nabla u \cdot \nabla u^\pm \, dx,$$
$$\Psi_\pm : E \to \mathbb{R}, \quad \Psi_\pm(u) = \int_\Omega f(x, u)u^\pm \, dx.$$

LEMMA 3.1.

(a) $Q_\pm$ *is differentiable at* $u \in H$ *with derivative* $Q'_\pm(u) \in E'$ *given by*

$$Q'_\pm(u)v = \int_{\pm u>0} ((-\Delta u)v + \nabla u \nabla v)\, dx.$$

(b) $Q_\pm|_H \in \mathcal{C}^1(H)$.

(c) $\Psi_\pm \in \mathcal{C}^1(E)$ *with derivative given by*

$$\Psi'_\pm(u)v = \int_\Omega f'(x, u^\pm)u^\pm v\, dx + \int_\Omega f(x, u^\pm)v\, dx.$$

PROOF. (a) Let $u, v \in H$, and consider for $t \neq 0$ the characteristic functions χ_j^t, $j = 1, 2, 3$, resp. χ_4 associated with the sets

$$\begin{aligned}
C_1^t &:= \{x \in \Omega \mid u(x) + tv(x) \geq 0,\ u(x) > 0\},\\
C_2^t &:= \{x \in \Omega \mid u(x) + tv(x) \geq 0,\ u(x) < 0\},\\
C_3^t &:= \{x \in \Omega \mid u(x) + tv(x) < 0,\ u(x) > 0\},\\
C_4 &:= \{x \in \Omega \mid u(x) = 0\},
\end{aligned}$$

respectively. Since $\nabla u = 0$ and $-\Delta u = 0$ a.e. on C_4 (see [12, Lemma 7.7], for instance), we have

$$\begin{aligned}
&\frac{1}{t}(Q_+(u + tv) - Q_+(u))\\
&\quad= \frac{1}{t}\int_\Omega \big(\nabla(u + tv) \cdot \nabla(u + tv)^+ - \nabla u \nabla u^+\big)\\
&\quad= \frac{1}{t}\left(\int_\Omega \big((-\Delta u)(u + tv)^+ + (\Delta u)u^+\big) + t\int_\Omega \nabla v \cdot \nabla(u + tv)^+\right)\\
&\quad= \frac{1}{t}\int_\Omega \chi_1^t\big((-\Delta u)(u + tv) + (\Delta u)u\big) + \int_\Omega \chi_1^t \nabla v \cdot \nabla(u + tv)\\
&\qquad+ \frac{1}{t}\int_\Omega \chi_2^t(-\Delta u)(u + tv) + \int_\Omega \chi_2^t \nabla v \cdot \nabla(u + tv)\\
&\qquad+ \frac{1}{t}\int_\Omega \chi_3^t(\Delta u)u + t\int_\Omega \chi_4 \nabla v \cdot \nabla v^+\\
&\quad= \int_\Omega \chi_1^t(-\Delta u)v + \int_\Omega \chi_1^t \nabla u \cdot \nabla v + o(1)
\end{aligned}$$

as $t \to 0$. The last equality is a consequence of the fact that $\chi_2^t, \chi_3^t \to 0$ pointwise a.e. on Ω for $t \to 0$, hence Lebesgue's theorem yields

$$\int_\Omega \chi_2^t((-\Delta u)v + \nabla v \cdot \nabla u) \to 0$$

and, using the definition of χ_2^t, χ_3^t,

$$\left|\int_\Omega (\chi_3^t - \chi_2^t)(\Delta u)\frac{u}{t}\right| \leq \int_\Omega (\chi_2^t + \chi_3^t)|\Delta u| \cdot |v| \to 0 \quad \text{for } t \to 0.$$

We conclude that

$$Q_+'(u)v = \lim_{t\to 0}\biggl(\int_\Omega \chi_1^t(-\Delta u)v + \int_\Omega \chi_1^t \nabla u\cdot\nabla v\biggr) = \int_{u>0}((-\Delta u)v + \nabla u\cdot\nabla v),$$

as claimed. The proof for Q_- proceeds analogously.

(b) For a sequence $u_n \to u \in H$ we have

$$\begin{aligned}
&|(Q_+'(u_n) - Q_+'(u))v| \\
&\quad = \biggl|\int_{u_n>0}((-\Delta u_n)v + \nabla u_n\cdot\nabla v) - \int_{u>0}((-\Delta u)v + \nabla u\cdot\nabla v)\biggr| \\
&\quad \le 2\,\|u_n - u\|_H\|v\|_H \\
&\qquad + \biggl|\int_{u\le 0<u_n}((-\Delta u)v + \nabla u\cdot\nabla v) + \int_{u_n\le 0<u}((-\Delta u)v + \nabla u\cdot\nabla v)\biggr| \\
&\quad \le 2\,\|u_n - u\|_H\cdot\|v\|_H \\
&\qquad + \biggl(\int_{u\le 0<u_n}(|\Delta u|^2 + |\nabla u|^2) + \int_{u_n\le 0<u}(|\Delta u|^2 + |\nabla u|^2)\biggr)^{1/2}\|v\|_H \\
&\quad = o(1)\|v\|_H,
\end{aligned}$$

using again that $\nabla u = 0$ and $\Delta u = 0$ a.e. on the zero set of u. Thus $(Q_+|_H)'$ is continuous, and the proof is complete for Q_+. The proof for Q_- proceeds analogously.

(c) This can be proved similarly as part (a). □

LEMMA 3.2. *The set $M\cap H$ is a $\mathcal{C}^1$-manifold of codimension two in H.*

Note that we do not claim that $M\cap H$ is complete in H. In fact, we do not expect this to be true in case that $\lambda_1 < 0$.

PROOF OF LEMMA 2.3. Define

$$g_\pm : E \to \mathbb{R}, \quad g_\pm(u) = \Phi'(u)u^\pm,$$

so that $M\cap H = \{u\in H : u^+ \ne 0,\ u^- \ne 0,\ g_+(u) = 0 = g_-(u)\}$.

Lemma 3.1 implies that $g_\pm|_H \in \mathcal{C}^1(H)$. For $u \in M\cap H$ we obtain

$$\begin{aligned}
g_+'(u)u^+ &= \int_\Omega \bigl(|\nabla u^+|^2 - f'(x,u)(u^+)^2\bigr), \quad g_+'(u)u^- = 0, \\
g_-'(u)u^- &= \int_\Omega (|\nabla u^-|^2 - f'(x,u)(u^-)^2), \quad g_-'(u)u^+ = 0.
\end{aligned}$$

Hence (f$_3$) yields $g_+'(u)u^+ < 0$ and $g_-'(u)u^- < 0$ for $u\in M\cap H$. Approximating u^+ and u^- by functions in H, we conclude that $(g_+'(u), g_-'(u)) \in \mathcal{L}(H,\mathbb{R}^2)$ is onto for every $u\in M\cap H$. From this the assertion follows. □

In the following, if $u\in E$ is a critical point of Φ, we denote by $m(u)$ the Morse index of u.

PROPOSITION 3.3. *Let $u \in M$ be a critical point of Φ with $\Phi(u) = \beta$. Then $m(u) = 2$.*

PROOF. By (f_3) there holds

$$\Phi''(u)(u^\pm, u^\pm) = \int_\Omega \big(|\nabla u^\pm|^2 - f'(x,u)(u^\pm)^2\big) < 0,$$

hence $m(u) \geq 2$. To show $m(u) \leq 2$, note first that $u \in H$ by elliptic regularity. Denote by $T \subset H$ the tangent space of the manifold $M \cap H$ at u. We show that

$$\Phi''(u)(v,v) \geq 0 \quad \text{for all } v \in T.$$

Indeed, by Lemma 3.2 there exists for every $v \in T$ a C^1-curve $\gamma: [-1,1] \to M \cap H$ such that $\gamma(0) = u$ and $\dot\gamma(0) = v$. Since $\Phi'(u)v = 0$, we calculate that $\Phi \circ \gamma: [-1,1] \to \mathbb{R}$ is even twice differentiable at $t = 0$ with derivative

$$\frac{\partial^2}{\partial t^2} \Phi \circ \gamma|_{t=0} = \Phi''(u)(v,v).$$

Recalling that $\Phi(u) = \min_{v \in M \cap H} \Phi(v)$, we infer that $\partial^2(\Phi \circ \gamma)/\partial t^2|_{t=0} \geq 0$, and hence (3.1) follows. Since $T \subset H$ has codimension two and H is dense in E, we conclude $m(u) \leq 2$, as required. $\square$

4. Proofs of Theorems 1.1 and 1.2

As in the last section we assume that (f_1)–(f_4) are in force. We first recall that

$$\lim_{t \to \infty} \Phi(tu) = -\infty \tag{4.1}$$

for every $u \in E \setminus \{0\}$. Indeed this is a well known consequence of (f_4).

Next we recall some notation from [1]. We set $X := \{u \in \mathcal{C}^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$ and

$$\Phi^c := \{u \in E : \Phi(u) \leq c\}, \quad \Phi^c_X := X \cap \Phi^c$$

and

$$\mathcal{S}_c := \{u \in E : \Phi(u) = c,\ \Phi'(u) = 0\}.$$

Let P denote the closed cone of nonnegative functions in X. As in [1] we consider the sets

$$\begin{aligned} \mathcal{SC}^- &:= \{u \in X : u \text{ is a sign changing subsolution of (1.1)}\}, \\ \mathcal{SC}^+ &:= \{u \in X : u \text{ is a sign changing supersolution of (1.1)}\}, \end{aligned}$$

as well as

$$I := \{(0,0)\} \cup \{(u,v) \in \mathcal{SC}^- \times \mathcal{SC}^+ : u < v\} \subset X \times X$$

and

$$A := \bigcup_{(u,v)\in I} ((u+P)\cup(v-P)) \subset X.$$

Note that any critical point $u \notin A$ of Φ is a minimal element of $\mathcal{SC}^-$ and a maximal element of $\mathcal{SC}^+$, i.e. it has the properties (c) and (d) of Theorem 1.1.

It is well known that the gradient of Φ has the form $\nabla\Phi = Id - K$ with $K\colon E \to E$ being compact and strongly order preserving. Due to (f_3) we may use the usual scalar product $\langle u,v\rangle_E = \langle \nabla u, \nabla v\rangle_{L^2} + \langle u,v\rangle_{L^2}$ in $E = H_0^1(\Omega)$ for the definition of the gradient vector field. Integrating $-\nabla\Phi$ we obtain a flow $\phi\colon \mathcal{O} \to E$ defined on an open subset $\mathcal{O} \subset \mathbb{R}\times E$ and satisfying

$$\begin{cases} \dfrac{\partial}{\partial t}\phi(t,u) = -\nabla\Phi(\phi(t,u)), \\ \phi(0,u) = u, \end{cases}$$

for all $(t,u)\in\mathcal{O}$. We shall sometimes write ϕ^t instead of $\phi(t,\,\cdot\,)$. With the help of this flow the following deformation type lemma can be shown (see [1, p. 136]):

Lemma 4.1. *Suppose that $\mathcal{S}_c \subset A$ for some $c > 0$. Then there is an $\varepsilon > 0$ and a homotopy $h\colon \big(\Phi_X^{c+\varepsilon}\cup A\big)\times[0,1] \to \Phi_X^{c+\varepsilon}\cup A$ such that*

(a) $h_t(\Phi_X^d \cup A) \subset \Phi_X^d \cup A$ *for all* $d \le c+\varepsilon$, $t\in[0,1]$.
(b) $h_1(\Phi_X^{c+\varepsilon}\cup A) \subset \Phi_X^{c-\varepsilon}\cup A$.

Now we have the necessary tools for the

Proof of Theorem 1.1. Let $e_1 \in P$ be the (up to normalization unique) first Dirichlet eigenfunction of $-\Delta u - f'(x,0)$ on Ω. First note that, since $\lambda_2 > 0$, there exists $r > 0$ and a $\mathcal{C}^1$-map $g^{st}\colon \langle e_1\rangle^\perp \cap B_r(0) \to \langle e_1\rangle$ such that for every $u = w + g^{st}(w) \in \mathrm{Graph}(g^{st})$ we have

$$\phi^t(u) \to 0 \quad \text{as } t\to\infty.$$

In fact, if $\lambda_1 > 0$ we may take $g^{st}\equiv 0$, whereas, in case $\lambda_1 \le 0$. $\mathrm{Graph}(g^{st})$ is the E-local stable manifold of 0 (which might be contained in a larger stable set if $\lambda_1 = 0$). Set

$$S^{st} := \{u = w + g^{st}(w) : w\in\langle e_1\rangle^\perp,\ \|w\|_E = r\} \subset E.$$

Observe that $\alpha := \inf\Phi(S^{st}) > 0$ and that

$$S^{st}\cap A = \emptyset, \tag{4.2}$$

the last fact being proved in [1, Lemma 4.5]. Put $\gamma := \alpha/2$ and consider the inclusion

$$j_c\colon (\Phi_X^c \cup A, \Phi_X^\gamma \cup A) \hookrightarrow (E, E\setminus S^{st}) \quad \text{for any } c \ge \gamma,$$

which is well defined by (4.2). It induces a homomorphism

$$j_c^*\colon H^2(E, E\setminus S^{st}) \to H^2(\Phi_X^c\cup A, \Phi_X^\gamma\cup A).$$

Here and in the following $H^*(C,D)$ stands for the Alexander–Spanier cohomology of the pair $D\subset C$ with integer coefficients. Next we prove that

$$H^2(E,E\setminus S^{st})\cong\mathbb{Z}. \tag{4.3}$$

Setting $E_1:=\mathbb{R}e_1$ and $S_rE_1^\perp:=\{u\in E_1^\perp:\|u\|_E=r\}$, it is easy to see that the pair $(E,E\setminus S^{st})$ is homeomorphic to the pair $(E,E\setminus S_rE_1^\perp)$, hence

$$H^2(E,E\setminus S^{st})\cong H^2(E,E\setminus S_rE_1^\perp).$$

Now the pair $(E,E\setminus S_rE_1^\perp)$ is the same as the product pair $(E_1,E_1\setminus\{0\})\times(E_1^\perp,E_1^\perp\setminus S_rE_1^\perp)$. The Künneth theorem shows that

$$H^2(E,E\setminus S_rE_1^\perp)\cong H^1(E_1^\perp,E_1^\perp\setminus S_rE_1^\perp)\cong\widetilde{H}^0(E_1^\perp\setminus S_rE_1^\perp)\cong\mathbb{Z}$$

which proves (4.3).

Now we can define $\overline{c}:=\inf\{c\ge\gamma: j_c^* \text{ is injective}\}$. Then $\overline{c}\ge\alpha$, since $\Phi_X^c\cup A\subset E\setminus S^{st}$, hence $j_c^*=0$ for $c<\alpha$. Next we show

$$\overline{c}\le\beta \tag{4.4}$$

with β given by (1.3). For this let $\varepsilon>0$, and choose $u\in M$ such that $\Phi(u)<\beta+\varepsilon/2$. By Lemma 2.1 we have

$$\Phi(\lambda u^++\mu u^-)\le\Phi(u)\quad\text{for every } \lambda,\mu\ge 0.$$

By (4.1) there exists some number $R>0$ such that

$$\Phi(\lambda u^++\mu u^-)\le 0\quad\text{whenever }\max\{\lambda,\mu\}\ge R.$$

Approximating u^+ and $-u^-$ with suitable functions $v_1,v_2\in P$, we can achieve that

$$\begin{aligned}\Phi(\lambda v_1-\mu v_2)&\le\beta+\varepsilon &&\text{for } 0\le\lambda,\mu\le R,\\ \Phi(\lambda v_1-\mu v_2)&\le\gamma &&\text{if }\max\{\lambda,\mu\}\ge R.\end{aligned}$$

Now we consider the sets

$$\begin{aligned}C&:=\{\lambda v_1-\mu v_2:0\le\lambda,\ \mu\le R\}\subset\operatorname{span}\{v_1,v_2\}\subset X,\\ \partial C&:=\{\lambda v_1-\mu v_2\in C:\min\{\lambda,\mu\}=0\text{ or }\max\{\lambda,\mu\}=R\}.\end{aligned}$$

We have the following inclusions:

$$(C,\partial C)\overset{i}{\hookrightarrow}(\Phi_X^{\beta+\varepsilon}\cup A,\Phi_X^\gamma\cup A)\overset{j_{\beta+\varepsilon}}{\hookrightarrow}(E,E\setminus S^{st})$$

We claim that the induced map $i^*\circ j_{\beta+\varepsilon}^*\colon H^2(E,E\setminus S^{st})\to H^2(C,\partial C)$ is an isomorphism. Using the notation $E_1=\mathbb{R}e_1$ from above it is easy to construct a homeomorphism

$$h\colon(E,E\setminus S^{st})\to(E,E\setminus S_rE_1^\perp),$$

so that the map $h \circ j_{\beta+\varepsilon} \circ i$ is homotopic to the inclusion

$$i_0 \colon (C, \partial C) \hookrightarrow (E, E \setminus S_r E_1^\perp).$$

Next we choose $e_2 \in E_1^\perp$ with $\|e_2\|_E = 1$ and consider the sets

$$\begin{aligned} C_1 &:= \{\lambda e_1 + \mu e_2 : |\lambda| \le R,\ 0 \le \mu \le R\} = B_R E_1 \times [0, R] \cdot e_2, \\ \partial C_1 &= \{\lambda e_1 + \mu e_2 : |\lambda| = R \text{ or } \mu \in \{0, R\}\}. \end{aligned}$$

Clearly $(C, \partial C)$ may be deformed to $(C_1, \partial C_1)$ within $(E, E \setminus S_r E_1^\perp)$. This shows that $i^* \circ j^*_{\beta+\varepsilon}$ is an isomorphism if, and only if, the inclusion

$$i_1 \colon (C_1, \partial C_1) \hookrightarrow (E, E \setminus S_r E_1^\perp)$$

induces an isomorphism. Now

$$\begin{aligned} (C_1, \partial C_1) &\cong (B_R E_1, S_R E_1) \times ([0, R] \cdot e_2, \{0, R \cdot e_2\}), \\ (E, E \setminus S_r E_1^\perp) &\cong (E_1, E_1 \setminus \{0\}) \times (E_1^\perp, E_1^\perp \setminus S_r E_1^\perp). \end{aligned}$$

Since the inclusions

$$\begin{aligned} (B_R E_1, S_R E_1) &\hookrightarrow (E_1, E_1 \setminus \{0\}), \\ ([0, R] \cdot e_2, \{0, R \cdot e_2\}) &\hookrightarrow (E_1^\perp, E_1^\perp \setminus S_r E_1^\perp) \end{aligned}$$

induce isomorphisms on cohomology levels, the claim follows by the naturality of the Künneth maps.

Now since $i^* \circ j^*_{\beta+\varepsilon}$ is an isomorphism, $j^*_{\beta+\varepsilon}$ is injective, and thus $\overline{c} \le \beta + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, (4.4) holds true.

Next we assert that

$$\mathcal{S}_{\overline{c}} \not\subset A. \tag{4.5}$$

Indeed, if on the contrary $\mathcal{S}_{\overline{c}} \subset A$, then Lemma 4.1 yields $\varepsilon > 0$ and a homotopy $h \colon \big(\Phi_X^{\overline{c}+\varepsilon} \cup A\big) \times [0,1] \to \Phi_X^{\overline{c}+\varepsilon} \cup A$ such that $h_1^* \circ j^*_{\overline{c}-\varepsilon} = j^*_{\overline{c}+\varepsilon}$, where

$$h_1^* \colon H^2(\Phi_X^{\overline{c}-\varepsilon} \cup A, \Phi_X^\gamma \cup A) \to H^2(\Phi_X^{\overline{c}+\varepsilon} \cup A, \Phi_X^\gamma \cup A)$$

is induced by $h_1 \colon (\Phi_X^{\overline{c}+\varepsilon} \cup A, \Phi_X^\gamma \cup A) \to (\Phi_X^{\overline{c}-\varepsilon} \cup A, \Phi_X^\gamma \cup A)$. Hence, since $j^*_{\overline{c}+\varepsilon}$ is injective, $j^*_{\overline{c}-\varepsilon}$ has to be injective as well. This however contradicts the definition of $\overline{c}$, and thus (4.5) is proved.

Now let $\overline{u} \in \mathcal{S}_{\overline{c}} \setminus A$. Then $\overline{u}$ is a sign changing solution of (1.1) having the properties (c) and (d) of Theorem 1.1. In particular $\overline{u} \in M$, and therefore $\overline{c} = \Phi(\overline{u}) \ge \beta$. In fact, equality holds by (4.4), and hence the remaining properties (a) resp. (b) are established by Theorem 1.3. □

Proof of Theorem 1.2. The proof is strongly based on [1, Section 6], and we need to recall the definition of the critical values from [1]. First we set

$$k_0 := \min\{l : \lambda_l > 0\} - 1.$$

If $k_0 > 0$, then put $W := V^\perp$, where V is the generalized Dirichlet eigenspace of $-\Delta + f'(x,0)$ associated with the eigenvalues $\lambda_1, \dots, \lambda_{k_0}$. If $k_0 = 0$ set $W := \langle e_1 \rangle^\perp$, where e_1 is the eigenvector associated with λ_1. Then $d_0 := \operatorname{codim} W = \max\{1, k_0\}$. By the stable manifold theorem there exists a (Lipschitz) continuous map $g^{st}\colon B_r W = W \cap B_r(0) \to W^\perp$ for $r > 0$ small such that

$$S^{st} := \{u = w + g^{st}(w) : w \in W, \|w\|_E = r\} \tag{4.6}$$

is contained in the local stable manifold of 0. As in (4.2) we have

$$S^{st} \cap A = \emptyset; \tag{4.7}$$

cf. [1, Lemma 4.5]. Observe furthermore that $S^{st} = -S^{st}$ because Φ is even, hence g^{st} is odd.

Let h^* denote the Borel cohomology for the group $G = \mathbb{Z}/2$ with coefficient ring $h^*(pt) \cong \mathbb{F}_2[\omega]$. If $B \subset A$ are G-spaces, $A' \subset A$, $B' \subset B$ are invariant subspaces and $\xi \in h^*(A,B)$ then we write $\xi|_{(A',B')}$ for the image of ξ under the homomorphism $h^*(A,B) \to h^*(A',B')$ induced from the inclusion. Setting $\alpha := (1/2) \inf \Phi(S^{st}) > 0$ and using (4.7), we have an inclusion

$$j_c\colon (\Phi_X^c \cup A, \Phi_X^\alpha \cup A) \hookrightarrow (X, X \setminus S^{st}) \overset{\simeq}{\hookrightarrow} (E, E \setminus S^{st}) \tag{4.8}$$

for $c \geq \alpha$. According to [1, Lemma 6.1] there exists an element $\eta \in h^{d_0+1}(E, E \setminus S^{st})$ with the following property. If $R > 0$ satisfies $S^{st} \subset \operatorname{int}_E B_R(0)$ and if $Y \subset E$ is a finite-dimensional subspace with $d = \dim Y > \operatorname{codim} W = d_0$ then

$$0 \neq \omega^{d-d_0-1} \cdot \eta|_{(B_R Y, \{0\} \cup S_R Y)} \in h^d(B_R Y, \{0\} \cup S_R Y). \tag{4.9}$$

Using this cohomology class we may consider the values

$$c_k := \inf \left\{ c \geq \alpha : j_c^*(\omega^{k-d_0-1} \cdot \eta) \neq 0 \in h^k(\Phi_X^c \cup A, \Phi_X^\alpha \cup A) \right\} \tag{4.10}$$

for $k \geq d_0 + 1$. In [1] it is shown that there is a sequence of critical points $(u_k)_{k \geq d_0+1}$ of Φ satisfying properties (a), (b), (d) and (e) of Theorem 1.2 and such that $\Phi(u_k) = c_k$ for $k \geq d_0 + 1$. In view of Theorem 1.4 it therefore remains to prove

$$c_k \leq \beta_k \quad \text{for } k \geq d_0 + 1. \tag{4.11}$$

We fix k and recall that, as a consequence of (4.1), we find for any given k-dimensional subspace $Y \subset X$ a positive number $R > 0$ such that

$$\Phi(u) \leq 0 \quad \text{for } u \in Y,\ \|u\| \geq R.$$

Hence for $\beta := \max \Phi(Y)$ we have $(B_R Y, \{0\} \cup S_R Y) \subset (\Phi_X^\beta \cup A, \Phi_X^\alpha \cup A) \subset (X, X \setminus S^{st})$, and this implies $c_k \leq \beta$ by (4.9) and (4.10). Since $X \subset E$ is dense we conclude (4.11), as required. $\square$

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